Extension of the 1981 Arakawa and Lamb Scheme to Arbitrary Grids

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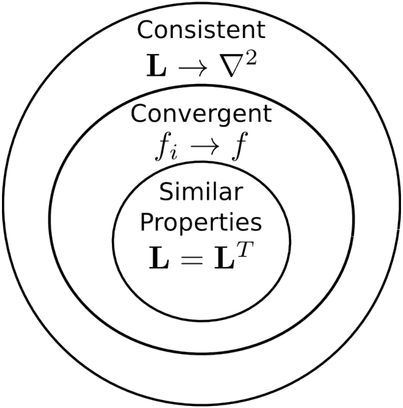
Intro

Introduction

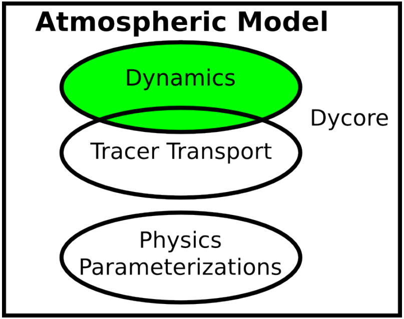
Key Principles of Numerical Modeling

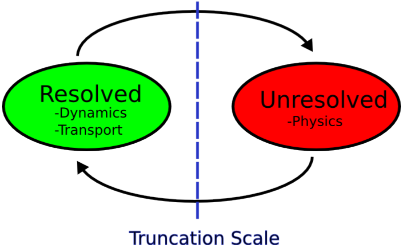
1. Not solving arbitrary PDEs- physical system
2. No analytic solutions
3. Differential equations $\rightarrow$ algebraic equations
4. **Do algebraic solutions have the same properties as the differential solutions?**



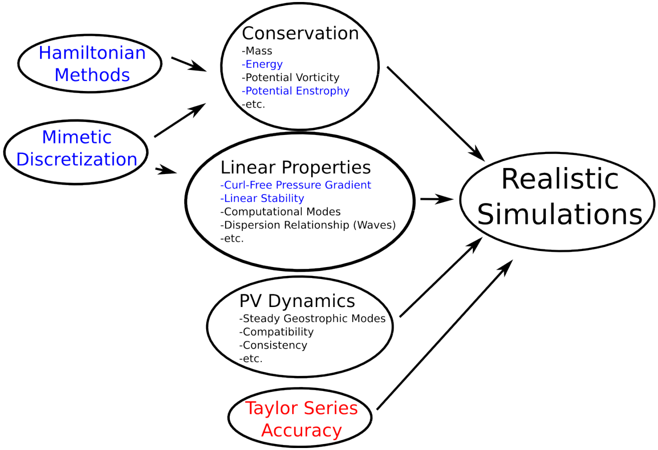
$$\Delta x, \Delta t \rightarrow 0$$

Broad Research Overview: Dynamical Cores





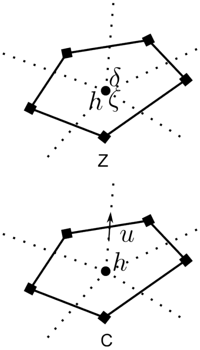
Elements of Dynamical Core Design



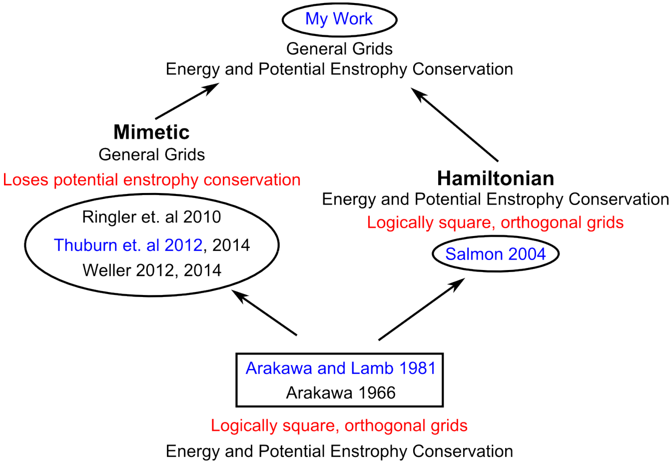
C Grid and Z Grid Schemes

C and Z Grid Comparison

1. C grid = staggered mass and wind
2. Z grid = collocated mass, vorticity and divergence
3. A major motivation of Z grid is to **avoid computational modes**; and have good wave dispersion properties **independent of Rossby radius**
4. Comes at the cost of solving elliptic (global) equations



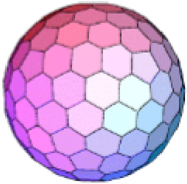
Standing on the Shoulders of Giants



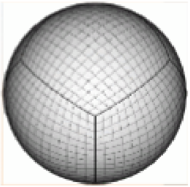
My Research

Dissertation Overview

1. **C Grid Scheme on Arbitrary Grids (Icosahedral and Cubed Sphere)**
2. Z Grid Scheme on Orthogonal Grids (Icosahedral)
3. Linear Modes: Stationary and Propagating (Inertia-Gravity Waves and Rossby Waves)
4. Test Cases- Williamson TC2 (Solid Body Rotation), TC5 (Flow over a Mountain), TC6 (Rossby-Haurwitz Wave); **Galewsky, Thuburn Forced-Dissipative; Order of Accuracy**



Icosahedral-hexagons



Cubed Sphere

Key Papers

Arakawa and Lamb 1981

A Potential Enstrophy and Energy Conserving Scheme for the Shallow Water Equations

AKIO ARAKAWA AND VIVIAN R. LAMB[1]

Salmon 2004

Poisson-Bracket Approach to the Construction of Energy- and Potential-Enstrophy-Conserving Algorithms for the Shallow-Water Equations

RICK SALMON

Thuburn, Cotter and Dubos 2012

A FRAMEWORK FOR MIMETIC DISCRETIZATION OF THE ROTATING SHALLOW-WATER EQUATIONS ON ARBITRARY POLYGONAL GRIDS*

J. THUBURN† AND C. J. COTTER‡

AL81 on arbitrary grids

Extension of AL81 to Arbitrary Grids

Shallow Water Equations

Vector Invariant Shallow Water Equations

Vector Calculus Formulation

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h\vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + q\hat{k} \times (h\vec{u}) + \vec{\nabla}\Phi = 0$$

Hamiltonian Formulation

$$\vec{x} = (h, \vec{u}) \; ; \; \frac{\partial \vec{x}}{\partial t} = \mathbb{J}\frac{\delta \mathcal{H}}{\delta \vec{x}}$$

$$\mathbb{J} = \begin{pmatrix} 0 & -\vec{\nabla}\cdot \\ -\vec{\nabla} & -q\hat{k}\times \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2}g(h, h) + \frac{1}{2}(h\vec{u}, \vec{u})$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi \\ h\vec{u} \end{pmatrix}$$

Shallow Water Equations: AL81 Scheme and Properties

Mimetic (Discrete Exterior Calculus, Thuburn et. al 2012)

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \qquad \bar{D}_2 \bar{D}_1 = 0$$

$$(\vec{\nabla})^* = -\vec{\nabla}\cdot \qquad D_2^T = -\bar{D}_1$$

Conservation (Hamiltonian Methods, Salmon 2004)

$$\mathcal{H} = \int_\Omega \frac{gh^2}{2} + \frac{h|\vec{u}|^2}{2} \qquad \mathbb{J} = -\mathbb{J}^T$$

$$\mathcal{Z} = \int_\Omega h\frac{q^2}{2} \qquad \mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

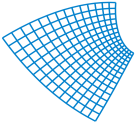
Arakawa and Lamb 1981 Scheme: Limitations and Extensions

(A Subset of) AL81 Shortcomings

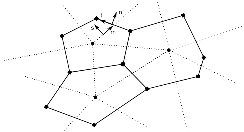
1. Restricted to logically square, orthogonal grids

TRiSK: Ringler, Skamarok, Klemp, Thuburn, Cotter, Dubos

1. Extension of AL81 to general, non-orthogonal polygonal grids
2. **Choose between total energy and potential enstrophy conservation**



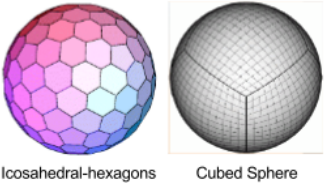
Logically square, orthogonal grid

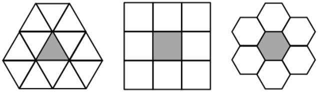


General, non-orthogonal grid

What am I trying to accomplish?

1. How can Arakawa and Lamb 1981 be extended to **arbitrary, non-orthogonal polygonal grids**?
2. Can the above be done in a way that preserves all of its desirable properties, and does not add new limitations?



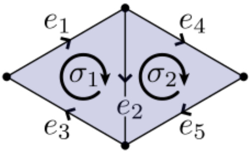


Mimetic Methods (Discrete Exterior Calculus)

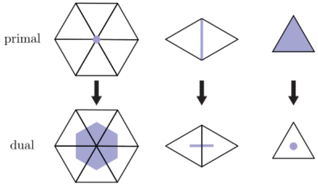
Mimetic Operators

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \iff \bar{D}_2 \bar{D}_1 = 0$$

$$(\vec{\nabla})^* = -\vec{\nabla}\cdot \iff D_2 = -\bar{D}_1{}^T$$



(Discrete) Exterior Derivative



Primal-Dual Grid
Mass lives on primal grid

Conservation Laws (Hamiltonian Mechanics)

Hamiltonian (Energy)

$$\mathbb{J}^T = -\mathbb{J}$$

$\mathcal{H}$ is positive definite

Casimirs (Potential Enstrophy)

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

Discrete Conservation

Enforce these conditions in discrete case $\rightarrow$ conservation!

Recap: Conservative, Mimetic Methods

Conservation Laws

$\mathcal{H}$ is positive definite

$$\mathbb{J} = -\mathbb{J}^T$$

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

Mimetic Properties

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \iff \bar{D}_2 \bar{D}_1 = 0$$

$$(\vec{\nabla})^* = -\vec{\nabla}\cdot \iff D_2 = -\bar{D}_1{}^T$$

Conservative, Mimetic Methods

1. Use mimetic operators to build a discrete (quasi-)Hamiltonian system

Generalized C Grid Discretization: Hamiltonian-DEC

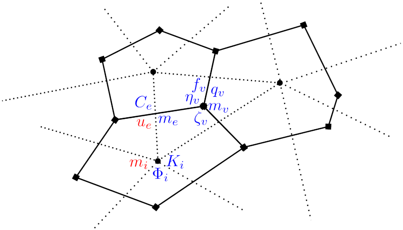
- Discrete variables are $m_i = \int h dA$ and $u_e = \int \vec{u} \cdot \vec{dl}$
- C grid staggering (m_i at cell centers, u_e at edges)

$$\vec{x} = (m_i, u_e)$$

$$\mathbb{J} = \begin{pmatrix} 0 & D_2 \\ \bar{D}_1 & \mathbf{Q} \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g(m_i, m_i)_{\mathbf{I}} + \frac{1}{2}(C_e, u_e)_{\mathbf{H}}$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \mathbf{I}\Phi_i \\ F_e \end{pmatrix} = \begin{pmatrix} \mathbf{I}K_i + g\mathbf{I}m_i \\ \mathbf{H}C_e \end{pmatrix}$$



Grid Staggering

Q operator is the remaining hurdle

What is **Q**?

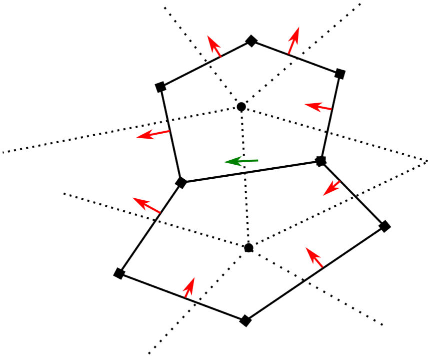


Diagram of Q operator action

1. $$q\hat{k}\times \rightarrow \mathbf{Q}$$
2. Given mass fluxes normal to primal edges and potential vorticities at vertices (black diamonds), produces PV fluxes normal to dual edges

General Form of Q

Following Salmon 2004, set

$$\mathbf{Q}F_e = \sum_{e' \in ECP(e)} \sum_{v \in VC(i)} q_v \alpha_{e,e',v} F_{e'}$$

What are $\alpha_{e,e',v}$'s?
Each $\alpha_{e,e',v}$ is associated with one green/red edge pair; and one blue vertex

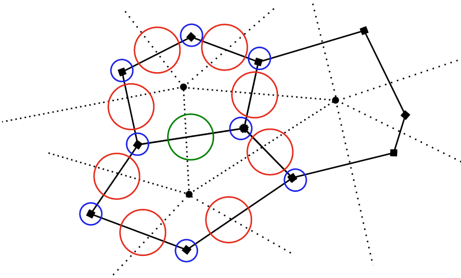


Diagram of Q operator stencil

Discrete Conservation

Energy

$$\mathbb{J}^T = -\mathbb{J} \longrightarrow \mathbf{Q} = -\mathbf{Q}^T \longrightarrow \alpha_{e,e',v} = -\alpha_{e',e,v}$$

Potential Enstrophy

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0 \longrightarrow \text{ linear system of equations } \longrightarrow \mathbf{A}\vec{\alpha} = \vec{b}$$

Also want $\mathbf{Q}$ to give steady geostrophic modes when q_v is constant

Solving $\mathbf{A}\vec{\alpha} = \vec{b}$

Issue: System is too large

- Geodesic grid: 90 coefficients per cell, all coefficients are interdependent $\rightarrow$ not feasible for realistic grids
- Cubed sphere grid is similar (24 coefficients per cell)

Solution: Subsystem Splitting

$$\mathbf{A}\vec{\alpha} = \vec{b} \longrightarrow \sum_i \mathbf{A}_i \vec{\alpha}_i = \vec{b}_i$$

Split into independent subsystems for each cell!

System has been solved for various planar and spherical grids

Gives AL81 on a uniform square grid

Recap: What have I accomplished?

What has been done?

- Arakawa and Lamb 1981 extended to arbitrary grids via new **Q**
- Coefficients can be precomputed (efficiently)

Hamiltonian

$\mathcal{H}$ is positive definite

$$\mathbb{J} = -\mathbb{J}^T$$

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

Mimetic

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \Longleftrightarrow \bar{D}_2 \bar{D}_1 = 0$$

$$(\vec{\nabla})^* = -\vec{\nabla}\cdot \Longleftrightarrow D_2 = -\bar{D}_1{}^T$$

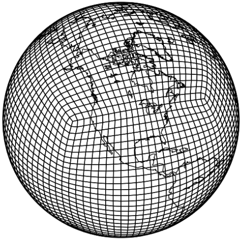
Test Results

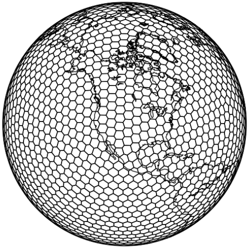
Test Case Results

Model Configuration

Settings

- Cubed Sphere: 6x384x384, 880K cells (25km resolution)
- Geodesic: G8, 640K cells (30km resolution)
- 3rd Order Adams Bashford (15s CS, 22.5s Geodesic)
- 3 Variants of **Q**



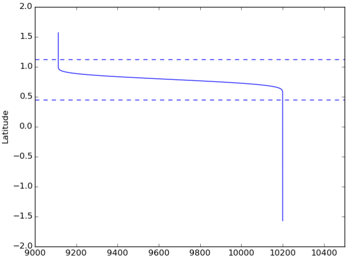


Galewsky et. al

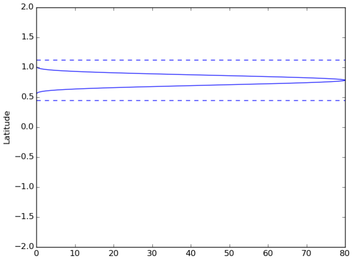
Galewsky (Unstable Jet)

Galewsky et. al- Description

Initial Height



Initial Wind (Zonal)

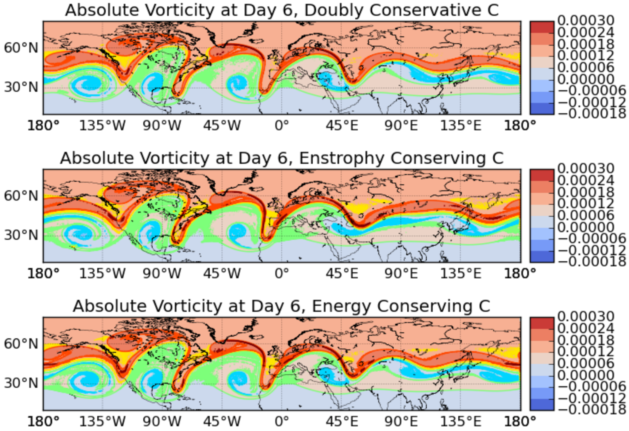


No dissipation

Run for 10 days, absolute vorticity at 6 days is shown

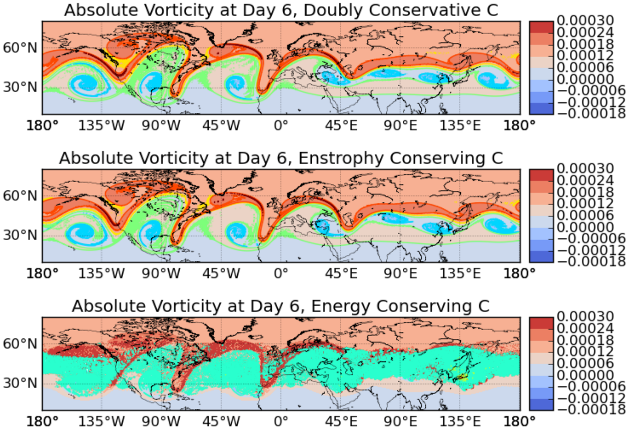
Perturbation added to (balanced) initial height field

Galewsky et. al (Unstable Jet)- C Grid Geodesic



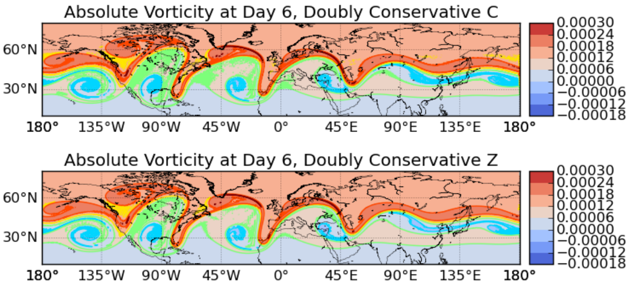
1. Inactive portion of jet differs
2. Minor differences in active portion of jet

Galewsky et. al (Unstable Jet)- C Grid Cubed Sphere



1. Inactive portion of jet differs
2. Total energy variant is unstable

Galewsky et. al (Unstable Jet)- Z Grid Geodesic

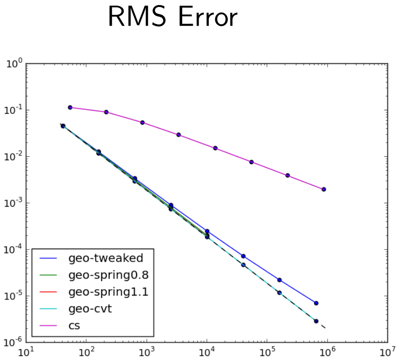


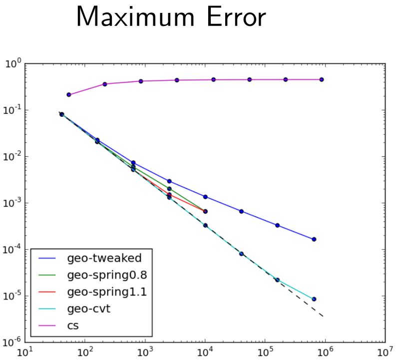
1. Almost identical results

Order of Accuracy

Order of Accuracy (Taylor Series Sense)

Order of Accuracy- Laplacian on Primal (C Grid)





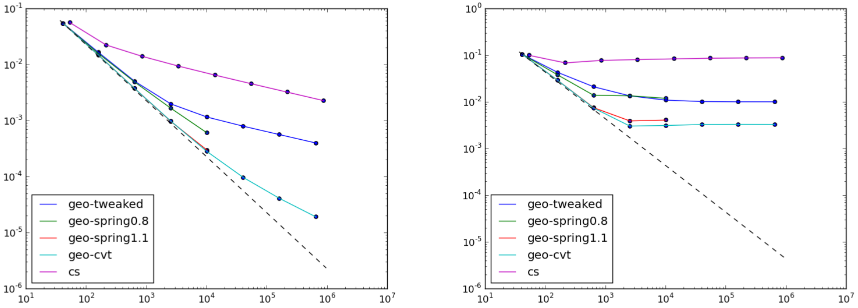
Computed for $\psi = \cos(\theta)\sin(\lambda)$

Cubed sphere is inconsistent, geodesic is $\approx$1st order

Order of Accuracy- **Q**

RMS Error

Maximum Error



Computed for $\psi = q = \cos(\theta)\sin(\lambda)$

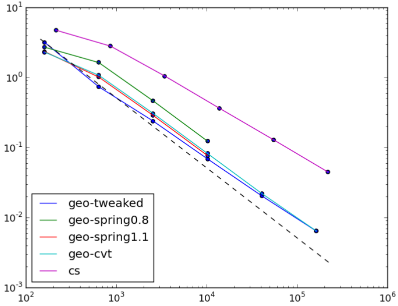
error $= q_e \bar{D}_1 \psi_e - \mathbf{Q}(q_v, D_1 \psi_v)$

error = exact PV flux from streamfunction vs computed PV flux

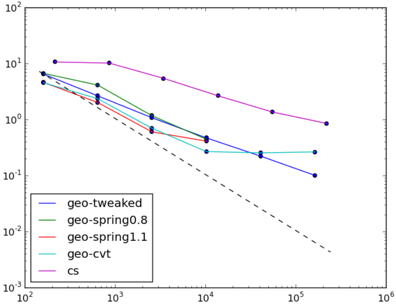
All grids are inconsistent

Order of Accuracy- Zonal Jet (Williamson Test Case 2, C Grid)

RMS Error



Maximum Error



Height errors
≈ 1st order for all grids

Thuburn Test

Thuburn (Forced-Dissipative Turbulence)

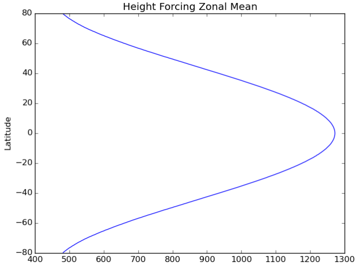
Thuburn (Forced-Dissipative Turbulence)- Description

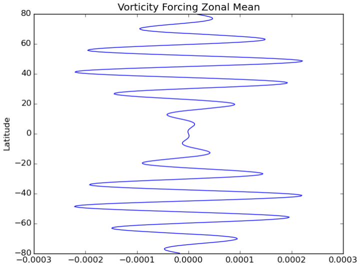
$$\frac{\partial h}{\partial t} = \cdots + \frac{h - h_{eqm}}{\tau_h}$$

Height Forcing

$$\frac{\partial \vec{u}}{\partial t} = \cdots + \frac{\vec{u} - \vec{u}_{eqm}}{\tau_h}$$

Vorticity Forcing

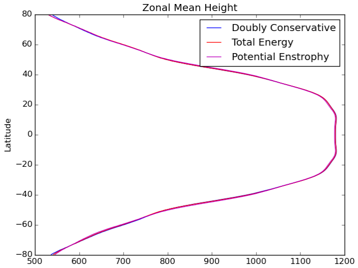


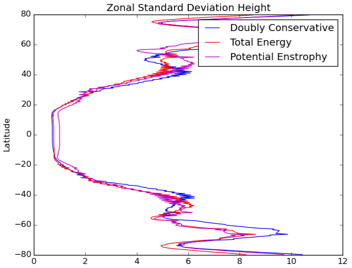


2400 days total run time: 400 days spin-up, 2000 days simulation

Run at C6/G6 resolution ($\approx$ 120km resolution)

Thuburn (Forced-Dissipative Turbulence)- Height (C Grid Geodesic)

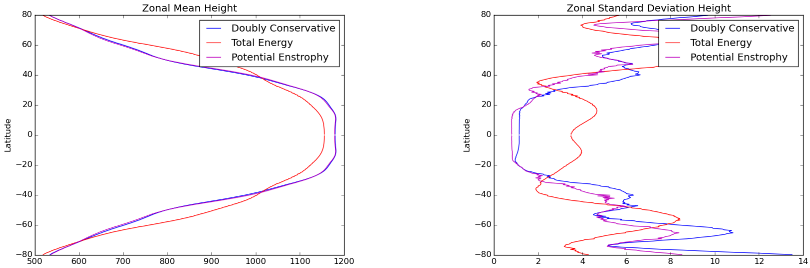




Zonal means are basically identical

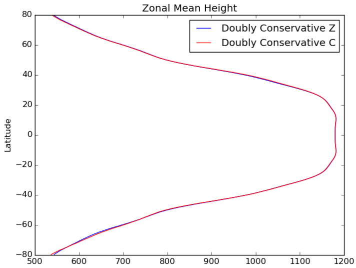
Differences (mid-latitudes) in zonal standard deviations

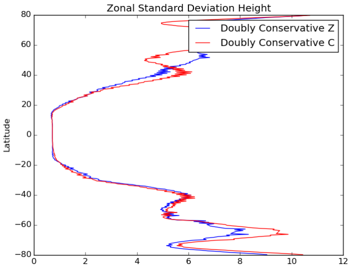
Thuburn (Forced-Dissipative Turbulence)- Height (C Grid Cubed Sphere)



Total energy variant has issues in zonal mean
Zonal standard deviations are different

Thuburn (Forced-Dissipative Turbulence)- Height (Z Grid Geodesic)

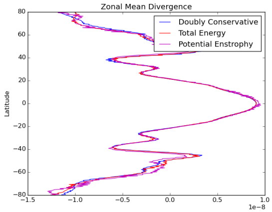




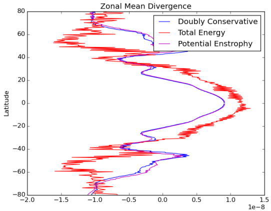
Identical zonal means

Differences in zonal standard deviations at mid-latitudes

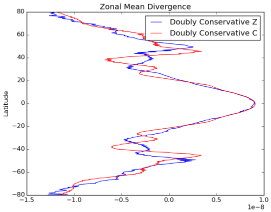
Thuburn (Forced-Dissipative Turbulence)- Divergence (Zonal Means, C and Z Grid)



C Grid Geodesic



C Grid Cubed Sphere



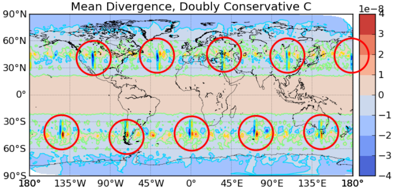
Z Grid Geodesic

Geodesic zonal means are almost identical

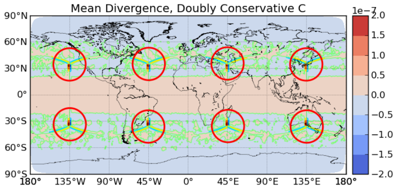
Again, cubed-sphere total energy variant is different

Z Grid and C grid markedly different

Thuburn (Forced-Dissipative Turbulence)- Divergence (C Grid)



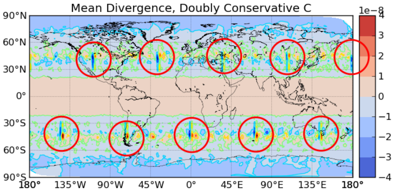
C Grid Geodesic



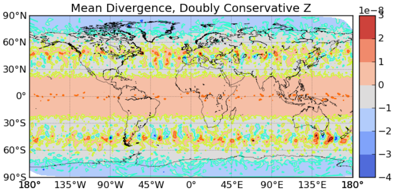
C Grid Cubed Sphere

Strong grid imprinting for ALL grids

Thuburn (Forced-Dissipative Turbulence)- Divergence (Z Grid)



C Grid Geodesic



Z Grid Geodesic

Z grid does not show the same grid imprinting

Possibly due to better behaviour of inconsistent Z grid operator (Jacobian)?

Summary and Conclusions

Conclusions

Summary

Summary

1. Galewsky results indicate all geodesic variants, and all but total energy cubed-sphere variants are producing good results
2. **Q** is inconsistent for all grids; Laplacians are inconsistent for cubed-sphere
3. Despite this, TC2 (Solid Body Rotation) is $\approx$ 1st order for all grids
4. Long time tests (Thuburn forced-dissipative) show strong grid imprinting NOT present in shorter (Galewsky) tests
5. The Z grid scheme does not appear to have any of these issues, but it is also inconsistent (Jacobian operator)

Fixing the C Grid Scheme

Primal-Dual Finite Elements

1. Recent work by John Thuburn and Colin Cotter
2. Low order, compound polygonal finite elements (same degrees of freedom) $\rightarrow$ all operators are consistent
3. Current version does not conserve energy or potential enstrophy
4. Can fix this with Hamiltonian approach

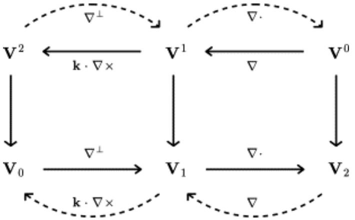
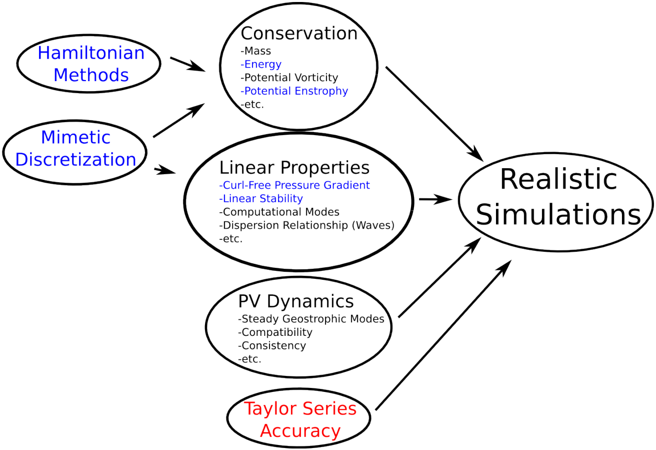


Figure from Thuburn and Cotter 2015. Bottom row is primal function spaces, top row is dual function spaces.

Elements of Dynamical Core Design- Redux



Conclusions

Conclusions

1. Hamiltonian + Discrete Exterior Calculus → AL81 on arbitrary polygonal grids
2. This C grid scheme on geodesic and cubed-sphere grids does not appear suitable as a basis for dynamical core development
3. This Z grid scheme does appear suitable, but inconsistency is troubling, although it can likely be fixed

Future Work

1. Consistent $\mathbf{Q}$ (Primal-Dual Finite Element)
2. Consistent Jacobian operator for Z grid (Finite Element)
3. Cubed sphere accuracy- Purser optimized grid
4. Fully conservative (implicit) time stepping

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Questions?