Total Energy and Potential Enstrophy Conserving Scheme on a Z grid

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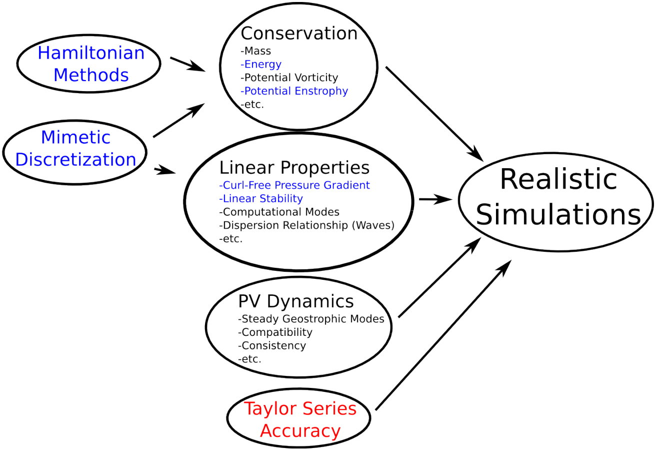
October 19th, 2015

Intro

Introduction

Elements of Dynamical Core Design



Standing on the Shoulders of Giants

Sommer and Nevir, 2009

Energy and Potential Enstrophy Conservation
Icosahedral grids
Staggered (ZC Grid)

My Work

Energy and Potential Enstrophy Conservation
Icosahedral grids
Unstaggered (Z Grid)

Salmon 2005, 2007

Energy and Potential Enstrophy Conservation
Planar Grids
Unstaggered (Z Grid)

Heikes and Randall 1994

No conservation
Icosahedral grids
Unstaggered (Z Grid)

Hamiltonian Approach to Conservation

Hamiltonian Approach to Conservation

Vorticity-Divergence Form of Equations

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h\vec{u}) = 0$$

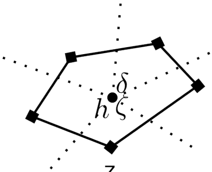
$$\frac{\partial \vec{u}}{\partial t} + q\hat{k} \times (h\vec{u}) + \vec{\nabla}\Phi = 0$$

$$\frac{\partial h}{\partial t} + \vec{\nabla}^2 \chi = 0$$

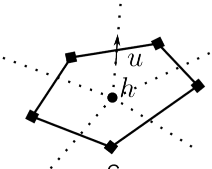
$$\frac{\partial \zeta}{\partial t} + \vec{\nabla} \cdot (q\vec{\nabla}\chi) - J(q,\psi) = 0$$

$$\frac{\partial \delta}{\partial t} - \vec{\nabla} \cdot (q\vec{\nabla}\psi) - J(q,\chi) + \vec{\nabla}^2\Phi = 0$$

$$h\vec{u} = \vec{\nabla}\chi + \vec{\nabla}^T\psi$$



Z



C

Nambu Bracket Form- Evolution Equations

$$\frac{d\mathcal{F}}{dt} = \{\mathcal{F},\mathcal{H},\mathcal{Z}\}_{\zeta\zeta\zeta} + \{\mathcal{F},\mathcal{H},\mathcal{Z}\}_{\delta\delta\zeta} + \{\mathcal{F},\mathcal{H},\mathcal{Z}\}_{\delta\zeta h}$$

$$\{\mathcal{F},\mathcal{H},\mathcal{Z}\}_{\zeta\zeta\zeta} = \int_\Omega d\Omega \mathcal{Z}_\zeta J(\mathcal{F}_\zeta,\mathcal{H}_\zeta)$$

$$\{\mathcal{F},\mathcal{H},\mathcal{Z}\}_{\delta\delta\zeta} = \int_\Omega d\Omega \mathcal{Z}_\zeta J(\mathcal{F}_\delta,\mathcal{H}_\delta)$$

$$\{\mathcal{F},\mathcal{H},\mathcal{Z}\}_{\delta\zeta h} = \int_\Omega d\Omega \Big(\vec{\nabla}\mathcal{Z}_h \cdot \vec{\nabla}\mathcal{F}_\delta \cdot \vec{\nabla}\mathcal{H}_\zeta \cdot \frac{1}{\vec{\nabla} q} - \vec{\nabla}\mathcal{Z}_h \cdot \vec{\nabla}\mathcal{F}_\zeta \cdot \vec{\nabla}\mathcal{H}_\delta \cdot \frac{1}{\vec{\nabla} q}\Big) + \mathrm{cyc}(\mathcal{F},\mathcal{H},\mathcal{Z})$$

All of these brackets are triply anti-symmetric

Nambu Bracket Form- Auxiliary Equations

Hamiltonian

$$-\chi = \frac{\delta \mathcal{H}}{\delta \delta}$$

$$-\psi = \frac{\delta \mathcal{H}}{\delta \zeta}$$

$$\Phi = \frac{\delta \mathcal{H}}{\delta h}$$

Potential Enstrophy

$$0 = \frac{\delta \mathcal{Z}}{\delta \delta}$$

$$q = \frac{\delta \mathcal{Z}}{\delta \zeta}$$

$$-\frac{q^2}{2} = \frac{\delta \mathcal{Z}}{\delta h}$$

Helmholtz Decomposition

$$\zeta = \vec{\nabla} \cdot (\frac{1}{h}\vec{\nabla}\psi) + J(\frac{1}{h}, \chi)$$

$$\delta = \vec{\nabla} \cdot (\frac{1}{h}\vec{\nabla}\chi) - J(\frac{1}{h}, \psi)$$

Conservation in Nambu Bracket Form

$$\frac{d\mathcal{H}}{dt} = \{\mathcal{H}, \mathcal{H}, \mathcal{Z}\} = -\{\mathcal{H}, \mathcal{H}, \mathcal{Z}\} = 0$$

$$\frac{d\mathcal{Z}}{dt} = \{\mathcal{Z}, \mathcal{H}, \mathcal{Z}\} = -\{\mathcal{Z}, \mathcal{H}, \mathcal{Z}\} = 0$$

1. Energy and potential enstrophy conservation rely ONLY on anti-symmetry of brackets
2. Independent of choice of $\mathcal{H}$ and $\mathcal{Z}$
3. Caveat: $\mathcal{Z}$ must cancel singularity in $\{\mathcal{F}, \mathcal{H}, \mathcal{Z}\}_{\delta\delta\zeta}$ bracket

What has been done?

Discretization Procedure

1. **Discretize Nambu Brackets**: enforce anti-symmetry $\rightarrow$ gives **L**,**FD** and **J** that conserve energy and potential enstrophy independent of discrete $\mathcal{H}$ and $\mathcal{Z}$
2. **Discretize $\mathcal{H}$ and $\mathcal{Z}$**: gives auxiliary equations for q_i and Φ_i, and elliptic problem for χ_i,ψ_i
3. **1 and 2 are done completely independently, with caveat about singularity cancellation in mixed bracket**

Discretization- Evolution Equations

Enforcing anti-symmetry in the discrete case yields (after a lot of algebra)

$$\frac{\partial h_i}{\partial t} + \mathbf{L}\chi_i = 0$$

$$\frac{\partial \zeta_i}{\partial t} + \mathbf{FD}(q_i, \chi_i) - \mathbf{J}(q_i, \psi_i) = 0$$

$$\frac{\partial \delta_i}{\partial t} - \mathbf{FD}(q_i, \psi_i) - \mathbf{J}(q_i, \chi_i) + \mathbf{L}\Phi_i = 0$$

- **L**, **FD** and **J** are the operators from Heikes and Randall 1995
-This approach conserves total energy and potential enstrophy INDEPENDENT of definitions for $\mathcal{H}$ and $\mathcal{Z}$
-Restricted to grids with orthogonal duals (no cubed sphere)
-Define auxiliary quantities χ_i, ψ_i, Φ_i and q_i from $\mathcal{H}$ and $\mathcal{Z}$

Discretization- Auxiliary Equations

Hamiltonian

$$-\chi_i = \frac{\delta \mathcal{H}}{\delta \delta_i}$$

$$-\psi_i = \frac{\delta \mathcal{H}}{\delta \zeta_i}$$

$$\Phi_i = \frac{\delta \mathcal{H}}{\delta h_i}$$

Potential Enstrophy

$$0 = \frac{\delta \mathcal{Z}}{\delta \delta_i}$$

$$\frac{\zeta_i}{h_i} = q_i = \frac{\delta \mathcal{Z}}{\delta \zeta_i}$$

$$-\frac{\zeta_i^2}{2h_i^2} = -\frac{q_i^2}{2} = \frac{\delta \mathcal{Z}}{\delta h_i}$$

Helmholtz Decomposition

$$\mathbf{A}\vec{x} = \vec{y}$$

with $\vec{x} = (\chi_i, \psi_i)^T$ and $\vec{y} = (\delta_i, \zeta_i)^T$

What has been done?

Discretization Procedure

1. **Discretize Nambu Brackets**: enforce anti-symmetry $\rightarrow$ gives **L**,**FD** and **J** that conserve energy and potential enstrophy independent of discrete $\mathcal{H}$ and $\mathcal{Z}$
2. **Discretize** $\mathcal{H}$ **and** $\mathcal{Z}$: gives auxiliary equations for q_i and Φ_i, and elliptic problem for χ_i,ψ_i
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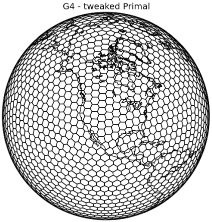
Test Results

Test Case Results

Model Configuration

Settings

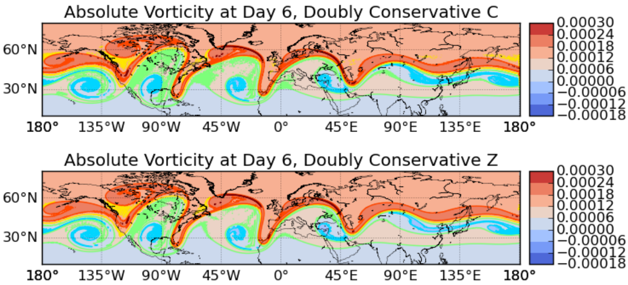
- Geodesic: G8, 640K cells (30km resolution), HR95 opt.
- 3rd Order Adams Bashford (22.5s time step)
- Multigrid used to solve elliptic problem
- Compared to a doubly conservative C grid scheme on the same grid



Galewsky et. al

Galewsky (Unstable Jet)

Galewsky et. al (Unstable Jet)- Z Grid Geodesic

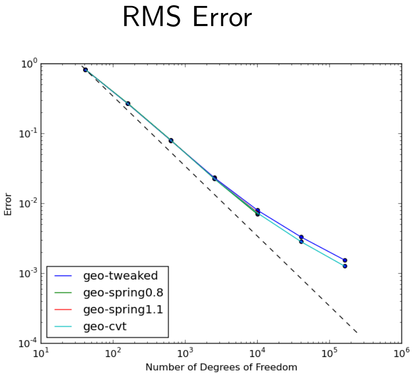


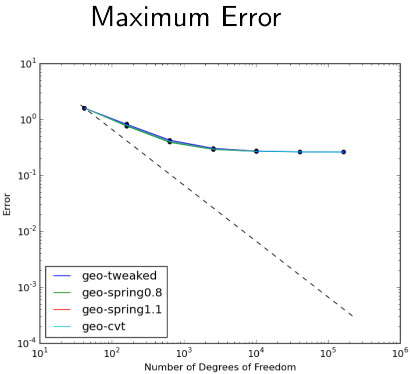
1. Almost identical results

Order of Accuracy

Order of Accuracy (Taylor Series Sense)

Order of Accuracy- Jacobian





Computed for $J(\alpha, \beta)$ where $\alpha = \cos^3\theta \sin 5\lambda$ and $\beta = \frac{-a^2}{2}\cos^3\theta \sin 3\lambda$

All grids are inconsistent

Thuburn Test

Thuburn (Forced Turbulence)

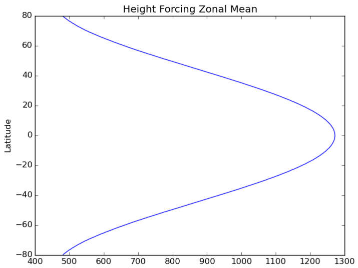
Thuburn (Forced Turbulence)- Description

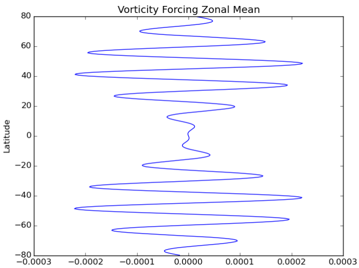
$$\frac{\partial h}{\partial t} = \cdots + \frac{h - h_{eqm}}{\tau_h}$$

Height Forcing

$$\frac{\partial \vec{u}}{\partial t} = \cdots + \frac{\vec{u} - \vec{u}_{eqm}}{\tau_u}$$

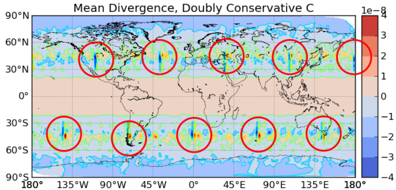
Vorticity Forcing



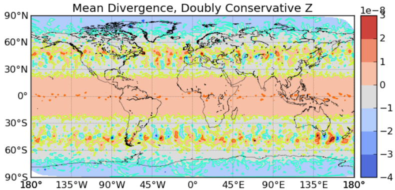


2400 days total run time: 400 days spin-up, 2000 days simulation
Run at C6/G6 resolution ($\approx$ 120km resolution)

Thuburn (Forced Turbulence)- Divergence (Z Grid)



C Grid Geodesic



Z Grid Geodesic

Z grid does not show the same grid imprinting

Possibly due to better behaviour of inconsistent Z grid operator (Jacobian)?

Summary and Conclusions

Conclusions

Summary

Summary

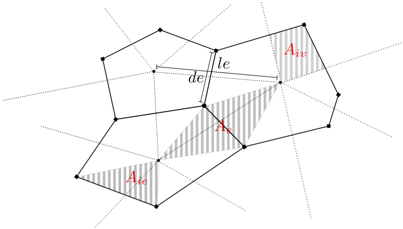
1. Galewsky results show little difference between C and Z grid schemes
2. Jacobian is inconsistent for all grids- Ross Heikes has preliminary work showing Laplacian and Flux Divergence are inconsistent as well
3. Long time tests (Thuburn forced turbulence) did not show strong grid imprinting that was present for C grid scheme

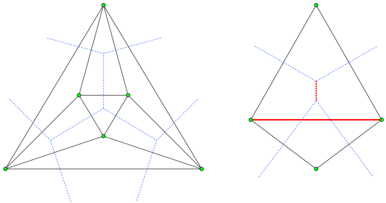
Future Work

1. Finite element version of Z grid scheme to fix accuracy issues, and extend scheme to more general grids

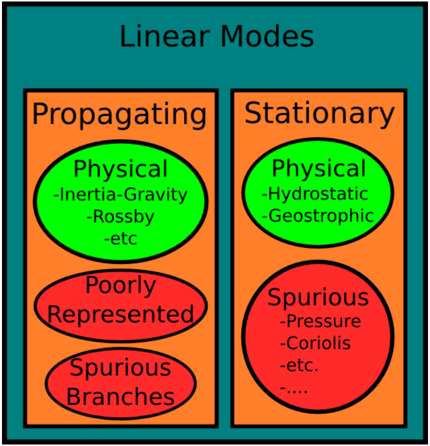
Appendix

Grid Geometry





Linear Modes



Actual Operators on Orthogonal Grid with Triangular Dual

$$\mathbf{L}(X) = \frac{1}{A_0} \sum_{e \in EC} \frac{le}{de} (X_i - X_0)$$

$$\mathbf{FD}(Y, X) = \frac{1}{A_0} \sum_{e \in EC} \frac{le}{de} (X_i - X_0) \frac{Y_i + Y_0}{2}$$

$$\mathbf{J}(Y, X) = \frac{1}{A_0} \sum_{e \in EC} \frac{Y_{i+1} - Y_{i-1}}{3} \frac{X_0 + X_i}{2}$$