Structure Preserving Discretization of the Rotating Shallow Water Equations

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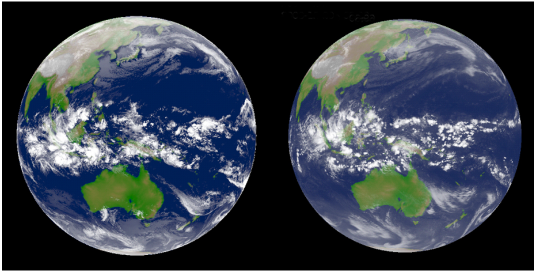
Broad Research Overview: What are dynamical cores?

1. Develop discrete models of the atmosphere (dynamical cores)
2. Dynamical core: deals with "resolved processes"
3. Parameterizations: deals with "unresolved processes"
4. Model: dynamical core + parameterizations



Which is the model, and which is reality? (from Miura et. al 2007)

Philosophy of Dynamical Core Development

1. Model development is a series of choices: equation sets, predicted variables, numerical methods, many others
2. Numerical model: Continuous equations $\rightarrow$ algebraic equations
3. Are solutions between these two the same?
4. $\rightarrow$ Mimetic methods, conservation properties

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \Longleftrightarrow \mathbf{CG}\vec{\phi} = 0$$

Mimetic Methods

Discrete Exterior Calculus

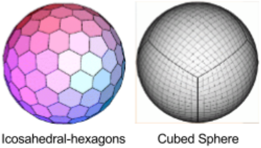
$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \iff \bar{D}_2 \bar{D}_1 = 0$$

$$\vec{\nabla} \cdot \vec{\nabla}^T \phi = 0 \iff D_2 D_1 = 0$$

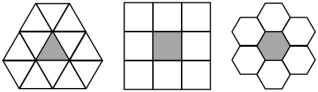
$$(\vec{\nabla})^* = -\vec{\nabla}\cdot \iff D_2 = \bar{D}_1{}^T$$

$$(\vec{\nabla}^T)^* = -\vec{\nabla}\times \iff \bar{D}_2 = -D_1^T$$

These are purely topological relations



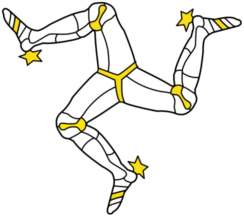
Spherical Grids



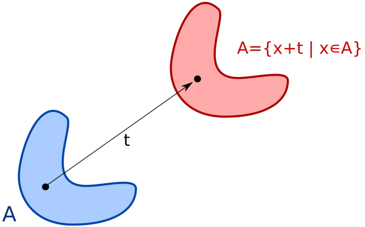
Planar Grids

Conservation Laws

1. Why are certain quantities conserved?
2. $\rightarrow$ Symmetry!
3. Ex. Translational symmetry $\leftrightarrow$ momentum conservation
4. Ex. Time symmetry $\leftrightarrow$ energy conservation
5. Noether's theorem: every continuous symmetry leads to conserved quantity (and visa-versa)



Rotational Symmetry



Translation Symmetry

Discrete Conservation Laws

1. **Problem:** Computers don't understand continuous symmetry
2. **Solution**: Computers understand anti-symmetry

$$\{A, B\} = -\{B, A\}$$

3. $\rightarrow$ Non-Canonical Hamiltonian Mechanics

$$\frac{\partial \vec{x}}{\partial t} = \mathbb{J} \frac{\delta \mathcal{H}}{\delta \vec{x}}$$

$$\mathbb{J} \frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

$\mathcal{H}$ - Hamiltonian (Energy)

$\mathcal{J}$ - Casimirs (PV, Potential Enstrophy)

$\mathbb{J}$- Antisymmetric bilinear operator (symplectic)

$\vec{x}$- state variables

Conservative, Mimetic Methods

Conservation Laws

1. Hamiltonian Mechanics
2. Salmon, Dubos, Gassmann, Sommer, Nevir, others

Mimetic Properties

1. Discrete Exterior Calculus
2. Thuburn, Cotter, others

Conservative, Mimetic Methods

1. Use DEC to build a discrete (quasi-)Hamiltonian system
2. Unifies two important lines of research

Shallow Water Equations

Vector Invariant Shallow Water Equations

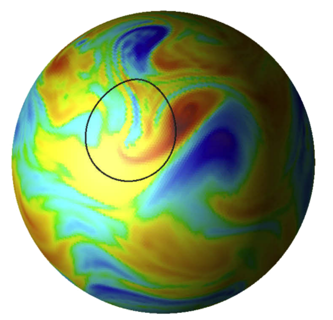
$$\vec{x} = (h, \vec{u})$$

$$\mathbb{J} = \begin{pmatrix} 0 & \vec{\nabla}\cdot \\ \vec{\nabla} & q\hat{k}\times \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g(h,h) + \frac{1}{2}(\vec{F}, \vec{u})$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi \\ \vec{F} \end{pmatrix} = \begin{pmatrix} gh + K \\ h\vec{u} \end{pmatrix}$$

$$\mathcal{Z} = \int_{\Omega} d\Omega h C(q)$$



Shallow water simulation from Ringler et. al 2010

Generalized C Grid Discretization

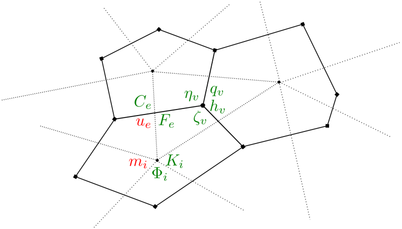
- Discrete variables are $m_i = \int h dA$ (discrete primal 2-form) and $u_e = \int \vec{u} \cdot \vec{dl}$ (discrete dual 1-form)
- C grid staggering (m_i at cell centers, u_e at edges)
- General formulation is:

$$\vec{x} = (m_i, u_e)$$

$$\mathbb{J} = \begin{pmatrix} 0 & D_2 \\ \bar{D}_1 & Q \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g(m_i, m_i)_{\mathbf{I}} + \frac{1}{2}(F_e, u_e)_{\mathbf{H}}$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \mathbf{I}\Phi_i \\ \mathbf{H} F_e \end{pmatrix}$$



Non-orthogonal primal-dual C grid

Summary

1. General discrete framework can conserve **mass, potential vorticity, total energy and potential enstrophy** on **general, non-orthogonal polygonal meshes**
2. Framework cleanly splits **topological** $(\bar{D}_1, D_2$, etc.) and **metrical** (**I**,**H**,etc.) aspects
3. Hamiltonian/DEC framework also has useful mimetic properties (linear stability, no spurious vorticity production, etc.)
4. **Any questions?**

Mass and Potential Vorticity Conservation

Mass Conservation

$$\frac{\partial m_i}{\partial t} + D_2 \mathbf{H} F_e = 0$$

Local and global conservation by form alone ("flux form", independent of how F_e is formulated)

Potential Vorticity Conservation

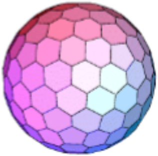
$$\frac{\partial m_v q_v}{\partial t} + \bar{D}_2 \mathbf{Q} \mathbf{H} F_e = 0$$

Local and global conservation by form alone ("flux form", independent of how $\mathbf{Q}$ is formulated)

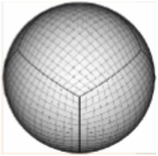
Discrete Energy Conservation

Requires two things:

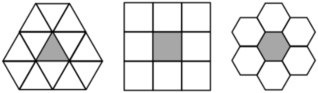
1. $\mathbb{J}$ is anti-symmetric: $D_2^T = -\bar{D}_1$, and $\mathbf{Q} = \mathbf{Q}^T$
2. $\mathcal{H}$ is positive definite: $\mathbf{I}$ and $\mathbf{H}$ are symmetric positive definite



Icosahedral-hexagons



Cubed Sphere

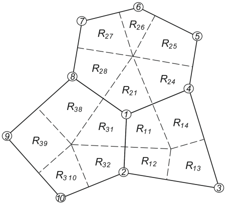


Planar Grids

Discrete Potential Enstrophy

$$\mathcal{Z}_{\mathcal{C}} = \frac{1}{2}(\eta_v, \mathbf{J}^{-1} q_v)_{\mathbf{J}}$$

$$\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = \begin{pmatrix} -\mathbf{R}^T \frac{q_v^2}{2} \\ \bar{D}_2{}^T q_v \end{pmatrix} = \begin{pmatrix} -\mathbf{R}^T \frac{q_v^2}{2} \\ D_1 q_v \end{pmatrix}$$



Construction of **R** from Thuburn et. al 2009

- Discrete vorticity is $\zeta_v = \bar{D}_2 u_e$
- Mass-weighted potential vorticity $m_v q_v = \zeta_v + f = \eta_v$
- $m_v = \mathbf{R} m_i$; **R** maps primal 2-forms to dual 2-forms

Discrete Potential Enstrophy Conservation

Conservation

Casimir conservation requires

$$\mathbb{J}\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = 0$$

which gives

$$D_2 D_1 q_v = 0 \qquad (1)$$

$$-\bar{D}_1 \mathbf{R}^T \frac{q_v^2}{2} + \mathbf{Q} D_1 q_v = 0 \qquad (2)$$

Operator Requirements

(1) is automatic ($D_2 D_1 = 0$)
(2) is tricky- depends on R^T

- TRiSK and Arakawa and Lamb 1981 schemes both construct $\mathbf{Q}$ such that (2) is satisfied
- Only Arakawa and Lamb 1981 also has $\mathbf{Q} = \mathbf{Q}^T$ (energy conservation)

- Form of (2) gives hope that Arakawa and Lamb 1981 scheme can be extended to non-orthogonal, arbitrary polygonal meshes