Benchmark on 2D St Venant-Exner Model

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1 Presentation of the test case

This is a standard test-case used in [5, 1, 2, 7] among others.
For weak interactions between the bed and the flow, it is very demanding in term of CPU time due to the large discrepancy between the time scales of the bed evolution and those of the water flow. Therefore, it is a good test to judge the practical usefulness of numerical schemes with respect to a balance between accuracy and robustness/cost.

1.1 The Mathematical model

The mathematical model is composed of the standard two-dimensional shallow water equations

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} &= 0 \\ \frac{\partial hu}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) + \frac{\partial}{\partial y}(huv) &= -gh\frac{\partial Z}{\partial x} \\ \frac{\partial hv}{\partial t} + \frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}\left(hv^2 + \frac{1}{2}gh^2\right) &= -gh\frac{\partial Z}{\partial y} \end{aligned} \tag{1}$$

completed by an Exner morphodynamical equation modeling the evolution of the sediment bed

$$(1-p)\frac{\partial Z}{\partial t} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0 \tag{2}$$

where p is the (constant) sediment porosity and Q_x and Q_y are the bed-load sediment transport fluxes in the x and y directions. This benchmark uses the Grass model [4] to define these fluxes

$$Q_x = Au\left(u^2+v^2\right)^{\frac{m-1}{2}}, \quad Q_y = Av\left(u^2+v^2\right)^{\frac{m-1}{2}} \tag{3}$$

where A and $1 \leq m \leq 4$ are experimental constants depending on the particular problem under consideration.

1.2 Parameters

We consider both weak and strong interactions of the bed and the flow and the parameters appearing in the Grass model are :

- Strong interaction : $m = 3$, $p = 0.4$, $A = 1$
- Weak interaction : $m = 3$, $p = 0.4$, $A = 0.001$,

1.3 Initial conditions

The initial bottom topography is defined by

$$Z(0,x) = \begin{cases} \sin^2\left(\dfrac{(x-300)\pi}{200}\right)\sin^2\left(\dfrac{(y-400)\pi}{200}\right) & \text{if } (x,y) \in Q_h \\ \\ 0 & \text{elsewhere} \end{cases} \tag{4}$$

where $Q_h = [300, 500] \times [400, 600]$. Initial condition for the flow height and velocity fields can be generated running a simulation without bed evolution (standard Shallow Water) until a steady state is reached with the initial conditions:

$$h(0,x,y) = 10 - Z(0,x,y) \quad u(0,x,y) = \frac{10}{h(0,x,y)} \quad v(0,x,y) = 0 \tag{5}$$

1.4 Boundary conditions

Let η be the exterior unit normal to the computational domain. At inflow the flux $(hu, hv).\eta = 10$ and the sediment layer height $Z = 0.1$ are imposed, free boundary conditions have to be used at the outflow while slip conditions $(hu, hv).\eta = 0$ are imposed on the lateral boundaries.

1.5 Meshes

The domain of interest is a square of $1000 \times 1000\ m^2$. In order to allow a fair comparison between the schemes, this square have to be meshed with regular space steps $\Delta x = \Delta y = 20m$. However in order to minimize the possible influence of the boundary conditions on the results, larger computational domains are allowed provided the central part respects the above requirement.

2 Required results

According to the classification proposed in [6], the formulations for approximating the mathematical model (1-2) can be classified as

- Steady approach (SA) consisting of a :
  - a fixed bottom stage where the shallow water equations (1) are iterated to an equilibrium while the bed is kept fixed
  - a changing bottom stage where the bed is updated whilst keeping all other variables fixed.

  The time steps used in the two stage can be different.

- Operator splitting (OS) approach using the same time step for the shallow water and bed evolution equations and where the systems (1) and (2) are iterated sequentially. This approach is similar to the previous one except that the water flow is no longer iterated to an equilibrium after each bed update.

- Strongly coupled approach (SC) : The two systems (1) and (2) are written as a single first order system and approximated simultaneously.

The required results for this benchmark are :

- the classification of the method used as Steady approach (SA), Operator splitting (OS) or Strongly coupled (SC).
- The contour plot of the bottom elevation, contour or elevation plot of the total water level $h + Z$ and velocity at $t = 500s$ for $A = 1$ and $t = 100h$ for $A = 0.001$.
- The total number of time steps (including the time steps needed to reach equilibrium in approach SA) and CPU time needed to reach $t = 500s$ for $A = 1$ and $t = 100h$ for $A = 0.001$.
- The sediment layer evolves towards a star-shaped pattern expanding along the time with a given spreading angle. Assuming that the interaction between the sediment layer and fluid is small, the following theoretical approximation of this angle is proposed in [3]

$$tan\theta = \frac{3\sqrt{3}(m-1)}{9m-1}$$

For m=3 this corresponds to a spreading angle of $\theta = 21.786789^o$. For the weak interaction case $A = 0.001$, the contributors should give an estimate of this angle.

References

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