Frobenius structure for inertia stacks

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Received *****; accepted after revision +++++

Presented by

Abstract

We introduce a string coproduct structure on the homology groups of the inertia stack $\Lambda\mathfrak{X}$ and prove that $H_\bullet(\Lambda\mathfrak{X})$ with the string product and coproduct becomes a (not necessarily unital or counital) Frobenius algebra. As an example, we explicitly describe the Frobenius algebra structure in the case when $\mathfrak{X}$ is $[*/G]$ for a connected compact Lie group G.

To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Résumé

La structure Frobenius pour un champ d'inertie. On construit un coproduit sur l'homologie $H_\bullet(\Lambda\mathfrak{X})$ du champ d'inertie d'un champ différentiel $\mathfrak{X}$ qui munit cette dernière d'une structure d'algèbre de Frobenius non (co)-unitaire en général. On explicite cette structure dans le cas d'un champ $[*/G]$ où G est un groupe de Lie compact simplement connexe.

Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Version française abrégée

Dans cette note, on construit un coproduit sur l'homologie du champ d'inertie $\Lambda\mathfrak{X}$, où $\mathfrak{X}$ est un champ différentiel, analogue au coproduit qui existe sur l'homologie de l'espace des lacets libres d'une variété [4].

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[1] Research partially supported by grants DMS-0306665 and DMS-0605725 & NSA grant H98230-06-1-0047

Dans un premier temps, on définit des morphismes d'évaluation $\mathrm{ev}_0,\ \mathrm{ev}_{1/2} : \Lambda\mathfrak{X} \to \mathfrak{X}$ analogues aux applications évaluant un lacet en 0 et $\frac{1}{2}$. Pour ce faire, à un groupoide Γ représentant le champ $\mathfrak{X}$, on associe un groupoide $\widetilde{\Lambda\Gamma}$ Morita équivalent à $\Lambda\Gamma$. Les objets de $\widetilde{\Lambda\Gamma}$ sont les diagrammes

$$x \xleftarrow{g_1} y \xleftarrow{g_2} x \tag{1}$$

dans la catégorie sous-jacente à Γ. Les flèches de $\widetilde{\Lambda\Gamma}$ sont données par les diagrammes commutatifs

$$\begin{array}{ccccc} x & \xleftarrow{g_1} & y & \xleftarrow{g_2} & x \\ {\scriptstyle h_0}\uparrow & & \uparrow{\scriptstyle h_{1/2}} & & \uparrow{\scriptstyle h_0} \\ x' & \longleftarrow & y' & \longleftarrow & x' \end{array}$$

Les morphismes d'évaluation sont définis par $\mathrm{ev}_0 : (g_1, g_2, h_0, h_{1/2}) \mapsto h_0$, $\mathrm{ev}_{1/2} : (g_1, g_2, h_0, h_{1/2}) \mapsto h_{1/2}$. On en déduit un diagramme cartésien

$$\begin{array}{ccc} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \longrightarrow & \Lambda\mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_{1/2})} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array}$$

et donc un morphisme de Gysin $G_\Delta^{(\mathrm{ev}_0,\mathrm{ev}_{1/2})} : H_\bullet(\Lambda\mathfrak{X}) \longrightarrow H_{\bullet-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$ lorsque $\mathfrak{X}$ est orienté. L'application composée

$$\delta : H_\bullet(\Lambda\mathfrak{X}) \xrightarrow{G_\Delta^{(\mathrm{ev}_0,\mathrm{ev}_{1/2})}} H_{\bullet-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{j} H_{\bullet-d}(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \cong \bigoplus_{i+j=\bullet-d} H_i(\Lambda\mathfrak{X}) \otimes H_j(\Lambda\mathfrak{X})$$

définit alors un coproduit coassociatif cocommutatif gradué sur $H_{\bullet-d}(\Lambda\mathfrak{X})$, appelé le coproduit fantôme de $\Lambda\mathfrak{X}$. Le résultat principal de cette note est

Théorème 0.1 *Soit $\mathfrak{X}$ un champ différentiel orienté de dimension d. L'homologie $H_\bullet(\Lambda\mathfrak{X})$ du champ d'inertie de $\mathfrak{X}$ munie du produit et coproduit fantôme est une algèbre de Frobenius (non nécessairement unitaire ou co-unitaire).*

Le champ d'inertie de $[*/G]$ est isomorphe au champ quotient $[G/G]$ où G agit sur lui même par conjugaison. Sa cohomologie est $H^\bullet([G/G]) \cong (S^*(\mathfrak{g}^*))^G \otimes (\Lambda\mathfrak{g}^*)^G \cong S(x_1, x_2, \cdots, x_l) \otimes \Lambda(y_1, y_2, \cdots, y_l)$, où les x_i et y_i sont respectivement les générateurs de $(S^*(\mathfrak{g}^*))^G$ et $(\Lambda\mathfrak{g}^*)^G$. On obtient

Théorème 0.2 *Soit G un groupe de Lie compact connexe. Le dual du produit fantôme est trivial sur $H^\bullet([G/G])$. Le dual du coproduit fantôme est donné sur $H^\bullet([G/G])$, pour tout élément $P(x_1 \dots x_l)y_1^{\epsilon_1} ... y_l^{\epsilon_l}$ et $Q(x_1 \dots x_l)y_1^{\epsilon'_1} ... y_l^{\epsilon'_l}$ dans $H^\bullet([G/G])$, par la formule*

$$(P(x_1 \dots x_l)y_1^{\epsilon_1} ... y_l^{\epsilon_l}) \star \left(Q(x_1 \dots x_l){y_1}^{\epsilon'_1} \dots {y_l}^{\epsilon'_l}\right) = (PQ)(x_1 \dots x_r)y_1^{\epsilon_1+\epsilon'_1-1} ... y_r^{\epsilon_r+\epsilon'_r-1}$$

avec la convention $y_j^{-1} = 0$.

1. Introduction

The purpose of this Note is to introduce a coproduct on the homology groups of the inertia stack $\Lambda\mathfrak{X}$ of an oriented differential stack in analogue with the coproduct on the homology groups of loop manifolds [4]. The later is the crucial ingredient in proving that string topology in the sense of [3], [4] forms a (open-closed) 2-dimensional topological field theory. As the first step, we introduce the evaluation maps $\mathrm{ev}_0,\ \mathrm{ev}_{1/2} : \Lambda\mathfrak{X} \to \mathfrak{X}$, analogous to the usual evaluation maps of a loop at 0 and $\frac{1}{2}$. To achieve this, in Section 2 we consider, for a groupoid Γ representing $\mathfrak{X}$, a canonical groupoid $\widetilde{\Lambda\Gamma}$ Morita equivalent to $\Lambda\Gamma$ and we define maps $\mathrm{ev}_0, \mathrm{ev}_{1/2} : \widetilde{\Lambda\Gamma} \to \Gamma$. In this way, we obtain a cartesian square

$$\begin{array}{ccc} \Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X} & \longrightarrow & \Lambda\mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle (ev_0, ev_{1/2})} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X}. \end{array} \tag{2}$$

Thus, if $\mathfrak{X}$ is an oriented differential stack of dimension d, Diagram (2) induces a Gysin map $G_\Delta^{(\mathrm{ev}_0, \mathrm{ev}_{1/2})} : H_\bullet(\Lambda\mathfrak{X}) \longrightarrow H_{\bullet-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X})$. Then the composition

$$\delta : H_\bullet(\Lambda\mathfrak{X}) \xrightarrow{G_\Delta^{(\mathrm{ev}_0, \mathrm{ev}_{1/2})}} H_{\bullet-d}(\Lambda\mathfrak{X} \times_{\mathfrak{X}} \Lambda\mathfrak{X}) \xrightarrow{j} H_{\bullet-d}(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \cong \bigoplus_{i+j=\bullet-d} H_i(\Lambda\mathfrak{X}) \otimes H_j(\Lambda\mathfrak{X})$$

defines a *string coproduct*. We show that the homology groups $H_\bullet(\Lambda\mathfrak{X})$ with the string product and coproduct becomes a (not necessarily unital or counital) Frobenius algebra.

As an example, for a compact connected Lie group G, we describe explicitly the Frobenius algebra structure of the inertia stack of $[*/G]$, which is isomorphic to the quotient stack $[G/G]$ with G acting on G by conjugation.

2. Inertia stack and evaluation maps

Let $\mathfrak{X}$ be a differential stack of dimension d and Γ a Lie groupoid representing $\mathfrak{X}$. The inertia stack $\Lambda\mathfrak{X}$ of $\mathfrak{X}$ is the stack associated to the inertia groupoid $\Lambda\Gamma : S\Gamma \rtimes \Gamma \rightrightarrows S\Gamma$, where $S\Gamma = \{g \in \Gamma_1 |\ s(g) = t(g)\}$ is the space of closed loops. The action of Γ on $S\Gamma$ is by conjugation. It is folklore to think of $\Lambda\mathfrak{X}$ as a stack of ghost loops on $\mathfrak{X}$ (i.e. constant on the coarse space). Any loop $S^1 \to X$ on a topological space X can not only be evaluated at 0 but also at 1/2. In this section we construct the analogues of these evaluation maps for the inertia stack. To achieve this we introduce another groupoid $\widetilde{\Lambda\Gamma}$ which is Morita equivalent to $\Lambda\Gamma$.

Objects of $\widetilde{\Lambda\Gamma}$ consist of all diagrams

$$x \xleftarrow{g_1} y \xleftarrow{g_2} x \tag{3}$$

in Γ. Note that the composition $g_1 g_2$ is a loop over x. Arrows of $\widetilde{\Lambda\Gamma}$ consist of commutative diagrams

$$\begin{array}{ccccc} x & \xleftarrow{g_1} & y & \xleftarrow{g_2} & x \\ {\scriptstyle h_0}\uparrow & & \uparrow{\scriptstyle h_{1/2}} & & \uparrow{\scriptstyle h_0} \\ x' & \longleftarrow & y' & \longleftarrow & x' \end{array}$$

Note that the left and right vertical arrows are the same. The target map is the top row

$$x \xleftarrow{g_1} y \xleftarrow{g_2} x$$

and the source map is the bottom row. The unit maps are obtained by taking identities as vertical arrows and the composition is obtained by superposing two diagrams and deleting the middle row of the diagram thus obtained. In other words $\widetilde{\Lambda\Gamma}$ is the transformation groupoid $\widetilde{S\Gamma} \rtimes_{\Gamma_0\times\Gamma_0} (\Gamma_1 \times \Gamma_1)$, where $\widetilde{S\Gamma} = \{(g_1, g_2) \in \Gamma_1 \times_{\Gamma_0} \Gamma_1 | t(g_1) = s(g_2)\}$, the momentum map $\widetilde{S\Gamma} \to \Gamma_0 \times \Gamma_0$ is (t, t), and the action is given, for all compatible $(h_0, h_{1/2}) \in \Gamma_1 \times \Gamma_1$, $(g_1, g_2) \in \widetilde{S\Gamma}$, by

$$(g_1, g_2) \cdot (h_0, h_{1/2}) = (h_0^{-1} g_1 h_{1/2}, h_{1/2}^{-1} g_2 h_0).$$

The evaluation maps are defined by the vertical arrows of $\widetilde{S\Gamma}$, i.e. for every $(g_1, g_2, h_0, h_{1/2}) \in \widetilde{\Lambda\Gamma}_1$,

$$ev_0 : (g_1, g_2, h_0, h_{1/2}) \mapsto h_0, \quad ev_{1/2} : (g_1, g_2, h_0, h_{1/2}) \mapsto h_{1/2}.$$

Lemma 2.1 *Both evaluation maps $ev_0 : \widetilde{\Lambda\Gamma} \to \Gamma$ and $ev_{1/2} : \widetilde{\Lambda\Gamma} \to \Gamma$ are groupoid morphisms.*

There is a natural map $p : \widetilde{\Lambda\Gamma} \to \Lambda\Gamma$ obtained by sending a diagram in $\widetilde{\Lambda\Gamma}_1$ to the composition of the horizontal arrows, i.e. $(g_1, g_2, h_0, h_{1/2}) \overset{p}{\mapsto} (g_1 g_2, h_0)$.

Lemma 2.2 *The map $p : \widetilde{\Lambda\Gamma} \to \Lambda\Gamma$ is a Morita morphism.*

Therefore $\widetilde{\Lambda\Gamma}$ also represents the inertia stack $\Lambda\mathfrak{X}$, and Lemma 2.1 implies that there are two stack maps $ev_0, ev_{1/2} : \Lambda\mathfrak{X} \to \mathfrak{X}$.

3. String coproduct and Frobenius structure

Let k be a field and A a k-vector space. Recall that A is said to be a *Frobenius algebra* if there is an associative commutative multiplication $\mu : A^{\otimes 2} \to A$ and a coassociative cocommutative comultiplication $\delta : A \to A^{\otimes 2}$ satisfying the following compatibility condition $\delta \circ \mu = (\mu \otimes 1) \circ (1 \otimes \delta) = (1 \otimes \mu) \circ (\delta \circ 1)$ in $\mathrm{Hom}(A^{\otimes 2}, A^{\otimes 2})$. Here we do not require the existence of unit or counit.

As shown in Theorem 3.1 [2], there is an associative and commutative string product on $H_\bullet(\Lambda\mathfrak{X})$. We now define a coproduct. Write $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ for the diagonal. The evaluation maps of Section 2 yield

Lemma 3.1 *The stack $\Lambda\mathfrak{X} \times_\mathfrak{X} \Lambda\mathfrak{X}$ fits into a cartesian square*

$$\begin{array}{ccc} \Lambda\mathfrak{X} \times_\mathfrak{X} \Lambda\mathfrak{X} & \longrightarrow & \Lambda\mathfrak{X} \\ \downarrow & & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_{1/2})} \\ \mathfrak{X} & \xrightarrow{\Delta} & \mathfrak{X} \times \mathfrak{X} \end{array}.$$

By Proposition 2.2 of [2], when $\mathfrak{X}$ is oriented, the cartesian square of Lemma 3.1 yields a Gysin map

$$G_\Delta^{(\mathrm{ev}_0, \mathrm{ev}_{1/2})} : H_\bullet(\Lambda\mathfrak{X}) \longrightarrow H_{\bullet - d}(\Lambda\mathfrak{X} \times_\mathfrak{X} \Lambda\mathfrak{X}).$$

There is also an inclusion $\Lambda\mathfrak{X} \times_\mathfrak{X} \Lambda\mathfrak{X} \to \Lambda\mathfrak{X} \times \Lambda\mathfrak{X}$.

Theorem 3.2 *Let $\mathfrak{X}$ be an oriented differential stack of dimension d. The composition*

$$\delta : H_\bullet(\Lambda\mathfrak{X}) \overset{G_\Delta^{(\mathrm{ev}_0, \mathrm{ev}_{1/2})}}{\longrightarrow} H_{\bullet - d}(\Lambda\mathfrak{X} \times_\mathfrak{X} \Lambda\mathfrak{X}) \longrightarrow H_{\bullet - d}(\Lambda\mathfrak{X} \times \Lambda\mathfrak{X}) \cong \bigoplus_{i+j = \bullet - d} H_i(\Lambda\mathfrak{X}) \otimes H_j(\Lambda\mathfrak{X})$$

defines a degree d coassociative graded cocommutative coproduct on the homology $H_\bullet(\Lambda\mathfrak{X})$, called the string coproduct on $\Lambda\mathfrak{X}$. Moreover, $H_\bullet(\Lambda\mathfrak{X})$ together with the string product and string coproduct becomes a Frobenius algebra.

Remark 1 *If $\mathfrak{X}$ has finitely generated homology groups in each degree, then by universal coefficient theorem, $H^\bullet(\Lambda\mathfrak{X})$ inherits a Frobenius coalgebra structure which is unital iff $(H_\bullet(\Lambda\mathfrak{X}), \delta)$ is counital.*

If $\mathfrak{X}$ is an oriented orbifold, the Frobenius structure on $H_\bullet(\Lambda\mathfrak{X})$ has a unit given by its fundamental class. Moreover if $\mathfrak{X}$ is a manifold, then its coalgebra structure is almost trivial; it maps the fundamental class to $1 \otimes 1$. If G is a finite group and $\mathfrak{X} = [*/G]$, we have $H_\bullet([*/G]) \cong Z(\mathbb{R}[G])$, the center of the group algebra of G as an algebra [2]. The coproduct is given by $\delta([g]) = \sum_{hk=g} [h] \otimes [k]$.

4. Gysin maps for compact connected Lie group actions

From now on, we assume that G is a compact and connected Lie group of dimension n. Let M be an oriented manifold with a smooth $G \times G$-action. Consider G as a subgroup of $G \times G$ by embedding it diagonally. In this way, M becomes a G-space and we have a morphism of stacks $[M/G] \to [M/G \times G]$. Therefore, by Proposition 2.2 of [2], there is a cohomology Gysin map $\Delta_! : H^\bullet[M/G] \to H^{\bullet-n}[M/G \times G]$. In this section we compute this Gysin map for cohomology with real coefficients.

Recall that when G is a compact connected Lie group, the real cohomology of a quotient stack $H^\bullet([M/G])$ can be computed using the Cartan model $(\Omega_G(M), d_G)$, where $\Omega_G(M) := (S(\mathfrak{g}^*) \otimes \Omega(M))^G$ is the space of equivariant polynomials $P : \mathfrak{g} \to \Omega(M)$, and $d_G(P)(\xi) := d(P(\xi)) - \iota_\xi P(\xi), \quad \forall \xi \in \mathfrak{g}$. Here d is the de Rham differential and ι_ξ is the contraction by the generating vector field of ξ.

Given a Lie group K and a Lie subgroup $G \subset K$, let G act on K from the right by multiplication and K act on itself from the left by multiplication. The submersion $K \to K/G$ is a principal K-equivariant right G-bundle. There is an isomorphism of stacks $[M/G] \xrightarrow{\sim} [K \times_G M/K]$ which induces an isomorphism in cohomology. It is known [5] that, on the Cartan model, this isomorphism can be described by an induction map $\mathrm{Ind}_K^G : \Omega_G(M) \to \Omega_K(K \times_G M)$. Here G acts on $K \times M$ by $(k, m) \cdot g = (k \cdot g, g^{-1} \cdot m)$. The induction map is the composition

$$\Omega_G(M) \stackrel{\mathrm{Pul}}{\to} \Omega_{K\times G}(K \times M) \stackrel{\mathrm{Car}}{\to} \Omega_K(K \times_G M),$$

where $\Omega_G(M) \stackrel{\mathrm{Pul}}{\to} \Omega_{K\times G}(K \times M)$ is the natural pullback map, and $\Omega_{K\times G}(K \times M) \xrightarrow{\mathrm{Car}} \Omega_K(K \times_G M)$ is the Cartan map corresponding to a K-invariant connection for the G-bundle $K \to K/G$ [5].

Now let K be the cartesian product group $G \times G$. We view G as the diagonal subgroup of K. The K action on itself by left multiplication commutes with the right G-action. The left Maurer-Cartan form $\Theta^L_{MC} \in \Omega^1(G) \otimes \mathfrak{g}$ on G yields a K-invariant form $\Theta \in \Omega^1(K) \otimes \mathfrak{g}$ by pullback along the projection on the second factor. Let M be a K $(= G \times G)$ space. It is then a G-space. Moreover there is a canonical isomorphism $(G \times G) \times_G M \xrightarrow{\sim} G \times M$. Thus we have an induction map

$$\mathrm{Ind}^G_{G\times G} : \Omega_G(M) \to \Omega_{G\times G}((G \times G) \times_G M) \cong \Omega_{G\times G}(G \times M),$$

where $G \times G$ acts on $G \times M$ by

$$(k_1, k_2).(g, m) = \big(k_1.g.k_2^{-1}, (k_1, k_2).m\big).$$

To obtain the Gysin map $H^\bullet([M/G]) \to H^{\bullet-n}([M/G \times G])$, one simply composes the above induction map $\mathrm{Ind}^G_{G\times G} : H^\bullet([M/G]) \to H^\bullet([G \times M/G \times G])$ with the equivariant fiber integration map (see [1]) $H^\bullet([G \times M/G \times G]) \to H^{\bullet-n}([M/G \times G])$ over the first factor G.

Proposition 4.1 *Given a $G \times G$-manifold M, the Gysin map $H^\bullet_G(M) \to H^{\bullet-n}_{G\times G}(M)$ is given, on the Cartan model, by the chain map $\Psi : \Omega_G(M) \to \Omega_{G\times G}(M)$: $\forall P \otimes \omega \in (S(\mathfrak{g}^*) \otimes \Omega(M))^G$,*

$$\Psi(P \otimes \omega) = \left((\xi_1, \xi_2) \mapsto \int_G P(-\xi_2)\varphi^*(\omega) \right),$$

where $\int_G$ is the fiber integration over the first factor G and $\varphi : G \times M \to M$ is the map $(g, m) \stackrel{\varphi}{\mapsto} (g^{-1}, 1)\cdot m$.

5. The Frobenius structure for compact connected Lie groups

Let $\mathfrak{X}$ be the differential stack $\mathfrak{X} = [*/G]$ with G being a connected compact Lie group. Its inertia stack $\Lambda[*/G]$ is the quotient stack $[G/G]$, where G acts on itself by conjugation, and $\Lambda[*/G] \times_{[*/G]} \Lambda[*/G]$ is isomorphic to the quotient stack $[G \times G/G]$, where G acts on $G \times G$ by the diagonal conjugation. In this section we investigate the Frobenius algebra structure of the homology $H_\bullet(\Lambda[*/G])$ with real coefficients. By universal coefficient theorem, it is equivalent to compute the Frobenius algebra structure of its dual $H^\bullet(\Lambda[*/G])$. It is known [5] that the cohomology of $[G/G]$ is

$$H^\bullet([G/G]) \cong (S^*(\mathfrak{g}^*))^G \otimes (\Lambda\mathfrak{g}^*)^G \cong S(x_1, x_2, \cdots, x_l) \otimes \Lambda(y_1, y_2, \cdots, y_l),$$

where $l = rank(G)$, and the generators are of degree $deg(x_i) = 2d_i$ and $deg(y_i) = 2d_i + 1$ with $d_1, \ldots, d_l$ being the exponents of G.

We denote $m : G \times G \to G$ and $\Delta : G \to G \times G$ the group multiplication and the diagonal map respectively. The diagonal map induces a stack map $\Delta : [G \times G/G] \to [G \times G/G \times G]$ and thus a Gysin map

$$\Delta_! : H^\bullet([G \times G/G]) \to H^{\bullet - n}([G \times G/G \times G])$$

by Proposition 2.2 of [2]. Similarly the group multiplication m induces a stack map $m : [G \times G/G] \to [G/G]$ and thus a Gysin map $m_! : H^\bullet([G \times G/G]) \to H^{\bullet-n}([G/G])$.

Lemma 5.1 *For the dual Frobenius algebra $H^\bullet(\Lambda[*/G])$, the coproduct is given by the composition*

$$\Delta : H^\bullet([G/G]) \xrightarrow{m^*} H^\bullet([G \times G/G]) \xrightarrow{\Delta_!} H^{*-n}([G \times G/G \times G]) \to \bigoplus_{i+j=\bullet-n} H^i([G/G]) \otimes H^j([G/G]),$$

while the product is given by the composition:

$$\star : H^i([G/G]) \otimes H^j([G/G]) \cong H^{i+j}([G \times G/G \times G]) \xrightarrow{\Delta^*} H^{i+j}([G \times G/G]) \xrightarrow{m_!} H^{i+j-n}([G/G]).$$

Applying Proposition 4.1, we have

Theorem 5.2 *The dual string coproduct on $H^\bullet([G/G])$ is trivial, while the dual string product on $H^\bullet([G/G])$ is given, $\forall\ P(x_1 \ldots x_r)y_1^{\epsilon_1}...y_l^{\epsilon_l}$ and $Q(x_1 \ldots x_l)y_1^{\epsilon'_1}...y_r^{\epsilon'_l} \in H^\bullet([G/G])$, by*

$$(P(x_1 \ldots x_l)y_1^{\epsilon_1}...y_l^{\epsilon_l}) \star \left(Q(x_1 \ldots x_l){y_1}^{\epsilon'_1} \ldots {y'_l}^{\epsilon'_l}\right) = (PQ)(x_1 \ldots x_l)y_1^{\epsilon_1+\epsilon'_1-1}...y_l^{\epsilon_l+\epsilon'_l-1}$$

with the convention that $y_j^{-1} = 0$.

Remark 2 It follows that the string product on $H_\bullet([G/G])$ is trivial while the string coproduct has a counit given by the fundamental class of G, which is dual of the cohomology class $y_1 \ldots y_l$.

Acknowledgements

We would like to thank several institutions for their hospitality while work on this project was being done: Erwin Schroedinger International Institute for Mathematical Physics (Ginot, Noohi and Xu), Université Pierre et Marie Curie (Xu) and Zhejiang University (Xu). We would also like to thank E. Meinrenken for useful discussions.

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