Efficiency of finite volume solvers for inhomogeneous systems.

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Summary of the talk

- Introduction
- Some non homogeneous systems
- Recent result of existence for inhomogeneous systems
- Construction of a new finite volume method: the $SRNH$ scheme
- Analysis: deduction of $SRNHS$ scheme
- Application of $SRNHS$ scheme to multidimensional Shallow water equations with source terms, and to a two phase flow problem
- Conclusion and future work
Introduction: Complex fluid flow phenomena such as combustion, multiphase flows or flows submitted to external forces, are represented by stiff or ill posed inhomogeneous systems (e.g. multiphase systems can have non hyperbolic regions). It is therefore not easy to extend the usual Riemann solvers based on system eigenvalues and eigenvectors computations. To propose an alternative, we consider in this work a particular class of non conservative systems. We assume that the solution of the associated Riemann problem is self-similar. Assuming this hypothesis, a new Non Homogeneous Riemann Solver (SRNH), using approximate states instead of approximate fluxes, was developed. The new scheme depends on a local parameter allowing to control numerical diffusion. The stability analysis of the scheme, first in the scalar case then in the case of systems of conservation laws, leads to a new formulation of the scheme which is based on the sign of genuine or approximate jacobian of the system considered.
1 Some non homogeneous systems

1.1 1D Two phase flows (non hyperbolic model):

\[
\begin{aligned}
\frac{\partial W(x,t)}{\partial t} + \frac{\partial F(W(x,t))}{\partial x} + S_1(x,W) &= S_2(x,W) \\
W(x,0) &= W_0(x),
\end{aligned}
\]

\(W(x,t) = (\alpha_v \rho_v, \alpha_v \rho_v u_v, \alpha_l \rho_l, \alpha_l \rho_l u_l)^T\)

\(F(W(x,t)) = (\alpha_v \rho_v u_v, \alpha_v \rho_v u_v^2, \alpha_l \rho_l u_l, \alpha_l \rho_l u_l^2)^T\)

\(S_1(x,W) = \left(0, \alpha_v \frac{\partial p}{\partial x}, 0, \alpha_l \frac{\partial p}{\partial x}\right)^T\)

\(S_2(x,W) = (0, \alpha_v \rho_v g, 0, \alpha_l \rho_l g)^T, \quad p = C_v \rho_v^\gamma = C_l \rho_l^\beta\)
1.2 2D Shallow Water equations with irregular topography:

\[
\begin{align*}
    h,_{t} + (hu),_{x} + (hv),_{y} & = 0 \\
    (hu),_{t} + (hu^2),_{x} + (huv),_{y} + g\left(\frac{h^2}{2}\right),_{x} & = -gh(Z_f),_{x} \\
    (hv),_{t} + (huv),_{x} + (hv^2),_{y} + g\left(\frac{h^2}{2}\right),_{y} & = -gh(Z_f),_{y},
\end{align*}
\]

where $h$ is the water elevation, $u =^t (u, v)$ the velocity, and $Z_f$ the bottom function.
1.3 Non-isentropic Euler equations in a duct with variable section

\[
\frac{\partial}{\partial t} \begin{bmatrix}
\rho A \\
\rho A u \\
\rho A E
\end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix}
\rho A u \\
\rho A (u^2 + p/\rho) \\
\rho A u H
\end{bmatrix} = \begin{bmatrix}
0 \\
p \frac{dA}{dx} \\
0
\end{bmatrix}
\] (3)

\[\gamma = 1.4\]

\[E = \frac{p}{(\gamma - 1)\rho} + \frac{u^2}{2}\]
\[H = \frac{\gamma p}{(\gamma - 1)\rho} + \frac{u^2}{2}\] (4)

\[A(x)\] is the duct section.
2 Recent result of existence for inhomogeneous systems

Consider the system of balance laws in $\mathbb{R}^m$:

$$\begin{cases}
\frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} = Q(x, W), & \forall x \in D \subset \mathbb{R}, \quad t > 0 \\
W(x, 0) = W_0(x), & \forall x \in D,
\end{cases}$$

(5)

with $Q(x, W) = G(W) \frac{\partial E(x, W)}{\partial x}$.

We make the fundamental assumption that the corresponding Riemann problem:

$$W_0(x) = W_L \text{ if } x \leq 0, \quad \text{and } W_0(x) = W_R \text{ if } x > 0,$$

admits a selfsimilar solution: $W(x, t) = H\left(\frac{x}{t}\right)$. 
Recent result (see [10]) : in case $E = E(x)$ is a Lipschitz continuous function of $x$, the Cauchy problem (5) admits a weak solution.

Proof: use Glimm random scheme, construct solution of local Riemann problems of extended systems, and show convergence by regularisation.

Provided the system is strictly hyperbolic, and initial Riemann data close enough, the solution of a local Riemann problem, exists, is unique, and consists of connecting left and right states by shock waves, contact discontinuities, rarefaction waves and a stationary wave discontinuity.
References


3 Presentation of the scheme

Integrating first equation of system (5) in the square:

\[ R = \left[ x_{\frac{i-1}{2}}, x_{\frac{i+1}{2}} \right] \times [t_n, t_{n+1}], \text{ gives:} \]

\[
W_{i}^{n+1} = W_{i}^{n} - r_{n} \left[ F \left( R_{s}(0, W_{i}^{n}, W_{i+1}^{n}) \right) - F \left( R_{s}(0, W_{i-1}^{n}, W_{i}^{n}) \right) \right] + \Delta t_{n} Q_{i}^{n},
\]

where \( r_{n} = \frac{\Delta t_{n}}{\Delta x} \),

\( Q_{i}^{n} \) is an approximation of \( \frac{1}{\Delta t_{n} \Delta x} \int_{R} Q(x, W) \, dx \, dt \).
Figure 1: The Riemann problem solution $R_s$ at a cell interface.
Let $W_{i+\frac{1}{2}}^n$ be an approximation of $Rs\left(0, W_i^n, W_{i+1}^n\right)$.

Figure 2: Staggered box around the interface $x_{i+\frac{1}{2}}$
With the choice: $X^- = x_i$ and $X^+ = x_{i+1}$, one gets a generalized expression:

$$
W^n_{i+\frac{1}{2}} = \frac{1}{2}(W^n_i + W^n_{i+1}) - \frac{\theta}{\Delta x} \left[ F(W^n_{i+1}) - F(W^n_i) \right] + \theta \, Q^n_{i+\frac{1}{2}},
$$

where

$$
Q^n_{i+\frac{1}{2}} = G \left( W^n_i, W^n_{i+1} \right) \left[ \frac{E(x_{i+1}, W^n_{i+1}) - E(x_i, W^n_i)}{\Delta x} \right].
$$

A possible choice is: $\theta = \frac{\alpha_{i+\frac{1}{2}}}{2} \Delta t$ (see Benkhaldoun 02).
Here, to make the extension of \( SRNH \) scheme to 2D easier, one writes: \( \theta = \alpha_{i+\frac{1}{2}}^n \bar{\theta} \) where \( \bar{\theta} \) is defined by the local Rusanov velocity (see figure 2):

\[
\bar{\theta} = \frac{\Delta x}{2S_{i+\frac{1}{2}}^n},
\]

where

\[
S_{i+\frac{1}{2}}^n = \max_{p=1...m} \left( \max \left( |\lambda_{i,p}^n|, |\lambda_{i+1,p}^n| \right) \right).
\]

On gets the following expression of the intermediate state :

\[
W_{i+\frac{1}{2}}^n = \frac{1}{2} (W_i^n + W_{i+1}^n) - \frac{\alpha_{i+\frac{1}{2}}^n}{2S_{i+\frac{1}{2}}^n} \left[ F(W_{i+1}^n) - F(W_i^n) \right]
\]

\[
+ \frac{\alpha_{i+\frac{1}{2}}^n}{2S_{i+\frac{1}{2}}^n} G \left( W_i^n, W_{i+1}^n \right) \left[ E(x_{i+1}, W_{i+1}^n) - E(x_i, W_i^n) \right].
\]
4 Analysis of SRNH scheme in the scalar case

\[
\begin{align*}
\begin{cases}
u^n_{i+\frac{1}{2}} &= \frac{1}{2} (u^{n+1}_i + u^n_i) - \frac{\alpha^n_{i+\frac{1}{2}}}{2S^n_{i+\frac{1}{2}}} (f(u^{n+1}_i) - f(u^n_i)) + \frac{\alpha^n_{i+\frac{1}{2}}}{2S^n_{i+\frac{1}{2}}} \Delta x Q^n_{i+\frac{1}{2}} \\
u^{n+1}_i &= u^n_i - r (f(u^n_{i+\frac{1}{2}}) - f(u^n_{i-\frac{1}{2}})) + \Delta t Q^n_i
\end{cases}
\end{align*}
\]

where $S^n_{i+\frac{1}{2}}$ is the local Rusanov velocity given by:

\[S^n_{i+\frac{1}{2}} = \max \left( |f'(u^n_{i+1})|, |f'(u^n_i)| \right) \] and

\[u^n_0 = \frac{1}{\Delta t \Delta x} \int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,0) \, dx \, dt.\]
Let us note $a \perp b = \min(a, b)$ and $a \top b = \max(a, b)$, and for $\gamma \geq 1$, the set $X_\gamma = \{ u \in \mathbb{R} | |u| \leq \gamma \| u_0 \|_{L^\infty(\mathbb{R})} \}$.

**Proposition 1.** Suppose $f$ is a monotone $C^1$ function, and source term vanishes, then under the two conditions:

1) $\frac{S^n_{i+\frac{1}{2}}}{|f'(a^n_{i+\frac{1}{2}})|} \leq \alpha^n_{i+\frac{1}{2}} \leq \gamma \frac{S^n_{i+\frac{1}{2}}}{|f'(a^n_{i+\frac{1}{2}})|}$, $\forall (i, n) \in \mathbb{Z} \times \mathbb{N},$

2) $r \gamma A \leq 1$ with $A = \max_{u \in X_\gamma} |f'(u)|$

where $a^n_{i+\frac{1}{2}}$ is a Roe state (given by the mean value theorem).

The scheme (7) respects the local maximum principle.

$$\min_{i \in \mathbb{Z}} u^n_i \leq \min_{i \in \mathbb{Z}} u^{n+1}_i \leq \max_{i \in \mathbb{Z}} u^{n+1}_i \leq \max_{i \in \mathbb{Z}} u^n_i.$$
Proposition 2. Suppose $f$ is a monotone $C^1$ function, then under the two conditions:

1) $\alpha_{i+\frac{1}{2}}^n = \bar{\gamma} \frac{S_{i+\frac{1}{2}}^n}{\left| f'(a_{i+\frac{1}{2}}^n) \right|}$, $\forall (i, n) \in \mathbb{Z} \times \mathbb{N}$ where $\bar{\gamma} \in [1, \gamma]$, $\bar{\gamma}$ is constant,

2) $r \gamma A \leq 1$.

the scheme (7) is monotone.

Théorème 1. Let $w_0 \in L^\infty (\mathbb{R})$, under the conditions of proposition (2), the numerical solution given by SRNH scheme converges to the unique entropy solution.
How to fix $\alpha_{i+\frac{1}{2}}^n$?

The stability condition ((2) of proposition (1)) requires that $|f'(a_i^n)|$ be finite. A sufficient condition for this is $a_i^n$ and $u_{i+\frac{1}{2}}^n$ remain bounded.

Remark: a sufficient condition for $u_{i+\frac{1}{2}}^n$ to be bounded, is $u_{i+\frac{1}{2}}^n \in [u_i^n \perp u_{i+1}^n, u_i^n \mathbin{\top} u_{i+1}^n]$.

And $u_{i+\frac{1}{2}}^n = \frac{1}{2} \left( 1 + \delta_{i+\frac{1}{2}}^n \right) u_i^n + \frac{1}{2} \left( 1 - \delta_{i+\frac{1}{2}}^n \right) u_{i+1}^n$,

gives $\alpha_{i+\frac{1}{2}}^n \leq \frac{S_{i+\frac{1}{2}}^n}{|f'(a_{i+\frac{1}{2}}^n)|}$.

And then, with condition (1) of proposition (1), one has:

$$\alpha_{i+\frac{1}{2}}^n = \frac{S_{i+\frac{1}{2}}^n}{|f'(a_{i+\frac{1}{2}}^n)|}$$
Finally, extending to the nonhomogeneous case, we obtain the **SRNHS** scheme:

\[
\begin{align*}
    u_{i+\frac{1}{2}}^n &= \frac{1}{2} (u_{i+1}^n + u_i^n) - \frac{1}{\left| f'(a_{i+\frac{1}{2}}^n) \right|} (f(u_{i+1}^n) - f(u_i^n)) \\
        &\quad + \frac{\Delta x}{2|f'(a_{i+\frac{1}{2}}^n)|} Q_{i+\frac{1}{2}}^n \\
    u_{i}^{n+1} &= u_i^n - r \left( f(u_{i+\frac{1}{2}}^n) - f(u_{i-\frac{1}{2}}^n) \right) + \Delta t Q_i^n 
\end{align*}
\]

or quite simply

\[
\begin{align*}
    u_{i+\frac{1}{2}}^n &= \frac{1}{2} (u_{i+1}^n + u_i^n) - \frac{1}{2} \text{sgn} \left[ f'(a_{i+\frac{1}{2}}^n) \right] (u_{i+1}^n - u_i^n) \\
        &\quad + \frac{\Delta x}{2|f'(a_{i+\frac{1}{2}}^n)|} Q_{i+\frac{1}{2}}^n \\
    u_{i}^{n+1} &= u_i^n - r \left( f(u_{i+\frac{1}{2}}^n) - f(u_{i-\frac{1}{2}}^n) \right) + \Delta t Q_i^n 
\end{align*}
\]
4.0.1 Extension to non linear systems

Here one considers a local Roe linearisation, and again one obtains
\[ \alpha_{i+\frac{1}{2}}^n = S_{i+\frac{1}{2}}^n |\Lambda^*(V(W_i^n, W_{i+1}^n))|^{-1} \]
and the SRNHS scheme writes:

\[
\begin{cases}
W_{i+\frac{1}{2}}^n = \frac{1}{2} (W_{i+1}^n + W_i^n) - \frac{1}{2} \text{sgn} \left[ B_{i+\frac{1}{2}}^n \right] (W_{i+1}^n - W_i^n) \\
+ \frac{\Delta x}{2} \left[ B_{i+\frac{1}{2}}^n \right]^{-1} Q_{i+\frac{1}{2}}^n \\
W_{i+1}^{n+1} = W_i^n - r \left( F(W_{i+\frac{1}{2}}^n) - F(W_{i-\frac{1}{2}}^n) \right) + \Delta t Q_i^n
\end{cases}
\]

with: \( B_{i+\frac{1}{2}}^n = (R\Lambda^* R^{-1}) (V(W_i^n, W_{i+1}^n)) \) is a pseudo jacobian matrix calculated at the Roe state \( V(W_i^n, W_{i+1}^n) \).
5 Application to the 1D Shallow Water equations with irregular topography
Let us consider the Shallow water equations:

\[
\begin{align*}
\frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} &= Q(x, W), \quad (x, t) \in \mathcal{D} \times \mathbb{R}^+_x, \quad \mathcal{D} \subset \mathbb{R} \\
W(x, 0) &= W_0(x), \quad x \in \mathcal{D}
\end{align*}
\]

with
\[
W(x, t) = (h(x, t), hu(x, t))^T
\]

\[
F(W(x, t)) = \left( hu(x, t), hu^2(x, t) + \frac{1}{2}gh^2(x, t) \right)^T
\]

\[
Q(x, W(x, t)) = \left( 0, -gh(x, t) \frac{dz(x)}{dx} \right)^T.
\]
The SRNHS scheme for problem (11) may be written:

\[
\begin{cases}
W_{i+\frac{1}{2}}^n = \frac{1}{2} (W_{i+1}^n + W_i^n) - \frac{1}{2} \sgn \left( B_{i+\frac{1}{2}}^n \right) (W_{i+1}^n - W_i^n) \\
+ \frac{\Delta x}{2} \left| B_{i+\frac{1}{2}}^n \right|^{-1} \ Q_{i+\frac{1}{2}}^n \\
W_i^{n+1} = W_i^n - r \left( F(W_{i+\frac{1}{2}}^n) - F(W_{i-\frac{1}{2}}^n) \right) + \Delta t Q_i^n
\end{cases}
\]

with

\[
Q_{i+\frac{1}{2}}^n = -\frac{g}{2\Delta x} (h_i^n + h_{i+1}^n) \begin{bmatrix}
0 \\
z_{i+1} - z_i
\end{bmatrix},
\]

and

\[
Q_i^n = \frac{1}{\Delta t \Delta x} \int_{t^n}^{t^{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} Q(x, W(x, t)) \, dx \, dt.
\]
Définition 1. \( W(x, t) \) is a static stationary solution of the system if \( \frac{\partial W}{\partial t} = 0 \) and \( u(x, t) = 0 \). In this case, one has
\[
h(x, t) + z(x) = \text{constant}.
\]

Définition 2. A finite volume scheme is said to verify the exact \( \mathcal{C} \)-property (Bermudez & Vazquez 1999), if it preserves the equilibrium state:
\[
h^n_i + z_i = c \quad \text{and} \quad u^n_i = 0 \quad \forall (i, n) \in \mathbb{Z} \times \mathbb{N}.
\]

Proposition 3. If the source term, in the second step of the scheme, is discretized as follows : \((Q^n_i)_1 = 0\), and

i) \((Q^n_i)_2 = -\frac{g}{4\Delta x} \left( h^n_{i+\frac{1}{2}} + h^n_{i-\frac{1}{2}} \right) (z_{i+1} - z_{i-1})\), or

ii) \((Q^n_i)_2 = -\frac{g}{8\Delta x} \left( h^n_{i+1} + 2h^n_i + h^n_{i-1} \right) (z_{i+1} - z_{i-1})\)

then the scheme (12) respects the exact \( \mathcal{C} \)-property.
Bed and free surface, $t = 10s$  
Water momentum $t = 10s$
Water free surface $cfl=0.5$, 1000 points (vacuum occurrence)

Water momentum $cfl=0.5$, 1000 points
Free surface on a step, $t=1\text{s}$, CFL=0.75

Error plot, CFL=0.75, slope $\approx 0.65$
6 Application to the non-isentropic Euler equations in a duct of variable cross section

The governing (Euler) equations can be written:

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{bmatrix} \rho A \\ \rho A u \\ \rho A E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho A u \\ \rho A (u^2 + p/\rho) \\ \rho A u H \end{bmatrix} &= \begin{bmatrix} 0 \\ p \frac{dA}{dx} \\ 0 \end{bmatrix} \\
\end{align*}
\]  

(13)

where \( \rho, u \) and \( p \) are the gas density, velocity and pressure respectively. \( A(x) \) is the cross section of the duct, and \( E \) and \( H \) represent the total energy and total enthalpy.

\[
\begin{align*}
E &= \frac{p}{(\gamma - 1)\rho} + \frac{u^2}{2} \\
H &= \frac{\gamma p}{(\gamma - 1)\rho} + \frac{u^2}{2} \\
\end{align*}
\]  

(14)
Figure 3: Shock tube problem with discontinuous cross section computed with 200 cells. Density (left) and Mach number (right) at $t=2s$ (the duct cross section area is plotted also as a dotted line in the Mach plot).
Figure 4: Shock tube problem with discontinuous cross section computed with 200 cells. Mass flow (left) and entropy (right) at $t=2s$. Both quantities must be constant across the cross section discontinuity.
<table>
<thead>
<tr>
<th></th>
<th>Constant State 1</th>
<th></th>
<th>Constant State 2</th>
<th></th>
<th>Constant State 3</th>
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<tr>
<td></td>
<td>Exact</td>
<td>Num.</td>
<td>Exact</td>
<td>Num.</td>
<td>Exact</td>
</tr>
<tr>
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<td>1.427</td>
<td>1.285</td>
<td>1.287</td>
<td>2.208</td>
</tr>
<tr>
<td>$u$</td>
<td>0.661</td>
<td>0.666</td>
<td>1.105</td>
<td>1.107</td>
<td>1.105</td>
</tr>
<tr>
<td>$M$</td>
<td>0.345</td>
<td>0.347</td>
<td>0.589</td>
<td>0.590</td>
<td>0.772</td>
</tr>
</tbody>
</table>

Table 1: Comparison between exact and numerically computed constant states on a 200 cell mesh
Figure 5: $L_1$ Convergence plot of the density and velocity for the shock tube problem with cross section discontinuity. In this case the convergence decay is more clearly visible than for the Shallow Water Equations.), with both curves rapidly approaching a stagnation condition.
7 Application of $SRNHS$ scheme to a two-fluid model
Let us consider the two-fluid model:

\[
\begin{aligned}
\frac{\partial W(x,t)}{\partial t} + \frac{\partial F(W(x,t))}{\partial x} + S_1(x,W) &= S_2(x,W) \\
W(x,0) &= W_0(x),
\end{aligned}
\]  

(15)

\[
W(x,t) = (\alpha_v \rho_v, \alpha_v \rho_v u_v, \alpha_l \rho_l, \alpha_l \rho_l u_l)^T
\]

\[
F(W(x,t)) = (\alpha_v \rho_v u_v, \alpha_v \rho_v u_v^2, \alpha_l \rho_l u_l, \alpha_l \rho_l u_l^2)^T
\]

\[
S_1(x,W) = \left(0, \alpha_v \frac{\partial p}{\partial x}, 0, \alpha_l \frac{\partial p}{\partial x} + \delta(p - p^i_l) \frac{\partial \alpha_l}{\partial x} \right)^T
\]

\[
S_2(x,W) = (0, \alpha_v \rho_v g, 0, \alpha_l \rho_l g)^T, \quad p - p^i_l = \alpha_v \rho_l (u_l - u_v)^2
\]

the subscript \( k \) is either \( v \) for vapour or \( l \) for liquid, and \( \delta \) is a non negative real parametr.
Numerical algorithm:

We use the splitting strategy presented in [Benkhaldoun 02]. The gravity source term is treated in a first step, to get \( \hat{W} \) from \( W^n \)

\[
\begin{align*}
\begin{cases}
\frac{\partial \hat{W}}{\partial t} = S_2(\hat{W}) \\
\hat{W}(x, t^n) = W^n(x),
\end{cases}
\end{align*}
\]

and using \( SRNHS \) scheme, we solve

\[
\begin{align*}
\begin{cases}
\frac{\partial W(x, t)}{\partial t} + A(W) \frac{\partial W(x, t)}{\partial x} = 0 \\
W(x, 0) = \hat{W}_0(x),
\end{cases}
\end{align*}
\]

(16)

where \( A(W) = \nabla F(W) + C(W) \) and \( C(W) \frac{\partial W(x, t)}{\partial x} = S_1(x, W) \).
Case $\delta = 0$

\[
\mathcal{A}(W) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-u_v^2 + \frac{\gamma p}{\rho_v} & 2u_v & \frac{\gamma p}{\rho_l} & 0 \\
0 & 0 & 0 & 1 \\
\frac{\alpha_l}{\alpha_v} \frac{\gamma p}{\rho_v} & 0 & -u_l^2 + \frac{\alpha_l}{\alpha_v} \frac{\gamma p}{\rho_l} & 2u_l
\end{pmatrix}.
\]

The $SRNHS$ scheme writes

\[
\begin{cases}
W_{i+\frac{1}{2}}^n = \frac{1}{2} \left( W_i^n + W_{i+1}^n \right) - \frac{1}{2} \text{sgn} \left( \mathcal{A}(\bar{W}) \right) \left( W_{i+1}^n - W_i^n \right) \\
W_{i}^{n+1} = W_i^n - r \left( F \left( W_{i+\frac{1}{2}}^n \right) - F \left( W_{i-\frac{1}{2}}^n \right) \right) + \Delta t (S_1)^n_i.
\end{cases}
\]  

(17)

$\bar{W}$ is a Roe state.
The matrix sign can be computed using Alouges’s method (Alouges 98, Alouges 99) which is based on the Newton-Schultz algorithm:

\[
\begin{align*}
B_{n+1} &= P_{a_n}(B_n), \\
L_{n+1} &= \frac{2(a_n+1)}{3} \sqrt{\frac{a_n+1}{3}}, \\
a_{n+1} &= \frac{1}{L_{n+1}(L_{n+1}+1)},
\end{align*}
\]

with

\[
L_0 = \max \left( |\bar{u}_v| + \sqrt{\frac{\gamma \bar{p}}{\bar{\rho}_v}}, |\bar{u}_l| + \sqrt{\frac{\gamma \bar{p}}{\bar{\rho}_v}} \right), \quad a_0 = \frac{1}{L_0(L_0 + 1)}
\]

\[
B_0 = A(\bar{W}) \quad \text{et} \quad P_a(X) = -aX^3 + (a + 1)X.
\]
For complex flows (imaginary part of the eigenvalues too preponderant), the Newton-Shultz algorithm does not work, one then switches to a system with $\delta \neq 0$, and one applies the perturbation method (Toumi et al. 99). One introduces the new variables $\tilde{\rho}_v = \frac{\rho_v}{\rho_v^0}$ and $\tilde{\rho}_l = \frac{\rho_l}{\rho_l^0}$, where $\rho_v^0$ and $\rho_l^0$ are two characteristic densities, and one defines $\epsilon = \frac{\rho_v^0}{\rho_l^0}$. ($\epsilon << 1$) is a very small parameter (the vapour phase is very light compared to liquid phase).
Finally one writes \( A(W) = A_0(W) + \epsilon H(W) \).
\( A_0(W) \) is a diagonal matrix on \( \mathbb{R} \) and we have

\[
\lambda_1 = u_v - c_1, \quad \lambda_2 = u_v + c_1, \quad \lambda_3 = u_l - c_2 \quad \text{and} \quad \lambda_4 = u_l + c_2,
\]

where \( c_2 = \alpha_v p, 1 = \frac{\gamma p}{\rho_v} \) and \( c_2 = \delta \alpha_v (u_v - u_l)^2 \).

The \( SRNHS \) scheme writes:

\[
\begin{cases}
W^n_{i+1/2} = \frac{1}{2} (W^n_i + W^n_{i+1}) - \frac{1}{2} \text{sgn} (A_0(\bar{W})) (W^n_{i+1} - W^n_i) \\
W^{n+1}_i = W^n_i - r \left( F \left( W^n_{i+1/2} \right) - F \left( W^n_{i-1/2} \right) \right) + \Delta t (S_1)_i^n.
\end{cases}
\]
$\alpha_v = 0.2 \text{ m/s}$

$u_l = 10 \text{ m/s}$

$u_v = 0 \text{ m/s}$

$P = 10^5 \text{Pa}$

Figure 6: Ransom problem: variation of liquid quantity according to time
Void fraction \((\alpha_{v,0} = 0.6), \delta = 0\), Alouges’s method

Void fraction \((\alpha_{v,0} = 0.2), \delta = 5 \times 10^{-4}\), Toumi’s method
8 Extension of $SRNHS$ scheme to non-homogeneous multidimensional problems
Consider the multidimensional hyperbolic system:

\[
\begin{cases}
\frac{\partial u}{\partial t} + \text{div} (F(u(x,t))) = 0, \quad \forall x \in \mathbb{R}^2, \forall t \in \mathbb{R}_+,
\end{cases}
\]

\[u(x,0) = u_0(x), \quad \forall x \in \mathbb{R}^2\]  

(18)

with \(F = (F_1, F_2)\) is a \(C^1\) function from \(\mathbb{R}^m\) to \(\mathbb{R}^2\),
\(u : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^m\) and \(u_0 \in L^\infty(\mathbb{R}^2)\).
The SRNHS scheme for the problem (18) may be written:

\[
\begin{align*}
W_{ij}^n &= \frac{1}{2} (W_i^n + W_j^n) - \frac{1}{2} \text{sgn} \left( \tilde{A}(W_i^n, W_j^n; \eta_{ij}) \right) (W_j^n - W_i^n) \\
W_i^{n+1} &= W_i^n - \frac{\Delta t^n}{A_i} \sum_{j \in N_i} |e_{ij}| G(W_i^n, W_j^n; \eta_{ij}) \\
W_i^0 &= \frac{1}{A_i} \int_{C_i} W(x, y, 0) dx dy,
\end{align*}
\]

with \( \tilde{A}(W_i^n, W_j^n; \eta_{ij}) \) is a Roe matrix and \( G \) is a numerical flux function \( G(W_i^n, W_j^n; \eta_{ij}) = F(W_{ij}^n) \cdot \eta_{ij} \).
8.1 Application of \textit{SRNHS} scheme to Shallow water equations with irregular topography

The system considered may by written as follows :

\[
\begin{aligned}
&h_{,t} + (hu)_{,x} + (hv)_{,y} = 0 \\
&(hu)_{,t} + (hu^2)_{,x} + (huv)_{,y} + g \left( \frac{h^2}{2} \right)_{,x} = -gh(Z_f)_{,x} \\
&(hv)_{,t} + (huv)_{,x} + (hv^2)_{,y} + g \left( \frac{h^2}{2} \right)_{,y} = -gh(Z_f)_{,y},
\end{aligned}
\tag{20}
\]

where \( h \) is the water level, \( \mathbf{u} = \dot{t} (u, v) \) the water velocity and \( Z_f \) the bottom height.
Figure 7: River bed, $\Xi = h + Z_f$

$\Xi_l = 4 \text{ m}$

$Z_{fl} = 0 \text{ m}$

$\Xi_r = 1.1 \text{ m}$

$Z_{fr} = 1 \text{ m}$

$(u, v)_l = (0, 0) \text{ m/s}$

$(u, v)_r = (0, 0) \text{ m/s}$

20 m
To calculate the predictor phase of SRNHS scheme, one projects the equations on each interface \( e_{ij} \), and gets the following system ([Abgrall 03])

\[
(U_\eta)_t + (F_\eta),_\eta = Q(x, y, U_\eta)
\]  

(21)

with

\[
U_\eta = (h, hu_\eta, hu_\tau)^T, \quad F_\eta = \left( hu_\eta, hu_\eta^2 + g\frac{h^2}{2}, hu_\eta u_\tau \right)^T, \quad Q(x, y, U_\eta) = (0, -gh(Z_f),_\eta, 0)^T,
\]

\[u_\eta = \mathbf{u} \cdot \eta, \quad u_\tau = \mathbf{u} \cdot \tau, \quad \eta \text{ and } \tau \text{ the normal and the tangential vector to the interface, and } (\cdot),_\eta \text{ the derivate along the normal vector } \eta.\]
In this case, the predictor phase of scheme SRNHS may be written as follows:

\[ U_{ij}^n = \frac{1}{2} (U_i^n + U_j^n) - \frac{1}{2} \text{sgn} \left( \nabla F_\eta \left( \bar{U} \right) \right) (U_j^n - U_i^n) \]

\[ + \frac{1}{2} \left| \nabla F_\eta \left( \bar{U} \right) \right|^{-1} Q_{ij}^n, \]  

(22)

where

\[ Q_{ij}^n = -\frac{g}{2} (h_i + h_j) \left( Z_{f,j} - Z_{f,i} \right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \]  

(23)

and \( \bar{U} \) the Roe state.
The corrector phase may be written as follows:

\[ W_i^{n+1} = W_i^n - \frac{\Delta t^n}{A_i} \sum_{j \in N_i} G (W_i^n, W_j^n, Q_{ij}^n, \eta_{ij}^n) + \Delta t Q_i^n, \] (24)

with

\[ G (W_i^n, W_j^n, Q_{ij}^n, \eta_{ij}^n) = F (W_{ij}^n) \cdot \eta_{ij} \]

\[ W_{ij}^n = \left( h_{ij}^n, (hu\eta)_{ij}^n \eta_x - (hu\tau)_{ij}^n \eta_y, (hu\tau)_{ij}^n \eta_y + (hu\tau)_{ij}^n \eta_x, \right)^T. \]

with

\[ Q_i^n = -g \frac{h_i}{A_i} \left( \begin{array}{c} 0 \\ \sum_{j \in N_i} Z_{ij} \cdot (\eta_{ij})_x |e_{ij}| \\ \sum_{j \in N_i} Z_{ij} \cdot (\eta_{ij})_y |e_{ij}| \end{array} \right), \] where \[ Z_{ij} = \frac{Z_{fi} A_i + Z_{fj} A_j}{A_i + A_j} \]
2D dam break over a step, water level $t=1.2$ s

2D dam break over a step, water level, Cross 1D, $t=1.2$s
2D dam break over a step, water level $t=1.2$ s, isolines

Velocity field
9 The convergence stagnation problem

Let us consider the scalar equation:

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = -u \frac{dz}{dx} \]  \hspace{1cm} (25)

with \( a > 0 \) and the following source function:

\[ z(x) = \begin{cases} 
z_L \text{ if } x < 0 \\
z_R \text{ if } x > 0 
\end{cases} \]  \hspace{1cm} (26)

In the following we will call:

\[ \Delta z = z_R - z_L \]  \hspace{1cm} (27)
The exact value $u^*$ in terms of $u_L$ and the problem parameters is:

$$u^*_{exact} = u_L \cdot e^{-\Delta z/a} = u_L (1 - (\Delta z/a) + \frac{1}{2}(\Delta z/a)^2 - \frac{1}{6}(\Delta z/a)^3 + ...)$$
Application of the SRNHS scheme to linear equation (25) with \( a > 0 \) leads to:

\[
\begin{align*}
    u_j^{n+1} &= u_j^n - ar (u_j^n - u_{j-1}^n) \\
    &
    \quad + \frac{r}{4} [(u_{j+1}^n + u_j^n)(z_{j+1} - z_j) - (u_j^n + u_{j-1}^n)(z_j - z_{j-1})] \\
    &
    \quad - \frac{r}{8} (u_{j+1}^n + 2u_j^n + u_{j-1}^n)(z_{j+1} - z_{j-1})
\end{align*}
\]  

(28)

which converges to: \( u_{num}^* = u_L \frac{1 - \Delta z/2a + 3\Delta z^2/64a^2}{1 + \Delta z/2a + 3\Delta z^2/64a^2} \)

hence: \( u_{num}^* = u_L (1 - \frac{\Delta z}{a} + \frac{1}{2} (\frac{\Delta z}{a})^2 - \frac{13}{64} (\frac{\Delta z}{a})^3 + \ldots) \)
Figure 9: Riemann problem for linear scalar equation with $\Delta z/a = 1$. Initial discontinuity at $x = 20$. Exact versus numerical solution with 102400 nodes (left). Error convergence rate (right).
Another way of solving this problem avoiding the use of an exact Godunov method is to regularize the source term discretization (and correspondingly the initial data) to ensure that parameter \( \Delta z/a \) is small at each cell interface. This can be accomplished for instance by taking:

\[
\hat{z}(x) = \frac{z_R + z_L}{2} + \frac{z_R - z_L}{2} \cdot \tanh\left(\frac{x}{C\Delta x^p}\right)
\]  
(29)

and

\[
\hat{u}_0(x) = \frac{u_R + u_L}{2} + \frac{u_R - u_L}{2} \cdot \tanh\left(\frac{x}{C\Delta x^p}\right)
\]  
(30)
Figure 10: Riemann problem for linear scalar equation with $\Delta z/a = 1$. SRNHS scheme with smooth initialization. Initial discontinuity at $x = 20$. Exact versus numerical solution with 800 nodes (left). Error convergence rate (right).
Figure 11: Smoothed dam break problem over a smoothed step. Flow rate (left) and $L_1$ Convergence plot of the velocity and the depth (right).
Figure 12: Abrupt initialisation dam break problem over a step. Flow rate (left) and $L_1$ Convergence plot of the velocity and the depth (right).
10 Pollutant Transport in the Strait of Gibraltar

For simplicity in presentation we write the equations in a conservative form as:

$$\partial_t W + \partial_x \left( F(W) - \tilde{F}(W) \right) + \partial_y \left( G(W) - \tilde{G}(W) \right) = Q(W), \quad (31)$$

where \( W \) and \( Q \) are the vectors of conserved variables and source terms, \( F \) and \( G \) are the convection tensor fluxes, \( \tilde{F} \) and \( \tilde{G} \) are the diffusion tensor fluxes

\[
W = \begin{pmatrix}
h
hu
hv
hC
\end{pmatrix}, \quad Q(W) = \begin{pmatrix}
0
-gh(S_{0x} + S_{fx})
-gh(S_{0y} + S_{fy})
hQ
\end{pmatrix},
\]
\[
\begin{align*}
\mathbf{F}(W) &= \begin{pmatrix}
    hu \\
    hu^2 + \frac{1}{2} gh^2 \\
    huv \\
    huC
\end{pmatrix}, \\
\mathbf{G}(W) &= \begin{pmatrix}
    hv \\
    huv \\
    hv^2 + \frac{1}{2} gh^2 \\
    hvC
\end{pmatrix}, \\
\mathbf{\tilde{F}}(W) &= (0, 0, 0, D_{xx} \partial_x (hC) + D_{xy} \partial_y (hC))^T, \\
\mathbf{\tilde{G}}(W) &= (0, 0, 0, D_{yx} \partial_x (hC) + D_{yy} \partial_y (hC))^T
\end{align*}
\]

where \(D_{xx}, D_{xy}, D_{yx} \text{ and } D_{yy}\) are entries of the diffusion matrix \(\mathbf{D}\) assumed to be nonnegative. \(S_{0x} = \partial_x Z, S_{0y} = \partial_y Z\), with \(Z(x, y)\) denotes the bottom topography, while \(S_{fx}\) and \(S_{fy}\) are the friction losses along the \(x\)- and \(y\)-direction, and are defined by

\[
S_{fx} = \eta^2 \frac{u \sqrt{u^2 + v^2}}{h^{4/3}}, \quad S_{fy} = \eta^2 \frac{v \sqrt{u^2 + v^2}}{h^{4/3}},
\]

where \(\eta\) is the Manning roughness coefficient.
Figure 13: Definition of the strait of Gibraltar (left) and its bathymetry (right).
Figure 14: Adapted meshes (first row), velocity vectors (second row) and pollutant concentration (third row) at different simulation times. From left to right, \( t = 1, 2, 3 \) and 4.5 hours.
11 Conclusions and future

* Construction of a new finite volume scheme designed for non homogeneous systems
* The approximate intermediate state is upwind instead of the numerical flux
* Both homogeneous and non homogeneous part of the system are upwind
* Equilibrium for steady states is respected
* New applications were considered (Flow in a duct, problems of pollutant transport)
* More complex problems (Water on a moving bed, realistic pollutant problems) are under study