Numerical test: Sediment transport *

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For the 1D sediment transport problem we have the following coupled system,

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial q}{\partial x} = 0, \\ \dfrac{\partial q}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{q^2}{h} + \dfrac{1}{2}gh^2\right) = -gh\dfrac{\partial z_s}{\partial x} - gh\dfrac{\partial z_b}{\partial x} - ghS_f, \\ \dfrac{\partial z_b}{\partial x} + \xi\dfrac{\partial}{\partial x}q_s(h,q) = 0, \end{cases} \tag{1}$$

where h is the water height, q the discharge, z_b a fixed topography, z_s the sediment height and S_f is the friction term modeling the drag effects between the fluid and the bed. Moreover, $\xi = 1/(1-\psi_0)$, being ψ_0 the porosity of the sediment layer. q_s is the solid transport discharge. In this test we consider Grass formula:

$$q_s = A_g \frac{q}{h}\left|\frac{q}{h}\right|^{m_g-1}, \; 1 \leqslant m_g \leqslant 4, \tag{2}$$

This test consists on comparing a numerical solution with an asymptotic and analytical solution obtained by Hudson and Sweby in [2], for Grass model when interaction constant A_g (2) is smaller than 10^{-2}. In this case, the layer sediment $\tilde{z}_b$ is over all computational domain and fluid is moving slowly with a constant discharge $q = q_0 \leq 10$. Then, we use the hypothesis that the water surface is flat. That is,

$$h = A_r - z_b(x,t), \quad \text{and} \quad q = q_0$$

being A_r a fixed level of reference and $q_0 \leq 10$ a constant value.

If the water surface, A_r, is constant and the discharge q_0 is contant, we have that the fluid velocity is

$$u(x,t) = \frac{q_0}{A_r - z_b(x,t)}. \tag{3}$$

So, from the continuity equation for the bed evolution,

$$\frac{\partial z_b}{\partial t} + \xi\frac{\partial q_s}{\partial x} = 0, \tag{4}$$

and Grass formula (2), using (3), we can rewrite (4) as

$$\frac{\partial z_b}{\partial t} + \left(\xi m_g A_g q_0^{m_g}(A_r - z_b)^{-(m_g+1)}\right)\frac{\partial z_s}{\partial x} = 0. \tag{5}$$

Then, it can be solved by using the method of characteristics.

We consider the following data for the numerical test. We set a computational domain whose length is $L = 1000$ meters, discretized with 250 nodes. CFL condition is set to 0.8. The sediment porosity is set to $\rho_0 = 0.4$ and the constant A_g of Grass formula (2) is set to 0.001 which corresponds to a weak interaction.

*See: [1]

The initial conditions are (see Figure 1),

$$h(x,0) = 10 - z_b(x,0), \quad q(x,0) = 10,$$

$$z_b(x,0) = \begin{cases} 0.1 + \sin^2\left(\dfrac{\pi\,(x-300)}{200}\right) & \text{if } 300 \le x \le 500, \\ 0.1 & \text{otherwise.} \end{cases} \tag{6}$$

As boundary condition, the flux and the depth of the sediment is imposed upstream, while free boundary conditions are imposed downstream.

From (5), we obtain that the analytical solution can be expressed as

$$z_b(x,t) = \begin{cases} \sin^2\left(\dfrac{\pi\,(x_o-300)}{200}\right) & \text{if } 300 \le x_o \le 500, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where x_o is the solution of equation

$$\begin{cases} x = x_o + A_g\,\xi\,m_g\,q_0^{m_g}\,t\left(A_r - \sin^2\left(\dfrac{\pi\,(x_o-300)}{200}\right)\right)^{-(m_g+1)} & \text{if } 300 \le x_o \le 500 \\ x = x_o + A_g\,\xi\,m_g\,q_0^{m_g}\,t A_r^{-(m_g+1)} & \text{otherwise.} \end{cases} \tag{8}$$

The usual value of $m_g = 3$ is considered.

This solution is valid to $t < t_0$, being t_0 the instant in which characteristics cross. In [2] it is estimated $t_0 = 238079.124 \times A_g^{-1}$.

In the same way, it is verified that

$$\tilde{z}_b(x,t) = z_b(x,t) + c_0,$$

where c_0 is a constant value, it is also a solution of the problem.

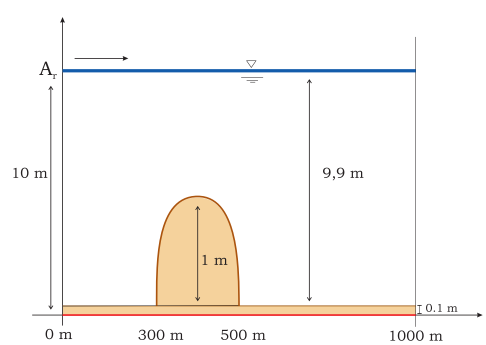


Figure 1: Initial condition.

To be able to compare numerical solution with analytical solution obtained in [2], without considering the behavior of numerical scheme in zones with-without sediment, we have compared the solution for the case,

$$\tilde{z}_b(x,t) = z_b(x,t) + 0.1.$$

The following numerical numerical results can be seen in [1]

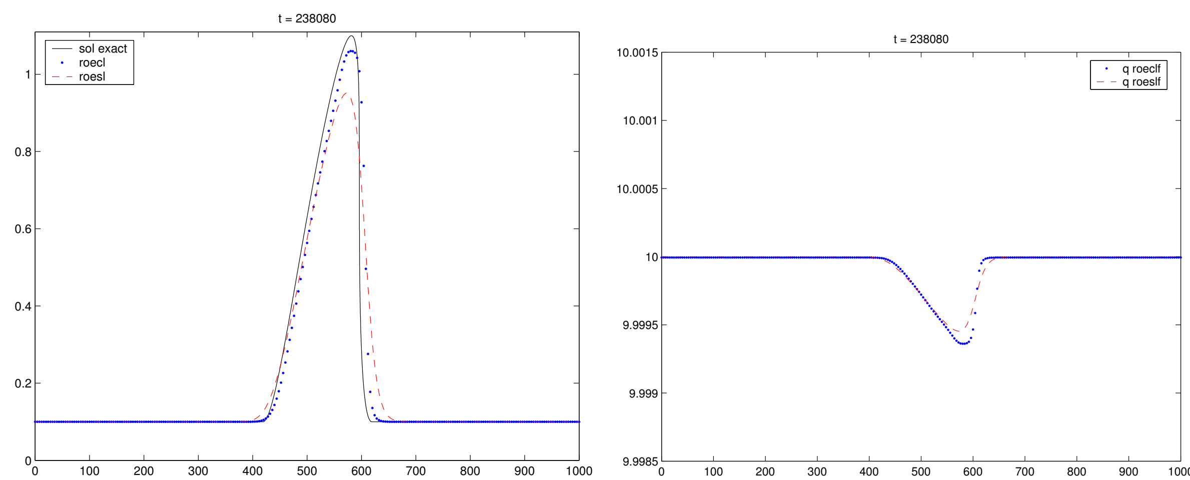
In Figures 2, 3, 4 we compare analytical solution (continuous line) and numerical solution obtained with the different schemes proposed in [1], corresponding to sediment layer evolution at instant $t = 238080\ s$; that is smaller than the maximal time in which analytical solution (8) is valid.

We can observe that all numerical schemes show a good sediment layer localization, being Roe scheme the most diffusive (Figure 2(a)). Moreover, the scheme that gives the best approximation is high order generalized Roe scheme with Weno state reconstructions of order 3, where time approximation is made using Runge-Kutta of order 3(Figure 4(a)).

In Figure 2 it is shown comparison between Roe method and linearized Lax-Wendroff method with flux limiters. In both methods we use Euler scheme for time discretization. In Figure 2(a), that describes layer sediment evolution, it is observed that Roe method is more diffusive than method with flux limiters, that is of second order in space and time for linear problems. In Figure 2(b) the discharge is presented. We observe that it is near 10 in both cases and almost constant in all domain, that is one of the hypothesis used to develop the analytical solution (8).

In Figure 3 it is compared flux limiter scheme with Roe-Weno of order 2. For time evolution it is used Euler for flux limiter scheme, because Lax-Wendroff scheme ensures order 2 in space and time. For scheme with Weno2 state reconstructions it is used TVD Runge-Kutta method of order 2. It must be observed that approximation of thickness sediment layer is similar in both schemes.

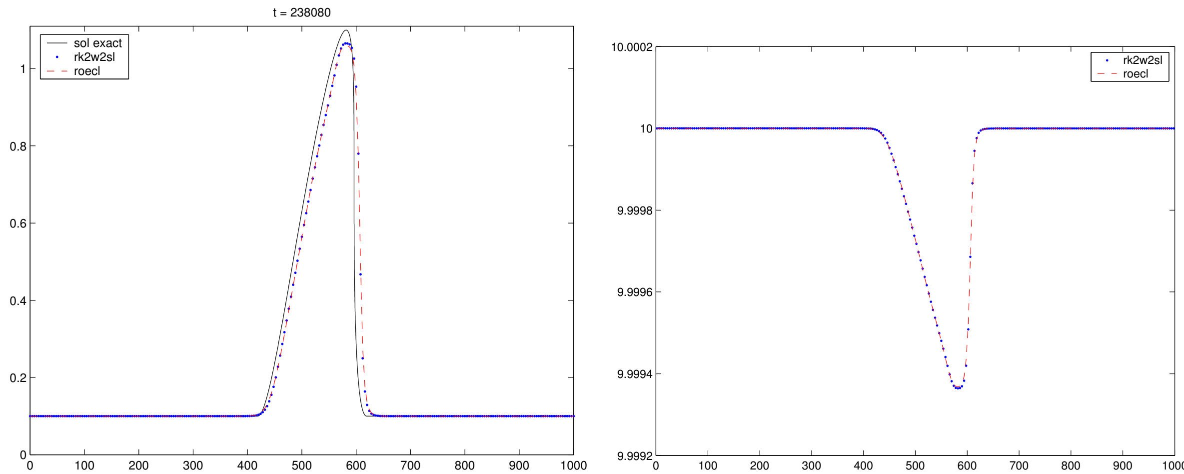
Finally in Figure 4 schemes with Weno2 and Weno3 state reconstructions are compared. For time evolution it is used Runge-Kutta2 and Runge-Kutta3, respectively. Higher order scheme approximates better the sediment layer thickness as expected.



(a) Sediment layer thickness.

(b) Mass-flow.

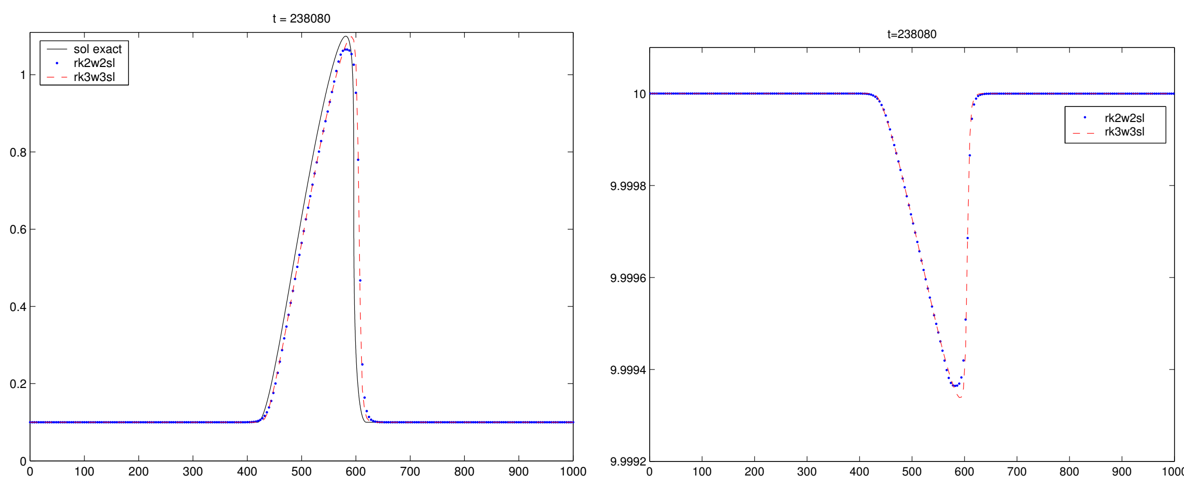
Figure 2: Roe-Flux limiters (dotted line). Euler-Roe (dash line).



(a) Sediment layer thickness.

(b) Mass-flow fluid.

Figure 3: Roe-Flux limiters (dash line). Weno2-Rk2 (dotted line).



(a) Sediment layer thickness.

(b) Mass-flow fluid.

Figure 4: Weno2-Rk2 (dotted line). Weno3-Rk3 (dash line).

References

[1] M.J. Castro Díaz, E.D. Fernández-Nieto, A.M. Ferreiro. *Sediment transport models in Shallow Water equations and numerical approach by high order finite volume methods.* Computers and Fluids, 37(3): 299–316, (2008).

[2] J. Hudson. *Numerical technics for morphodynamic modelling.* Ph.D. Thesis. University of Whiteknights, (2001).