Notes on factorization algebras, factorization homology and applications

Grégory Ginot

Institut Mathématiques de Jussieu Paris Rive Gauche,
Université Pierre et Marie Curie - Sorbonne Universités

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Abstract

These notes are an expanded version of two series of lectures given at the winter school in mathematical physics at les Houches and at the Vietnamese Institute for Mathematical Sciences. They are an introduction to factorization algebras, factorization homology and some of their applications, notably for studying E_n-algebras. We give an account of homology theory for manifolds (and spaces), which give invariant of manifolds but also invariant of E_n-algebras. We particularly emphasize the point of view of factorization algebras (a structure originating from quantum field theory) which plays, with respect to homology theory for manifolds, the role of sheaves with respect to singular cohomology. We mention some applications to the study of mapping spaces, in particular in string topology and for (iterated) Bar constructions and study several examples, including some over stratified spaces.

Contents

1 Introduction and motivations

These notes are an introduction to factorization algebras and factorization homology in the context of topological spaces and manifolds. The origin of factorization algebras and factorization homology, as defined by Lurie [L3] and Costello-Gwilliam [CG], are to be found in topological quantum field theories and conformal field theories. Indeed, they were largely motivated and influenced by the pioneering work of Beilinson-Drinfeld [BD] and also of Segal [S2, S3]. *Factorization homology* is a catchword to describe homology theories *specific* to say oriented manifolds of a fixed dimension n. There are also variant specific to many other classes of structured manifold of fixed dimension. Typically the structure in question would be a framing[1] or a spin structure or simply no structure at all.

Factorization algebras are algebraic structures which shed many similarities with (co)sheaves and were introduced to describe Quantum Field Theories much in the same way as the structure of a manifold or scheme is described by its sheaf of functions [BD, CG]. They are related to factorization homology in the same way as singular cohomology is related to sheaf cohomology.

Unlike classical singular homology for which any abelian group can be used as coefficient of the theory, in order to define factorization homology, one needs a more complicated piece of algebraic data: that of an E_n-algebra[2]. These algebras have been heavily studied in algebraic topology ever since the seventies where they were introduced to study iterated loop spaces and configuration spaces [BV, Ma, S1]. They have been proved to also have deep significance in mathematical physics [Ko, CG], string topology [CS, CV] and (derived) algebraic geometry [BD, FG, PTTV, L4]. E_1-algebras are essentially the same thing as A_∞-algebras, that is homotopy associative algebras. On the other hand, E_∞-algebras are homotopy commutative algebras. In general the E_n- structures form a hierarchy of more and more homotopy commutative algebra structures. In fact, an E_n-algebra is an homotopy associative algebra whose multiplication μ_0 is commutative up to an homotopy operator μ_1. This operator is itself commutative up to an homotopy operator μ_2 and so on up to μ_{n-1} which is no longer required to be homotopy commutative.

Since factorization homology depends on (some class of) both manifold and E_n-algebra, they also give rise to invariant of E_n-algebras. These invariants have proven useful as we illustrate in § 7. For instance, in dimension $n = 1$, factoriza-

[1] that is a trivialization of the tangent bundle

[2] More accurately, E_n-algebras are the piece of data needed in the case of framed manifolds. For other structured manifolds, one needs E_n-algebras equipped with additional structure; for instance an invariance under their natural $SO(n)$-action in the oriented manifold case

tion homology evaluated on a circle is the usual Hochschild homology of algebras (together with its circle action inducing cyclic homology as well). For $n = \infty$, factorization homology gives rise to an invariant of topological spaces[3] (sometimes called higher Hochschild homology [P]) which we recall in § 2. It is easier to study and interesting in its own since it is closely related to mapping spaces, their derived analogues and observables of classical topological field theories.

We give the precise axioms of homology theory of manifolds in § 3. Factorization homology can be computed using Čech complexes of factorization algebras, which, as previously alluded to, are a kind of "multiplicative, non-commutative" analogue of cosheaves. Definitions, properties and many examples of factorization algebras are discussed in § 4. Factorization algebras were introduced to describe observables of Quantum Field Theories [CG, BD] but they also are a very convenient way to encode and study many algebraic structures which arose in algebraic topology and mathematical physics as we illustrate in § 4. In particular, in § 6 we study in depth locally constant factorization algebras on stratified spaces and their link with various categories of modules over E_n-algebras, giving many examples. We also give a detail account of various operations and properties of factorization algebras in § 5. We then (§ 7) review several applications of the formalism of factorization algebras and homology. Notably to cohomology and deformations of E_n-algebras, (higher) Deligne conjecture and also in (higher) string topology and for Bar constructions of iterated loop spaces (and more generally to obtain models for iterated Bar constructions *with* their algebraic structure). In § 8 , we consider the case of commutative factorization algebras and prove their theory reduces to the one of cosheaves. In particular, we cover the pedagogical example of classical homology (with twisted coefficient) viewed as factorization homology.

1.1 Eilenberg-Steenrod axioms for homology theory of spaces

Factorization homology and factorization algebras generalize ideas from the axiomatic approach to classical homology of spaces (and (co)sheaf theory) which we now recall. We then explain how they can be generalized. The usual (co)homology groups of topological spaces are uniquely determined by a set of axioms. These are the *Eilenberg-Steenrod axioms* which were formulated in the 40s [ES].

Classically they express that an homology theory for spaces is uniquely determined by (ordinary) functors from the category of pairs (X,A) $(A \subset X)$ of spaces to the category of $(\mathbb{N})$-graded abelian groups satisfying some axioms. Such a functor splits as the direct sum $H_*(X,A) = \bigoplus_{i\geq 0} H_i(X,A)$ where $H_i(X,A)$ is the degree i homology groups of the pair. This homology group can in fact be defined as the homology of the mapping cone $cone(A \hookrightarrow X)$ of the inclusion of the pair. Further, the long exact sequence in homology relating the homology of the pair to the homology of A and X is induced by a short exact sequence of chain complexes $C_*(A) \hookrightarrow C_*(X) \twoheadrightarrow C_*(X,A)$ (where C_* is the singular chain complex). Similarly,

[3] and not just manifolds of a fix dimension

the Mayer-Vietoris exact sequence is induced by a short exact sequence of chain complexes.

This suggests that the classical Eilenberg-Steenrod axioms can be lifted at the *chain complex* level. That is, we can characterize classical homology as a functor from the category of spaces (up to homotopy) to the category of chain complexes (up to quasi-isomorphism).

Let us formalize a bit this idea. A *homology theory* $\mathscr{H}$ for *spaces* is a functor $\mathscr{H} : Top \to Chain(\mathbb{Z})$ from the category Top of topological spaces[4] to the category $Chain(\mathbb{Z})$ of chain complexes over $\mathbb{Z}$ (in other words differential graded abelian groups). This functor has to satisfy the following three axioms.

1. *(homotopy invariance)* The functor $\mathscr{H}$ shall send homotopies between maps of topological spaces to homotopies between maps of chain complexes.

2. *(monoidal)* The functor $\mathscr{H}$ shall be defined by its value on the connected components of a space. Hence we require it sends disjoint unions of topological spaces to direct sum, that is the canonical map $\oplus_{\alpha\in I}\mathscr{F}(X_\alpha) \to \mathscr{H}(\coprod_{\alpha\in I} X_\alpha)$ is an homotopy equivalence[5] (here I is any set).

3. *(excision)* There is another additional property encoding (given the other ones) the classical excision property as well as the Mayer-Vietoris principle. The additional property essentially stipulates the effect of gluing together two CW-complexes along a sub-complex. Let us formulate it this way: assume $i : Z \hookrightarrow X$ and $j : Z \hookrightarrow Y$ are inclusions of closed sub CW-complex of X and Y. Let $X \cup_Z Y \cong X \coprod Y/(i(z) = j(z), z \in Z)$ be the pushout of X, Y along Z. The functoriality of $\mathscr{H}$ gives maps $\mathscr{H}(Z) \stackrel{i_*}{\to} \mathscr{H}(X)$ and $\mathscr{H}(Z) \stackrel{j_*}{\to} \mathscr{H}(Y)$; hence a chain complex morphism $\mathscr{H}(Z) \stackrel{i_*-j_*}{\to} \mathscr{H}(X) \oplus \mathscr{H}(Y)$. Functoriality also yields a natural map $\mathscr{H}(X)\oplus\mathscr{H}(Y) \to \mathscr{H}(X\cup_Z Y)$ whose composition with $i_* - j_*$ is null.

   The *excision axioms requires that the canonical map*

   $$cone\Big(\mathscr{H}(Z) \stackrel{i_*-j_*}{\to} \mathscr{H}(X)\oplus\mathscr{H}(Y)\Big) \longrightarrow \mathscr{H}(X\cup_Z Y)$$

   is an homotopy equivalence. Here $\mathrm{cone}(f)$ is the mapping cone[6] (in $Chain(\mathbb{Z})$) of the map f of chain complexes.

We can state the following theorem (which follows from Corollary 20 and is the (pre-)dual of a result of Mandell [M1] for cochains).

Theorem 1 (Eilenberg-Steenrod)**.** *Let G be an abelian group. Up to natural homotopy equivalence, there is a unique homology theory for spaces, that is functor*

[4] for simplicity we assume that we consider only spaces homotopy equivalent to CW-complexes

[5] sometimes this map is required to be an actual isomorphism but this is not needed

[6] if we know that $f : C_* \to D_*$ is injective, then $cone(f)$ is quasi-isomorphic to the quotient chain complex D_*/C_*. See for instance [W] for mapping cones of general chain maps.

$\mathscr{H}: Top \to Chain(\mathbb{Z})$ satisfying axioms 1, 2 and 3 and further the dimension *axiom:*

$$\mathscr{H}(pt) \overset{\simeq}{\to} G.$$

The functor in Theorem 1 is of course given by the usual singular chain complex with value in G. We can even assume in the theorem that G is any chain complex, in which case we recover extraordinary homology theories[7].

Theorem 1 implies that the category of functors satisfying axioms 1, 2, 3 is (homotopy) equivalent to the category of chain complexes; the equivalence being given by the evaluation of a functor at the point. To assign to a chain complex V_* an homology theory, one consider the functor $X \mapsto C_*(X,\mathbb{Z}) \otimes V_*$.

For a CW-complex X, the singular cohomology $H^*(X,G)$ can be computed as sheaf cohomology of X with value in the constant sheaf G_X of locallly constant functions on X with values in G. In particular, the singular cochain complex is naturally quasi-isomorphic to the derived functor $\mathbb{R}\Gamma(G_X)$ of sections of G_X. Replacing G_X by a locally constant sheaf (with germs G) yields cohomology with local coefficient in G[8]. This point of view realizes singular cohomology (with local coefficient) as a special case of the theory of sheaf/Čech cohomology which also has other significance and applications in geometry when allowing more general sheaves.

Note that the homotopy invariance axiom can be reinterpreted as saying that the functor $\mathscr{H}$ is continuous. Indeed, there are natural topologies on the morphism sets of both categories. For instance, one can consider the compact-open topology on the set of maps $\mathrm{Hom}_{Top}(X,Y)$ (see Example 61 for $Chain(\mathbb{Z})$). Any continuous functor, that is a functor $\mathscr{H}$ such that the maps $\mathrm{Hom}_{Top}(X,Y) \to \mathrm{Hom}_{Chain(\mathbb{Z})}(\mathscr{H}(X),\mathscr{H}(Y))$ are continuous, sends homotopies to homotopies (and homotopies between homotopies to homotopies between homotopies and so on).

Note also that excision axiom really identifies $\mathscr{H}(X \cup_Z Y)$ with a homotopy colimit. It is precisely the homotopy coequalizer $hocoeq\Big(\mathscr{H}(Z) \underset{j_*}{\overset{i_*}{\rightrightarrows}} \mathscr{H}(X) \oplus \mathscr{H}(Y)\Big)$ which is computed by the mapping cone $cone(i_* - j_*)$. Further, in this axiom, we do not need $\mathscr{H}$ to be precisely the cone but any natural chain complex quasi-isomorphic to it will do the job. This suggests to actually use a more flexible model than topological categories. A convenient way to *deal simultaneously with topological categories, homotopy colimits* (in particular homotopy quotients) *and identification of chain complexes up to quasi-isomorphism* is to consider the ∞-categories associated to topological spaces and chain complexes and ∞-functors between them (see Appendix 10.1, Examples 60 and 61). The passage from topological categories to ∞-categories essentially allows to work in categories in which

[7] in this case, the uniqueness is not necessarily true if one works at the homology level instead of chain complexes.

[8] if G is a linear representation of a group H and X is the classifying space of H, then one recovers this way the group (co)homology of H with value in G

(weak) homotopy equivalences have been somehow "inverted" but which still retain enough information of the topology of the initial categories.

Furthermore, in the monoidal axiom, we can replace the direct sum of chain complexes by any symmetric monoidal structure, for instance by the tensor product $\otimes$ of chain complexes. This yields the notion of homology theory for spaces with values in $(Chain(\mathbb{Z}), \otimes)$ see § 2.1.1. The latter are not determined by a mere chain complex but by a (homotopy) commutative algebra A. This theory is called factorization homology for spaces and commutative algebras and its main properties are detailled in § 2. In fact, already at this level, we see that one needs to replace the cone construction in the excision axiom by an appropriate derived functor.

To produce invariant of manifolds which are not invariant of spaces, one needs to replace Top by another topological category of manifolds. For instance, fixing $n \in \mathbb{N}$, one can consider the category Mfld_n^{fr} whose objects are *framed manifolds of dimension n* and whose morphisms are framed embeddings. In that case, an homology theory is completely determined by an E_n-algebra. The precise definitions and variants of homology theories for various classes of structured manifolds (including the local coefficient ones) and the appropriate notion of coefficient is the content of § 3.

The (variants of) E_n-algebras which arise as coefficient of homology theory for manifolds can be seen as a special case of factorization algebras, and more precisely as locally constant factorization algebras, which are to factorization algebras what (acyclic resolutions of) locally constant sheaves are to sheaves. This point of view is detailed in § 4.2 and extended to stratified spaces in § 6. The latter case gives simple[9] description of several categories of modules over E_n-algebras as well as categories of E_n-algebras acting on E_m-algebras, which is used in the many applications of § 7.

1.2 Notation and conventions

1. Let k be a commutative unital ring. The ∞-category of differential graded k-modules (*i.e.* chain complexes) will be denoted $Chain(k)$. The (*derived*) tensor product over k will be denoted $\otimes$. The k-linear dual of $M \in Chain(k)$ will be denoted $M^\vee$.

2. All manifolds are assumed to be Hausdorff, second countable, paracompact and thus metrizable.

3. We write Top for the ∞-category of topological spaces (up to homotopy) and Top^f for its sub ∞-category spanned by the (spaces with the homotopy type of) finite CW-complexes. We also denote Top_*, resp. $sSet_*$, the ∞-categories of *pointed* topological spaces and simplicial sets. We write $C_*(X)$ *and* $C^*(X)$ *for the singular chain and cochain complex* of a space X. We write $sSet$ for

[9]in the sense that the cosheaf condition satisfied by factorization algebras encodes some topology which, from the classical E_n-operad point of view necessitates an heavier homotopical machinery.

the ∞-category of simplicial sets (up to homotopy) which is equivalent to Top.

4. The ∞-categories of unital commutative differential graded algebras (up tho homotopy) will be denoted by $CDGA$. We simply refer to unital commutative differential graded algebras as CDGAs.

5. Let $n \in \mathbb{N} \cup \{\infty\}$. By an E_n-algebra we mean an algebra over an E_n-operad. We write E_n**-Alg** for the ∞-category of (unital) E_n-algebras (in $Chain(k)$). See Appendix: § 10.2. We also write $E_n\text{-}\mathbf{Mod}_A$, resp. $E_1\text{-}\mathbf{LMod}_A$, resp. $E_1\text{-}\mathbf{RMod}_A$ the ∞-categories of E_n-A-modules, resp. left A-modules, resp. right A-modules.

6. The ∞-category of (small) ∞-categories will be denoted ∞**-Cat**.

7. We work with a cohomological grading (unless otherwise stated) for all our (co)homology groups and graded spaces, even when we use subscripts to denote the grading (so that our chain complexes have a funny grading). In particular, all differentials are of degree $+1$, of the form $d : A^i \to A^{i+1}$ and the homology groups $H_i(X)$ of a space X are concentrated in non-positive degree. If $(C^*, d_C) \in Chain(k)$, we denote $C^*[n]$ the chain complex given by $(C^*[n])^i := C^{i+n}$ with differential $(-1)^n d_C$.

8. We will denote $\mathbf{PFac}_X$, resp. $\mathbf{Fac}_X$, resp. $\mathbf{Fac}_X^{lc}$ the ∞-categories of prefactorization algebras, resp. factorization algebras, resp. locally constant factorization algebras over X. See Definition 15.

9. Usually, if C a (topological or simplicial or model) category, we will use the boldface letter $\mathbf{C}$ to denote the ∞-category associated to C (see Appendix 10). This is for instance the case for the categories of topological spaces or chain complexes or CDGAs mentionned above.

10. Despite their names, the values of Hochschild or factorization (co)homology will be (co)chain complexes (up to equivalences), *i.e.* objects of $Chain(k)$, or objects of another ∞-category such as E_∞**-Alg**.

These notes deal mainly with applications of factorization algebras in algebraic topology and homotopical algebra. However, there are very interesting applications to mathematical physics as described in the work of Costello *et al* [CG, C2, C3], [GrGw, Gw] and also beautiful applications in algebraic geometry and geometric representation theory, for instance see [BD, FG, Ga1, Ga2].

We almost always refer to the existing literature for proofs; though there are some exceptions to this rule, mainly in sections § 5, § 6 and § 8, where we treat several new (or not detailed in the literature) examples and results related to factorization algebras. To help the reader browsing through the examples in § 5 and § 6, the longer proofs are postponed to a dedicated appendix, namely § 9. Some other references concerning factorization algebras and factorization homology include [L3, L2] and [An, AFT, Ca2, F1, F2, GTZ2, GTZ3, Ta].

About ∞-categories: We use ∞-categories as a convenient framework for homotopical algebra and in particular as higher categorical *derived categories*. In our context, they will typically arise when one consider a topological category or a category $\mathscr{M}$ with a notion of (weak homotopy) equivalence. The ∞-category associated to that case will be a lifting of the homotopy category $Ho(\mathscr{M})$ (the category obtained by formally inverting the equivalences). It has *spaces* of morphisms and composition and associativity laws are defined up to coherent homotopies. We recall some basic examples and definitions in the Appendix: § 10.1.

Topological categories and continuous functors between them are actually a model for ∞-categories and (∞-)functors between them. By a topological category we mean a category $\mathscr{C}$ endowed with a *space* of morphisms $\mathrm{Map}_{\mathscr{C}}(x,y)$ between objects such that the composition $\mathrm{Map}_{\mathscr{C}}(x,y)\times \mathrm{Map}_{\mathscr{C}}(y,z)\to \mathrm{Map}_{\mathscr{C}}(x,z)$ is continuous. A continuous functor $F:\mathscr{C}\to\mathscr{D}$ between topological categories is a functor (of the underlying categories) such that for all objects x, y, the map $\mathrm{Map}_{\mathscr{C}}(x,y)\stackrel{F}{\to}\mathrm{Map}_{\mathscr{D}}(F(x),F(y))$ is continuous. In fact, every ∞-category admits a strict model, in other word is equivalent to a topological category (though finding a strict model can be hard in practice). One can also replace, in the previous paragraph, topological categories by simplicially enriched categories, which are the same thing as topological categories where spaces are replaced by simplicial sets (and continuous maps by maps of simplicial sets). In practice many topological categories we consider are geometric realization of simplicially enriched categories.

The reader can thus substitute topological category to ∞-category in every statement of these notes (or also simplicially enriched or even differential graded[10]), but modulo the fact that one may have to replace the topological or ∞-category in question by another equivalent topological one. The same remark applies to functors between ∞-categories. Furthermore, many constructions involving factorization algebras are actually carried out in (topological) categories (which provide concrete models to homotopy equivalent (derived) ∞-category of some algebraic structures).

If $\mathscr{C}$ is an ∞-category, we will denote $\mathrm{Map}_{\mathscr{C}}(x,y)$ its space of morphisms from x to y while we will simply write $\mathrm{Hom}_{\mathscr{D}}(x,y)$ for the morphism *set* of an ordinary category $\mathscr{D}$ (that is a topological category whose space of morphisms are discrete).

Many derived functors of homological algebra have natural extensions to the setting of ∞-categories. In that case we will use the usual derived functor notation to denote their canonical lifting to ∞-category and to emphasize that they can be computed using the *usual resolutions* of homological algebra. For instance, we will denote $(M,N)\mapsto M\otimes^{\mathbb{L}}_A N$ for the functor E_1-$\mathbf{RMod}_A\times E_1$-$\mathbf{LMod}_A\to Chain(k)$ lifting the usual tensor product of left and right modules to their ∞-categories.

[10]in this case, we refer to [To] for the needed homotopy categorical framework on dg-categories

There is a slight exception to this notational rule. We denote $M \otimes N$ the *derived* tensor products of complexes in $Chain(k)$. We do not use a derived tensor product notation since it will be too cumbersome and since in practice it will often be applied in the case where k is a field or M, N are projective over k.

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2 Factorization homology for commutative algebras and spaces and derived higher Hochschild homology

Factorization homology restricted to commutative algebras is also known as *higher Hochschild homology* and has been studied (in various guise) since at least the end of the 90s (see the approach of [EKMM, MCSV] to topological Hochschild homology, or, the work of Pirashvili [P] which is closely related to Γ-homology). Though its axiomatic description is an easy corollary of the description of Top as a symmetric monoidal category with pushouts, it has a lot of nice properties and appealing combinatorial description in characteristic zero (related to rational homotopy theory à la Sullivan). We review some of its main properties in this Section.

2.1 Homology theory for spaces and derived Hochschild homology

2.1.1 Axiomatic presentation

Let us first start by defining the axioms of an homology theory for spaces with values in the symmetric monoidal ∞-category $(Chain(k),\otimes)$ (instead of $(Chain(\mathbb{Z}),\oplus)$). The (homotopy) commutative monoids in $(Chain(k),\otimes)$ are the E_∞-algebras (Definition 34). In characteristic zero, one can restrict to differential graded commutative algebras since the natural functor $CDGA \to E_\infty$**-Alg** is an (homotopy) equivalence.

The ∞-category Top has a symmetric monoidal structure given by disjoint union of spaces $X\coprod Y$, which is also the coproduct of X and Y in Top. The identity map $id_X : X \to X$ yields a canonical map $X\coprod X \stackrel{\coprod id_X}{\longrightarrow} X$ which is associative and commutative in the ordinary category of topological spaces. Hence X is *canonically* a commutative algebra object in $(Top,\coprod)$. And so is its image by a symmetric monoidal functor. We thus have:

Lemma 1. *Let $(\mathscr{C},\otimes)$ be a symmetric monoidal ∞-category. Any symmetric monoidal functor $F : Top \to \mathscr{C}$ has a canonical lift $\tilde{F} : Top \to E_\infty$-**Alg**$(\mathscr{C})$.*

In particular, any homology theory $Top \to Chain(k)$ shall have a canonical factorization $Top \to E_\infty$-**Alg**. This motivates the following definition.

Definition 1. An homology theory for spaces with values in the symmetric monoidal ∞-category $(Chain(k),\otimes)$ is an ∞-functor $\mathscr{CH} : Top \times E_\infty$-**Alg** $\to E_\infty$-**Alg** (denoted $(X,A) \mapsto CH_X(A)$ on the objects), satisfying the following axioms:

i) **(value on a point)** there is a natural equivalence $CH_{pt}(A) \xrightarrow{\simeq} A$ in E_∞-**Alg**;

ii) **(monoidal)** the canonical maps (induced by universal property of coproducts)
$$\bigotimes_{i\in I} CH_{X_i}(A) \xrightarrow{\simeq} CH_{\coprod_{i\in I} X_i}(A)$$
are equivalences (for any set I);

iii) **(excision)** The functor $\mathscr{CH}$ commutes with homotopy pushout of spaces, *i.e.*, the canonical maps (induced by the universal property of derived tensor product)
$$CH_X(A) \overset{\mathbb{L}}{\underset{CH_Z(A)}{\otimes}} CH_Y(A) \xrightarrow{\simeq} CH_{X\cup_Z^h Y}(A)$$
are natural equivalences.

Remark 1. If one replace Top by Top^f then axiom *ii)* is equivalent to saying that, the functors $X \mapsto CH_X(A)$ are symmetric monoidal.

Remark 2 (**Homology theory for spaces in an arbitrary symmetric monoidal ∞-category**). If $(\mathscr{C},\otimes)$ is a symmetric monoidal ∞-category, we define an homology theory with values in $(\mathscr{C},\otimes)$ in the same way, simply replacing $Chain(k)$ by $\mathscr{C}$ (and thus E_∞-**Alg** by E_∞-**Alg**$(\mathscr{C})$).

All results (in particular the existence and uniqueness Theorem 2) in Section 2.1, Section 2.2 and Section 2.3 still hold by just replacing the monoidal structure of $Chain(k)$ by the one of $\mathscr{C}$, provided that $\mathscr{C}$ has colimits and that its monoidal structure commutes with geometric realization.

Theorem 2. 1. *There is an unique*[11] *homology theory for spaces (in the sense of Definition 1).*

2. *This homology theory is given by* derived Hochschild chains*,* i.e.*, there are natural equivalences*
$$A \boxtimes X \cong CH_X(A)$$
where $A \boxtimes X$ is the tensor of the E_∞-algebra A with the space X (see Remark 37). In particular,
$$\mathrm{Map}_{Top}\big(X, \mathrm{Map}_{E_\infty\text{-}\boldsymbol{Alg}}(A,B)\big) \cong \mathrm{Map}_{E_\infty\text{-}\boldsymbol{Alg}}(CH_X(A),B). \tag{1}$$

[11] up to contractible choices

3. ***(generalized uniqueness)*** *Let $F : E_\infty$-**Alg** $\to E_\infty$-**Alg** be a functor. There is an unique functor $Top \times E_\infty$-**Alg** $\to E_\infty$-**Alg** satisfying axioms **ii), iii)** in Definition 1 and whose value on a point is $F(A)$. This functor is $(X,A) \mapsto CH_X(F(A))$.*

Remark 3. Theorem 2 still holds with Top^f instead of Top (and where in axiom *ii)* one restricts to finite sets I). In that case, it can be rephrased as follows:

Proposition 1. *The functor $F \mapsto F(pt)$ from the category of symmetric monoidal functors $Top^f \to Chain(k)$ satisfying excision*[12] *to the category of E_∞-algebras is a natural equivalence.*

Similarly, Theorem 2 can be rephrased in the following way:
"*The functor $F \mapsto F(pt)$ from the category of functors $Top \to Chain(k)$ preserving arbitrary coproducts and satisfying excision to the category of E_∞-algebras is an natural equivalence.*"

An immediate consequence of $A \boxtimes X \cong CH_X(A)$ and the identity (1) is the following natural equivalence

$$CH_{X\times Y}(A) \cong CH_X\big(CH_Y(A)\big) \tag{2}$$

in E_∞-**Alg**. This is the "exponential law" for derived Hochschild homology.

Another interesting consequence of (1) is that, for any spaces K and X and E_∞-algebra A, the identity map in $\mathrm{Map}_{E_\infty\text{-}\mathbf{Alg}}(CH_{X\times K}(A), CH_{X\times K}(A))$ yields a canonical element in $\mathrm{Map}_{Top}\big(K, \mathrm{Map}_{E_\infty\text{-}\mathbf{Alg}}(CH_X(A), CH_{X\times K}(A))\big)$ hence a canonical map of chain complexes

$$\text{tens} : C_*\big(K\big) \otimes CH_X(A) \longrightarrow CH_{K\times X}(A). \tag{3}$$

Similarly, let $f : K \times X \to Y$ be a map of topological spaces, then we get a canonical continuous map $K \longrightarrow Map_{E_\infty\text{-}Alg}(CH_X(A), CH_Y(A))$ or equivalently a chain map $f_* : C_*(K) \otimes CH_X(A) \longrightarrow CH_Y(A)$ in $Chain(k)$ which is just the composition

$$C_*(K) \otimes CH_X(A) \xrightarrow{\text{tens}} CH_{K\times X}(A) \xrightarrow{f_*} CH_Y(A)$$

where the last map is by functoriality of $\mathcal{CH}$ with respect to maps of topological spaces.

Remark 4 (**Group actions on derived Hochschild homology**). Since $\mathcal{CH}$ is a functor of both variables, $CH_X(A)$ has a natural action of the topological monoid $Map_{Top}(X,X)$ (and thus of the group $Homeo(X)$), *i.e.*, there is a monoid[13] map $Map_{Top}(X,X) \to Map_{E_\infty\text{-}\mathbf{Alg}}(CH_X(A), CH_X(A))$. By adjunction, we get a chain map[14] $C_*\big(Map_{Top}(X,X)\big) \otimes CH_X(A) \to CH_X(A)$ which exhibits $CH_X(A)$ as a module over $Map_{Top}(X,X)$ in E_∞-**Alg** .

[12] by Lemma 1, the excision axiom makes sense for any such functor

[13] here, monoid means an homotopy monoid, that is an E_1-algebra in the symmetric monoidal category $(Top, \times)$

[14] and higher homotopy coherences

2.1.2 Derived functor interpretation

We now explain a derived functor interpretation of derived Hochschild homology. Recall (Example 65) that the singular chain functor of a space X has a natural structure of E_∞-coalgebras. In other words, it is an object (abusively denoted $C_*(X)$) of $\mathbf{Fun}^{\otimes}(\mathbf{Fin}^{op}, Chain(k))$ the category of *contravariant* symmetric monoidal functor from finite sets to chain complexes.

We can identify an E_∞-coalgebra C, resp. an E_∞-algebra A, respectively, with a *right* module, resp. *left* module over the (∞-)operad $\mathbb{E}_\infty$; or equivalently with contravariant, resp. covariant, symmetric monoidal functors from **Fin** to $Chain(k)$). We can thus form their (derived) tensor products $C \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty}{\otimes}} A \in Chain(k)$ which is computed as a (homotopy) coequalizer:

$$C \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty^{\otimes}}{\otimes}} A \cong hocoeq\Big(\coprod_{f:\{1,\dots,q\}\to\{1,\dots,p\}} C^{\otimes p} \otimes \mathbb{E}_\infty(q,p) \otimes A^{\otimes q} \rightrightarrows \coprod_n C^{\otimes n} \otimes A^{\otimes n}\Big)$$

where the maps $f : \{1,\dots,q\} \to \{1,\dots,p\}$ are maps of sets. The upper map in the coequalizer is induced by the maps $f^* : C^{\otimes p} \otimes \mathbb{E}_\infty(q,p) \otimes A^{\otimes q} \to C^{\otimes q} \otimes A^{\otimes q}$ obtained from the coalgebra structure of C and the lower map is induced by the maps $f_* : C^{\otimes p} \otimes \mathbb{E}_\infty^{\otimes}(q,p) \otimes A^{\otimes q} \to C^{\otimes p} \otimes A^{\otimes p}$ induced by the algebra structure. One can define similarly $C \overset{\mathbb{L}}{\underset{\mathbf{Fin}}{\otimes}} A$ the derived tensor product of a covariant and contravariant **Fin**-modules.

Proposition 2. *Let X be a space and A be an E_∞-algebra. There is a natural equivalence (in $Chain(k)$)*

$$CH_X(A) \;\cong\; C_*(X) \overset{\mathbb{L}}{\underset{\mathbb{E}_\infty}{\otimes}} A.$$

If A has a structure of CDGA, then we further have $CH_X(A) \;\cong\; C_*(X) \overset{\mathbb{L}}{\underset{\mathbf{Fin}}{\otimes}} A$

Proof. Note that the E_∞-coalgebra structure on $C_*(X_\bullet)$ is given by the functor $\mathbf{Fin}^{op} \to Chain(k)$ defined by $I \mapsto k\big[Hom_{Fin}(I, X_\bullet)\big]$. The rest of the proof is the same as in [GTZ2, Proposition 4]. □ □

Remark 5 (Factorization homology of commutative algebras as derived mapping stacks). There is another nice interpretation of derived Hochschild homology in terms of derived (or homotopical) algebraic geometry. Let $\mathbf{dSt}_k$ be the ∞-category of derived stacks over the ground ring k described in details in [TV2, Section 2.2]. This category admits internal Hom's that we denote by $\mathbb{R}Map(F,G)$ following [TV2, TV3] and further is also an enrichment of the homotopy category of spaces. Indeed, any simplicial set $X_\bullet$ yields a constant simplicial presheaf $E_\infty\text{-}Alg \to sSet$ defined by $R \mapsto X_\bullet$ which, in turn, can be stackified. We denote $\mathfrak{X}$ the associated stack, *i.e.* the stackification of $R \mapsto X_\bullet$, which depends only on the (weak) homotopy type of $X_\bullet$. For a (derived) stack $\mathfrak{Y} \in \mathbf{dSt}_k$, we denote $\mathscr{O}_{\mathfrak{Y}}$ its functions, *i.e.*, $\mathscr{O}_{\mathfrak{Y}} := \mathbb{R}\underline{Hom}(\mathfrak{Y}, \mathbb{A}^1)$, (see [TV2]). A direct application of Theorem 2 is:

Corollary 1 ([GTZ2]). *Let $\mathfrak{R} = \mathbb{R}Spec(R)$ be an affine derived stack (for instance an affine stack) [TV2] and $\mathfrak{X}$ be the stack associated to a space X. Then the Hochschild chains over X with coefficients in R represent the mapping stack $\mathbb{R}Map(\mathfrak{X}, \mathfrak{R})$. That is, there are canonical equivalences*

$$\mathscr{O}_{\mathbb{R}Map(\mathfrak{X},\mathfrak{R})} \cong CH_X(R), \qquad \mathbb{R}Map(\mathfrak{X}, \mathfrak{R}) \cong \mathbb{R}Spec\big(CH_X(R)\big)$$

If a group G acts on X, the natural action of G on $CH_X(A)$ (Remark 4) identifies with the natural one on $\mathbb{R}Map(\mathfrak{X}, \mathfrak{R})$ under the equivalence given by Corollary 1.

2.2 Pointed spaces and higher Hochschild cohomology

In order to have a dual and relative versions of the construction of § 2.1, we consider the (∞-)category Top_* of *pointed* spaces. Let $\tau : pt \to X$ be a base point of $X \in Top_*$. The map τ yields a map of E_∞-algebras $A \cong CH_{pt}(A) \xrightarrow{\tau_*} CH_X(A)$ and thus makes $CH_X(A)$ an A-module. Let M be an E_∞-module over A; for instance, take M to be a module over a CDGA A. Note that M has induced left and right modules structures[15] over A.

Definition 2. Let A be an E_∞-algebra and M be an E_∞-module over A.

- The (derived) *Hochschild cochains* of A with values in M over (a pointed topological space) X is given by

  $$CH^X(A, M) := \mathbb{R}\mathrm{Hom}_A^{left}(CH_X(A), M),$$

  the (derived) chain complex of homomorphisms of underlying left E_1-modules over A (Definition 36).

- The (derived) *Hochschild homology* of A with values in M over (a pointed space) X is defined as

  $$CH_X(A, M) := M \overset{\mathbb{L}}{\underset{A}{\otimes}} CH_{X_\bullet}(A)$$

  the relative tensor product of (a left and a right) E_1-modules over A.

The two definitions above depend on the choice of the base point even though we do not write it explicitly in the definition.

Remark 6. One can also use the relative tensor products of E_∞-modules over A (as defined, for instance, in [L3, KM]) for defining the Hochschild homology $CH_X(A, M)$. This does not change the computation (and makes Lemma 2 below trivial) according to Proposition 40 (or [L3, KM]). The same remark applies to the definition of derived Hochschild cohomology.

[15]Note also that there is an equivalence of ∞-categories $E_1\text{-}\mathbf{LMod}_A \cong E_1\text{-}\mathbf{RMod}_A$ if $A \in E_\infty\text{-}\mathbf{Alg}$

Since the based point map $\tau_* : A \to CH_X(A)$ is a map of E_∞-algebras, the canonical module structure of $CH_X(A)$ over itself induces a $CH_X(A)$-module structure on $CH_X(A,M)$ after tensoring by A (see [KM, Part V], [L3]):

Lemma 2. *Let M be in E_∞-**Mod**$_A$, that is, M is an E_∞-A-module. Then $CH_X(A,M)$ is canonically a E_∞-module over $CH_X(A)$.*

The Lemma is obvious when A is a CDGA.

We have the ∞-category E_∞-**Mod** of pairs (A,M) with A an E_∞-algebra and M an A-module (Definition 35). Let $\pi_{E_\infty} : E_\infty$-**Mod** $\to E_\infty$-**Alg** be the canonical functor.

Proposition 3 ([GTZ2, GTZ3]).

- *The derived Hochschild chains (Definition 2) induces a functor of ∞-categories $CH : (X,M) \mapsto CH_X(\pi_{E_\infty}(M),M)$ from $Top_* \times E_\infty$-**Mod** to E_∞-**Mod** which fits into a commutative diagram*

$$\begin{tikzcd} Top_* \times E_\infty\text{-}\boldsymbol{Mod} \arrow[r, "CH"] \arrow[d, "for \times \pi_{E_\infty}"'] & E_\infty\text{-}\boldsymbol{Mod} \arrow[d, "\pi_{E_\infty}"] \\ Top \times E_\infty\text{-}\boldsymbol{Alg} \arrow[r, "\mathscr{CH}"] & E_\infty\text{-}\boldsymbol{Alg}. \end{tikzcd}$$

  Here $for : Top_* \to Top$ forget the base point.

- *The derived Hochschild cochains (Definition 2) induces a functor of ∞-categories $(X,M) \mapsto CH^{X_\bullet}(A,M)$ from $(Top_*)^{op} \times E_\infty$-**Mod**$_A$ to E_∞-**Mod**$_A$, which is further contravariant with respect to A.*

In particular, if $M = A$, then we have an natural equivalence $CH_X(A,A) \cong CH_X(A)$ in E_∞-**Mod**[16].

Remark 7 (**Functor homology point of view**). There is also a derived functor interpretation of the above functors as in § 2.1.2. Let **Fin**$_*$ be the ∞-category associated to the category of pointed finite sets (Example 57). If X is pointed, then we have a functor $\tilde{C}_*(X) : \mathbf{Fin}_*{}^{op} \to Chain(k)$ which sends a finite pointed set I to $C_*(\mathrm{Map}_{pointed}(I,X))$ the singular chain on the space of pointed maps from I to X. Further, let M be an E_∞-module. Similarly to § 2.1.2 we find a symmetric monoidal functor $\tilde{M} : \mathbf{Fin}_* \to Chain(k)$. When M is a module over a CDGA A, denoting $*$ the base point, this is simply the functor $\tilde{M}(\{*\} \coprod J) = M \otimes A^{\otimes J}$, see [Gi]. This functor actually factors through E_1-**LMod**$_A$.

We have a dual version of $\tilde{M}$, that we denote $\mathscr{H}(A,M) : \mathbf{Fin}_*{}^{op} \to Chain(k)$, defined as $\mathscr{H}(A,M)(J) := \mathrm{Hom}_A(\tilde{A}(J),M)$ (where $J \mapsto \tilde{A}(J)$ is the functor $\mathbf{Fin}_* \to E_1$-**LMod**$_A$ defined by the canonical E_∞-module structure of A). See [Gi] for an explicit construction when A is a CDGA and M a module.

A proof similar to the one of Proposition 2 yields:

[16] here we implicitly use the canonical functor E_∞-**Alg** $\to E_\infty$-**Mod** which sees an A-lagebra as a module over itself

Proposition 4. *There are natural equivalences*

$$CH_X(A,M) \cong \tilde{C}_*(X) \overset{\mathbb{L}}{\underset{\mathbf{Fin}_*}{\otimes}} \tilde{M}, \qquad CH^X(A,M) \cong \mathbb{R}Hom_{\mathbf{Fin}_*}(\tilde{C}_*(X), \mathscr{H}(A,M)).$$

2.3 Explicit model for derived Hochschild chains

Following Pirashvili [P], one can construct rather simple explicit chain complexes computing derived Hochschild chains when the input is a CDGA. We mainly deal with the unpointed case, the pointed one being similar and left to the reader.

In this section, we consider only CDGAs. Note that, if we assume k is of characteristic zero, E_∞-algebras are always homotopy equivalent to a CDGA so that we do not loose much generality. This construction, using simplicial sets as models for topological spaces, provides explicit semi-free resolutions for $CH_X(A)$ which makes them combinatorially appealing.

Let $(A = \bigoplus_{i\in\mathbb{Z}} A^i, d, \mu)$ be a differential graded, associative, commutative algebra and let n_+ be the set $n_+ := \{0,\dots,n\}$. We define $CH_{n_+}(A) := A^{\otimes n+1} \cong A^{\otimes n_+}$. Let $f: k_+ \to \ell_+$ be any set map, we denote by $f_*: A^{\otimes k_+} \to A^{\otimes \ell_+}$, the linear map given by

$$f_*(a_0\otimes a_1\otimes\cdots\otimes a_k) = (-1)^\varepsilon \cdot b_0\otimes b_1\otimes\cdots\otimes b_\ell, \tag{4}$$

where $b_j = \prod_{i\in f^{-1}(j)} a_i$ (or $b_j = 1$ if $f^{-1}(j) = \emptyset$) for $j = 0,\dots,\ell$. The sign ε in equation (4) is determined by the usual Koszul sign rule of $(-1)^{|x|\cdot|y|}$ whenever x moves across y. In particular, $n_+ \mapsto CH_{n_+}(A)$ is functorial. Extending the construction by colimit we obtain a well-defined functor

$$Y \mapsto CH_Y(A) := \varinjlim_{Fin\ni K\to Y} CH_K(A) \tag{5}$$

from sets to differential graded commutative algebras (since the tensor products of CDGAs is a CDGA). Now, if $Y_\bullet$ is a simplicial set, we get a simplicial CDGA $CH_{Y_\bullet}(A)$ and by the Dold-Kan construction a CDGA whose product is induced by the *shuffle product* which is defined (in simplicial degree p, q) as the composition

$$\begin{aligned} sh: CH_{Y_p}(A)\otimes CH_{Y_q}(A) \xrightarrow{sh^\times} CH_{Y_{p+q}}(A)\otimes CH_{Y_{p+q}}(A) \cong CH_{Y_{p+q}}(A\otimes A) \\ \xrightarrow{\mu_*} CH_{Y_{p+q}}(A). \end{aligned} \tag{6}$$

Here $\mu: A\otimes A\to A$ denotes the multiplication in A (which is a map of algebras) and, denoting s_i the degeneracies of the simplicial structure in $CH_{Y_\bullet}(A)$,

$$sh^\times(v\otimes w) = \sum_{(\mu,\nu)} sgn(\mu,\nu)(s_{\nu_q}\dots s_{\nu_1}(v)\otimes s_{\mu_p}\dots s_{\mu_1}(w)),$$

where (μ,ν) denotes a (p,q)-shuffle, i.e. a permutation of $\{0,\dots,p+q-1\}$ mapping $0\le j\le p-1$ to μ_{j+1} and $p\le j\le p+q-1$ to ν_{j-p+1}, such that $\mu_1<\cdots<\mu_p$ and $\nu_1<\cdots<\nu_q$. The differential $D: CH_{Y_\bullet}(A)\to CH_{Y_\bullet}(A)[1]$ is given as follows.

The tensor products of chain complexes $A^{\otimes Y_i}$ have an internal differential which we abusively denote as d since it is induced by the differential $d : A \to A[1]$. Then, the differential on $CH_{Y_\bullet}(A)$ is given by the formula:

$$D\big(\bigotimes_{i\in Y_i} a_i\big) \quad := \quad (-1)^i d\big(\bigotimes_{i\in Y_i} a_i\big) + \sum_{r=0}^{i} (-1)^r (d_r)_*\big(\bigotimes_{i\in Y_i} a_i\big),$$

where the $(d_r)_* : CH_{Y_i}(A) \to CH_{Y_{i-1}}(A)$ are induced by the corresponding faces $d_r : Y_i \to Y_{i-1}$ of the simplicial set $Y_\bullet$.

Definition 3. Let $Y_\bullet$ be a simplicial set. The Hochschild chains over $Y_\bullet$ of A is the commutative differential graded algebra $(CH_{Y_\bullet}(A), D, sh)$.

The rule $(Y_\bullet, A) \mapsto (CH_{Y_\bullet}(A), D, sh)$ is a bifunctor from the ordinary discrete categories of simplicial sets and CDGA to the ordinary discrete category of CDGA.

If $Y_\bullet$ is a *pointed* simplicial set, we have a canonical CDGA map $A \overset{\sim}{\to} CH_{pt_\bullet}(A) \to CH_{Y_\bullet}(A)$. This allows to mimick Definition 2:

Definition 4. Let $Y_\bullet$ be a simplicial set, A a CDGA and M an A-module (viewed as a symmetric bimodule).

- The *Hochschild chains* of A with values in M over $Y_\bullet$ are:

$$CH_{X_\bullet}(A,M) := M \underset{A}{\otimes} CH_{X_\bullet}(A).$$

- The *Hochschild cochains* of A with values in M over $Y_\bullet$ are:

$$CH^{X_\bullet}(A,M) = Hom_A(CH_{X_\bullet}(A), M).$$

The above definition computes the derived Hochschild homology of Theorem 2. Indeed, we have the adjunction $|-| : sSet \underset{\sim}{\overset{\sim}{\rightleftarrows}} Top : \Delta_\bullet(-)$ given by the geometric realization $|Y_\bullet|$ of a simplicial set and the singular set functor: $n \mapsto \Delta_n(X) := \mathrm{Hom}_{Top}(\Delta^n, X)$ (where $\Delta^n \in Top$ is the standard n-simplex). This adjunction is a Quillen adjunction hence induces an equivalence of ∞-categories. Further (by unicity Theorem 2) we have a commutative diagram (in $\mathrm{Fun}(sSet \times CDGA, E_\infty\text{-}\mathbf{Alg})$)

$$\begin{tikzcd} sSet \times CDGA \arrow[rr, "(Y_\bullet,A)\mapsto CH_{Y_\bullet}(A)"] \arrow[d, "|-|"', "\simeq"] & & CDGA \arrow[d] \\ Top \times CDGA \arrow[r] & Top \times E_\infty\text{-}\mathbf{Alg} \arrow[r, "\mathcal{CH}"] & E_\infty\text{-}\mathbf{Alg}, \end{tikzcd} \tag{7}$$

see [GTZ2, GTZ3] for more details. From there, we get

Proposition 5. *One has natural equivalences $CH_{X_\bullet}(A) \cong CH_{|X_\bullet|}(A)$ of E_∞-algebras as well as equivalences*

$$CH_{X_\bullet}(A,M) \cong CH_{|X_\bullet|}(A,M), \qquad CH^{X_\bullet}(A,M) \cong CH^{|X_\bullet|}(A,M)$$

of $CH_{|X_\bullet|}(A)$-modules.

We now demonstrate the above combinatorial definitions in a few examples (in which we assume, for simplicity, that A has a trivial differential).

Example 1 (The point and the interval). The point has a trivial simplicial model given by the constant simplicial set $pt_n = \{pt\}$. Hence

$$(CH_{pt_\bullet}(A), D) := A \overset{0}{\leftarrow} A \overset{id}{\leftarrow} A \overset{0}{\leftarrow} A \overset{id}{\leftarrow} A \cdots$$

which is a deformation retract of A (as a CDGA). A (pointed) simplicial model for the interval $I = [0,1]$ is given by $I_n = \{\underline{0}, 1 \cdots, n+1\}$, hence in simplicial degree n, $CH_{I_n}(A,M) = M \otimes A^{\otimes n+1}$ and the simplicial face maps are

$$d_i(a_0 \otimes \cdots a_{n+1}) = a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_{n+1}.$$

An easy computation shows that $CH_{I_\bullet}(A,M) = Bar(M,A,A)$ is the standard Bar construction[17] which is quasi-isomorphic to M.

Example 2 (The circle). The circle $S^1 \cong I/(0 \sim 1)$ has (by Example 1) a simplicial model $S^1_\bullet$ which is the quotient $S^1_n = I_n/(0 \sim n+1) \cong \{0, \ldots, n\}$. One computes that the face maps $d_i : S^1_n \to S^1_{n-1}$, for $0 \leq i \leq n-1$ are given by $d_i(j)$ is equal to j or $j-1$ depending on $j = 0, \ldots, i$ or $j = i+1, \ldots, n$ and $d_n(j)$ is equal to j or 0 depending on $j = 0, \ldots, n-1$ or $j = n$. For $i = 0, \ldots, n$, the degeneracies $s_i(j)$ is equal to j or $j+1$ depending on $j = 0, \ldots, i$ or $j = i+1, \ldots, n$. This is the standard simplicial model of S^1 cf. [L, 6.4.2]. Thus, $CH_{S^1_\bullet}(A) = \bigoplus_{n \geq 0} A \otimes A^{\otimes n}$ and the differential agrees with the *usual one on the Hochschild chain complex* $C_\bullet(A)$ of A (see [L]).

It can be proved that the S^1 action on $CH_{S^1}(A)$ given by Remark 4 agrees with the canonical mixed complex structure of $CH_{S^1_\bullet}(A)$ (see [TV4]).

Example 3 (The torus). The torus $\mathbb{T}$ is the product $S^1 \times S^1$. Thus, by Example 2, it has a simplicial model given by $(S^1 \times S^1)_\bullet$ the diagonal simplicial set associated to the bisimplicial set $S^1_\bullet \times S^1_\bullet$, i.e. $(S^1 \times S^1)_k = S^1_k \times S^1_k = \{0, \ldots, k\}^2$. We may write $(S^1 \times S^1)_k = \{(p,q) \,|\, p,q = 0, \ldots, k\}$ which we equipped with the lexicographical ordering. The face maps $d_i : (S^1 \times S^1)_k \to (S^1 \times S^1)_{k-1}$ and degeneracies $s_i : (S^1 \times S^1)_k \to (S^1 \times S^1)_{k+1}$, for $i = 0, \ldots, k$, are given as the products of the differentials and degeneracies of $S^1_\bullet$, i.e. $d_i(p,q) = (d_i(p), d_i(q))$ and $s_i(p,q) = (s_i(p), s_i(q))$.

We obtain $CH_{(S^1 \times S^1)_\bullet}(A,A) = \bigoplus_{k \geq 0} A \otimes A^{\otimes (k^2+2k)}$. The face maps d_i can be described more explicitly, when placing the tensor $a_{(0,0)} \otimes \cdots \otimes a_{(k,k)}$ in a $(k+1) \times (k+1)$ matrix. For $i = 0, \ldots, k-1$, we obtain $d_i(a_{(0,0)} \otimes \cdots \otimes a_{(k,k)})$ by multiplying the ith and $(i+1)$th rows and the ith and $(i+1)$th columns simultaneously, *i.e.*,

[17] the two-sided one, with values in the two A-modules A and M

$d_i(a_{(0,0)} \otimes \cdots \otimes a_{(k,k)})$ is equal to:

$$\begin{array}{ccccc}
a_{(0,0)} & \cdots & (\mathbf{a_{(0,i)}a_{(0,i+1)}}) & \cdots & \mathbf{a_{(0,k)}} \\
\vdots & & \vdots & & \vdots \\
\mathbf{a_{(i-1,0)}} & \cdots & (a_{(i-1,i)}a_{(i-1,i+1)}) & \cdots & \otimes a_{(i-1,k)} \\
(\mathbf{a_{(i,0)}a_{(i+1,0)}}) & \cdots & (a_{(i,i)}a_{(i,i+1)}a_{(i+1,i)}a_{(i+1,i+1)}) & \cdots & (a_{(i,k)}a_{(i+1,k)}) \\
\mathbf{a_{(i+2,0)}} & \cdots & (a_{(i+2,i)}a_{(i+2,i+1)}) & \cdots & a_{(i+2,k)} \\
\vdots & & \vdots & & \vdots \\
\mathbf{a_{(k,0)}} & \cdots & (a_{(k,i)}a_{(k,i+1)}) & \cdots & a_{(k,k)}
\end{array}$$

The differential d_k is obtained by multiplying the kth and 0th rows and the kth and 0th columns simultaneously, *i.e.*, $d_k(a_{(0,0)} \otimes \cdots \otimes a_{(k,k)})$ equals

$$\begin{array}{cccc}
(a_{(0,0)}a_{(0,k)}a_{(k,0)}a_{(k,k)}) & (\mathbf{a_{(0,1)}a_{(k,1)}}) & \cdots & (\mathbf{a_{(0,k-1)}a_{(k,k-1)}}) \\
(\mathbf{a_{(1,0)}a_{(1,k)}}) & a_{(1,1)} & \cdots & a_{(1,k-1)} \\
\vdots & \vdots & & \vdots \\
(\mathbf{a_{(k-1,0)}a_{(k-1,k)}}) & a_{(k-1,1)} & \cdots & a_{(k-1,k-1)}
\end{array}$$

Example 4 (The Riemann sphere S^2). The sphere S^2 has a simplicial model $S^2_\bullet = I^2_\bullet/\partial I^2_\bullet$ i.e. $S^2_n = \{(0,0)\} \coprod \{1 \cdots n\}^2$. Thus $CH_{S^2_\bullet}(A) = \bigoplus_{n\geq 0} A \otimes A^{\otimes n^2}$.

Here the face and degeneracies maps are the diagonal ones as for $(S^1 \times S^1)_\bullet$ in Example 3. In particular, the ith differential is also obtained from the previous examples by setting $d_i^{S^2_\bullet}(p,q) = (0,0)$ in the case that $d_i(p) = 0$ or $d_i(q) = 0$ (where d_i is the i^{th}-face map of $S^1_\bullet$), or setting otherwise $d_i(p,q) = (d_i(p), d_i(q))$. For $i \leq n-1$, we obtain $d_i(a_{(0,0)} \otimes \cdots \otimes a_{(k,k)})$ is equal to:

$$\begin{array}{cccc}
a_{(0,0)} & & & \\
 & (a_{(i-1,i)}a_{(i-1,i+1)}) & \cdots & a_{(i-1,n)} \\
 & (a_{(i,i)}a_{(i,i+1)}a_{(i+1,i)}a_{(i+1,i+1)}) & \cdots & (a_{(i,n)}a_{(i+1,n)}) \\
 & (a_{(i+2,i)}a_{(i+2,i+1)}) & \cdots & a_{(i+2,n)} \\
 & \vdots & & \vdots \\
 & (a_{(n,i)}a_{(n,i+1)}) & \cdots & a_{(n,n)}
\end{array}$$

which is similar to the one of Example 3 without the "boldface" tensors.

Example 5 (higher spheres). Similarly to S^2, we have the *standard model* $S^d_\bullet := (I_\bullet)^d/\partial(I_\bullet)^d \cong S^1_\bullet \wedge \cdots \wedge S^1_\bullet$ (d-factors) for the sphere S^d. Hence $S^d_n \cong \{\underline{0}\} \coprod \{1 \cdots n\}^d$ and $CH_{S^d_\bullet}(A) = \bigoplus_{n\geq 0} A \otimes A^{\otimes n^d}$. The face operators are similar to those of Example 4 (except that, instead of a matrix, we have a dimension d-lattice) and face maps are obtained by simultaneously multiplying each i^{th}-hyperplane with $(i+1)^{\text{th}}$-hyperplane in each dimension. The last face d_n is obtained by multiplying all tensors of all n^{th}-hyperplanes with a_0.

We also have the *small model* $S^d_{sm\bullet}$ which is the simplicial set with exactly two *non-degenerate* simplices, one in degree 0 and one in degree d. Then $S^d_{smn} \cong$

$\{1, \ldots, \binom{n}{d}\}$. Using this model, it is straightforward to check the following computation of the first homology groups of $CH_{S^d}(A)$:

$$H_n(CH_{S^d}(A)) \cong H_n(CH_{S^d_{sm_\bullet}}(A)) = \begin{cases} = A & \text{if } n = 0 \\ = 0 & \text{if } 0 < n < d \\ = \Omega^1_A & \text{if } n = d \end{cases}$$

where Ω^1_A is the A-module of Kähler differentials (see [L, W]).

Example 6 (Hochschild-Kostant-Rosenberg). Let A be a smooth commutative algebra. The classical Hochschild-Kostant-Rosenberg Theorem states that its (standard) Hochschild homology is given by the algebra of Kähler forms $\wedge^\bullet_A(\Omega^1_A) \cong S^\bullet_A(\Omega^1_A[1])$, where Ω^1_A is the A-module of Kähler differentials; here a i-form is viewed as having cohomological degree $-i$ and $S^\bullet_A$ is the free graded commutative algebra functor (in the category of graded A-modules). This theorem extends to Hochschild homology over all spheres:

Theorem 3 (Generalized HKR). *Let A be a smooth algebra and X be an affine smooth scheme or a smooth manifold. Let $n \geq 1$ and Σ^g be a genus g surface.*

1. ***(Pirashvili [P])** There is a quasi-isomorphism of CDGAs: $CH_{S^n}(A) \cong S^\bullet_A\big(\Omega^1_A[n]\big)$.*
2. ***([GTZ])** There is an equivalence $CH_{\Sigma^g}(A) \cong S^\bullet_A\big(\Omega^1_A[2] \oplus (\Omega^1_A[1])^{\oplus 2g}\big)$ of CDGAs.*
3. *There are equivalences $CH_{S^n}(\mathscr{O}_X) \cong S^\bullet_{\mathscr{O}_X}\big(\Omega^1_X[n]\big)$ and*

$$CH_{\Sigma^g}(\mathscr{O}_X) \cong S^\bullet_{\mathscr{O}_X}\big(\Omega^1_X[2] \oplus (\Omega^1_X[1])^{\oplus 2g}\big)$$

of sheaves of CDGAs[18].

The third assertion in Theorem 3 follows from 1 and 2 after sheafifying in an appropriate way the derived Hochschild chains.

2.4 Relationship with mapping spaces

We have seen the relationship between derived Hochschild chains and derived mapping spaces (Remark 5). It is also classical that the usual Hochschild homology of de Rham forms of a simply connected manifold M is a model for the de Rham forms on the free loop space $LM := \mathrm{Map}(S^1, M)$ of M (see [Ch]). There is a generalization of this result for spaces where the forms are replaced by the singular cochains with their E_∞-algebra structure ([Jo]). These two results extend to derived Hochschild chains in general to provide algebraic models of mapping spaces.

First, we sketch a generalization of Chen iterated integrals (studied in [GTZ]). Let M be a compact, oriented manifold, denote by $\Omega^\bullet_{dR}(M)$ the differential graded algebra of differential forms on M, and let $Y_\bullet$ be a simplicial set with geometric realization $Y := |Y_\bullet|$. Denote $M^Y := \mathrm{Map}_{sm}(Y, M)$ the space of continuous maps

[18] here the differentials on the right hand sides are zero; they are *not* the de Rham differential

from Y to M, which are smooth on the interior of each simplex in Y. Recall from Chen [Ch, Definition 1.2.1], that a differentiable structure on M^Y is specified by the set of plots $\phi : U \to M^Y$, where $U \subset \mathbb{R}^n$ for some n, which are those maps whose adjoint $\phi_\sharp : U \times Y \to M$ is continuous on $U \times Y$, and smooth on the restriction to the interior of each simplex of Y, i.e. $\phi_\sharp|_{U\times(\text{simplex of }Y)^\circ}$ is smooth. Following [Ch, Definition 1.2.2], a p-form $\omega \in \Omega^p_{dR}(M^Y)$ on M^Y is given by a p-form $\omega_\phi \in \Omega^p_{dR}(U)$ for each plot $\phi : U \to M^Y$, which is invariant with respect to smooth transformations of the domain.

We now define the space of Chen (generalized) *iterated integrals* $\mathscr{C}hen(M^Y)$ of the mapping space M^Y. Let $\eta : Y_\bullet \to \Delta_\bullet|Y_\bullet|$ be the canonical simplicial map (induced by adjunction) which is given for $i \in Y_k$ by maps $\eta(i) : \Delta^k \to Y$ in the following way,

$$\eta(i)(t_1 \le \dots \le t_k) := [(t_1 \le \dots \le t_k) \times \{i\}] \in \Big(\coprod \Delta^\bullet \times Y_\bullet / \sim\Big) = Y. \tag{8}$$

The map η allows to define, for any plot $\phi : U \to M^Y$, a map $\rho_\phi := ev \circ (\phi \times id)$,

$$\rho_\phi : U \times \Delta^k \xrightarrow{\phi\times id} M^Y \times \Delta^k \xrightarrow{ev} M^{Y_k}, \tag{9}$$

where ev is defined as the evaluation map,

$$ev(\gamma : Y \to M, \underline{t})(i) = \gamma\big(\eta(i)(\underline{t})\big). \tag{10}$$

Now, if we are given a form $\bigotimes_{y\in Y_k} a_y \in \big(\Omega_{dR}(M)\big)^{\otimes Y_k}$ (with only finitely many $a_i \neq 1$), the pullback $(\rho_\phi)^*\big(\bigotimes_{y\in Y_k} a_y\big) \in \Omega^\bullet(U \times \Delta^k)$, may be integrated along the fiber Δ^k, and is denoted by

$$\left(\int_{\mathscr{C}} \bigotimes_{y\in Y_k} a_y\right)_\phi := \int_{\Delta^k} (\rho_\phi)^*\big(\bigotimes_{y\in Y_k} a_y\big) \quad \in \Omega^\bullet_{dR}(U).$$

The resulting $p = (\sum_i deg(a_i) - k)$-form $\int_{\mathscr{C}} \Big(\bigotimes_{y\in Y_k} a_y\Big) \in \Omega^p_{dR}(M^Y$ is called the (generalized) *iterated integral* of $a_0, \dots, a_{y_k}$. The subspace of the space of De Rham forms $\Omega^\bullet(M^Y)$ generated by all iterated integrals is denoted by $\mathscr{C}hen(M^Y)$. In short, we may picture an iterated integral as the pullback composed with the integration along the fiber Δ^k of a form in M^{Y_k},

$$M^Y \xleftarrow{\int_{\Delta^k}} M^Y \times \Delta^k \xrightarrow{ev} M^{Y_k}$$

Definition 5. We define $\mathscr{I}t^{Y_\bullet} : CH_{Y_\bullet}(\Omega^\bullet_{dR}(M)) \cong (\Omega^\bullet_{dR}(M))^{\otimes Y_\bullet} \to \mathscr{C}hen(M^Y)$ by

$$\mathscr{I}t^{Y_\bullet}\left(\bigotimes_{y\in Y_k} a_y\right) := \int_{\mathscr{C}} \Big(\bigotimes_{y\in Y_k} a_y\Big). \tag{11}$$

Interesting applications of iterated integrals to study gerbes and higher holonomy are given in [AbWa, TWZ].

Theorem 4 ([GTZ])**.** *The iterated integral map $\mathscr{I}t^{Y_\bullet}: CH_{Y_\bullet}(\Omega^\bullet_{dR}(M)) \to \Omega^\bullet_{dR}(M^Y)$ is a (natural) map of CDGAs.*

Further, assume that $Y = |Y_\bullet|$ is n-dimensional, i.e. the highest degree of any non-degenerate simplex is n, and assume that M is n-connected. Then, $\mathscr{I}t^{Y_\bullet}$ is a quasi-isomorphism.

There is also a purely topological and characteristic free analogue of this result using singular cochains instead of forms.

Theorem 5 ([F2, GTZ3])**.** *Let X,Y be topological spaces. There is a natural map of E_∞-algebras*

$$CH_Y(C^*(X)) \longrightarrow C^*(\mathrm{Map}(Y,X))$$

which is an equivalence when $Y = |Y_\bullet|$ is n-dimensional and X is connected, nilpotent with finite homotopy groups in degree less or equal to n (for instance when X is n-connected).

Example 7. We compute the iterated integral map (11) in the case of $S^1_\bullet$ (Example 2) and $\mathbb{T}$ (Example 3). Since S^1 is the interval $I = [0,1]$ where the endpoints 0 and 1 are identified, the map $\eta(i) : S^1_k = \{0,1\dots,k\} \to \Delta_k(S^1) = \mathrm{Map}(\Delta^k, S^1)$ defined via (8) is given by $\eta(i)(0 \le t_1 \le \cdots \le t_k \le 1) = t_i$, where we have set $t_0 = 0$. Thus, the evaluation map (10) becomes

$$ev(\gamma : S^1 \to M, t_1 \le \cdots \le t_k) = (\gamma(0), \gamma(t_1), \dots, \gamma(t_k)) \in M^{k+1}.$$

Furthermore, this recovers the classical Chen iterated integrals $It^{S^1_\bullet} : CH_\bullet(A,A) \to \Omega^\bullet(M^{S^1})$ as follows. For a plot $\phi : U \to M^{S^1}$ we have,

$$\begin{aligned} It^{S^1_\bullet}(a_0 \otimes \cdots \otimes a_k)_\phi &= \left(\int_{\mathscr{C}} a_0 \dots a_k\right)_\phi = \int_{\Delta^k} (\rho_\phi)^*(a_0 \otimes \cdots \otimes a_k) \\ &= (\pi_0)^*(a_0) \wedge \int_{\Delta^k} (\widetilde{\rho_\phi})^*(a_1 \otimes \cdots \otimes a_k) = (\pi_0)^*(a_0) \wedge \int a_1 \dots a_k, \end{aligned}$$

where $\widetilde{\rho_\phi} : U \times \Delta^k \xrightarrow{\phi \times id} M^{S^1} \times \Delta^k \xrightarrow{\widetilde{ev}} M^k$ is the classical Chen integral $\int a_1 \dots a_k$ from [Ch] and $\pi_0 : M^{S^1} \to M$ is the evaluation at the base point $\pi_0 : \gamma \mapsto \gamma(0)$.

In the case of the torus $\mathbb{T} = S^1 \times S^1$, the map $\eta(p,q) : (S^1 \times S^1)_k \to \mathrm{Map}(\Delta^k, S^1 \times S^1)$ is given by $\eta(p,q)(0 \le t_1 \le \cdots \le t_k \le 1) = (t_p, t_q) \in S^1 \times S^1$, for $p,q = 0,\dots,k$ and $t_0 = 0$. Thus, the evaluation map (10) becomes

$$ev(\gamma : \mathbb{T} \to M, t_1 \le \cdots \le t_k) = \begin{pmatrix} \gamma(0,0), \gamma(0,t_1), \cdots, \gamma(0,t_k), \\ \gamma(t_1,0), \gamma(t_1,t_1), \cdots, \gamma(t_1,t_k), \\ \dots \\ \gamma(t_k,0), \gamma(t_k,t_1), \cdots, \gamma(t_k,t_k) \end{pmatrix} \in M^{(k+1)^2}$$

According to definition 5, the iterated integral $It^{S^1 \times S^1}(a_{(0,0)} \otimes \cdots \otimes a_{(k,k)})$ is given by a pullback under the above map $M^{S^1 \times S^1} \times \Delta^k \xrightarrow{ev} M^{(k+1)^2}$, and integration along the fiber Δ^k.

2.5 The *wedge product* of higher Hochschild cohomology

Let $A \xrightarrow{f} B$ be a map of CDGAs. Note that it makes B into an A-algebra as well as an $A\otimes A$-algebra (since the multiplication $A\otimes A \to A$ is an algebra morphism). The excision axiom (Theorem 2) implies

Lemma 3. *Let M be an A-module and X,Y be pointed topological spaces. There is a natural equivalence*

$$\mu : Hom_{A\otimes A}(CH_X(A)\otimes CH_Y(A), M) \xrightarrow{\simeq} CH^{X\vee Y}(A,M)$$

We use Lemma 3 to obtain

Definition 6 ([Gi])**.** The *wedge product* of (derived) Hochschild cochains is the linear map

$$\begin{aligned}\mu_\vee : CH^X(A,B)\otimes CH^Y(A,B) &\longrightarrow Hom_{A\otimes A}\Big(CH_X(A)\otimes CH_Y(A), B\otimes B\Big)\\ &\xrightarrow{(m_B)_*} Hom_{A\otimes A}\Big(CH_X(A)\otimes CH_Y(A), B\Big) \cong CH^{X\vee Y}(A,B) \qquad (12)\end{aligned}$$

where the first map is the obvious one: $f\otimes g \mapsto \big(x\otimes y \mapsto f(x)\otimes g(y)\big)$.

Example 8. If $X_\bullet, Y_\bullet$ are finite simplicial sets models of X,Y, the map $\mu_\vee$ can be combinatorially described as the composition of the linear map $\tilde{\mu}$ given, for any $f\in CH^{X_n}(A,B)\cong Hom_A(A^{\otimes \# X_n},B)$, $g\in CH^{X_n}(A,B)\cong Hom_A(A^{\otimes \# Y_n},B))$ by

$$\tilde{\mu}(f,g)(a_0,a_2,\dots a_{\# X_n}, b_2,\dots,b_{\# Y_n}) = a_0.f(1,a_2,\dots a_{\# X_n}).g(1,b_2,\dots,b_{\# Y_n})$$

(where a_0 corresponds to the element indexed by the base point of $X_n\vee Y_n$) with the Eilenberg-Zilber quasi-isomorphism from $CH^{X_\bullet}(A,B)\otimes CH^{Y_\bullet}(A,B)$ to the chain complex associated to the diagonal cosimplicial space $\big(CH^{X_n}(A,B)\otimes CH^{Y_n}(A,B)\big)_{n\in\mathbb{N}}$.

Proposition 6. *The wedge product (of Definition 6) is associative*[19]*. In particular, if there is a diagonal $X \xrightarrow{\delta} X\vee X$ making X an E_1-coalgebra (in $(Top_*,\vee)$), then $(CH^X(A,B),\delta^*\circ\mu_\vee)$ is an E_1-algebra.*

Example 9. A standard example of space with a diagonal is a sphere S^d. For $d=1$, we obtain a cup product on the usual Hochschild cochain complex which is (homotopy) equivalent to the standard cup-product for Hochschild cochains from [Ge].

The little d-dimensional little cubes operad Cube_d acts continuously on S^d by the pinching map

$$pinch : \mathrm{Cube}_d(r)\times S^d \longrightarrow \bigvee_{i=1\dots r} S^d. \qquad (13)$$

[19] precisely, it means that $\mu_\vee$ makes $X\mapsto CH^X(A,B)$ into a lax monoidal functor $((Top_*)^{op},\vee)\to (Chain(k),\otimes)$

given, for any $c \in \mathrm{Cube}_d(r)$, by the map $pinch_c : S^d \to \bigvee S^d$ collapsing the complement of the interiors of the r rectangles to the base point. We thus get a map

$$\tilde{pinch} : \mathrm{Cube}_d(r) \longrightarrow \mathrm{Map}_{CDGA}(CH_{S^d}(A,B), CH_{\bigvee S^d}(A,B)) \tag{14}$$

Applying the contravariance of Hochschild cochains and the wedge product (Definition 6), we get, for all $d \geq 1$, a morphism

$$pinch^*_{S^d,r} : C_*\big(\mathrm{Cube}_d(r)\big) \otimes \Big(CH^{S^d}(A,B)\Big)^{\otimes r} \\ \xrightarrow{(\mu_\vee)^{(d-1)}} C_*\big(\mathrm{Cube}_d(r)\big) \otimes CH^{\bigvee_{i=1}^r S^d}\big(A,B\big) \xrightarrow{\tilde{pinch}^*} CH^{S^d}(A,B). \tag{15}$$

The map (15) has a canonical extension to the case of E_∞-algebras. We find

Proposition 7 ([Gi, GTZ3])**.** *Let $A \xrightarrow{f} B$ be a CDGA (or E_∞-algebra) map. The collection of maps $(pinch_{S^d,k})_{k\geq 1}$ makes $CH^{S^d}(A,B)$ an E_d-algebra.*

The algebra structure is natural with respect to CDGA maps, meaning that given a commutative diagram
$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "\varphi"] \\ A' \arrow[u, "\psi"] \arrow[r, "f'"] & B' \end{tikzcd}$$
, the canonical map $h' \mapsto \varphi \circ h' \circ \psi$ is an E_d-algebras morphism $CH^{S^d}(A',B') \to CH^{S^d}(A,B)$.

Remark 8. If $f : A \to B$ is a CDGA map, it is possible to describe this E_d-algebra structure by giving an explicit action of the filtration $F_d\mathrm{BE}$ of the Barrat-Eccles operad on $CH^{S^d_\bullet}(A,B)$ using the standard simplicial model of S^d (Example 5).

Example 10. If $A = k$, $CH^{S^n}(k,B) \cong B$ (viewed as E_n-algebras). If $B = k$, then the E_n-algebra structure of $CH^{S^n}(A,k)$ is the dual of the E_n-coalgebra structure given by the n-times iterated Bar construction $Bar^{(n)}(A)$, see § 7.4.

3 Homology Theory for manifolds

3.1 Categories of structured manifolds and variations on E_n-algebras

In order to specify what is a homology theory for manifolds, we need to specify an interesting category of manifolds.

Definition 7. Let Mfld_n be the ∞-category associated[20] to the topological category with objects topological manifolds of dimension n and with morphism space

$$\mathrm{Map}_{\mathrm{Mfld}_n}(M,N) := \mathrm{Emb}(M,N)$$

the space of all embeddings of M into N (viewed as a subspace of the space $\mathrm{Map}(M,N)$ of all continuous maps from M to N endowed with the compact-open topology).

[20] by Example 58

In the above definition, the manifolds can be closed or open, but have no boundary[21].

Remark 9. It is important to consider embeddings instead of smooth maps. Indeed, the category of all manifolds and all (smooth) maps is weakly homotopy equivalent to Top so that, in that case, one would obtain a homology theory which extends to spaces.

Remark 10 (smooth manifolds). One can also restrict to *smooth* manifolds in which case it makes sense to equip $\mathrm{Emb}(M,N)$ with the weak Whitney C^∞-topology; this gives us *the* ∞*-category* Mfld_n^{un} *of smooth manifolds of dimension n*. This latter category embeds in Mfld_n and this embedding is an equivalence onto the full subcategory of Mfld_n spanned by the smooth manifolds.

One can also consider categories of more structured manifolds, such as oriented, spin or framed manifolds, as follows. Let $E \to X$ be a topological n-dimensional vector bundle, which is the same as a space X together with a (homotopy class of) map $e : X \to B\mathrm{Homeo}(\mathbb{R}^n)$ from X to the classifying space of the group of homeomorphisms of $\mathbb{R}^n$. *An* (X,e)*-structure on a manifold* $M \in \mathrm{Mfld}_n$ is a map $f : M \to X$ such that TM is the pullback $f^*(E)$ which is the same as a factorization $M \xrightarrow{f} X \xrightarrow{e} B\mathrm{Homeo}(\mathbb{R}^n)$ of the map $M \xrightarrow{e_M} B\mathrm{Homeo}(\mathbb{R}^n)$ classifying the tangent (micro-)bundle of M.

Definition 8. Let $\mathrm{Mfld}_n^{(X,e)}$ be the (homotopy) pullback (in ∞-**Cat**)

$$\mathrm{Mfld}_n^{(X,e)} := \mathrm{Mfld}_n \times^h_{Top_{/B\mathrm{Homeo}(\mathbb{R}^n)}} Top_{/X}.$$

In other words $\mathrm{Mfld}_n^{(X,e)}$ is the ∞-category with objects n-dimensional topological manifolds with an (X,e)-structure and with morphism the embeddings preserving the (X,e)-structure. The latter morphisms are made into a topological space by identifying them with the homotopy pullback space

$$\begin{aligned}\mathrm{Map}_{\mathrm{Mfld}_n^{(X,e)}}(M,N) &:= \mathrm{Emb}^{(X,e)}(M,N)\\ &\cong \mathrm{Emb}(M,N) \times^h_{\mathrm{Map}_{/B\mathrm{Homeo}(\mathbb{R}^n)}(M,N)} \mathrm{Map}_{/X}(M,N).\end{aligned}$$

Example 11. We list our main examples of study.

- Let $X = pt$, then E is trivial (here e is induced by the canonical base point of $B\mathrm{Homeo}(\mathbb{R}^n)$) and an (X,e)-structure on M is a framing, that is, a trivialization of the tangent (micro-)bundle of M. In that case, *we denote* $\mathrm{Mfld}_n^{fr} := \mathrm{Mfld}_n^{(pt,e)}$ *the* ∞*-category of framed manifolds*. Note that this ∞-category is equivalent to the one associated to the topological category with objects the framed manifolds of dimension n and morphism spaces from M to N the framed embeddings, that is the pairs (f,h) where $f \in \mathrm{Emb}(M,N)$ and h is an homotopy between the two trivialisation of TM induced by the framing of M and the framing of N pulled-back along f.

[21] though homology theory for manifolds can be extended to stratified manifolds, see [AFT]

- Let $X = BO(n)$ and $BO(n) \stackrel{e}{\to} B\mathrm{Homeo}(\mathbb{R}^n)$ be the canonical map. Then $\mathrm{Mfld}_n^{(BO(n),e)}$ is (equivalent to) the *∞-category of smooth n-manifolds* of Remark 10. This essentially follows because the map $O(n) \to \mathrm{Diffeo}(\mathbb{R}^n)$ is a deformation retract and the characterization of smooth manifolds in terms of their micro-bundle structure [KiSi].

- Let $X = BSO(n)$ and $BSO(n) \stackrel{e}{\to} B\mathrm{Homeo}(\mathbb{R}^n)$ be the canonical map induced by the inclusion of $SO(n) \hookrightarrow \mathrm{Homeo}(\mathbb{R}^n)$. Then a $(BSO(n),e)$-structure on M is an orientation of M. We denote $\mathrm{Mfld}_n^{or} := \mathrm{Mfld}_n^{(BSO(n),e)}$ *the ∞-category of oriented smooth n-manifolds*. Similarly to the framed case, it has a straightforward description as the ∞-category associated a topological category with morphisms the space of oriented embeddings.

- If X is a n-dimensional manifold, we can take $e_X : X \to B\mathrm{Homeo}(\mathbb{R}^n)$ to be the map corresponding to the tangent bundle $TX \to X$ of X. We simply denote $\mathrm{Mfld}_n^{(X,TX)}$ the associated ∞-category of manifolds. Every open subset of X is canonically an object of $\mathrm{Mfld}_n^{(X,TX)}$.

The (topological) coproduct $M \coprod N$ (that is disjoint union) of two (X,e)-manifolds M,N has a canonical structure of (X,e)-manifold (given by $M \coprod N \stackrel{f\coprod g}{\to} X$ where $M \stackrel{f}{\to} X$ and $N \stackrel{g}{\to} X$ define the (X,e)-structures). Note that in general there are no embeddings $M \coprod M \to M$ so that the disjoint union of manifolds is *not* a coproduct (in the sense of category theory) in $\mathrm{Mfld}_n^{(X,e)}$. Nevertheless

Lemma 4. $(\mathrm{Mfld}_n^{(X,e)}, \coprod)$ *is a symmetric monoidal ∞-category.*

There is a canonical choice of framing of $\mathbb{R}^n$ which induces a canonical (X,e)-structure on $\mathbb{R}^n$ for any pointed space X. Unlike other manifolds, there are interesting framed embeddings $\coprod \mathbb{R}^i \to \mathbb{R}^i$ for any integer i. Indeed, in view of Example 11 and Definition 51, the space of embeddings $\mathrm{Emb}^{fr}(\coprod_{\{1,\dots,r\}} \mathbb{R}^n, \mathbb{R}^n) = \mathrm{Disk}_n^{fr}(r,1)$ is homotopy equivalent to $\mathrm{Cube}_n(r)$ the arity r space of the little cube operad and thus is homotopy equivalent to the configuration space of r unordered points in $\mathbb{R}^n$.

This motivates the following (X,e)-structured version of E_n-algebras.

Definition 9. Let $\mathrm{Disk}_n^{(X,e)}$ be the full subcategory of $\mathrm{Mfld}_n^{(X,e)}$ spanned by disjoints union of standard euclidean disks $\mathbb{R}^n$. The ∞-category of $\mathrm{Disk}_n^{(X,e)}$-algebras[22] is the category

$$\mathbf{Fun}^{\otimes}(\mathrm{Disk}_n^{(X,e)}, \mathit{Chain}(k))$$

of symmetric monoidal (∞-)functors from $(\mathrm{Disk}_n^{(X,e)}, \coprod)$ to $(\mathit{Chain}(k), \otimes)$.

The *underlying object* of a $\mathrm{Disk}_n^{(X,e)}$-algebra A is its value $A(\mathbb{R}^n)$ on a single disk $\mathbb{R}^n$. We will often abusively denote in the same way the $\mathrm{Disk}_n^{(X,e)}$-algebra and its underlying object.

[22] which we also referred to as the category of (X,e)-structured E_n-algebras

We denote $\mathrm{Disk}^{(X,e)}$**-Alg** *the* ∞*-category of* $\mathrm{Disk}_n^{(X,e)}$*-algebras* and $\mathrm{Disk}^{(X,e)}$**-Alg**$(\mathscr{C})$ the one of $\mathrm{Disk}_n^{(X,e)}$-algebras with values in a symmetric monoidal category $\mathscr{C}$ (whose definition are the same as Definition 9 with $(Chain(k),\otimes)$ replaced by $(\mathscr{C},\otimes)$). The underlying object induces a functor

$$\mathrm{Disk}^{(X,e)}\textbf{-Alg}(\mathscr{C}) \longrightarrow \mathscr{C} \tag{16}$$

Example 12.

- For $X = pt$, the category of $\mathrm{Disk}_n^{(X,e)}$-algebras will be *denoted* Disk_n^{fr}***-Alg***. It is equivalent to the usual category of E_n-algebras and corresponds to the case of framed manifolds.
- The category of $\mathrm{Disk}_n^{(BSO(n),e)}$-algebras is equivalent to the category of algebras over the operad $\big(\mathrm{Cube}_n(r) \ltimes SO(n)^r\big)_{r\geq 1}$ introduced in [SW] (and called the framed little disk operad). Since these algebras corresponds to the case of oriented manifolds, we call them *oriented* E_n*-algebras* and we simply write Disk_n^{or} for $\mathrm{Disk}_n^{(BSO(n),e)}$. It can be shown that Disk_n^{or}-algebras are homotopy fixed points of the E_n-algebras with respect to the action of $SO(n)$ on the operad Disk_n^{fr}.
- Similarly, the category of $\mathrm{Disk}_n^{(BO(n),e)}$-algebras is equivalent to the category of algebras over the operad $\big(\mathrm{Cube}_n(r) \ltimes O(n)^r\big)_{r\geq 1}$. We also call them *unoriented* E_n*-algebras* and simply write Disk_n^{un} for $\mathrm{Disk}_n^{(BO(n),e)}$.
- Let $U \cong \mathbb{R}^n$ be a disk in X. By restriction to sub-disks of U, we have a canonical functor $\mathrm{Disk}_n^{(X,TX)}$**-Alg** $\to \mathrm{Disk}_n^{(\mathbb{R}^n,T\mathbb{R}^n)}$**-Alg** $\cong E_n$**-Alg** (see Theorem 9 below). It follows that a $\mathrm{Disk}_n^{(X,TX)}$-algebra is simply a family of E_n-algebras over X.

Example 13 (Commutative algebras as $\mathrm{Disk}_n^{(X,e)}$-algebras). The canonical functor $\mathrm{Disk}_n^{(X,e)} \to$ **Fin** (where **Fin** is the ∞-category associated to the category of finite sets) shows that any E_∞-algebra (Definition 34) has a canonical structure of $\mathrm{Disk}_n^{(X,e)}$-algebras. Thus we have canonical functors

$$CDGA \longrightarrow E_\infty\textbf{-Alg} \longrightarrow \mathrm{Disk}_n^{(X,e)}\textbf{-Alg}.$$

For A a differential graded commutative algebra, this structure is the symmetric monoidal functor defined by $A(\coprod_{i\in I}\mathbb{R}^n) := A^{\otimes I}$ and, for an (X,e)-preserving embedding $\coprod_I \mathbb{R}^n \hookrightarrow \mathbb{R}^n$, by the (iterated) multiplication $A^{\otimes I} \to A$.

Example 14 (Opposite of an E_n-algebra). There is a canonical $\mathbb{Z}/2\mathbb{Z}$-action on E_n**-Alg** induced by the antipodal map $\tau : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto -x$ acting on the source of $\mathbf{Fun}^{\otimes}(Disk_n^{fr}, Chain(k))$. If A is an E_n-algebra, then the result of this action $A^{op} := \tau^*(A)$ is its opposite algebra. If $n = \infty$, the antipodal map is homotopical to the identity so that A^{op} is equivalent to A as an E_∞-algebra.

3.2 Factorization homology of manifolds

We now explain what is a Homology Theory for Manifolds (Definition 10) in a way parallel to the presentation of the Eilenberg-Steenrod axioms. We first need an analogue of Lemma 1 for monoidal functors out of manifolds (instead of spaces) in order to formulate the correct excision axiom.

Observe that $\mathbb{R}^n$ is canonically an E_n-algebra object in Mfld_n. Let N be an $(n-s)$-dimensional manifold such that $N\times\mathbb{R}^s$ has an (X,e)-structure. Then, similarly, $N\times\mathbb{R}^s$ is also an E_s-algebra object in $\mathrm{Mfld}_n^{(X,e)}$. Let us describe more precisily this structure: for finite sets I,J, we have continuous maps

$$\gamma_{I,J}^N:\mathrm{Emb}^{fr}\Big(\coprod_I\mathbb{R}^s,\coprod_J\mathbb{R}^s\Big)\to\mathrm{Emb}^{(X,e)}\Big(\coprod_I(N\times\mathbb{R}^s),\coprod_J(N\times\mathbb{R}^s)\Big)$$

induced by the composition

$$\coprod_I(N\times\mathbb{R}^s)\cong N\times\Big(\coprod_I\mathbb{R}^s\Big)\overset{id_N\times f}{\to}N\times\Big(\coprod_J\mathbb{R}^s\Big)\cong\coprod_J(N\times\mathbb{R}^s)$$

for any $f\in\mathrm{Emb}^{fr}\big(\coprod_I\mathbb{R}^s,\coprod_J\mathbb{R}^s\big)$. In particular, taking $s=n$, $N=pt$, the above maps induce a canonical map of operads

$$\gamma:\mathrm{Disk}_n^{fr}\to\mathrm{Disk}_n^{(X,e)}$$

and thus we have an underlying functor $\gamma^*:\mathrm{Disk}_n^{(X,e)}$**-Alg** $\longrightarrow E_n$**-Alg**. And more generally we obtain functors: $(\gamma^N)^*:\mathrm{Disk}_n^{(N\times\mathbb{R}^s,T(N\times\mathbb{R}^s))}$**-Alg** $\longrightarrow E_s$**-Alg**.

The main consequence is that any symmetric monoidal functor (from $\mathrm{Mfld}_n^{(X,e)}$) maps $N\times\mathbb{R}^s$ to an E_s-algebra object of the target category. More precisely:

Lemma 5. *Let* $\big(\mathrm{Mfld}_n^{(X,e)},\coprod\big)\overset{\mathscr{F}}{\longrightarrow}(Chain(k),\otimes)$ *be a symmetric monoidal functor.*

1. *For any manifold $N\times\mathbb{R}^s$ with an (X,e)-structure, $\mathscr{F}(N\times\mathbb{R}^s)$ has a canonical E_s-algebra structure.*

2. *Let M be an (X,e)-structured manifold with an end*[23] *trivialized as $N\times\mathbb{R}$ (where N is of codimension 1 and the open part of M lies in the neighborhood of $N\times\{-\infty\}$, see Figure 1). Then $\mathscr{F}(M)$ has a canonical left*[24] *module structure over the E_1-algebra $\mathscr{F}(N\times\mathbb{R})$.*

3. *$\mathscr{F}(\mathbb{R}^n)$ has a natural structure of* $\mathrm{Disk}_n^{(X,e)}$*-algebra.*

[23] *i.e.* open boundary component

[24] If $N\times\mathbb{R}$ is trivialized so that the open part of M is in the neighborhood of $N\times\{+\infty\}$, then $\mathscr{F}(M)$ has a canonical *right* module structure.

Proof. Endow $\mathbb{R}^n$ with its canonical framing. It automatically inherits an (X,e)-structure for every connected component[25] of X (since every vector bundle is locally trivial). Now, since $\mathscr{F}$ is symmetric monoidal, then $\mathscr{F}(\mathbb{R}^n)$ also has an induced structure of $\mathrm{Disk}_n^{(X,e)}$-algebra. Let us describe the structure mentioned in 1. and 2.

1. For any manifold $N \times \mathbb{R}^s$ with an (X,e)-structure, the E_s-algebra structure on $\mathscr{F}(N \times \mathbb{R}^i)$ is given by the structure maps

$$\begin{aligned}
\mathrm{Emb}^{fr}\Big(\coprod_I \mathbb{R}^s, \mathbb{R}^s\Big) &\xrightarrow{\gamma^N_{I,pt}} \mathrm{Emb}^{(X,e)}\Big(\coprod_I (N \times \mathbb{R}^i), N \times \mathbb{R}^s\Big) \\
&\xrightarrow{\mathscr{F}(\mathrm{Emb}^{(X,e)}(-,-))} \mathrm{Map}\Big(\mathscr{F}\Big(\coprod_I (N \times \mathbb{R}^s)\Big), \mathscr{F}(N \times \mathbb{R}^s)\Big) \\
&\xrightarrow{\simeq} \mathrm{Map}\Big(\big(\mathscr{F}(N \times \mathbb{R}^s)\big)^I, \mathscr{F}(N \times \mathbb{R}^s)\Big).
\end{aligned}$$

The fact that $\mathscr{F}$ is monoidal ensures it defines an E_s-algebra structure. The latter can also be simply at the chain level as the composition

$$\begin{aligned}
&C_*\Big(\mathrm{Emb}^{fr}\Big(\coprod_I \mathbb{R}^s, \mathbb{R}^s\Big)\Big) \otimes \big(\mathscr{F}(N \times \mathbb{R}^s)\big)^I \\
&\quad \xrightarrow{\gamma^N_{I,pt}} C_*\Big(\mathrm{Emb}^{(X,e)}\Big(\coprod_I (N \times \mathbb{R}^i), N \times \mathbb{R}^s\Big)\Big) \otimes \big(\mathscr{F}(N \times \mathbb{R}^s)\big)^I \\
&\longrightarrow C_*\Big(\mathrm{Emb}^{(X,e)}\Big(\coprod_I (N \times \mathbb{R}^s), N \times \mathbb{R}^s\Big)\Big) \otimes \mathscr{F}\Big(\coprod_I (N \times \mathbb{R}^s)\Big) \xrightarrow{\mathscr{F}(\mathrm{Emb}^{(X,e)}(-,-))} \mathscr{F}(N \times \mathbb{R}^s).
\end{aligned}$$

From the definition, it is clear that $\gamma^*(\mathscr{F}(\mathbb{R}^n)) \cong \mathscr{F}(\mathbb{R}^n)$ as an E_n-algebra (where the two structures are given by 1. and 3.).

2. Now, let M be an (X,e)-structured manifold with an end trivialized as $N \times \mathbb{R}$; $\mathscr{F}(N \times \mathbb{R})$ is an E_1-algebra by 1. The left module structure of $\mathscr{F}(M)$ is given

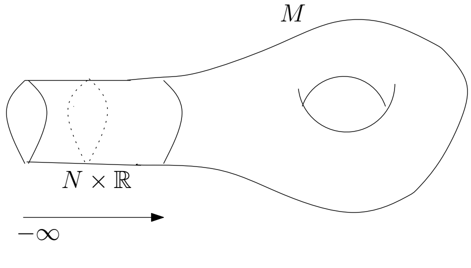

Figure 1: A manifold M with a trivialization $N \times \mathbb{R}$ of its open boundary.

[25] In practice, X will almost always be connex so that the structure will be canonical

at the chain level by the maps

$$C_*\Big(\mathrm{Emb}^{fr}\Big((\coprod_I \mathbb{R})\coprod(0,1],(0,1]\Big)\Big)\otimes\Big(\mathscr{F}(N\times\mathbb{R})\Big)^I\times\mathscr{F}(M)$$

$$\xrightarrow{\gamma^N_{I\coprod\{*\},pt}} C_*\Big(\mathrm{Emb}^{(X,e)}\Big((\coprod_I N\times\mathbb{R})\coprod M,M\Big)\Big)\otimes\big(\mathscr{F}(N\times\mathbb{R}^i)\big)^I\times\mathscr{F}(M)$$

$$\longrightarrow C_*\Big(\mathrm{Emb}^{(X,e)}\Big(\coprod_I(N\times\mathbb{R})\coprod M,M\Big)\Big)\otimes\mathscr{F}\Big((\coprod_I(N\times\mathbb{R}))\coprod M\Big)$$

$$\xrightarrow{\mathscr{F}(\mathrm{Emb}^{(X,e)}(-,-))}\mathscr{F}(N\times\mathbb{R}^i)$$

which are deduced from the follwoing maps in Top:

$$\mathrm{Emb}^{fr}\Big((\coprod_I\mathbb{R})\coprod(0,1],(0,1]\Big)\xrightarrow{\gamma^N_{I\coprod\{*\},pt}}\mathrm{Emb}^{(X,e)}\Big((\coprod_I N\times\mathbb{R})\coprod M,M\Big)$$

$$\xrightarrow{\mathscr{F}(\mathrm{Emb}^{(X,e)}(-,-))}\mathrm{Map}\Big(\mathscr{F}\Big((\coprod_I(N\times\mathbb{R}))\coprod M\Big),\mathscr{F}(N\times\mathbb{R}^i)\Big)$$

$$\xrightarrow{\simeq}\mathrm{Map}\Big(\big(\mathscr{F}(N\times\mathbb{R}^i)\big)^I\times\mathscr{F}(M),\mathscr{F}(N\times\mathbb{R}^i)\Big).$$

□

The above lemma is crucial in order to formulate the excision property.

Definition 10. An homology theory for (X,e)-manifolds (with values in the symmetric monoidal ∞-category $(Chain(k),\otimes)$) is a functor

$$\mathscr{F}:\mathrm{Mfld}_n^{(X,e)}\times\mathrm{Disk}_n^{(X,e)}\textbf{-Alg}\to Chain(k)$$

(denoted $(M,A)\mapsto\mathscr{F}_M(A)$) satisfying the following axioms:

i) **(dimension)** there is a natural equivalence $\mathscr{F}_{\mathbb{R}^n}(A)\cong A$ in $Chain(k)$;

ii) **(monoidal)** the functor $M\mapsto\mathscr{F}_M(A)$ is symmetric lax-monoidal and, for any set I, the following induced maps are equivalences (naturally in A)

$$\bigotimes_{i\in I}\mathscr{F}_{M_i}(A)\xrightarrow{\simeq}\mathscr{F}_{\coprod_{i\in I}M_i}(A).\tag{17}$$

iii) **(excision)** Let M be an (X,e)-manifold. Assume there is a codimension 1 submanifold N of M with a trivialization $N\times\mathbb{R}$ of its neighborhood such that M is decomposable as $M=R\cup_{N\times\mathbb{R}}L$ where R,L are submanifolds of M glued along $N\times\mathbb{R}$. By Lemma 5, $\mathscr{F}_{N\times\mathbb{R}}(A)$ is an E_1-algebra and $\mathscr{F}_R(A)$,

$\mathscr{F}_L(A)$ are respectively right and left modules. The *excision axiom*[26] is that the canonical map

$$\mathscr{F}_L(A) \overset{\mathbb{L}}{\underset{\mathscr{F}_{N\times\mathbb{R}}(A)}{\otimes}} \mathscr{F}_R(A) \xrightarrow{\simeq} \mathscr{F}_M(A)$$

(induced by the universal property of the right hand side) is an equivalence.

Remark 11. The symmetric lax-monoidal condition in axiom ii) means that there are natural (in A, M) transformations like (17) compatible with composition for any finite I and invariant under the action of permutations. The axiom ii) thus implies that $M \mapsto \mathscr{F}_M(A)$ is symmetric monoidal. When I is not finite, the right hand side in (17) is the colimit $\varinjlim_{F\to I} \bigotimes_{j\in F} \mathscr{F}_{M_j}(A)$ over all finite sets F and the map is induced by the universal property of the colimit and the lax monoidal property.

Theorem 6 (Francis [F2])**.** *There is an unique*[27] *homology theory for* (X,e)*-manifolds (in the sense of Definition 10), which is called factorization homology*[28].

Factorization homology is defined in [L3] and *its value on a* (X,e)*-manifold* M *and* $Disk_n^{(X,e)}$*-algebra* A *is denoted* $\int_M A$.

Remark 12 (**other coefficients**). In Definition 10 and Theorem 6, one can replace the symmetric monoidal category $(Chain(k),\otimes)$ by any symmetric monoidal ∞-category $(\mathscr{C},\otimes)$ which has all colimits and whose monoidal structure commutes with geometric realization and filtered colimits, see [F1, F2, AFT].

Remark 13 (finite variant). If one restricts to the full subcategory of $\mathrm{Mfld}_n^{(X,e)}$ spanned by the manifolds which have finitely many connected components which are the interior of closed manifolds, then axiom ii) becomes equivalent to asking $\mathscr{F}$ to be naturally symmetric monoidal and Theorem 6 stills holds in this context.

Note that factorization homology depends on the (X,e)-structure not the underlying topological manifold structure of M in general. For instance, if $M = \mathbb{R}$ is equipped with its standard framing and A is an associative algebra (hence E_1), then $\int_{\mathbb{R}} A \cong A$ as an E_1-algebra. However, if $N = \mathbb{R}$ is equipped with the opposite framing (pointing toward $-\infty$), then $\int_N A \cong A^{op}$ (where A^{op} is the algebra with opposite multiplication) as an E_1-algebra (see [F1, L3] for more general statements).

Remark 14. In particular, Theorem 6 implies that the functor $\mathscr{F} \mapsto \mathscr{F}_{\mathbb{R}^n}$ from, the category of symmetric monoidal functors $\mathrm{Mfld}_n^{(X,e)} \to Chain(k)$ satisfying excision[29], to the category of $\mathrm{Disk}_n^{(X,e)}$-algebras (which is a well defined functor by Lemma 5) is a natural equivalence.

[26] or Mayer-Vietoris principle

[27] up to contractible choices

[28] the name comes from the fact that it satisfies the factorization property (Remark 20). Another name is topological chiral homology.

[29] i.e. axiom iii) in Definition 10

Definition 11. Let $A \in \mathrm{Disk}_n^{(X,e)}$**-Alg**. The homology theory for (X,e)-manifolds defined[30] by A will be called factorization homology (or homology theory) with coefficient in A.

Example 15 (Hochschild homology). Let A be a differential graded associative algebra (or even an A_∞-algebra) and choose a framing of $S^1 = SO(2)$ induced by its Lie group structure. We can use excision to evaluate the factorization homology with value in A on the framed manifold S^1. Here, we see the circle as being obtained by gluing two intervals: $S^1 = \mathbb{R} \cup_{\{1,-1\}\times\mathbb{R}} \mathbb{R}$, see Figure 2 .

Note that the induced framing on $\{1,-1\}\times\mathbb{R}$ correspond to the standard framing of $\mathbb{R}$ on the component $\{1\}\times\mathbb{R}$ and the opposite framing on the component $\{-1\}\times\mathbb{R}$ so that $\int_{\{-1\}\times\mathbb{R}} A = A^{op}$ (see Example 27). Thus, by excision we find that

$$\int_{S^1} A \cong \int_{\mathbb{R}} A \overset{\mathbb{L}}{\underset{\int_{\{1,-1\}\times\mathbb{R}} A}{\otimes}} \int_{\mathbb{R}} A \cong A \overset{\mathbb{L}}{\underset{A\otimes A^{op}}{\otimes}} A \cong HH(A) \tag{18}$$

where $HH(A)$ is the usual Hochschild homology[31] of A with value in itself.

Example 16. Let Free_n be the free E_n-algebra on k, which is naturally a Disk_n^{un}-algebra (Example 12). It can thus be evaluated on any manifold.

Proposition 8 ([AFT]). *Let M be a manifold. Then $\int_M \mathrm{Free}_n \cong C_*\big(\coprod_{n\in\mathbb{N}} \mathrm{Conf}_n(M)\big)$ where $\mathrm{Conf}_n(M)$ is the space of configurations of n -unordered points in M.*

In particular, factorization homology is not an homotopy invariant of manifolds (since configurations spaces of unordered points are not, see [LS]). By considering configuration spaces of points with labels, one has a similar result for the free E_n-algebra $\mathrm{Free}_n(V)$ associated to $V \in \mathit{Chain}(k)$, see [AFT].

Example 17 (**Non-abelian Poincaré duality**). Let us now mention another important example of computation of factorization homology. Let (Y, y_0) be a pointed space and $\Omega^n(Y) := \{f : [0,1]^n \to Y, f(\partial[0,1]^n) = y_0\}$ be its n-fold based loop

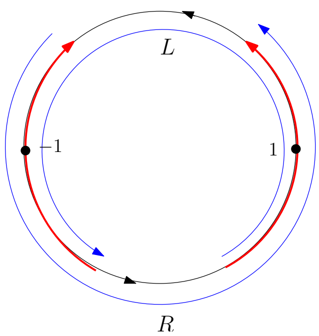

Figure 2: The decomposition of the circle S^1 into 2 intervals (pictured in blue just across the circle) $L \cong \mathbb{R}$ and $R \cong \mathbb{R}$ along a trivialization $\{1,-1\}\times\mathbb{R}$ (pictured in red). The arrows are indicating the orientations/framing of the circle and other various pieces of the decomposition.

[30] in the sense of Remark 14

[31] at least if A is projective over k; if A is not projective over k, there are several variants of Hochschild homology, the one we are considering is the derived version and correspond to what is sometimes called shukla homology [Sh, Q]

space. Then the singular chains $C_*(\Omega^n(Y))$ has a natural structure of unoriented E_n-algebra.

Theorem 7 (non-abelian Poincaré duality, Lurie [L3]). *If M is a manifold of dimension n and Y an $n-1$-connective pointed space, then*

$$\int_M C_*(\Omega^n(Y)) \cong C_*(\mathrm{Map}_c(M,Y))$$

where $\mathrm{Map}_c(M,Y)$ *is the space of compactly supported maps from M to Y.*

If $n=1$ and Y is connected, Theorem 7 reduces to Goodwillie's quasi-isomorphism [G] $HH(C_*(\Omega(Y))) \cong C_*(LY)$ where $LY = \mathrm{Map}(S^1,Y)$ is the free loop space of Y.

Remark 15 (**Derived functor definition**). One possible way for defining factorization homology is similar to the one of § 2.1.2. Indeed, let A be a $\mathrm{Disk}_n^{(X,e)}$-algebra. Then A defines a covariant functor $\mathrm{Disk}_n^{(X,e)} \to Chain(k)$. Similarly, if M is in $\mathrm{Mfld}_n^{(X,e)}$, then it defines a contravariant functor $E_M^{(X,e)} : \big(\mathrm{Disk}_n^{(X,e)}\big)^{op} \to Top$, given by the formula

$$E_M^{(X,e)}\Big(\coprod_{i\in I}\mathbb{R}^n\Big) := \mathrm{Emb}^{(X,e)}\Big(\coprod_{i\in I}\mathbb{R}^n, M\Big).$$

The data of A and M thus gave a functor

$$E_M^{(X,e)}\otimes A : \big(\mathrm{Disk}_n^{(X,e)}\big)^{op}\times \mathrm{Disk}_n^{(X,e)} \xrightarrow{E_M^{(X,e)}\times A} Top\times Chain(k) \xrightarrow{\otimes} Chain(k).$$

Here $Top\times Chain(k) \overset{\otimes}{\to} Chain(k)$ means the tensor of a space with a chain complex which is equivalent to $(X,D_*)\mapsto C_*(X)\otimes D_*$ where $C_*(X)$ is the singular chain functor of X (with value in k).

Proposition 9 ([F2]). *The factorization homology $\int_M A$ is the (homotopy) coend of $E_M^{(X,e)}\otimes A$. In other words:*

$$\begin{aligned}\int_M A &\cong E_M^{(X,e)} \underset{\mathrm{Disk}_n^{(X,e)}}{\overset{\mathbb{L}}{\otimes}} A\\ &\cong hocolim\Bigg(\coprod_{f:\{1,\dots,q\}\to\{1,\dots,p\}} C_*\big(E_M^{(X,e)}(\mathbb{R}^n)\big)^{\otimes p}\otimes \mathrm{Disk}_n^{(X,e)}(q,p)\otimes A^{\otimes q}\\ &\qquad\qquad \rightrightarrows \coprod_m C_*\big(E_M^{(X,e)}(\mathbb{R}^n)\big)^{\otimes m}\otimes A^{\otimes m}\Bigg)\end{aligned}$$

The Proposition remains true with $(Chain(k),\otimes)$ replaced by any symmetric monoidal ∞-category satisfying the assumptions of Remark 12.

4 Factorizations algebras

In this section we will give a Čech type construction of Factorization homology which plays for Factorization homology the same role as sheaf cohomology plays for singular cohomology[32]. This analogue of cosheaf theory is given by factorization algebras which we describe in length here.

4.1 The category of factorization algebras

We start by describing various categories of (pre)factorization algebras (including the locally constant ones).

Following Costello Gwilliam [CG], given a topological space X, a *prefactorization* algebra over X is an algebra over the colored operad whose objects are open subsets of X and whose morphisms from $\{U_1, \cdots, U_n\}$ to V are empty unless when U_i's are mutually disjoint subsets of U, in which case they are singletons. Unfolding the definition, we find

Definition 12. A prefactorization algebra on X (with value in chain complexes) is a rule that assigns to any open set U a chain complex $\mathscr{F}(U)$ and, to any finite family of pairwise disjoint open sets $U_1, \dots, U_n \subset V$ included in an open V, a chain map

$$\rho_{U_1,\dots,U_n,V} : \mathscr{F}(U_1) \otimes \cdots \otimes \mathscr{F}(U_n) \longrightarrow \mathscr{F}(V).$$

These structure maps are required to satisfy obvious *associativity and symmetry* conditions (see [CG]): the map $\rho_{U_1,\dots,U_n,V}$ is invariant with respect to the action of the symmetric group S_n by permutations of the factors on its domain (in other words, the map $\rho_{U_1,\dots,U_n,V}$ depends only of the collection $U_1, \dots, U_n, V$ not on the particular choice of ordering of the open sets) and $\rho_{U,U}$ is the identity[33] of $\mathscr{F}(U)$. Further, the associativity condition is that: for any finite collection of pairwise disjoint open subsets $(V_j)_{j\in J}$ lying in an open subset W together with, for all $j \in J$, a finite collections $(U_{i,j})_{i\in I_j}$ of pairwise disjoint open subset lying in V_j, the following diagram

$$\begin{tikzcd}
\bigotimes_{(i,j)\in\coprod_r I_r} \mathscr{F}(U_{ij}) \arrow[rr, "\rho_{(U_{ij}),W}"] \arrow[dr, "\bigotimes_j \rho_{(U_{ij})_{i\in I_j},V_j}"'] & & \mathscr{F}(W) \\
& \bigotimes_{j\in J} \mathscr{F}(V_j) \arrow[ur, "\rho_{(V_j)_{j\in J},W}"'] &
\end{tikzcd} \tag{19}$$

[32] singular cohomology of a paracompact space X can be computed as the cohomology of the constant sheaf $\mathbb{Z}_X$ on X while singular cohomology with twisted coefficient is computed by sheaf cohomology with value in a locally constant sheaf

[33] we could weaken this condition to be only a weak-equivalence or actually just a chain map. In the latter case, we will obtain a (homotopy) strictly weaker notion of prefactorization algebras; however, this will *not* change the notion of factorization algebras since the condition of being a factorization algebra (Definition 13) will imply that $\rho_{U,U}$ is an equivalence as U is always a factorizing cover of itself; since it is idempotent by associativity, we will get that it is homotopy equivalent to the identity

is commutative.

If $\mathscr{U}$ is an open cover of X, we define a *prefactorization algebra on $\mathscr{U}$*, also denoted a *$\mathscr{U}$-prefactorization algebra*, to be the same thing as a prefactorization algebra except that $\mathscr{F}(U)$ is defined only for $U \in \mathscr{U}$.

Remark 16. One can define a prefactorization algebra with value in any symmetric monoidal category $(\mathscr{C}, \otimes)$ by replacing chain complexes by objects of $\mathscr{C}$.

Remark 17. Prefactorization algebras are *pointed* since the inclusion $\emptyset \hookrightarrow U$ of the empty set in any open induces a canonical map $\mathscr{F}(\emptyset) \to \mathscr{F}(U)$. Further, the structure maps of a prefactorization algebra exhibit $\mathscr{F}(\emptyset)$ as a commutative algebra in $(\mathscr{C}, \otimes)$ (non necessarily unital) and $\mathscr{F}(U)$ as a $\mathscr{F}(\emptyset)$-module.

There is a *Čech-complex* associated to a cover $\mathscr{U}$ of an open set U. Denoting $P\mathscr{U}$ the set of finite pairwise disjoint open subsets $\{U_1, \dots, U_n \,|\, U_i \in \mathscr{U}\}$ (n is not fixed), it is, by definition the realization of the simplicial chain complex

$$\check{C}_\bullet(\mathscr{U}, \mathscr{F}) = \bigoplus_{\alpha \in P\mathscr{U}} \left(\bigotimes_{U \in \alpha} \mathscr{F}(U) \right) \leftleftarrows \bigoplus_{(\alpha,\beta) \in P\mathscr{U} \times P\mathscr{U}} \left(\bigotimes_{(U,V) \in \alpha \times \beta} \mathscr{F}(U \cap V) \right) \Lleftarrow \cdots$$

where the horizontal arrows are induced by the natural inclusions as for the usual Čech complex of a cosheaf (see [CG]).

Let us describe the simplicial structure more precisely. In simplicial degree i, we get the chain complex $\check{C}_i(\mathscr{U}, \mathscr{F}) := \bigoplus_{\alpha \in P\mathscr{U}^{i+1}} \mathscr{F}(\alpha)$ where, for $\alpha = (\alpha_0, \dots, \alpha_i) \in P\mathscr{U}^i$,we denote $\mathscr{F}(\alpha)$ the tensor product of chain complexes (with its natural differential) :

$$\mathscr{F}(\alpha) = \bigotimes_{U_j \in \alpha_j} \mathscr{F}\Big(\bigcap_{j=0}^{i} U_j \Big).$$

We write $d_{in} : \bigoplus_{\alpha \in P\mathscr{U}^{i+1}} \mathscr{F}(\alpha) \to \bigoplus_{\alpha \in P\mathscr{U}^{i+1}} \mathscr{F}(\alpha)$ the induced differential. The face maps $\partial_s : \bigoplus_{\alpha \in P\mathscr{U}^{n+1}} \mathscr{F}(\alpha) \longrightarrow \bigoplus_{\beta \in P\mathscr{U}^{n}} \mathscr{F}(\beta)$ $(s = 0 \dots n)$ are the direct sum of maps $\widehat{\rho}^s_\alpha : \mathscr{F}(\alpha) \to \mathscr{F}(\widehat{\alpha}^s)$ where $\widehat{\alpha}^s = (\alpha_0, \dots, \alpha_{s-1}, \alpha_{s+1}, \dots, \alpha_n)$ is obtained by discarding the s^{th}-collection of opens in $P\mathscr{U}^{n+1}$. Precisely $\widehat{\rho}^s_\alpha$ is the tensor product

$$\bigotimes_{U_j \in \alpha_j} \mathscr{F}\Big(\bigcap_{j=0}^{n} U_j \Big) \longrightarrow \bigotimes_{\substack{U_k \in \alpha_k, \\ k \neq s}} \mathscr{F}\Big(\bigcap_{\substack{k=0 \\ k \neq s}}^{n} U_k \Big)$$

of the structure maps associated to the inclusion of opens $\bigcap_{j=0 \dots n} U_j$ into $\bigcap_{j \neq s} U_j$. The degeneracies are similarly given by operations $(\alpha_0, \dots, \alpha_n) \mapsto (\alpha_0, \dots, \alpha_j, \alpha_j, \dots, \alpha_n)$ doubling a set α_j.

The simplicial chain-complex $\check{C}_\bullet(\mathscr{U}, \mathscr{F})$ can be made into a chain complex (which is the total complex of a bicomplex):

$$\check{C}(\mathscr{U}, \mathscr{F}) = \bigoplus_{\alpha \in P\mathscr{U}} \mathscr{F}(\alpha) \leftarrow \bigoplus_{\beta \in P\mathscr{U} \times P\mathscr{U}} \mathscr{F}(\beta)[1] \leftarrow \cdots$$

where the horizontal arrows are induced by the alternating sum of the faces ∂_j in the standard way. In other words, $\check{C}(\mathscr{U},\mathscr{F}) = \bigoplus_{i\geq 0}\Big(\bigoplus_{\alpha\in P\mathscr{U}^{i+1}} \mathscr{F}(\alpha)[i]\Big)$ with differential the sum of $\mathscr{F}(\alpha)[i] \stackrel{(-1)^i d_{in}}{\longrightarrow} \mathscr{F}(\alpha)[i]$ and

$$\sum_{j=0}^{n}(-1)^j\partial_j : \bigoplus_{\alpha\in P\mathscr{U}^{n+1}} \mathscr{F}(\alpha)[n] \to \bigoplus_{\beta\in P\mathscr{U}^{n}} \mathscr{F}(\beta)[n-1].$$

Remark 18. If a cover $\mathscr{U}$ is stable under finite intersections, we only need $\mathscr{F}$ to be a prefactorization algebra on $\mathscr{U}$, to define the Čech-complex $\check{C}(\mathscr{U},\mathscr{F})$.

If $\mathscr{U}$ is a cover of an open set U, then the structure maps of $\mathscr{F}$ yield canonical maps $\mathscr{F}(\alpha)\to\mathscr{F}(U)$ which commute with the simplicial maps. Thus, we get a natural map of simplicial chain complexes $\big(\check{C}_i(\mathscr{U},\mathscr{F})\to\mathscr{F}(U)\big)_{i\geq 0}$ to the constant simplicial chain complex $\big(\mathscr{F}(U)\big)_{i\geq 0}$. Passing to geometric realization, we obtain a canonical chain complex homomorphism:

$$\check{C}(\mathscr{U},\mathscr{F}) \longrightarrow \mathscr{F}(U). \tag{20}$$

Remark 19 (**Čech complexes in** $(\mathscr{C},\otimes)$). If $(\mathscr{C},\otimes)$ is a symmetric monoidal category with coproducts, we define the Čech complex of a prefactorization algebra with values in $\mathscr{C}$ in the same way, replacing the direct sum by the coproduct in order to get a simplicial object $\check{C}_\bullet(\mathscr{U},\mathscr{F})$ in $\mathscr{C}$. If further, $\mathscr{C}$ has a geometric realization, then we obtain the Čech complex $\check{C}(\mathscr{U},\mathscr{F})\in\mathscr{C}$ exactly as for chain complexes above and the canonical map (20) is also well defined.

Definition 13. An open cover of $\mathscr{U}$ is *factorizing* if, for all finite collections $x_1,\dots,x_n$ of distinct points in U, there are pairwise disjoint open subsets $U_1,\dots,U_k$ in $\mathscr{U}$ such that $\{x_1,\dots,x_n\}\subset\bigcup_{i=1}^k U_i$.

A prefactorization algebra $\mathscr{F}$ on X is said to be a *homotopy*[34] *factorization algebra* if, for all open subsets $U\in Op(X)$ and for every factorizing cover $\mathscr{U}$ of U, the canonical map $\check{C}(\mathscr{U},\mathscr{F})\to\mathscr{F}(U)$ is a quasi-isomorphism (see [C2, CG]).

Note that we do *not* consider the lax version of homotopy factorization algebra defined in [CG].

Remark 20 (**Factorization property**). If $\mathscr{F}$ is a factorization algebra and $U_1,\dots,U_i$ are disjoint open subsets of X, the factorization condition implies that the structure map

$$\mathscr{F}(U_1)\otimes\cdots\otimes\mathscr{F}(U_i) \longrightarrow \mathscr{F}(U_1\cup\cdots\cup U_i) \tag{21}$$

is a quasi-isomorphism. In particular, this implies that our definition of factorization algebra agrees with the one of [CG].

Another consequence is that $\mathscr{F}(\emptyset)\cong k$ (or, more generally, is the unit of the symmetric monoidal category $\mathscr{C}$ if $\mathscr{F}$ has values in $\mathscr{C}$).

[34]we can also say *derived factorization algebra*. Unless otherwise specified, the word factorization algebra will always mean a homotopy factorization algebra in these notes

The fact that the map (21) is an equivalence is called the *factorization property* in the terminology of Beilinson-Drinfeld [BD], in the sense that the value of $\mathscr{F}$ on disjoint opens factors through its value on each connected component.

Example 18 (the trivial factorization algebra). The ***trivial*** prefactorization algebra k is the *constant* prefactorization algebra given by the rule $U \mapsto k(U) := k$, with structure maps given by multiplication. It is a (homotopy) factorization algebra. It is in particular locally constant over any stratified space X (Definitions 14 and 21).

One defines similarly the trivial factorization algebra over X with values in a symmetric monoidal ∞-category $(\mathscr{C}, \otimes)$ by the rule $U \mapsto \mathbf{1}_{\mathscr{C}}$ where $\mathbf{1}_{\mathscr{C}}$ is the unit of the monoidal structure.

Remark 21 (genuine factorization algebras). The notion of homotopy (or derived) factorization algebra in Definition 13 is a homotopy version of a more naive, underived, version of factorization algebra. This version is a prefactorization algebra such that the following sequence

$$\Big(\bigoplus_{\alpha \in P\mathscr{U}^2} \mathscr{F}(\alpha)\Big) \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \Big(\bigoplus_{\beta \in P\mathscr{U}} \mathscr{F}(\beta)\Big) \to \mathscr{F}(U)$$

is (right) exact for any factorizing cover $\mathscr{U}$ of U. In other words we ask for a similar condition as in Definition 13 but with the truncated Čech complex. We refer to prefactorization algebras satisfying this condition as *genuine factorization algebras* (they are also called *strict* in [CG]). Note that a genuine factorization algebra is *not* a (homotopy) factorization algebra in general. Homotopy factorization algebras are to genuine factorization algebras what homotopy cosheaves are to cosheaves; that is they are obtained by replacing the naive version by an acyclic resolution.

When X is a manifold we have the notion of locally constant factorization algebra which roughly means that the structure maps do not depend on the size of the open subsets but only their relative shapes:

Definition 14. Let X be a topological manifold of dimension n. We say that an open subset U of X is a *disk* if U is homeomorphic to a standard euclidean disk $\mathbb{R}^n$. A (pre-)factorization algebra over X is *locally constant* if for any inclusion of open disks $U \hookrightarrow V$ in X, the structure map $\mathscr{F}(U) \to \mathscr{F}(V)$ is a quasi-isomorphism.

Let us mention that a locally constant prefactorization algebra is automatically a (homotopy) factorization algebra, see Remark 24.

Definition 15. A *morphism* $\mathscr{F} \to \mathscr{G}$ of (pre)factorization algebras over X is the data of chain complexes morphisms $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ for every open set $U \subset X$ which commute with the structures maps; that is the following diagram

$$\begin{tikzcd}
\mathscr{F}(U_1)\otimes\cdots\otimes\mathscr{F}(U_i) \arrow[r, "\rho_{U_1,\dots,U_i,V}"] \arrow[d, "\otimes\phi_j(U_j)"'] & \mathscr{F}(V) \arrow[d, "\phi_V"] \\
\mathscr{G}(U_1)\otimes\cdots\otimes\mathscr{G}(U_i) \arrow[r, "\rho_{U_1,\dots,U_i,V}"] & \mathscr{G}(V)
\end{tikzcd}$$

is commutative for any pairwise disjoint finite family $U_1,\dots,U_i$ of open subsets of an open set V. Morphisms of (pre)factorization algebras are naturally enriched over topological space. Indeed, we have *mapping spaces* $\mathrm{Map}(\mathscr{F},\mathscr{G})$ defined as the geometric realization of the simplicial set

$$n \mapsto \mathrm{Map}(\mathscr{F},\mathscr{G})_n := \{\text{prefactorization algebras morphisms from } \mathscr{F} \text{ to } C^*(\Delta^n)\otimes\mathscr{G}\}$$

where $C^*(\Delta^n)\otimes\mathscr{G}$ is the prefactorization algebra whose value on an open set U is $C^*(\Delta^n)\otimes\mathscr{G}(U)$. We obtain in this way ∞-categories of (pre)factorization algebras (as in § 10.1, Example 58).

The ∞*-category of prefactorization algebras over X is denoted* $\boldsymbol{PFac}_X$ and similarly we write $\boldsymbol{Fac}_X$ *for the* ∞*-categories of factorization algebras over X* and $\boldsymbol{Fac}_X^{lc}$ *for the locally constant ones* (which is a full subcategory). Also if $(\mathscr{C},\otimes)$ is a symmetric monoidal (∞-)category (with coproducts and geometric realization), we will denote $\mathbf{Fac}_X(\mathscr{C})$, $\mathbf{Fac}_X^{lc}(\mathscr{C})$ the ∞-categories of *factorization algebras in* $\mathscr{C}$.

Note that the embedding $\mathbf{Fac}_X \to \mathbf{PFac}_X$ is a fully faithful embedding.

The underlying tensor product[35] of chain complexes induces a tensor product of factorization algebras which is computed pointwise: for $\mathscr{F},\mathscr{G}\in\mathbf{PFac}_X$ and an open set U, we have

$$\big(\mathscr{F}\otimes\mathscr{G}\big)(U) := \mathscr{F}(U)\otimes\mathscr{G}(U) \tag{22}$$

and the structure maps are just the tensor product of the structure maps. If $(\mathscr{C},\otimes)$ is symmetric monoidal, the same construction yields a monoidal structure on $\mathbf{PFac}_X(\mathscr{C})$. Its unit is the trivial factorization algebra with values in $\mathscr{C}$ (Example 18).

Proposition 10 (Costello-Gwilliam [CG])**.** *The (∞-)categories* $\boldsymbol{PFac}_X(\mathscr{C})$*,* $\boldsymbol{Fac}_X(\mathscr{C})$*,* $\boldsymbol{Fac}_X^{lc}(\mathscr{C})$[36] *are symmetric monoidal with tensor product given by* (22)*.*

Remark 22 (**Restrictions**). If $Y\subset X$ is an open subspace, then we have natural restriction functors $\mathbf{PFac}_X(\mathscr{C})\to\mathbf{PFac}_Y(\mathscr{C})$, $\mathbf{Fac}_X(\mathscr{C})\to\mathbf{Fac}_Y(\mathscr{C})$. When X is a manifold, the same holds for locally constant factorization algebras.

Definition 16. If U is an open subset of X and $\mathscr{A}\in\mathbf{PFac}(X)$, we write $\mathscr{A}_{|U}\in$ $\boldsymbol{PFac}_U$ *for the restriction of* $\mathscr{A}$ *to U* and similarly for (possibly locally constant) factorization algebras.

A homeomorphism $f: X\overset{\simeq}{\to} Y$ induces isomorphisms $\mathbf{PFac}_X\cong\mathbf{PFac}_Y$ and $\mathbf{Fac}_X\cong\mathbf{Fac}_Y$ (or $\mathbf{Fac}_X^{lc}\cong\mathbf{Fac}_Y^{lc}$ when X is a manifold) realized by the functor f_* see § 5.1.

[35] recall our convention that if k is not a field, the tensor product really means derived tensor product

[36] the latter is defined when X is a manifold

4.2 Factorization homology and locally constant factorization algebras

We now explain the relationship between the Čech complex of a factorization algebras and factorization homology. We first start to express $\mathrm{Disk}_n^{(X,TX)}$-algebras in terms of factorization algebras. For simplicity, we assume in § 4.2 that manifolds are smooth. For topological manifolds one obtains the same result as below by replacing geodesic convex neighborhoods by families of embeddings $\mathbb{R}^n \to M$ wich preserves the (M,TM)-structure and whose images form a basis of open of M.

Let M be a manifold with an (X,e)-structure. Every open subset U of M inherits a canonical (X,e)-structure given by the factorization $U \hookrightarrow M \stackrel{f}{\to} X \stackrel{e}{\to} B\mathrm{Homeo}(\mathbb{R}^n)$ of the map $e_U : U \to B\mathrm{Homeo}(\mathbb{R}^n)$ classifying the tangent bundle of U. This construction extends canonically into a functor

$$f_* : \mathrm{Disk}_n^{(M,TM)} \longrightarrow \mathrm{Disk}_n^{(X,e)}$$

and (by Definition 9) we have

Lemma 6. *An (X,e)-structure $M \stackrel{f}{\to} X \stackrel{e}{\to} B\mathrm{Homeo}(\mathbb{R}^n)$ on a manifold M induces a functor $f^* : \mathrm{Disk}_n^{(X,e)}$-**Alg** $\to \mathrm{Disk}_n^{(M,TM)}$-**Alg**.*

Now let A be a $\mathrm{Disk}_n^{(X,e)}$-algebra and choose a metric on M. A family of pairwise disjoint open convex geodesic neighborhoods $U_1,\dots,U_i$ which lies in a convex geodesic neighborhood[37] V, defines an (X,e)-structure preserving embedding $i_{U_1,\dots,U_i,V} \in \mathrm{Emb}^{(X,e)}\big(\coprod_{\{1,\dots,i\}}\mathbb{R}^n,\mathbb{R}^n\big)$ so that the $\mathrm{Disk}_n^{(X,e)}$-algebra structure of A yields a structure map

$$\mu_{U_1,\dots,U_i,V} : A^{\otimes i} \stackrel{i_{U_1,\dots,U_i,V}}{\longrightarrow} \mathrm{Emb}^{(X,e)}\Big(\coprod_{\{1,\dots,i\}}\mathbb{R}^n,\mathbb{R}^n\Big)\otimes A^{\otimes i} \to A.$$

This allows us to define a *prefactorization algebra $\mathscr{F}_A$ on open convex geodesic subsets* by the formula $\mathscr{F}_A(V) := A$. Since, the convex geodesic neighborhoods form a basis of open which is stable by intersection, for any open set $U \subset M$, we have the Čech complex[38]

$$\check{C}(\mathscr{CV}(U),\mathscr{F}_A)$$

where $\mathscr{CV}(U)$ is the factorizing cover of U given by the geodesic convex open subsets of U. The following result shows that $\mathscr{F}_A$ is actually (the restriction of) a factorization algebra and computes factorization homology.

Theorem 8 ([GTZ2])**.** *Let A be a $\mathrm{Disk}_n^{(X,e)}$-algebra.*

[37] which is thus canonically homeomorphic to an euclidean disk

[38] the construction is actually the extension of a factorization algebra on $\mathscr{CV}(U)$ as in Section 5

- *The rule $M \mapsto \check{C}(\mathscr{CV}(M), \mathscr{F}_A)$ is a homology theory for (X,e)-manifolds. In particular the Čech complex is independent of the choice of the metric and computes factorization homology of M:*

$$\check{C}(\mathscr{CV}(M), \mathscr{F}_A) \simeq \int_M A.$$

- *The functor $(U,A) \mapsto \check{C}(\mathscr{CV}(U), \mathscr{F}_A)$ induces an equivalence of ∞-categories $Disk_n^{(M,TM)}$-**Alg** $\xrightarrow{\simeq}$ $\textbf{Fac}^{lc}_M$.*

Since we have a preferred choice of framing for $\mathbb{R}^n$, the projection map $\mathbb{R}^n \to pt$ induces an equivalence of ∞-categories $\mathrm{Disk}_n^{(\mathbb{R}^n, T\mathbb{R}^n)} \xrightarrow{\simeq} \mathrm{Disk}_n^{(pt,e)}$ and thus equivalences $\mathrm{Disk}_n^{(X,TX)}$-**Alg** $\cong \mathrm{Disk}_n^{fr}$-**Alg** $\cong E_n$-**Alg** (see Example 12). Hence Theorem 8 is a slight generalization of the following beautiful result.

Theorem 9 (Lurie [L3]). *There is a natural equivalence of ∞-categories*

$$E_n\text{-}\boldsymbol{Alg} \;\cong\; \boldsymbol{Fac}^{lc}_{\mathbb{R}^n}.$$

*The functor $\boldsymbol{Fac}^{lc}_{\mathbb{R}^n} \to E_n$-**Alg** is given by the global section (i.e. the pushforward p_* where $p : \mathbb{R}^n \to pt$, see § 5.1) and the inverse functor is precisely given by factorization homology.*

Locally constant factorization algebras on $\mathbb{R}^n$ are thus *a model for E_n-algebras*. More generally, locally constant factorization algebras are a model for $\mathrm{Disk}_n^{(X,TX)}$-algebras in which the cosheaf property replaces[39] some of the higher homotopy machinery needed for studying these algebras (at the price of working with “lax” algebras).

Remark 23. Theorem 9 is the key example of the relationship between factorization algebras and factorization homology so we now explain the equivalence in more depth. Recall that E_n-**Alg** is the ∞-category of algebras over the operad Cube_n of little cubes. It is equivalent to the ∞-category Disk_n^{fr}-**Alg** since we have an equivalence of operads $\mathrm{Cube}_n \xrightarrow{\simeq} \mathrm{Disk}_n^{fr}$ induced by a choice of diffeomorphism $\theta : (0,1)^n \cong \mathbb{R}^n$. Consider the open cover $\mathscr{D}$ of $\mathbb{R}^n$ consisting of all open disks and denote $\textbf{PFac}^{lc}_{\mathscr{D}}$[40] the category of $\mathscr{D}$-prefactorization algebras which satisfy the locally constant condition (Definition 14 and Definition 12). Evaluation of a Disk_n^{fr}-algebra on an open disk yields a functor Disk_n^{fr}-**Alg** $\to \textbf{PFac}^{lc}_{\mathscr{D}}$ which is an equivalence by [L3, §5.2.4].

Similarly, let $\mathscr{R}$ be the cover of $(0,1)^n$ by open rectangles and $\textbf{PFac}^{lc}_{\mathscr{R}}$ be the category of locally constant $\mathscr{R}$-prefactorization algebras. Evaluation of a Cube_n-algebra on rectangles yields a functor E_n-**Alg** $= \mathrm{Cube}_n$-**Alg** $\to \textbf{PFac}^{lc}_{\mathscr{R}}$. Denote

[39] note that factorization algebras are described by operads in *discrete* space together with the Čech condition, see Remark 24

[40] Note that the category of $\mathscr{D}$-prefactorization algebras is the category of algebras over the colored operad $N(\mathrm{Disk}(\mathbb{R}^n))$, see Remark 24.

$\mathbf{Fac}^{lc}_{\mathscr{D}}$, $\mathbf{Fac}^{lc}_{\mathscr{R}}$ the category of locally constant factorization algebras over the covers $\mathscr{D}$, $\mathscr{R}$ respectively (see § 5.2). We have two commutative diagram and an equivalence between them induced by the diffeomorphism θ:

$$\begin{tikzcd}
\mathrm{Disk}_n^{fr}\text{-}\mathbf{Alg} \arrow[r, "\simeq"] \arrow[dr, dotted] & \mathbf{PFac}^{lc}_{\mathscr{D}} & \mathbf{Fac}^{lc}_{\mathbb{R}^n} \arrow[l] \arrow[dl, "\simeq"] \\
& \mathbf{Fac}^{lc}_{\mathscr{D}} \arrow[u, hook] &
\end{tikzcd} \tag{23}$$

$$\begin{tikzcd}
E_n\text{-}\mathbf{Alg} \arrow[r, "\simeq"] \arrow[dr, dotted] & \mathbf{PFac}^{lc}_{\mathscr{R}} & \mathbf{Fac}^{lc}_{(0,1)^n} \arrow[l] \arrow[dl, "\simeq"] \\
& \mathbf{Fac}^{lc}_{\mathscr{R}} \arrow[u, hook] &
\end{tikzcd} \tag{24}$$

where the dotted arrows exists by Theorem 8 and the diagonal right equivalences are given by Proposition 17. Since the embedding of factorization algebras in prefactorization algebras (over any cover or space) is fully faithful, we obtain that all maps in Diagrams (23) and (24) are equivalences so that we recover the equivalence $E_n\text{-}\mathbf{Alg} \cong \mathbf{Fac}^{lc}_{\mathbb{R}^n} \cong \mathbf{Fac}^{lc}_{(0,1)^n}$ of Theorem 9, also see [Ca2].

Example 19 (Constant factorization algebra on framed manifolds). Let M be a framed manifold of dimension n. By Theorem 8 or [L3, GTZ2], any E_n-algebra A yields a locally constant factorization algebra $\mathscr{A}$ on M which is defined by assigning to any geodesic disk D the chain complex $\mathscr{A}(D) \cong A$. We call such a factorization algebra the *constant factorization algebra* on M associated to A since it satisfies the property that there is a (globally defined) E_n-algebra A together with natural (with respect to the structure map of the factorization algebra) quasi-isomorphism $\mathscr{A}(D) \stackrel{\simeq}{\to} A$ for every disk D.

In particular, for $n = 0, 1, 3, 7$, there is a faithful embedding of E_n-algebras into constant factorization algebras over the n-sphere S^n.

If a manifold X is not framable, we can obtain constant factorization algebras on X by using (un)oriented E_n-algebras instead of plain E_n-algebras:

Example 20 (Constant factorization algebra on (oriented) smooth manifolds). Let A be an unoriented E_n-algebra (i.e. a Disk_n^{un}-algebra, Example 12). Then A yields a (locally) constant factorization $\mathscr{A}$ algebra on any smooth manifold of dimension n which is defined by assigning to any geodesic disk D the chain complex $\mathscr{A}(D) \cong A$.

Similarly, an oriented E_n-algebra (i.e. a Disk_n^{or}-algebra) yields a (locally) constant factorization algebra on any oriented dimension n manifold.

Example 21 (Commutative factorization algebras). The canonical functor $E_\infty\text{-}\mathbf{Alg} \to \mathrm{Disk}_n^{(X,e)}\text{-}\mathbf{Alg}$ (see Example 13) shows that any E_∞-algebras induces a canonical structure of (locally) constant factorization algebra on any (topological) manifold M. In that case, the factorization homology reduces to the derived Hochschild chains according to Theorem 8 and Theorem 10 below. See § 8.2 for more details.

Theorem 10. *If A is an E_∞-algebra*[41]*, then, for every topological manifold M, there is an natural equivalence $CH_M(A) \cong \int_M A$. In particular, factorization homology of E_∞-algebras extends uniquely as an homology theory for spaces (see Definition 1).*

Proof. This is proved in [GTZ2], also see [F1]. The result essentially follows by uniqueness of the homology theories (Theorem 6). Namely, if $\mathscr{H}_A$ is an homology theory for spaces whose value on a point is A, then $\mathscr{H}_A(\mathbb{R}^n) = A$ ($\mathbb{R}^n$ is contractible) and further $\mathscr{H}_A$ satisfies the monoidal and excision axioms of a homology theory for manifolds. □ □

Example 22 (pre-cosheaves). Let $\mathscr{P}$ be a pre-cosheaf on X (with values in vector spaces or chain complexes). For any open $U \subset X$, set $\mathscr{F}(U) := S^\bullet(\mathscr{P}(U)) = \bigoplus_{n\geq 0}(\mathscr{P}(U)^{\otimes n})_{S_n}$ where $S^\bullet$ is the free (differential graded) commutative algebra functor. Then $\mathscr{F}$ is a prefactorization algebra with structure maps given by the algebra structure of $S^\bullet(\mathscr{P}(V))$:

$$\begin{aligned}\mathscr{F}(U_1)\otimes\cdots\otimes\mathscr{F}(U_i) &\cong S^\bullet(\mathscr{P}(U_1))\otimes\cdots\otimes S^\bullet(\mathscr{P}(U_i))\\ &\stackrel{\otimes S^\bullet(\mathscr{P}(U_i\to V))}{\longrightarrow} S^\bullet(\mathscr{P}(V))\otimes\cdots\otimes S^\bullet(\mathscr{P}(V)) \longrightarrow S^\bullet(\mathscr{P}(V)) = \mathscr{F}(V).\end{aligned}$$

Proposition 11 (*cf.* [CG]). *If $\mathscr{F}$ is a homotopy cosheaf, then $\mathscr{F}$ is a factorization algebra (not necessarily locally constant).*

In characteristic zero, if $\mathscr{P}$ is a homotopy cosheaf, then $\mathscr{F}$ is a factorization algebra.

Example 23 (Observables). Several examples of (pre-)factorization algebras arising from theoretical physics (more precisely from perturbative quantum field theories) are described in the beautiful work [CG, C2]. They arose as deformations of those obtained as in the previous Example 22. For instance, let $E \to X$ be a (possibly graded) vector bundle over a smooth manifold X. Let $\mathscr{E}$ be the sheaf of smooth sections of E (which may be endowed with a differential which is a differential operator) and $\mathscr{E}'$ be its associated distributions. The above construction yields a (homotopy) factorization algebra $U \mapsto S(\mathscr{E}'(U)) = \bigoplus_{n\geq 0}\mathrm{Hom}_{\mathscr{O}_X(U)}(\mathscr{E}(U)^{\otimes n}, \mathbb{R})$. In [CG], Costello-Gwilliam have been refining this example to equip the classical observables of a classical field theory with the structure of a factorization algebra (with values in P_0-algebras, Example 64). Their construction is a variant of the classical AKSZ formalism [AKSZ]. Related constructions are studied in [Pa].

Further, the quantum observables of a quantization of the classical field theory, when they exist, also form a factorization algebra (not necessarily locally constant). A very nice example of this procedure arises when X is an elliptic curve, see [C2].

These factorization algebras (with values in lax P_0-algebras) encode the algebraic structure governing observables of the field theories (in the same way as the observables of classical mechanics are described by the algebra of smooth functions on a manifold together with its Poisson bracket). Very roughly speaking, the

[41] for instance a (differential graded) commutative algebra

locally constant factorization algebras correspond to observables of topological field theories.

Example 24 (Enveloping factorization algebra of a dg-Lie algebra). Let $\mathscr{L}$ be a homotopy cosheaf of differential graded Lie algebras on a Hausdorff space X, over a characteristic zero ring. For instance, $\mathscr{L}$ can be the cosheaf of *compactly supported* forms $\Omega^{\bullet,c}_{dR,M} \otimes \mathfrak{g}$ with value in $\mathfrak{g}$ where $\mathfrak{g}$ is a differential graded Lie algebra and $\Omega^{\bullet}_{dR,M}$ is the (complex of) sheaf on a manifold M given by the de Rham complex. If M is a complex manifold, another interesting example is obtained by substituting the Dolbeaut complex to the de Rham complex.

For any open $U \subset X$, we can form the Chevalley-Eilenberg chain complex $C^{CE}_{\bullet}\big(\mathscr{L}(U)\big)$ of the (dg-)Lie algebra $\mathscr{L}(U)$. Its underlying k-module (see [W]) is given by

$$C^{CE}_{\bullet}\big(\mathscr{L}(U)\big) := S^{\bullet}\big(\mathscr{L}(U)[1]\big)$$

and its differential is induced by the Lie bracket and inner differential of $\mathscr{L}$. The structure maps of Example 22 (applied to $\mathscr{F} = \mathscr{L}[1]$) are maps of chain complexes (since $\mathscr{L}$ is a precosheaf of dg-Lie algebras), hence make $C^{CE}_{\bullet}\big(\mathscr{L}(-)\big)$ a prefactorization algebra over X, which we denote $C^{CE}(\mathscr{L})$ (note that this construction only requires $\mathscr{L}$ to be a precoheaf of dg-Lie algebras). As a corollary of Proposition 11, one obtains

Corollary 2 (Theorem 4.5.3, [Gw]). *If $\mathscr{L}$ is a homotopy cosheaf of dg-Lie algebras, the prefactorization algebra $C^{CE}(\mathscr{L})$ is a (homotopy) factorization algebra.*

The above corollary extends to homotopy cohseaves of L_∞-algebras as well.

This construction actually generalizes the construction of the *universal enveloping algebra* of a Lie algebra which corresponds to the case $\mathscr{L} = \Omega^{\bullet,c}_{dR,\mathbb{R}} \otimes \mathfrak{g}$ ([Gw]).

More generally the observables of *Free Field Theories* can be obtained this way, see [Gw] for many examples.

Remark 24 (Algebras over disks in X). Assume X has a cover by euclidean neighborhoods. One can define a colored operad whose objects are open subsets of X that are homeomorphic to $\mathbb{R}^n$ and whose morphisms from $\{U_1, \cdots, U_n\}$ to V are empty except when the U_i's are mutually disjoint subsets of V, in which case they are singletons. We can take the monoidal envelope of this operad (as in Appendix 10.2 or [L3, §2.4]) to get a symmetric monoidal ∞-category $\mathrm{Disk}(X)$ (also see [L3], Remark 5.2.4.7). For any symmetric monoidal ∞-category $\mathscr{C}$, we thus get the ∞-category $\mathrm{Disk}(X)$**-Alg** $:= \mathbf{Fun}^{\otimes}(\mathrm{Disk}(X), \mathscr{C})$ of $\mathrm{Disk}(X)$-algebras. Unfolding the definition we find that a $\mathrm{Disk}(X)$-algebra is precisely a $\mathscr{D}_{isk}$-prefactorization algebra over X where $\mathscr{D}_{isk}$ is the set of all open disks in X.

A $\mathrm{Disk}(X)$-algebra is *locally constant* if for any inclusion of open disks $U \hookrightarrow V$ in X, the structure map $\mathscr{F}(U) \to \mathscr{F}(V)$ is a quasi-isomorphism (see [L3]). By Theorem 8 and [L3, §5.2.4], locally constant $N(\mathrm{Disk}(M))$-algebras are the same as locally constant factorization algebras. Hence we have

Proposition 12. *A locally constant $\mathscr{D}_{isk}$-prefactorization algebra has an unique extension as a locally constant homotopy factorization algebra. In fact, the functor $\boldsymbol{Fac}_X^{lc} \to \boldsymbol{PFac}_{\mathscr{D}_{isk}}^{lc} \cong Disk(X)\text{-}\boldsymbol{Alg}^{lc}$ is an equivalence.*

In particular, a *locally constant prefactorization algebra $\mathscr{F}$ has an unique extension as a locally constant factorization algebra $\mathscr{F}^\circ$* taking the same values as $\mathscr{F}$ on any disk.

Example 25. The unique extension $\mathscr{F}^\circ$ of $\mathscr{F}$ as a factorization algebra can have different values than $\mathscr{F}$ and two different prefactorization algebras on X can have the same values on the open cover $\mathscr{D}_{isk}$. As a trivial example, let $X = \{x,y\}$ be a two points discrete set. Then a $\mathscr{D}_{isk}$-prefactorization algebra is given by two pointed chain complexes $\mathscr{F}(\{x\}) = V_x$, $\mathscr{F}(\{y\}) = V_y$. It is locally constant and the associated factorization algebra is given by $\mathscr{F}^\circ(\{x,y\}) \cong V_x \otimes V_y$. However, if W is any chain complex with pointed maps $V_x \to W$, $V_y \to W$ then we have a prefactorization algebras $\mathscr{G}$ defined by $\mathscr{G}(X) = W$ and is otherwise the same as $\mathscr{F}^\circ$. In particular, $\mathscr{G}$ is a prefactorization algebra on X with the same values as $\mathscr{F}^\circ$ on the disks of X but is different from $\mathscr{F}^\circ$.

The notion of being locally constant for a *factorization algebra* is indeed a local property (though its definition is about all disks) as proved by the following result.

Proposition 13. *Let M be a topological manifold and $\mathscr{F}$ be a factorization algebra on M. Assume that there is an open cover $\mathscr{U}$ of M such that for any $U \in \mathscr{U}$ the restriction $\mathscr{F}_{|U}$ is locally constant. Then $\mathscr{F}$ is locally constant on M.*

See § 9.1 for a proof.

Remark 25 (**Ran space**). Factorization algebras on X can be seen as a certain kind of cosheaf on the Ran space of X. This definition is actually the correct one to deal with factorization algebras in the algebraic geometry context (see [BD, FG]). The *Ran space $Ran(X)$ of a manifold X* is the space of finite non-empty subsets of X. Its topology[42] is the coarsest topology on $\mathrm{Ran}(X)$ for which the sets $\mathrm{Ran}(\{U_i\}_{i\in I})$ are open for every non-empty finite collection of pairwise disjoint opens subsets U_i ($i \in I$) of X. Here, the set $\mathrm{Ran}(\{U_i\}_{i\in I})$ is the collection of all finite subsets $\{x_j\}_{j\in J} \subset X$ such that $\{x_j\}_{j\in J} \cap U_i$ is non empty for every $i \in I$.

Two subsets U, V of $\mathrm{Ran}(X)$ are said to be independent if the two subsets $\left(\bigcup_{S\in U} S\right) \subset X$ and $\left(\bigcup_{T\in V} T\right) \subset X$ are disjoint (as subsets of X). If U, V are subsets of $\mathrm{Ran}(X)$, one denotes $U \star V$ the subset $\{S \cup T, S \in U, T \in V\}$ of $\mathrm{Ran}(X)$.

It is proved in [L3] that a factorization algebra on X is the same thing as a constructible cosheaf F on $\mathrm{Ran}(X)$ which satisfies in addition the *factorizing condition*, that is, that satisfies that, for every family of pairwise independent open subsets, the canonical map

$$F(U_1) \otimes \cdots \otimes F(U_n) \longrightarrow F(U_1 \star \cdots \star U_n)$$

[42] this topology is closely related (but different) to the final topology on $\mathrm{Ran}(X)$ making the canonical applications $X^n \to \mathrm{Ran}(X)$ ($n > 0$) continuous

is an equivalence (the condition is similar to Remark 20).

This characterization explain why there is a similarity between cosheaves and factorization algebras. However, the factorization condition is *not* a purely cosheaf condition and is not compatible with every operations on cosheaves.

5 Operations for factorization algebras

In this section we review many properties and operations available for factorization algebras.

5.1 Pushforward

If $\mathscr{F}$ is a prefactorization algebra on X, and $f : X \to Y$ is a continuous map, one can define the *pushforward* $f_*(\mathscr{F})$ by the formula $f_*(\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$. If $\mathscr{F}$ is an (homotopy) factorization algebra then so is $f_*(\mathscr{F})$, see [CG].

Proposition 14. *The pushforward is a symmetric monoidal functor $f_* : \boldsymbol{Fac}_X \to \boldsymbol{Fac}_Y$ and further $(f \circ g)_* = f_* \circ g_*$.*

Let us abusively *denote $\mathscr{G}$ for the global section $\mathscr{G}(pt)$ of a factorization algebra over the point pt*. Let $p : X \to pt$ be the canonical map. By Theorem 8, when X is a manifold and $\mathscr{F}_A$ is a locally constant factorization algebra associated to a $\mathrm{Disk}_n^{(X,TX)}$-algebra, then the factorization homology of M is

$$\int_X A \cong \mathscr{F}_A(A) \cong p_*(\mathscr{F}_A). \tag{25}$$

This analogy legitimates to call $p_*(\mathscr{F})$, that is the (derived) global sections of $\mathscr{F}$, its factorization homology:

Definition 17. The *factorization homology*[43] *of a factorization algebra* $\mathscr{F} \in \mathbf{Fac}_X$ is $p_*(\mathscr{F})$ and is also denoted $\int_X \mathscr{F}$.

Proposition 15. *Let $f : X \to Y$ be a locally trivial fibration between smooth manifolds. If $\mathscr{F} \in \boldsymbol{Fac}_X$ is locally constant, then $f_*(\mathscr{F}) \in \boldsymbol{Fac}_Y$ is locally constant.*

Proof. Let $U \hookrightarrow V$ be an inclusion of an open sub-disk U inside an open disk $V \subset Y$. Since V is contractible, it can be trivialized so we can assume $f^{-1}(V) = V \times F$ with F a smooth manifold. Taking a stable by finite intersection and factorizing cover $\mathscr{V}$ of F by open disks, we have a factorizing cover $\{V\} \times \mathscr{V}$ of $f^{-1}(V)$ consisting of open disks in X. Similarly $\{U\} \times \mathscr{V}$ is a factorizing cover of $f^{-1}(U)$ consisting of open disks. In particular for any $D \in \mathscr{V}$, the structure map $\mathscr{F}(U \times D) \to \mathscr{F}(V \times D)$ is a quasi-isomorphism since $\mathscr{F}$ is locally constant. Thus the induced map $\check{C}(\{U\} \times \mathscr{V}, \mathscr{F}) \to \check{C}(\{V\} \times \mathscr{V}, \mathscr{F})$ is a quasi-isomorphism as well which implies that $f_*(\mathscr{F})(U) \to f_*(\mathscr{F})(V)$ is a quasi-isomorphism. □ □

[43] recall that we are considering homotopy factorization algebras, which are already derived objects. If $\mathscr{G}$ is a genuine factorization algebra (Remark 21), then its factorization homology would be $\mathbb{L}p_*(\mathscr{G}) := p_*(\tilde{G})$ where $\tilde{G}$ is an acyclic resolution of $\mathscr{G}$ (as genuine factorization algebra)

Example 26. Locally constant factorization algebras induced on a submanifold Let $i : X \hookrightarrow \mathbb{R}^n$ be an embedding of a manifold X into $\mathbb{R}^n$ and NX be an open tubular neighborhood of X in $\mathbb{R}^n$. We write $q : NX \to X$ for the bundle map. If A is an E_n-algebra, then it defines a factorisation algebra $\mathscr{F}_A$ on $\mathbb{R}^n$. Then the pushforward $q_*(\mathscr{F}_{\mathscr{A}|NX})$ is a locally constant (by Proposition 15) factorization algebra on X, which is *not* constant in general if the normal bundle NX is not trivialized.

Since a continuous map $f : X \to Y$ yields a factorization $X \xrightarrow{f} Y \to pt$ of $X \to pt$, Proposition 14 and the equivalence (25) imply the following pushforward formula of factorization homology.

Proposition 16 (pushforward formula)**.** *Let $X \xrightarrow{f} Y$ be continuous and $\mathscr{F}$ be in $\boldsymbol{Fac}_X$. The factorization homology of $\mathscr{F}$ over X is the same as the factorization homology of $f_*(\mathscr{F})$ over Y:*

$$\int_X \mathscr{F} \cong p_*(\mathscr{F}) \cong p_*\big(f_*(\mathscr{F})\big) \cong \int_Y f_*(\mathscr{F}). \tag{26}$$

5.2 Extension from a basis

Let $\mathscr{U}$ be a *basis* stable by finite intersections for the topology of a space X and which is also a *factorizing cover*. Let $\mathscr{F}$ be a (homotopy) *$\mathscr{U}$-factorization algebra*, that is a $\mathscr{U}$-prefactorization algebra (Definition 12) such that, for any $U \in \mathscr{U}$ and factorizing cover $\mathscr{V}$ of U consisting of open sets in $\mathscr{U}$, the canonical map $\check{C}(\mathscr{V},\mathscr{F}) \to \mathscr{F}(U)$ is a quasi-isomorphism.

Proposition 17 (Costello-Gwilliam [CG])**.** *There is an unique*[44] *(homotopy) factorization algebra $i_*^{\mathscr{U}}(\mathscr{F})$ on X extending $\mathscr{F}$ (that is equipped with a quasi-isomorphism of $\mathscr{U}$-factorization algebras $i_*^{\mathscr{U}}(\mathscr{F}) \to \mathscr{F}$).*

Precisely, for any open set $V \subset X$, one has

$$i_*^{\mathscr{U}}(\mathscr{F})(V) := \check{C}(\mathscr{U}_V,\mathscr{F})$$

where $\mathscr{U}_V$ is the open cover of V consisting of all open subsets of V which are in $\mathscr{U}$.

Note that the uniqueness is immediate since, if $\mathscr{G}$ is a factorization algebra on X, then for any open V the canonical map $\check{C}(\mathscr{U}_V,\mathscr{G}) \to \mathscr{G}(V)$ is a quasi-isomorphism and, further, the Čech complex $\check{C}(\mathscr{U}_V,\mathscr{G})$ is computed using only open subset in $\mathscr{U}$.

Proposition 17 gives a way to construct (locally constant) factorization algebras as we now demonstrate.

Example 27. By Example 19, we know that an associative unital algebra (possibly differential graded) gives a locally constant factorization algebra on the interval

[44] up to contractoble choices

$\mathbb{R}$. It can be explicitly given by using extension along a basis. Indeed, the collection $\mathscr{I}$ of intervals (a,b) $(a<b)$ is a factorizing basis of opens, which is stable by finite intersections. Then one can set a $\mathscr{I}$-prefactorization algebra $\mathscr{F}_A$ by setting $\mathscr{F}_A((a,b)) := A$. For pairwise disjoints open interval $I_1,\dots,I_n \subset I$, where the indices are chosen so that $\sup(I_i) \le \inf(I_{i+1})$, the structure maps are given by

$$
\begin{aligned}
A^{\otimes n} = \mathscr{F}_A(I_1)\otimes\cdots\otimes\mathscr{F}_A(I_n) \quad &\longrightarrow \quad \mathscr{F}_A(I) = A && (27)\\
a_1\otimes\cdots\otimes a_n \quad &\longmapsto \quad a_1\cdots a_n\,. && (28)
\end{aligned}
$$

To extend this construction to a full homotopy factorization algebra on $\mathbb{R}$, one needs to check that $\mathscr{F}_A$ is a $\mathscr{I}$-factorization algebra which is the content of Proposition 27 below.

In the construction, we have chosen an implicit orientation of $\mathbb{R}$; namely, in the structure map (27), we have decided to multiply the elements (a_i) by choosing to order the intervals in increasing order from left to right.

Let $\tau:\mathbb{R}\to\mathbb{R}$ be the antipodal map $x\mapsto -x$ reversing the orientation. One can check that $\tau_*(\mathscr{F}_A) = \mathscr{F}_{A^{op}}$ where A^{op} is the algebra A with opposite multiplication $(a,b)\mapsto b\cdot a$. In other words, *choosing the opposite orientation* (that is the decreasing one) *of* $\mathbb{R}$ amounts to *replacing the algebra by its opposite algebra*.

Example 28 (back to the circle). The circle S^1 also has a (factorizing) basis given by the open (embedded) intervals (of length less than half of the perimeter of the circle in order to be stable by intersection). Choosing an orientation on the circle, one can define a (homotopy) factorization algebra $\mathscr{S}_A$ on the circle using again the structure maps (27). This gives an explicit construction of the factorization algebra associated to a framing of S^1 from Example 19 (since they agree on a stable by finite intersection basis of open sets). The global section of $\mathscr{S}_A$ are thus the Hochschild chains of A by the computation (18).

Similarly to Example 27, choosing the opposite framing on the circle amounts to considering the factorization algebra $\mathscr{S}_{A^{op}}$. However, *unlike on* $\mathbb{R}$, there is an equivalence $\mathscr{S}_A \cong \mathscr{S}_{A^{op}}$ induced by the fact that there is an orientation preserving diffeomorphism between the to possible orientations of the circle and that further, the value of $\mathscr{S}_A$ on any interval is constant.

5.3 Exponential law: factorization algebras on a product

Let $\pi_1 : X\times Y\to X$ be the canonical projection. By Proposition 15, we have the *pushforward* functor $\pi_{1*} : \mathbf{Fac}^{lc}_{X\times Y}\to\mathbf{Fac}^{lc}_X$. This functor has an natural lift into $\mathbf{Fac}^{lc}(Y)$. Indeed, if U is open in X and V is open in Y, we have a chain complex:

$$\underline{\pi_1}_*(\mathscr{F})(U,V) := \mathscr{F}(\pi_1{}^{-1}_{|X\times V}(U)) = \mathscr{F}(U\times V).$$

Let $V_1,\dots,V_i$ be pairwise disjoint open subsets in an open set $W\subset Y$ and consider

$$
\begin{aligned}
\underline{\pi_1}_*(\mathscr{F})(U,V_1)\otimes\cdots\otimes\underline{\pi_1}_*(\mathscr{F})(U,V_i) &\cong \mathscr{F}(U\times V_1)\otimes\cdots\otimes\mathscr{F}(U\times V_i)\\
&\overset{\rho_{U\times V_1,\dots,U\times V_i,U\times W}}{\longrightarrow} \mathscr{F}(U\times W) = \underline{\pi_1}_*(\mathscr{F})(U,W). \quad (29)
\end{aligned}
$$

The map (29) makes $\underline{\pi_1}_*(\mathscr{F})(U)$ a prefactorization algebra on Y. If $U_1,\dots,U_j$ are pairwise disjoint open inside an open $O\subset X$, the collection of structure maps

$$\underline{\pi_1}_*(\mathscr{F})(U_1,V)\otimes\cdots\otimes\underline{\pi_1}_*(\mathscr{F})(U_j,V)\cong\mathscr{F}(U_1\times V)\otimes\cdots\otimes\mathscr{F}(U_j\times V)$$
$$\stackrel{\rho_{U_1\times V,\dots,U_j\times V,O\times V}}{\longrightarrow}\mathscr{F}(O\times V)=\underline{\pi_1}_*(\mathscr{F})(O,V)\quad(30)$$

indexed by opens $V\subset Y$ is a map $\underline{\pi_1}_*(\mathscr{F})(U_1)\otimes\cdots\otimes\underline{\pi_1}_*(\mathscr{F})(U_j)\longrightarrow\underline{\pi_1}_*(\mathscr{F})(O)$ of prefactorization algebras over Y.

Combining the two constructions, we find that the structure maps (29) and (30) make $\pi_{1*}(\mathscr{F})$ a prefactorization algebra over X with values in the category of prefactorization algebras over Y. In other words we have just defined a functor:

$$\underline{\pi_1}_*:\mathbf{PFac}_{X\times Y}\longrightarrow\mathbf{PFac}_X(\mathbf{PFac}_Y)\qquad(31)$$

fitting into a commutative diagram

$$\begin{tikzcd}
\mathbf{PFac}_{X\times Y} \arrow[r, "\underline{\pi_1}_*"] \arrow[dr, "(\pi_1)_*"'] & \mathbf{PFac}_X(\mathbf{PFac}_Y) \arrow[d, "\mathbf{PFac}_X(p_*)"] \\
 & \mathbf{PFac}_X
\end{tikzcd}$$

where p_* is given by Definition 17.

Proposition 18. *Let $\pi_1: X\times Y\to X$ be the canonical projection. The pushforward* (31) *by π_1 induces a functor*

$$\underline{\pi_1}_*:\boldsymbol{Fac}_{X\times Y}\longrightarrow\boldsymbol{Fac}_X(\boldsymbol{Fac}_Y)$$

and, if X, Y are smooth manifolds, an equivalence $\underline{\pi_1}_*:\boldsymbol{Fac}^{lc}_{X\times Y}\xrightarrow{\simeq}\boldsymbol{Fac}^{lc}_X(\boldsymbol{Fac}^{lc}_Y)$ of ∞-categories .

See § 9.1 for a proof.

The above Proposition is a slight generalization of (and relies on) the following ∞-category version of the beautiful Dunn's Theorem [Du] proved under the following form by Lurie [L3] (see [GTZ2] for the pushforward interpretation):

Theorem 11 (Dunn's Theorem)**.** *There is an equivalence of ∞-categories*

$$E_{m+n}\text{-}\boldsymbol{Alg}\xrightarrow{\simeq}E_m\text{-}\boldsymbol{Alg}(E_n\text{-}\boldsymbol{Alg}).$$

Under the equivalence E_n-$\boldsymbol{Alg}\cong\boldsymbol{Fac}^{lc}_{\mathbb{R}^n}$ (Theorem 9), the above equivalence is realized by the pushforward $\underline{\pi}_*:\boldsymbol{Fac}^{lc}_{\mathbb{R}^m\times\mathbb{R}^n}\to\boldsymbol{Fac}^{lc}_{\mathbb{R}^m}(\boldsymbol{Fac}^{lc}_{\mathbb{R}^n})$ associated to the canonical projection $\pi:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m$.

Example 29 (PTVV construction). There is a derived geometry variant of the AKSZ formalism introduced recently in [PTTV] which leads to factorization algebras similarly to Example 23. We briefly sketch it. The main input is a (derived Artin) stack X (over a characteristic zero field) which is assumed to be compact and equipped with an orientation (write d_X for the dimension of X) and a stack Y with an n-shifted symplectic structure ω (for instance take Y to be the shifted cotangent complex $Y = T^*[n]Z$ of a scheme Z). The natural evaluation map $ev : X \times \mathbb{R}Map(X,Y)$ allows to pullback the symplectic structure on the space of fields $\mathbb{R}Map(X,Y)$. Precisely, $\mathbb{R}Map(X,Y)$ carries an natural $(n-d_X)$-shifted symplectic structure roughly given by the integration $\int_{[X]} ev^*(\omega)$ of the pullback of ω on the fundamental class of X. It is expected that the observables $\mathscr{O}_{\mathbb{R}Map(X,Y)}$ carries a structure of P_{1+n-d_X}-algebra. Assume further that X is a Betti stack, that is in the essential image of $Top \to \mathbf{dSt}_k$. It will then follow from Corollary 1 and Example 21 that $\mathscr{O}_{\mathbb{R}Map(X,Y)}$ belongs to $\mathbf{Fac}^{lc}_X(P_{1+n-d_X}\text{-}\mathbf{Alg})$. Using the formality of the little disks operads in dimension ≥ 2, Proposition 18 then will give $\mathscr{O}_{\mathbb{R}Map(X,Y)}$ the structure of a locally constant factorization algebra on $X \times \mathbb{R}^{1+n-d_X}$ when $n > d_X$. It is also expected that the quantization of such shifted symplectic stacks shall carries canonical locally constant factorization algebras structures.

Note that the Pantev-Toën-Vaquié-Vezzosi construction was recently extended by Calaque [Ca] to add boundary conditions. The global observables of such *relative* mapping stacks shall be naturally endowed with the structure of locally constant factorization algebra on stratified spaces (as defined in § 6).

Proposition 18 and Theorem 8 have the following consequence

Corollary 3 (Fubini formula [GTZ2]). *Let M, N be manifolds of respective dimension m, n and let A be a $\mathrm{Disk}_{n+m}^{(M\times N, T(M\times N))}$-algebra. Then, $\int_N A$ has a canonical structure of $\mathrm{Disk}_m^{(M,TM)}$-algebra and further,*

$$\int_{M\times N} A \cong \int_M \Big(\int_N A\Big).$$

Example 30. Let A be a smooth commutative algebra. By Hochschild-Kostant-Rosenberg theorem (see Example 6), $CH_{S^1}(A) \xrightarrow{\simeq} S^\bullet_A(\Omega^1(A)[1])$; this algebra is also smooth. Since A is commutative it defines a factorization algebra on the torus $S^1 \times S^1$. By Corollary 3, we find that

$$\int_{S^1\times S^1} A \cong \int_{S^1} \big(S^\bullet_A(\Omega^1(A)[1])\big) \cong S^\bullet_A\Big(\Omega^1(A)[1] \oplus \Omega^1(A)[1] \oplus \Omega^1(A)[2]\Big).$$

5.4 Pullback along open immersions and equivariant factorization algebras

Let $f : X \to Y$ be an open immersion and let $\mathscr{G}$ be a factorization algebra on Y. Since $f : X \to Y$ is an open immersion, the set

$$\mathscr{U}_f := \{U \text{ open in } X \text{ such that } f_{|U} : U \to Y \text{ is an homeomorphism}\}$$

is an open cover of X as well as a factorizing basis. For $U \in \mathscr{U}_f$, we define $f^*(\mathscr{G})(U) := \mathscr{G}(f(U))$. The structure maps of $\mathscr{G}$ make $f^*(\mathscr{G})$ a $\mathscr{U}_f$-factorization algebra in a canonical way. Thus by Proposition 17, $i_*^{\mathscr{U}_f}(f^*(\mathscr{G}))$ is the factorization algebra on X extending $f^*(\mathscr{G})$. We (abusively) denote $f^*(\mathscr{G}) := i_*^{\mathscr{U}_f}(f^*(\mathscr{G}))$ and call it the pullback along f of the factorization algebra $\mathscr{G}$.

Proposition 19 ([CG]). *The pullback along open immersion is a functor $f^* : \mathbf{Fac}_Y \to \mathbf{Fac}_X$. If $f : X \to Y$ and $g : Y \to Z$ are open immersions, then $(g \circ f)^* = f^* \circ g^*$.*

If $U \in \mathscr{U}_f$, then U is an open subset of the open set $f^{-1}(f(U))$. Thus if $\mathscr{F}$ is a factorization algebra on X, we have the natural map

$$\mathscr{F}(U) \overset{\rho_{U,f^{-1}(f(U))}}{\longrightarrow} \mathscr{F}(f^{-1}(f(U))) \cong f^*(f_*(\mathscr{F}))(U) \tag{32}$$

which is a map of $\mathscr{U}_f$-factorization algebras. Since $\mathscr{F}$ and $f^*(f_*(\mathscr{F}))$ are factorization algebras on X, the above map extends uniquely into a map of factorization algebras on X. We have proved:

Proposition 20. *Let $f : X \to Y$ be an open immersion. There is an natural transformation $Id_{\mathbf{Fac}_X} \to f^* f_*$ induced by the maps* (32).

Example 31. Let $X = \{c, d\}$ be a discrete space with two elements and consider the projection $f : X \to pt$. A factorization algebra $\mathscr{G}$ on pt is just the data of a chain complex G with a distinguished cycle g_0 while a factorization algebra $\mathscr{F}$ on X is given by two chain complexes C, D (with distinguished cycles c_0, d_0) and the rule $\mathscr{F}(\{c\}) = C$, $\mathscr{F}(\{d\}) = D$, $\mathscr{F}(\{c\}) = C \overset{id\otimes\{d_0\}}{\longrightarrow} C \otimes D = \mathscr{F}(X)$ and $\mathscr{F}(\{d\}) = C \overset{\{c_0\}\otimes id}{\longrightarrow} C \otimes D = \mathscr{F}(X)$. In that case we have that

$$f^*(f_*(\mathscr{F})(\{x\}) = C \otimes D = f^*(f_*(\mathscr{F})(\{y\}) = f^*(f_*(\mathscr{F})(X)$$

while $f_*(f^*(\mathscr{G}))(pt) = F \otimes F$. Note that there are no natural transformation of chain complexes $F \otimes F \to F$ in general; in particular f^* *and* f_* *are not adjoint*.

In fact f_* does *not* have any adjoint in general; indeed as the above example of $F : \{c, d\} \to pt$ demonstrates, f_* does not commute with coproducts nor products.

We now turn on to a descent property of factorization algebras. Let G be a discrete group acting on a space X. For $g \in G$, we write $g : X \to X$ the homeomorphism $x \mapsto g \cdot x$ induced by the action.

Definition 18. A G-equivariant factorization algebra on X is a factorization algebra $\mathscr{G} \in \mathbf{Fac}_X$ together with, for all $g \in G$, (quasi-)isomorphisms of factorization algebras

$$\theta_g : g^*(\mathscr{G}) \overset{\simeq}{\to} \mathscr{G}$$

such that $\theta_1 = \mathrm{id}$ and

$$\theta_{gh} = \theta_h \circ h^*(\theta_g) : h^*(g^*(\mathscr{G})) \to \mathscr{G}.$$

We write $\mathbf{Fac}_X^G$ for the category of G-equivariant factorization algebras over X.

Assume G acts properly discontinuously and X is Hausdorff so that the quotient map $q : X \to X/G$ is an open immersion. If $\mathscr{F}$ is a factorization algebra over X/G, then $q^*(\mathscr{F})$ is G-equivariant (since $q(g\cdot x) = q(x)$). We thus have a functor $q^* : \mathbf{Fac}_{X/G} \longrightarrow \mathbf{Fac}_X^G$.

Proposition 21 (Costello-Gwilliam [CG]). *If the discrete group G acts properly discontinuously on X, then the functor $q^* : \mathbf{Fac}_{X/G} \longrightarrow \mathbf{Fac}_X^G$ is an equivalence of categories.*

The proof essentially relies on considering the factorization basis given by trivialization of the principal G-bundle $X \to X/G$ to define an inverse to q^*.

Proposition 22. *If the discrete group G acts properly discontinuously on a smooth manifold X, then the equivalence $q^* : \mathbf{Fac}_{X/G} \longrightarrow \mathbf{Fac}_X^G$ factors as an equivalence $q^* : \mathbf{Fac}^{lc}_{X/G} \longrightarrow (\mathbf{Fac}^{lc}_X)^G$ between the subcategories of locally constant factorization algebras.*

Proof. Let U be an open set such that $q_{|U} : U \to X/G$ is an homeomorphism onto its image. Then, for every open subset V of U, $q^*(\mathscr{F})(V) = \mathscr{F}(q(V))$. Thus, if $\mathscr{F}$ satisfies the condition of being locally constant for disks included in U, then so does $q^*(\mathscr{F})$ for disks included in $q(U)$. Hence, by Proposition 13, $q^*(\mathscr{F})$ is locally constant if $\mathscr{F}$ is locally constant.

Now, assume $\mathscr{G} \in \mathbf{Fac}_X^G$ is locally constant. Then $(q^*)^{-1}(\mathscr{G})$ is the factorization algebra defined on every section $X/G \supset U \to \sigma(U) \subset X$ of q (with U open) by $(q^*)^{-1}(\mathscr{G})(U) = \mathscr{G}(\sigma(U))$. Since every disk D is contractible, we always have a section $D \to X$ of $q_{|D}$. Thus, if $\mathscr{G}$ is locally constant, then so is $(q^*)^{-1}(\mathscr{G})$. □ □

Remark 26 (General definition of equivariant factorization algebras). Definition 18 can be easily generalized to topological groups as follows. Indeed, if G acts continuously on X, then the rule $(g,\mathscr{F}) \mapsto g^*(\mathscr{F})$ induces a right action of G on $\mathbf{Fac}_X$[45].

The ∞-category of G-equivariant factorization algebras is the ∞-category of homotopy G-fixed points of $\mathbf{Fac}_X$:

$$\mathbf{Fac}_X^G := (\mathbf{Fac}_X)^{hG}.$$

This ∞-category is equivalent to the one of Definition 18 for discrete groups. It is the ∞-category consisting of a factorization algebra $\mathscr{G}$ on X together with quasi-isomorphisms of factorization algebras $\theta_g : g^*(\mathscr{G}) \to \mathscr{G}$ (inducing a ∞-functor $BG \to \mathbf{Fac}_X$, where BG is the ∞-category associated to the topological category with a single object and mapping space of morphisms given by G) and equivalences $\theta_{gh} \sim \theta_h \circ h^*(\theta_g)$ satisfying some higher coherences.

[45]that is a map of E_1-algebras (in *Top*) from G^{op} to $\mathbf{Fun}(\mathbf{Fac}_X, \mathbf{Fac}_X)$

5.5 Example: locally constant factorization algebras over the circle

Let $q : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ be the universal cover of S^1 and let $\mathscr{F}$ be a locally constant factorization algebra on S^1. By Proposition 22, $\mathscr{F}$ is equivalent to the data of a $\mathbb{Z}$-equivariant locally constant factorization algebra on $\mathbb{R}$ which is the same as a locally constant factorization algebra over $\mathbb{R}$ together with an equivalence of factorization algebras, the equivalence being given by $\theta_1 : 1^*(q^*(\mathscr{F})) \stackrel{\simeq}{\to} q^*(\mathscr{F})$. By Theorem 9, the category of locally constant factorization algebras on $\mathbb{R}$ is the same as the category of E_1-algebras, and thus is equivalent to its full subcategory of *constant* factorization algebras. It follows that we have a canonical equivalence $1^*(\mathscr{G}) \cong \mathscr{G}$ for $\mathscr{G} \in \mathbf{Fac}^{lc}_{\mathbb{R}}$ (in particular, for any open interval I, the structure map $1 * (\mathscr{G}(I)) = \mathscr{G}(1+I) \to \mathscr{G}(\mathbb{R})$ is a quasi-isomorphism).

Definition 19. We denote $mon : q^*(\mathscr{F}) \cong 1^*(q^*(\mathscr{F})) \stackrel{\theta_1}{\to} q^*(\mathscr{F})$ the self-equivalence of $q^*(\mathscr{F})$ induced by θ_1 and call it the *monodromy* of $\mathscr{F}$.

We thus get the following result

Corollary 4. *The category $\mathbf{Fac}^{lc}_{S^1}$ of locally constant factorization algebra on the circle is equivalent to the ∞-category $\mathbf{Aut}(E_1\text{-}\mathbf{Alg})$ of E_1-algebras equipped with a self-equivalence.*

Remark 27. Using Proposition 18, it is easy to prove similarly that $\mathbf{Fac}^{lc}_{S^1\times S^1}$ is equivalent to the category of E_2-algebras equipped with two commuting monodromies (i.e. self-equivalences).

It seems harder to describe the categories of locally constant factorization algebras over the spheres S^3, S^7 in terms of E_3 and E_7-algebras (due to the complicated homotopy groups of the spheres). However, for $n = 3,7$, there shall be an embedding of the categories of E_n-algebras equipped with an n-gerbe[46] into $\mathbf{Fac}^{lc}_{S^n}$.

Let $\mathscr{F}$ be a locally constant factorization algebra on S^1 (identified with the unit sphere in $\mathbb{C}$). We wish to compute the *global section of* $\mathscr{F}$ (i.e. its factorization homology $\int_{S^1} \mathscr{F}$). Let $\mathscr{B} \cong \mathscr{F}(S^1 \setminus \{1\})$ be its underlying E_1-algebra (with monodromy $mon : \mathscr{B} \stackrel{\simeq}{\to} \mathscr{B}$). We use the orthogonal projection $\pi : S^1 \to [-1,1]$ from S^1 to the real axis. The equivalence (26) yields $\mathscr{F}(S^1) \cong \pi_*(\mathscr{F})([-1,1])$ and by Proposition 15 and Proposition 27, we are left to compute the E_1-algebra $\pi_*(\mathscr{F})\big((-1,1)\big)$ and left and right modules $\pi_*(\mathscr{F}\big((-1,1]\big)$, $\pi_*(\mathscr{F})\big([-1,1)\big)$. From Example 27, we get $\pi_*(\mathscr{F})\big((-1,1)\big) \cong \mathscr{B} \otimes \mathscr{B}^{op}$. Further,

$$\pi_*(\mathscr{F})\big([-1,1)\big) \cong \mathscr{F}(S^1 \setminus \{1\}) = \mathscr{B} \quad \text{(as a } \mathscr{B}\otimes\mathscr{B}^{op}\text{-module).}$$

Similarly $\pi_*(\mathscr{F})\big((-1,1]\big) \cong \mathscr{B}^{mon}$, that is $\mathscr{B}$ viewed as a $\mathscr{B}\otimes\mathscr{B}^{op}$-module through the monodromy. When $\mathscr{B}$ is actually a differential graded algebra, then the bimodule stucture of $\mathscr{B}^{mon}$ boils down to $a \cdot x \cdot b = mon(b) \cdot m \cdot a$. This proves the following which is also asserted in [L3, §5.3.3].

[46] by an n-gerbe over $A \in E_n\text{-}\mathbf{Alg}$, we mean a monoid map $\mathbb{Z} \to \Omega^{n-1}\mathrm{Map}_{E_n\text{-}\mathbf{Alg}}(A,A)$.

Corollary 5. *Let $\mathscr{B}$ be a locally constant factorization algebra on S^1. Let B be a differential graded algebra and $mon : B \overset{\simeq}{\to} B$ be a quasi-isomorphism of algebras so that (B, mon) is a model for the underlying E_1-algebra of $\mathscr{B}$ and its monodromy. Then the factorization homology*

$$\int_{S^1} \mathscr{B} \cong B \underset{B\otimes B^{op}}{\overset{\mathbb{L}}{\otimes}} B^{mon} \cong HH(B, B^{mon})$$

is computed by the (standard) Hochschild homology[47] *$HH(B)$ of B with value in B twisted by the monodromy.*

Example 32 (the circle again). Let $p : \mathbb{R} \to S^1$ be the universal cover of S^1. By Example 27, an unital associative algebra A defines a locally constant factorization algebra, denoted $\mathscr{A}$, on $\mathbb{R}$. By Proposition 15, the pushforward $p_*(\mathscr{A})$ is a locally constant factorization algebra on S^1, which, on any interval $I \subset S^1$ is given by $p_*(\mathscr{A})(I) = \mathscr{A}(I \times \mathbb{Z}) = A^{\otimes \mathbb{Z}}$. It is however *not a constant* factorization algebra since the global section of $p_*(\mathscr{A})$ is different from the Hochschild homology of A:

$$p_*(\mathscr{A})(S^1) = \mathscr{A}(\mathbb{R}) \cong A \not\cong HH(A^{\otimes \mathbb{Z}})$$

(for instance if A is commutative the Hochschild homology of $A^{\otimes \mathbb{Z}}$ is $A^{\otimes \mathbb{Z}}$ in degree 0.) Indeed, the monodromy of $\mathscr{A}$ is given by the automorphism σ of A which sends the element a_i in the tensor index by an integer i into the tensor factor indexed by $i+1$, that is $\sigma\big(\bigotimes_{i\in\mathbb{Z}} a_i\big) = \bigotimes_{i\in\mathbb{Z}} a_{i-1}$.

However, by Corollary 5, we have that, for any E_1-algebra A,

$$HH\big(A^{\otimes \mathbb{Z}}, \big(A^{\otimes \mathbb{Z}}\big)^{mon}\big) \cong A.$$

5.6 Descent

There is a way to glue together factorization algebras provided they satisfy some descent conditions which we now explain.

Let $\mathscr{U}$ be an open cover of a space X (which we assume to be equipped with a factorizing basis). We also assume that all intersections of infinitely many different opens in $\mathscr{U}$ are empty. For every finite subset $\{U_i\}_{i\in I}$ of $\mathscr{U}$, let $\mathscr{F}_I$ be a factorization algebra on $\bigcap_{i\in I} U_i$. For any $i \in I$, we have an inclusion $s_i : \bigcap_{i\in I} U_i \hookrightarrow \bigcap_{j\in I\setminus\{i\}} U_j$.

Definition 20. A gluing data is a collection, for all finite subset $\{U_i\}_{i\in I} \subset \mathscr{U}$ and $i \in I$, of quasi-isomorphisms $r_{I,i} : \mathscr{F}_I \longrightarrow \big(\mathscr{F}_{I\setminus\{i\}}\big)_{|\mathscr{U}_I}$ such that, for all I, $i, j \in I$, the following diagram commutes:

$$\begin{tikzcd}
\mathscr{F}_I \arrow[r, "r_{I,j}"] \arrow[d, "r_{I,i}"'] & \big(\mathscr{F}_{I\setminus\{j\}}\big)_{|\mathscr{U}_I} \arrow[d, "r_{I\setminus\{j\},i}"] \\
\big(\mathscr{F}_{I\setminus\{i\}}\big)_{|\mathscr{U}_I} \arrow[r, "r_{I\setminus\{i\},j}"] & \big(\mathscr{F}_{I\setminus\{i,j\}}\big)_{|\mathscr{U}_I}.
\end{tikzcd}$$

[47] in particular by the standard Hochschild complex (see [L]) $C_*(B, B^{mon})$ when B is flat over k

Given a gluing data, one can define a factorizing basis $\mathcal{V}_{\mathcal{U}}$ given by the family of all opens which lies in some $U \in \mathcal{U}$. For any $V \in \mathcal{V}_{\mathcal{U}}$, set $\mathcal{F}(V) = \mathcal{F}_{I_V}(V)$ where I_V is the largest subset of I such that $V \in \bigcap_{j \in I_V} U_j$. The maps $R_{I,i}$ induce a structure of $\mathcal{V}_{\mathcal{U}}$-prefactorization algebra.

Proposition 23 ([CG])**.** *Given a gluing data, the $\mathcal{V}_{\mathcal{U}}$-prefactorization $\mathcal{F}$ extends uniquely into a factorization algebra $\mathcal{F}$ on X whose restriction $\mathcal{F}_{|\mathcal{U}_I}$ on each $\mathcal{U}_I$ is canonically equivalent to $\mathcal{F}_I$.*

Note that if the $\mathcal{F}_I$ are the restrictions to $\mathcal{U}_I$ of a factorization algebra $\mathcal{F}$, then the collection of the $\mathcal{F}_I$ satisfy the condition of a gluing data.

6 Locally constant factorization algebras on stratified spaces and categories of modules

There is an interesting variant of locally constant factorization algebras over (topologically) stratified spaces which can be used to *encode categories of E_n-algebras and their modules* for instance. Note that by Remark 20, all our categories of modules will be pointed, that is coming with a preferred element. We gave the definition and several examples in this Section. An analogue of Theorem 8 for stratified spaces shall provide the link between the result in this section and results of [AFT].

6.1 Stratified locally constant factorization algebras

In this paper, by a *stratified space* of dimension n, we mean a Hausdorff paracompact topological space X, which is filtered as the union of a sequence of closed subspaces $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that any point $x \in X_i \setminus X_{i-1}$ has a neighborhood $U_x \overset{\phi}{\simeq} \mathbb{R}^i \times C(L)$ in X where $C(L)$ is the (open) cone on a stratified space of dimension $n-i-1$ and the homeomorphism preserves the filtration[48]. We further require that $X \setminus X_{n-1}$ is dense in X. In particular, a stratified space of dimension 0 is simply a topological manifold of dimension 0 and $X_i \setminus X_{i-1}$ is a topological manifold of dimension i (possibly empty or non-connected).

The connected components of $X_i \setminus X_{i-1}$ are called the dimension i-strata of X. We always assume that X has at most countable strata.

Definition 21. An open subset D of X is called a (stratified) *disk* if it is homeomorphic to $\mathbb{R}^i \times C(L)$ with L stratified of dimension $n-i-1$, the homeomorphism preserves the filtration and further $D \cap X_i \neq \emptyset$ and $D \subset X \setminus X_{i-1}$. We call i the *index of* the (stratified disk) D. It is the *smallest integer* j such that $D \cap X_j \neq \emptyset$

We say that a (stratified) disk D is a *good neighborhood at X_i* if i is the index of D and D intersects only one connected component of $X_i \setminus X_{i-1}$.

[48] that is $\phi(U_x \cap X_{i+j+1}) = \mathbb{R}^i \times C(L_j)$ for $0 \leq j \leq n-i-1$

A *factorization algebra* $\mathscr{F}$ over a stratified space X is called *locally constant* if for any inclusion of (stratified) disks $U \hookrightarrow V$ such that both U and V are good neighborhoods at X_i (for the same $i \in \{0,\dots,n\}$)[49], the structure map $\mathscr{F}(U) \to \mathscr{F}(V)$ is a quasi-isomorphism.

The underlying space of almost all examples of stratified spaces X arising in these notes will be a manifold (with boundary or corners). In that cases, all (stratified) disk are homeomorphic to to a standard euclidean (half-)disk $\mathbb{R}^{n-j} \times [0,+\infty)^j$.

Let X be a manifold (without boundary) and let X^{str} be the same manifold endowed with some stratification. A locally constant factorization algebra on X is also locally constant with respect to the stratification. Thus, we have a fully faithful embedding

$$\mathbf{Fac}_X^{lc} \longrightarrow \mathbf{Fac}_{X^{str}}^{lc}. \tag{33}$$

Several general results on locally constant factorization algebras from § 4 have analogues in the stratified case. We now list three useful ones.

Proposition 24. *Let X be a stratified manifold and $\mathscr{F}$ be a factorization algebra on X such that there is an open cover $\mathscr{U}$ of X such that for any $U \in \mathscr{U}$ the restriction $\mathscr{F}_{|U}$ is locally constant. Then $\mathscr{F}$ is locally constant on X.*

The functor (33) generalizes to inclusion of any stratified subspace.

Proposition 25. *Let $i : X \hookrightarrow Y$ be a stratified (that is filtration preserving) embedding of stratified spaces in such a way that $i(X)$ is a reunion of strata of Y. Then the pushforward along i preserves locally constantness, that is lift as a functor*

$$\boldsymbol{Fac}_X^{lc} \to \boldsymbol{Fac}_Y^{lc}.$$

Proof. Let $\mathscr{F}$ be in $\mathbf{Fac}_X^{lc}$ and $U \subset D$ be good disks of index j at a neigborhood of a strata in $i(X)$. The preimage $i^{-1}(U) \cong i(X) \cap U$ is a good disk of index j in X and so is $i^{-1}(D)$. Hence $i_*(\mathscr{F}(U)) \to i_*(\mathscr{F}(D))$ is a quasi-isomorphism. On the other hand, if $V \subset Y \setminus i(X)$ is a good disk, then $i_*(\mathscr{F}(V)) \cong k$. Since the constant factorization algebra with values k (example 18) is locally constant on every stratified space, the result follows. □

Let $f : X \to Y$ be a locally trivial fibration between stratified spaces. We say that f is *adequatly stratified* if Y has an open cover by trivializing (stratified) disks V which are good neighborhoods satisfying that:

- $f^{-1}(V) \overset{\psi}{\simeq} V \times F$ has a cover by (stratified) disks of the form $\psi^{-1}(V \times D)$ which are good neighborhoods in X;
- for sub-disks $T \subset U$ which are good neighborhoods (in V) with the same index, then $\psi^{-1}(T \times D)$ is a good neighborhood of X of same index as $\psi^{-1}(U \times D)$.

[49] in other words are good neighborhoods of same index

Obvious examples of adequatly stratified maps are given by locally trivial stratified fibrations; in particular by proper stratified submersions according to Thom first isotopy lemma [Th, GMcP].

Proposition 26. *Let $f : X \to Y$ be adequatly stratified. If $\mathscr{F} \in \boldsymbol{Fac}_X$ is locally constant, then $f_*(\mathscr{F}) \in \boldsymbol{Fac}_Y$ is locally constant.*

Proof. Let $U \hookrightarrow V$ be an inclusion of open disks which are both good neighborhoods at Y_i; by proposition 24, we may assume V lies in one of the good trivializing disk in the definition of an adequatly stratified map so that we have a cover by opens homeomorphic to $(\psi^{-1}(V \times D_j))_{j\in J}$ which are good neigborhood such that $(\psi^{-1}(U \times D_j))_{j\in J}$ is also a good neighborhood of same index. This reduces the proof to the same argument as the one of Proposition 15. $\square$

If X, Y are stratified spaces with finitely many strata, there is a natural stratification on the product $X \times Y$, given by $(X\times Y)_k := \bigcup_{i+j=k} X_i \times Y_j \subset X \times Y$. The natural projections on X and Y are adequatly stratified.

Corollary 6. *Let X, Y be stratified spaces with finitely many strata. The push-forward $\underline{\pi_1}_* : \boldsymbol{Fac}_{X\times Y} \longrightarrow \boldsymbol{Fac}_X(\boldsymbol{Fac}_Y)$ (see Proposition 18) induces a functor $\underline{\pi_1}_* : \boldsymbol{Fac}^{lc}_{X\times Y} \longrightarrow \boldsymbol{Fac}^{lc}_X(\boldsymbol{Fac}^{lc}_Y)$.*

We conjecture that $\underline{\pi_1}_*$ is an equivalence under rather weak conditions on X and Y. We will give a couple of examples.

Proof. Since the projections are adequatly stratified, the result follows from Proposition 18 together with Proposition 26 applied to both projections (on X and Y). $\square$

Remark 28. Let $\mathscr{F}$ be a stratified locally constant factorization algebra on X. Let $U \subset V$ be stratified disks of same index i, but not necessarily good neighborhoods at X_i. Assume all connected component of $V \cap X_i$ contains exactly one connected component of $U \cap X_i$. Then the structure maps $\mathscr{F}(U) \to \mathscr{F}(V)$ is a quasi-isomorphism. Indeed, we can take a factorizing cover $\mathscr{V}$ of V by good neighborhoods D such that $D \cap U$ are good neigborhoods. Then, $\mathscr{F}(D\cap U) \to \mathscr{F}(D)$ is a quasi-isomorphism and thus we get a quasi-isomorphism $\check{C}(\mathscr{F}, \mathscr{V}\cap U) \overset{\simeq}{\to} \check{C}(\mathscr{F}, \mathscr{V})$.

6.2 Factorization algebras on the interval and (bi)modules

Let us consider an important example: the closed interval $I = [0,1]$ viewed as a stratified space[50] with two dimension 0-strata given by $I_0 = \{0,1\}$.

The disks at I_0 are the half-closed intervals $[0,s)$ $(s<1)$ and $(t,1]$ $(0<t)$ and the disks at I_1 are the open intervals (t,u) $(0<t<u<1)$. The disks of (the stratified space) I form a (stable by finite intersection) factorizing basis denoted $\mathscr{I}$.

[50] note that this stratification is just given by looking at $[0,1]$ as a manifold with boundary

An example of stratified locally constant factorization algebra on I is obtained as follows. Let A be a differential graded associative unital algebra, M^r a pointed differential graded right A-module (with distinguished element denoted $m^r \in M^r$) and M^ℓ a pointed differential graded left A-module (with distinguished element $m^\ell \in M^\ell$). We define a $\mathscr{I}$-prefactorization algebra by setting, for any interval $J \in \mathscr{I}$

$$\mathscr{F}(J) := \begin{cases} M^r \text{ if } 0 \in J \\ M^\ell \text{ if } 1 \in J \\ A \text{ else.} \end{cases}$$

We define its structure maps to be given by the following[51]:

- $\mathscr{F}(\emptyset) \to \mathscr{F}([0,s))$ is given by $k \ni 1 \mapsto m^r$, $\mathscr{F}(\emptyset) \to \mathscr{F}((t,s))$ is given by $1 \mapsto 1_A$ and $\mathscr{F}(\emptyset) \to \mathscr{F}((t,1]))$ is given by $k \ni 1 \mapsto m^\ell$;
- For $0 < s < t_1 < u_1 < \cdots < t_i < u_i < v < 1$ one sets

$$\begin{array}{rcl} M^r \otimes A^{\otimes i} = \mathscr{F}([0,s)) \otimes \mathscr{F}((t_1,u_1)) \otimes \cdots \otimes \mathscr{F}((t_i,u_i)) & \longrightarrow & \mathscr{F}([0,v)) = M^r \\ m \otimes a_1 \otimes \cdots \otimes a_i & \longmapsto & m \cdot a_1 \cdots a_i \,; \end{array}$$

$$\begin{array}{rcl} A^{\otimes i} \otimes M^\ell = \mathscr{F}((t_1,u_1)) \otimes \cdots \otimes \mathscr{F}((t_i,u_i)) \otimes \mathscr{F}((v,1]) & \longrightarrow & \mathscr{F}((s,1]) = M^\ell \\ a_1 \otimes \cdots \otimes a_i \otimes n & \longmapsto & a_1 \cdots a_i \cdot n \,; \end{array}$$

  and also

$$\begin{array}{rcl} A^{\otimes i} = \mathscr{F}((t_1,u_1)) \otimes \cdots \otimes \mathscr{F}((t_i,u_i)) & \longrightarrow & \mathscr{F}((s,v)) = A \\ a_1 \otimes \cdots \otimes a_i & \longmapsto & a_1 \cdots a_i \,. \end{array}$$

It is straightforward to check that $\mathscr{F}$ is a $\mathscr{I}$-prefactorization algebra and, by definition, it satisfies the locally constant condition.

Proposition 27 shows that $\mathscr{F}$ is indeed a locally constant factorization algebra on the closed interval I. Further, *any* locally constant factorization algebra on I is (homotopy) equivalent to such a factorization algebra.

Also note that the $\mathscr{I}$-prefactorization algebra induced by $\mathscr{F}$ on the open interval $(0,1)$ is precisely the $\mathscr{I}$-prefactorization algebra constructed in Example 27 (up to an identification of $(0,1)$ with $\mathbb{R}$). We denote it $\mathscr{F}_A$.

Proposition 27. *Let $\mathscr{F}$, $\mathscr{F}_A$ be defined as above.*

1. *The $\mathscr{I}$- prefactorization algebra $\mathscr{F}$ is an $\mathscr{I}$-factorization algebra hence extends uniquely into a factorization algebra (still denoted) $\mathscr{F}$ on the stratified closed interval $I = [0,1]$;*

[51] as in Example 27, we use the implicit orientation of I given by increasing numbers

2. *in particular, $\mathscr{F}_A$ also extends uniquely into a factorization algebra (still denoted) $\mathscr{F}_A$ on $(0,1)$.*

3. *There is an equivalence $\int_{[0,1]} \mathscr{F} = \mathscr{F}([0,1]) \cong M^r \overset{\mathbb{L}}{\underset{A}{\otimes}} M^\ell$ in $Chain(k)$.*

4. *Moreover, any locally constant factorization algebra $\mathscr{G}$ on $I=[0,1]$ is equivalent[52] to $\mathscr{F}$ for some A, M^ℓ, M^r, that is, it is uniquely determined by an E_1-algebra $\mathscr{A}$ and pointed left module $\mathscr{M}^\ell$ and pointed right module $\mathscr{M}^r$ satisfying*

$$\mathscr{G}([0,1)) \cong \mathscr{M}^r, \quad \mathscr{G}((0,1]) \cong \mathscr{M}^\ell, \quad \mathscr{G}((0,1)) \cong \mathscr{A}$$

with structure maps given by the E_1-structure similarly to those of $\mathscr{F}$.

For a proof, see § 9.2. The last statement restricted to the open interval $(0,1)$ is just Theorem 9 (in the case $n=1$).

Example 33. We consider the closed half-line $[0,+\infty)$ as a stratified manifold, with strata $\{0\} \subset [0,+\infty)$ given by its boundary. Namely, it has a 0-dimensional strata given by $\{0\}$ and thus one dimension 1 strata $(0,+\infty)$. Similarly, there is a stratified closed half-line $(-\infty,0]$. From Proposition 27 (and its proof) we also deduce

Proposition 28. *There is an equivalence of ∞-categories between locally constant factorization algebra on the closed half-line $[0,+\infty)$ and the category E_1**-RMod** of (pointed) right modules over E_1-algebras[53]. This equivalence sits in a commutative diagram*

$$\begin{tikzcd} \boldsymbol{Fac}^{lc}_{[0,+\infty)} \arrow[r, "\cong"] \arrow[d] & E_1\text{-}\boldsymbol{RMod} \arrow[d] \\ \boldsymbol{Fac}^{lc}_{(0,+\infty)} \arrow[r, "\cong"] & E_1\text{-}\boldsymbol{Alg} \end{tikzcd}$$

where the left vertical functor is given by restriction to the open line and the lower horizontal functor is given by Theorem 9.

*There is a similar equivalence (and diagram) of ∞-categories between locally constant factorization algebra on the closed half-line $(-\infty,0]$ and the category E_1**-LMod** of (pointed) left modules over E_1-algebras.*

Let X be a manifold and consider the stratified manifold $X \times [0,+\infty)$, with a $\dim(X)$ open strata $X \times \{0\}$. Using Corollary 6 and Proposition 28 one can get

Corollary 7. *The pushforward along the projection $X \times [0,+\infty) \to [0,+\infty)$ induces an equivalence*

$$\boldsymbol{Fac}^{lc}_{X\times[0,+\infty)} \xrightarrow{\simeq} E_1\text{-}\boldsymbol{RMod}(\boldsymbol{Fac}^{lc}_X).$$

[52] more precisely, taking A, M^ℓ, M^r be strictification of $\mathscr{A}$, $\mathscr{M}^\ell$ and $\mathscr{M}^r$, there is a quasi-isomorphism of factorization algebras from $\mathscr{F}$ (associated to A, M^ℓ, M^r) to $\mathscr{G}$.

[53] which, informally, is the category of pairs (A, M^r) where A is an E_1-algebra and M^r a (pointed) right A-module

6.3 Factorization algebras on pointed disk and E_n-modules

In this section we relate E_n-modules[54] and factorization algebras over the pointed disk.

Let $\mathbb{R}^n_*$ *denote the pointed disk* which we see as a stratified manifold with one 0-dimensional strata given by the point $0 \in \mathbb{R}^n$ and n-dimensional strata given by the complement $\mathbb{R}^n \setminus \{0\}$.

Definition 22. We denote $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$ the ∞-*category of locally constant factorization algebras on the pointed disk* $\mathbb{R}^n_*$ (in the sense of Definition 21).

Recall the functor (33) giving the obvious embedding $\mathbf{Fac}^{lc}_{\mathbb{R}^n} \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$.

Locally constant factorization algebras on $\mathbb{R}^n_*$ are related to those on the closed half-line (§ 6.2) as follows: let $N : \mathbb{R}^n \to [0,+\infty)$ be the euclidean norm map $x \mapsto \|x\|$. We have the pushforwards $N_* : \mathbf{Fac}_{\mathbb{R}^n_*} \to \mathbf{Fac}_{[0,+\infty)}$ and $(-N)_* : \mathbf{Fac}_{\mathbb{R}^n_*} \to \mathbf{Fac}_{(-\infty,0]}$.

Lemma 7. *If $\mathscr{F} \in \boldsymbol{Fac}^{lc}_{\mathbb{R}^n_*}$, then $N_*(\mathscr{F}) \in \boldsymbol{Fac}^{lc}_{[0,+\infty)}$ and $(-N)_*(\mathscr{F}) \in \boldsymbol{Fac}^{lc}_{(-\infty,0]}$.*

Proof. For $0 < \varepsilon < \eta$, the structure map $N_*(\mathscr{F})\big([0,\varepsilon)\big) \cong \mathscr{F}\big(N^{-1}([0,\varepsilon))\big) \to \mathscr{F}\big(N^{-1}([0,\eta))\big) \cong N_*(\mathscr{F})\big([0,\eta)\big)$ is an equivalence since $\mathscr{F}$ is locally constant and $N^{-1}([0,\alpha)$ is a euclidean disk centered at 0. Further, by Proposition 15, $N_*(\mathscr{F}_{|\mathbb{R}^n\setminus\{0\}})$ is locally constant from which we deduce that $N_*(\mathscr{F})$ is locally constant on the stratified half-line $[0,+\infty)$. The case of $(-N)_*$ is the same. □ □

Our next task is to define a functor E_n-$\mathbf{Mod} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$ from (pointed) E_n-modules (see § 10.2) to (locally constant) factorization algebras on the pointed disk. It is enough to associate (functorially), to any $M \in E_n$-$\mathbf{Mod}$, a $\mathscr{CV}(\mathbb{R}^n)$-factorization algebra $\mathscr{F}_M$ where $\mathscr{CV}(\mathbb{R}^n)$ is the (stable by finite intersection) factorizing basis of $\mathbb{R}^n$ of convex open subsets. It turns out to be easy: since any convex subset C is canonically an embedded framed disk, we can set $\mathscr{F}_M(C) := M(C)$. In other words, we assign the module to a convex neighborhood of 0 and the algebra to a convex neigborhood which does not contain the origin.

Then set the structure maps $\mathscr{F}_M(C_1)\otimes\cdots\otimes\mathscr{F}_M(C_i) \to \mathscr{F}_M(D)$, for any pairwise disjoint convex subsets C_k of D, to be given by the map $M(C_1)\otimes\cdots\otimes M(C_i) \to M(D)$ associated to the framed embedding $\coprod_{k=1\dots i}\mathbb{R}^n \cong \bigcup_{k=1\dots i} C_k \hookrightarrow D \hookrightarrow \mathbb{R}^n$.

Theorem 12. *The rule $M \mapsto \mathscr{F}_M$ induces a fully faithful functor $\psi : E_n$-$\boldsymbol{Mod} \to \boldsymbol{Fac}^{lc}_{\mathbb{R}^n_*}$ which fits in a commutative diagram*

$$\begin{tikzcd} E_n\text{-}\boldsymbol{Alg} \arrow[r, "\simeq"] \arrow[d, "can"'] & \boldsymbol{Fac}^{lc}_{\mathbb{R}^n} \arrow[d] \\ E_n\text{-}\boldsymbol{Mod} \arrow[r, "\psi"] & \boldsymbol{Fac}^{lc}_{\mathbb{R}^n_*}. \end{tikzcd}$$

[54] according to our convention in Appendix 10.2, all E_n-modules are pointed by definition

Here $can : E_n$*-**Alg*** $\to E_n$*-**Mod** is given by the canonical module structure of an algebra over itself.*

We will now identify E_n-modules in terms of factorization algebras on $\mathbb{R}^n_*$; that is the essential image of the functor $\psi : E_n\text{-}\mathbf{Mod} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$ given by Theorem 12. Recall the functor $\pi_{E_n} : E_n\text{-}\mathbf{Mod} \to E_n\text{-}\mathbf{Alg} \cong \mathbf{Fac}^{lc}_{\mathbb{R}^n}$ which, to a module $M \in E_n\text{-}\mathbf{Mod}_A$, associates $\pi_{E_n}(M) = A$.

By restriction to the open set $\mathbb{R}^n \setminus \{0\}$ we get that the two compositions of functors

$$E_n\text{-}\mathbf{Mod} \xrightarrow{\psi} \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}} \quad \text{and} \quad E_n\text{-}\mathbf{Mod} \xrightarrow{\pi_{E_n}} E_n\text{-}\mathbf{Alg} \cong \mathbf{Fac}^{lc}_{\mathbb{R}^n} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}$$

are equivalent. Hence, we get a factorization of (ψ, π_{E_n}) to the pullback

$$(\psi, \pi_{E_n}) : E_n\text{-}\mathbf{Mod} \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \mathbf{Fac}^{lc}_{\mathbb{R}^n}.$$

Informally, $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \mathbf{Fac}^{lc}_{\mathbb{R}^n}$ is simply the (∞-)category of pairs $(\mathscr{A}, \mathscr{M}) \in \mathbf{Fac}^{lc}_{\mathbb{R}^n} \times \mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$ together with a quasi-isomorphism $f : \mathscr{A}_{|\mathbb{R}^n\setminus\{0\}} \to \mathscr{M}_{|\mathbb{R}^n\setminus\{0\}}$ of factorization algebras.

Corollary 8. *The functor* E_n*-**Mod*** $\overset{(\psi,\pi_{E_n})}{\longrightarrow} \boldsymbol{Fac}^{lc}_{\mathbb{R}^n_*} \times^h_{\boldsymbol{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \boldsymbol{Fac}^{lc}_{\mathbb{R}^n}$ *is an equivalence.*

Now, if A is an E_n-algebra, we can see it as a factorization algebra on $\mathbb{R}^n$ (Theorem 9) and taking the (homotopy) fiber at $\{A\}$ of the right hand side of the equivalence in Corollary 8, we get

Corollary 9. *The functor* $E_n\text{-}\boldsymbol{Mod}_A \xrightarrow{\psi} \boldsymbol{Fac}^{lc}_{\mathbb{R}^n_*} \times^h_{\boldsymbol{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{A\}$ *is an equivalence.*

Example 34 (Locally constant factorization algebra on the pointed line $\mathbb{R}_*$). The case of the pointed line $\mathbb{R}_*$ is slightly special since it has two (and not one) strata of maximal dimension. The two embeddings $j_+ : (0, +\infty) \hookrightarrow \mathbb{R}$ and $j_- : (-\infty, 0) \hookrightarrow \mathbb{R}$ yields two restrictions functors: $j^*_\pm : \mathbf{Fac}^{lc}_{\mathbb{R}_*} \to \mathbf{Fac}^{lc}_{\mathbb{R}}$. Hence $\mathscr{F} \in \mathbf{Fac}^{lc}_{\mathbb{R}_*}$ determines two E_1-algebras $R \cong \mathscr{F}\big((0, +\infty)\big)$ and $L \cong \mathscr{F}\big((-\infty, 0)\big)$.

By Lemma 7, the pushforward $(-N)_* : \mathbf{Fac}^{lc}_{\mathbb{R}_*} \to \mathbf{Fac}^{lc}_{(-\infty,0]}$ along $-N : x \mapsto -|x|$ is well defined. Then Proposition 28 implies that $(-N)_*(\mathscr{F})$ is determined by a left module M over the E_1-algebra $A \cong (-N)_*(\mathscr{F})\big((0, +\infty)\big) \cong L \otimes R^{op}$, *i.e.* by a (L, R)-bimodule.

We thus have a functor $(j^*_\pm, (-N)_*) : \mathbf{Fac}^{lc}_{\mathbb{R}_*} \to \mathbf{BiMod}$ where $\mathbf{BiMod}$ is the ∞-category of bimodules (in $Chain(k)$) defined in [L3, §4.3] (*i.e.* the ∞-category of triples (L, R, M) where L, R are E_1-algebras and M is a (L, R)-bimodule).

Proposition 29. *The functor* $(j^*_\pm, (-N)_*) : \boldsymbol{Fac}^{lc}_{\mathbb{R}_*} \cong \boldsymbol{BiMod}$ *is an equivalence.*

Example 35 (From pointed disk to the n-dimensional annulus: $S^{n-1}\times\mathbb{R}$). Let $\mathscr{M}$ be a locally constant factorization algebra on the pointed disk $\mathbb{R}^n_*$. By the previous results in this Section (specifically Lemma 7 and Proposition 28), for any $\mathscr{A}\in\mathbf{Fac}^{lc}_{S^{n-1}\times\mathbb{R}}$, the pushforward along the euclidean norm $N:\mathbb{R}^n\to[0,+\infty)$ factors as a functor

$$N_*:\mathbf{Fac}^{lc}_{\mathbb{R}^n_*}\times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}}\{\mathscr{A}\}\longrightarrow E_1\text{-}\mathbf{RMod}_{\mathscr{A}(S^{n-1}\times\mathbb{R})}$$

from the category of locally constant factorization algebras on the pointed disk whose restriction to $\mathbb{R}^n\setminus\{0\}$ is (quasi-isomorphic to) $\mathscr{A}$ to the category of right modules over the E_1-algebra[55] $\mathscr{A}(S^{n-1}\times\mathbb{R})\cong\int_{S^{n-1}\times\mathbb{R}}\mathscr{A}$. Similarly, we have the functor $(-N)_*:\mathbf{Fac}^{lc}_{\mathbb{R}^n_*}\times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}}\{\mathscr{A}\}\longrightarrow E_1\text{-}\mathbf{LMod}_{\mathscr{A}(S^{n-1}\times\mathbb{R})}$.

Proposition 30. *Let $\mathscr{A}$ be in $\boldsymbol{Fac}^{lc}_{S^{n-1}\times\mathbb{R}}$.*

- *The functor $N_*:\boldsymbol{Fac}^{lc}_{\mathbb{R}^n_*}\times_{\boldsymbol{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}}\{\mathscr{A}\}\longrightarrow E_1\text{-}\boldsymbol{RMod}_{\mathscr{A}(S^{n-1}\times\mathbb{R})}$ is an equivalence of ∞-categories.*
- *$(-N)_*:\boldsymbol{Fac}^{lc}_{\mathbb{R}^n_*}\times_{\boldsymbol{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}}\{\mathscr{A}\}\longrightarrow E_1\text{-}\boldsymbol{LMod}_{\mathscr{A}(S^{n-1}\times\mathbb{R})}$ is an equivalence.*

See § 9.3 for a Proof.

The Proposition allows to reduce a category of“modules” over $\mathscr{A}$ to a category of modules over an (homotopy) dg-*associative* algebra.

We will see in section 7.1 that it essentially gives a concrete description of the enveloping algebra of an E_n-algebra.

6.4 More examples

We now give, without going into any details, a few more examples of locally constant factorization algebras on stratified spaces which can be studied along the same way as we did previously.

Example 36 (towards quantum mechanics). One can make the following variant of the interval example of Proposition 27 (see [CG] for details). Fix a one parameter group $(\alpha_t)_{t\in[0,1]}$ of invertible elements in an associative (topological) algebra A. Now, we define a factorization algebra on the basis $\mathscr{I}$ of open disks of $[0,1]$ in the same way as in Proposition 27 except that we add the element α_d to any hole of length d between two intervals (corresponding to the inclusion of the empty set: $\mathscr{F}(\emptyset)\to\mathscr{F}(I)$ into any open interval). Precisely, we set:

- the structure map $k=\mathscr{F}(\emptyset)\to\mathscr{F}\big(s,t)\big)=A$ is given by the element α_{t-s}, the structure map $k=\mathscr{F}(\emptyset)\to\mathscr{F}\big[0,t)\big)=M^r\ m^r\cdot\alpha_t$ and the structure map $k=\mathscr{F}(\emptyset)\to\mathscr{F}\big(u,1]\big)=M^\ell\ \alpha_{1-u}\cdot m^\ell$.

---
[55] the E_1-structure is given by Lemma 5

- For $0 < s < t_1 < u_1 < \cdots < t_i < u_i < v < 1$ one sets the map

  $$M^r \otimes A^{\otimes i} = \mathscr{F}([0,s)) \otimes \mathscr{F}((t_1,u_1)) \otimes \cdots \mathscr{F}((t_i,u_i)) \longrightarrow \mathscr{F}([0,v)) = M^r$$

  to be given by

  $$m \otimes a_1 \otimes \cdots \otimes a_i \longmapsto m \cdot \alpha_{t_1-s} a_1 \alpha_{t_2-u_1} a_2 \cdots \alpha_{t_i-u_{i-1}} a_i \alpha_{v-u_i},$$

  the map

  $$A^{\otimes i} \otimes M^\ell = \mathscr{F}((t_1,u_1)) \otimes \cdots \mathscr{F}((t_i,u_i)) \otimes \mathscr{F}((v,1]) \longrightarrow \mathscr{F}((s,1]) = M^\ell$$

  to be given by $a_1 \otimes \cdots \otimes a_i \otimes n \longmapsto \alpha_{t_1-s} a_1 \alpha_{t_2-u_1} a_2 \cdots \alpha_{t_i-u_{i-1}} a_i \alpha_{v-u_i} n$, and also the map $A^{\otimes i} = \mathscr{F}((t_1,u_1)) \otimes \cdots \otimes \mathscr{F}((t_i,u_i)) \longrightarrow \mathscr{F}((s,v)) = A$ to be given by

  $$a_1 \otimes \cdots \otimes a_i \longmapsto \alpha_{t_1-s} a_1 \alpha_{t_2-u_1} a_2 \cdots \alpha_{t_i-u_{i-1}} a_i \alpha_{v-u_i}.$$

One can check that these structure maps define a $\mathscr{I}$-factorization algebra and thus a factorization algebra on I.

One can replace chain complexes with topological $\mathbb{C}$-vector spaces (with monoidal structure the completed tensor product) and take V to be an Hilbert space. Then one can choose $A = (End^{cont}(V))^{op}$ and $M^r = V = M^\ell$ where the left A-module structure on M^ℓ is given by the action of adjoint operators. Let $\alpha_t := e^{it\varphi}$ where $\varphi \in End^{cont}(V)$. One has $\mathscr{F}([0,1]) = \mathbb{C}$. One can think of A as the algebra of observables where V is the space of states and $e^{it\varphi}$ is the time evolution operator. Now, given two consecutives measures O_1, O_2 made during the time intervals $]s,t[$ and $]u,v[$ $(0 < s < t < u < v < 1)$, the probability amplitude that the system goes from an initial state v^r to a final state v^ℓ is the image of $O_1 \otimes O_2$ under the structure map

$$A \otimes A = \mathscr{F}(]s,t[) \otimes \mathscr{F}(]u,v[) \longrightarrow \mathscr{F}([0,1]) = \mathbb{C},$$

and is usually denoted $\langle v^\ell | e^{i(1-v)\varphi} O_2 e^{i(u-t)\varphi} O_1 e^{is\varphi} | v^r \rangle$.

Example 37 (the upper half-plane). Let $H = \{z = x + iy \in \mathbb{C}, y \geq 0\}$ be the closed upper half-plane, viewed as a stratified space with $H_0 = \emptyset$, $H_1 = \mathbb{R}$ the real line as dimension 1 strata and $H_2 = H$. The orthogonal projection $\pi : H \to \mathbb{R}$ onto the imaginary axis $\mathbb{R}i \subset \mathbb{C}$ induces, by Proposition 6 (and Proposition 28), an equivalence

$$\mathbf{Fac}^{lc}_H \xrightarrow{\simeq} \mathbf{Fac}^{lc}_{[0,+\infty)}(\mathbf{Fac}^{lc}_{\mathbb{R}}) \cong E_1\text{-}\mathbf{RMod}(E_1\text{-}\mathbf{Alg}).$$

Using Dunn Theorem 11, we have (sketched a proof of the fact) that

Proposition 31. *The ∞-category of stratified locally constant factorization algebra on H is equivalent to the ∞-category of algebras over the swiss cheese operad, that is the ∞-category consisting of triples (A,B,ρ) where A is an E_2-algebra, B an E_1-algebra and $\rho : A \to \mathbb{R}Hom_B^{E_1}(B,B)$ is an action of A on B compatible with all the multiplications*[56].

[56]that is the map ρ is a map of E_2-algebras where $\mathbb{R}\mathrm{Hom}_B^{E_1}(B,B)$ is the E_2-algebra given by the (derived) center of B. In particular, ρ induces a map $\rho(1_B) : A \to B$

One can also consider another stratification $\tilde{H}$ on $\mathbb{R}\times[0,+\infty)$ given by adding a 0-dimensional strata to H, given by the point $0\in\mathbb{C}$. There is now 4 kinds of (stratified) disks in $\tilde{H}$: the half-disk containing 0, the half disk with a connected boundary component lying on $(-\infty,0)$, the half-disk containing 0, the half disk with a connected boundary component lying on $(0,+\infty)$ and the open disks in the interior $\{x+iy,\, y>0\}$ of H.

One proves similarly

Proposition 32. *Locally constant factorization algebras on the stratified space $\tilde{H}$ are the same as the category given by quadruples (M,A,B,ρ_A,ρ_B,E) where E is an E_2-algebra, (A,ρ_A), (B,ρ_B) are E_1-algebras together with a compatible action of E and (M,ρ_M) is a (A,B)-bimodule together with a compatible action*[57] *of E.*

Examples of such factorization algebras occur in deformation quantization in the presence of two branes, see [CFFR].

Note that the norm is again adequatly stratified so that, if $\mathscr{F}\in\mathbf{Fac}^{lc}_{\tilde{H}}$, then $(-N)_*(\mathscr{F})\in\mathbf{Fac}^{lc}\big((-\infty,0]\big)\cong E_1\textbf{-LMod}$. Using the argument of Proposition 30 and Proposition 27, we see that if $\mathscr{F}$ is given by a tuple (M,A,B,ρ_A,ρ_B,E), then the underlying E_1-algebra of $(-N)_*(\mathscr{F})$ is (the two-sided Bar construction) $A\overset{\mathbb{L}}{\underset{E}{\otimes}}B^{op}$.

Example 38 (the unit disk in $\mathbb{C}$). Let $D=\{z\in\mathbb{C},\,|z|\le 1\}$ be the closed unit disk (with dimension 1 strata given by its boundary). By Lemma 7, the norm gives us a pushforward functor $N_*:\mathbf{Fac}^{lc}_D\to\mathbf{Fac}^{lc}_{[0,1]}$. A proof similar to the one of Corollary 7 and Theorem 12 shows that the two restrictions of N_* to $D\setminus\partial D$ and $D\setminus\{0\}$ induces an equivalence

$$\mathbf{Fac}^{lc}_D\xrightarrow{\simeq}\mathbf{Fac}^{lc}_{D\setminus\partial D}\times^h_{\mathbf{Fac}^{lc}_{(0,1)}}E_1\textbf{-RMod}\big(\mathbf{Fac}^{lc}_{S^1}\big).$$

By Corollary 4 and Theorem 9, one obtains:

Proposition 33. *The ∞-category of locally constant factorization algebra on the (stratified) closed unit disk D is equivalent to the ∞-category consisting of quadruples (A,B,ρ,f) where A is an E_2-algebra, B an E_1-algebra, $\rho: A:\mathbb{R}Hom_{E_1\textbf{-Alg}}(B,B)$ is an action of A on B compatible with all the multiplications and $f:B\to B$ is a monodromy compatible with ρ.*

One can also consider a variant of this construction with dimension 0 strata given by the center of the disk. In that case, one has to add a E_2-A-module to the data in Proposition 33.

Example 39 (the closed unit disk in $\mathbb{R}^n$). Let D^n be the closed unit disk of $\mathbb{R}^n$ which is stratified with a single strata of dimension $n-1$ given by its boundary $\partial D^n = S^{n-1}$. We have restriction functors $\mathbf{Fac}^{lc}_{D^n}\longrightarrow\mathbf{Fac}^{lc}_{D^n\setminus\partial D^n}\cong E_n\textbf{-Alg}$ (Theorem 9),

$$E_n\textbf{-Alg}\cong\mathbf{Fac}^{lc}_{D^n\setminus\partial D^n}\to\mathbf{Fac}^{lc}_{S^{n-1}\times(0,1)}\cong E_1\textbf{-Alg}(\mathbf{Fac}^{lc}_{S^{n-1}})\quad\text{(by Proposition 18)}$$

[57] precisely, this means the choice of a factorization $A\otimes B^{op}\longrightarrow A\overset{\mathbb{L}}{\underset{E}{\otimes}}B^{op}\xrightarrow{\rho_M}\mathbb{R}\mathrm{Hom}(M,M)$ (in $E_1\textbf{-Alg}$) of the (A,B)-bimodule structure of M

and $\mathbf{Fac}^{lc}_{D^n} \longrightarrow \mathbf{Fac}^{lc}_{D^n\setminus\{0\}}$. From Corollary 7, we deduce

Proposition 34. *The above restriction functors induce an equivalence*

$$\boldsymbol{Fac}^{lc}_{D^n} \xrightarrow{\simeq} E_n\text{-}\boldsymbol{Alg} \times_{E_1\text{-}\boldsymbol{Alg}\left(\boldsymbol{Fac}^{lc}_{S^{n-1}}\right)} E_1\text{-}\boldsymbol{LMod}\left(\boldsymbol{Fac}^{lc}_{S^{n-1}}\right).$$

Let $f: A \to B$ be an E_n-algebra map. Since $S^{n-1}\times\mathbb{R}$ has a canonical framing, both A and B carries a structure of locally constant factorization algebra on S^{n-1}, which is induced by pushforward along the projection $q: S^{n-1}\times\mathbb{R}\to S^{n-1}$ (see Example 26). Further, B inherits an E_n-module structure over A induced by f and, by restriction, $q_*(B_{|S^{n-1}\times\mathbb{R}})$ is an E_1-module over $q_*(A_{|S^{n-1}\times\mathbb{R}})$. Thus, Proposition 34 yields a factorization algebra $\omega_{D^n}(A\xrightarrow{f} B) \in \mathbf{Fac}^{lc}_{D^n}$.

Proposition 35. *The map ω_{D^n} induces a faithful functor $\omega_{D^n}: Hom_{E_n\text{-}\boldsymbol{Alg}} \longrightarrow \boldsymbol{Fac}^{lc}_{D^n}$ where $Hom_{E_n\text{-}\boldsymbol{Alg}}$ is the ∞-groupoid of E_n-algebras maps.*

By the relative higher Deligne conjecture (Theorem 16), we also have the E_n-algebra $HH_{E_n}(A,B_f)$ and an E_n-algebra map $HH_{E_n}(A,B_f)\to B$ given as the composition

$$HH_{E_n}(A,B_f) \cong \mathfrak{z}(f) \xrightarrow{id\otimes 1_A} \mathfrak{z}(f)\otimes\mathfrak{z}(k\xrightarrow{1_A} A) \longrightarrow \mathfrak{z}(k\xrightarrow{1_B} B) \cong B.$$

Thus by Proposition 35, the pair $(HH_{E_n}(A,B_f),B)$ also defines a factorization algebra on D^n.

Let $\tau: D^n\to\mathbb{R}^n\cup\{\infty\}$ be the map collapsing ∂D^n to the point ∞. It is an adequatly stratified map if we see $\mathbb{R}^n\cup\{\infty\}$ as being stratified with one dimension 0 strata given by ∞. Denote $\widehat{\mathbb{R}^n}$ this stratified manifold. The composition of τ_* with ω_{D^n} gives us:

Corollary 10. *There is a faithful functor $\widehat{\omega}: Hom_{E_n\text{-}\boldsymbol{Alg}} \longrightarrow \boldsymbol{Fac}^{lc}_{\widehat{\mathbb{R}^n}}$.*

Example 40 (The Square). Consider the square $I^2 := [0,1]^2$, the product of the interval (with its natural stratification of § 6.2) with itself. This stratification agrees with the one given by seeing I^2 as a manifold with corners. Thus, there are four 0-dimensional strata corresponding to the vertices of I^2 and four 1-dimensional strata corresponding to the edges. Let us denote $\{1,2,3,4\}$ the set of vertices and $[i,i+1]$ the corresponding edge linking the vertices i and $i+1$ (ordered cyclically[58]). See Figure 3.

We can construct a factorization algebra on I^2 as follows. For every edge $[i,j]$ of I^2, let $A_{[i,j]}$ be an E_1-algebra; also let A be an E_2-algebra. For every vertex i, let M_i be a $(A_{[i-1,i]},A_{[i,i+1]})$-bimodule. Finally, assume that A acts on each $A_{[i,j]}$ and M_k in a compatible way with algebras and module structures. Compatible, here, means that for every vertex i, the data $(M_i,A_{[i-1,i]},A_{[i,i+1]},A)$ (together with the various module structures) define a locally constant stratified factorization algebra on a neighborhood[59] of the vertex i as given by Proposition 32.

Using Propositions 27 and 29 (and their proofs) and Remark 23, we see that we obtain a locally constant factorization algebra $\mathscr{I}$ on I^2 whose value on an open rectangle $R \subset I^2$ is given by

$$\mathscr{I}(R) := \begin{cases} M_i & \text{if } R \text{ is a good neighborhood at the vertex } i; \\ A_{[i,i+1]} & \text{if } R \text{ is a good neighborhood at the edge } [i,i+1]; \\ A & \text{if } R \text{ lies in } I^2 \setminus \partial I^2. \end{cases} \tag{34}$$

The structure maps being given by the various module and algebras structure[60].

By Corollary 6, we get the functor $\underline{\pi_1}_* : \mathbf{Fac}^{lc}_{[0,1]^2} \to \mathbf{Fac}^{lc}_{[0,1]}(\mathbf{Fac}^{lc}_{[0,1]})$. Combining the proof of Proposition 18 and Proposition 27 (similarly to the proof of Corollary 7), we get:

Proposition 36. *The functor* $\underline{\pi_1}_* : \boldsymbol{Fac}^{lc}_{[0,1]^2} \to \boldsymbol{Fac}^{lc}_{[0,1]}(\boldsymbol{Fac}^{lc}_{[0,1]})$ *is an equivalence of* ∞*-categories.*

Moreover, any stratified locally constant factorization algebra on I^2 *is quasi-isomorphic to a factorization algebra associated to a tuple* $(A, A_{[i,i+1]}, M_i, i = 1\dots 4)$ *as in the rule* (34) *above.*

Let us give some examples of computations of the global sections of $\mathscr{I}$ on various opens.

- The band $I^2 \setminus ([4,1] \cup [2,3])$ is isomorphic (as a stratified space) to $(0,1) \times [0,1]$. Then from the definition of factorization homology and Proposi-

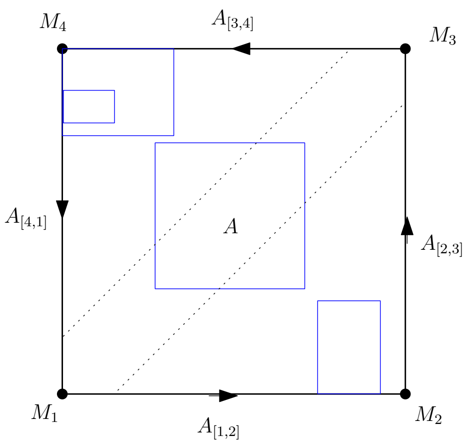

Figure 3: The stratification of the square. The (oriented) edges and vertices are decorated with their associated algebras and modules. Also 4 rectangles which are good neighborhoods at respectively 1, $[4,1]$, $[1,2]$ and the dimension 2-strata are depicted in blue. The band D_{13} is delimited by the dotted lines while the annulus T is the complement of the interior blue square.

[58] in other words we have the 4 edges $[1,2]$, $[2,3]$, $[3,4]$ and $[4,1]$

[59] for instance, take the complement of the closed sub-triangle of I^2 given by the three other vertices, which is isomorphic as a stratified space to the pointed half plane $\tilde{H}$ of Example 37

[60] note that we orient the edges accordingly to the ordering of the edges

tion 27.3, we get

$$\begin{aligned}\mathscr{I}\big(I^2\setminus([4,1]\cup[2,3])\big) &\cong p_*\big(\mathscr{I}_{|I^2\setminus([4,1]\cup[2,3])}\big)\cong p_*\big(\underline{\pi_1}_*(\mathscr{I}_{|I^2\setminus([4,1]\cup[2,3])})\big)\\ &= \int_{[0,1]}\underline{\pi_1}_*(\mathscr{I}_{|I^2\setminus([4,1]\cup[2,3])})\cong A_{[1,2]}\overset{\mathbb{L}}{\underset{A}{\otimes}}(A_{[3,4]})^{op}.\end{aligned}$$

In view of Corollary 7 in the case $X=\mathbb{R}$ (and Theorem 9); this is an equivalence of E_1-algebras.

- Consider a tubular neighborhood of the boundary, namely the complement[61] $T:=I^2\setminus[1/4,3/4]^2$ (see Figure 3). The argument of Proposition 27 and Example 15 shows (by projecting the square on $[0,1]\times\{1/2\}$) that

$$\mathscr{I}\big(T\big)\cong\Big(M_4\overset{\mathbb{L}}{\underset{A_{[4,1]}}{\otimes}}M_1\Big)\overset{\mathbb{L}}{\underset{A_{[1,2]}\otimes(A_{[3,4]})^{op}}{\bigotimes}}\Big(M_2\overset{\mathbb{L}}{\underset{A_{[2,3]}}{\otimes}}M_3\Big).$$

- Now, consider a diagonal band, say a tubular neigborhood D_{13} of the diagonal linking the vertex 1 to the vertex 3 (see Figure 3). Then projecting onto the diagonal $[1,3]$ and using again Proposition 27, we find,

$$\mathscr{I}\big(D_{13}\big)\cong M_1\overset{\mathbb{L}}{\underset{A}{\otimes}}M_3.$$

Iterating the above constructions to higher dimensional cubes, one finds

Proposition 37. *Let $[0,1]^n$ be the stratified cube. The pushforward along the canonical projections is an equivalence $\boldsymbol{Fac}^{lc}_{[0,1]^n}\overset{\simeq}{\to}\boldsymbol{Fac}^{lc}_{[0,1]}\big(\cdots(\boldsymbol{Fac}^{lc}_{[0,1]})\cdots\big)$.*

In other words, $\mathbf{Fac}^{lc}_{[0,1]^n}$ is a *tractable model* for an ∞-category consisting of the data of an E_n-algebra A_n together with E_{n-1}-algebras $A_{n-1,i_{n-1}}$ ($i_{n-1}=1\dots2n$) equipped with an action of A_n, E_{n-2}-algebras $A_{n2,i_{n-2}}$ ($i_{n-2}=1\dots2n(n-1)$) each equipped with a structure of bimodule over 2 of the $A_{n-1,j}$ compatible with the A actions, ..., E_k-algebras A_{k,i_k} ($i_k=1\dots2^{n-k}\binom{n}{k}$) equipped with structure of $n-k$-fold modules over ($n-k$ many of) the $A_{k+1,j}$ algebras, compatible with the previous actions, and so on ....

Similarly to the previous example 39, we also have a faithful functor

$$E_n\text{-}\mathbf{Alg}\cong\mathbf{Fac}^{lc}_{\mathbb{R}^n}\longrightarrow\mathbf{Fac}^{lc}_{[0,1]^n}.$$

Example 41 (Iterated categories of (bi)modules). We consider the n-fold product $(\mathbb{R}_*)^n$ of the pointed line (see Example 34) with its induced stratification. It has one 0-dimensional strata given by the origins, $2n$ many 1-dimensional strata given by half of the coordinate axis, ..., and 2^n open strata.

[61] note that the obvious radial projection $T\to\partial I^2$ is *not* adequatly stratified

Locally constant (stratified) factorization algebras on $(\mathbb{R}_*)^n$ are a model for iterated categories of bimodules objects. Indeed, by Corollary 6, the iterated first projections on $\mathbb{R}_*$ yields a functor $\underline{\pi}_* : \mathbf{Fac}^{lc}_{(\mathbb{R}_*)^n} \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}_*}\Big(\mathbf{Fac}^{lc}_{\mathbb{R}_*}\big(\dots(\mathbf{Fac}^{lc}_{\mathbb{R}_*})\dots\big)\Big)$. From Proposition 29 (and its proof) combined with the arguments of the proofs of Corollary 7 and Proposition 30, we get

Corollary 11. *The functor*

$$\underline{\pi}_* : \boldsymbol{Fac}^{lc}_{(\mathbb{R}_*)^n} \longrightarrow \boldsymbol{Fac}^{lc}_{\mathbb{R}_*}\Big(\boldsymbol{Fac}^{lc}_{\mathbb{R}_*}\big(\dots(\boldsymbol{Fac}^{lc}_{\mathbb{R}_*})\dots\big)\Big) \cong \boldsymbol{BiMod}\Big(\boldsymbol{BiMod}\big(\dots(\boldsymbol{BiMod})\dots\big)\Big)$$

is an equivalence.

An interesting consequence arise if we assume that the restriction to $\mathbb{R}^n\setminus\{0\}$ of $\mathscr{F}\in\mathbf{Fac}^{lc}_{(\mathbb{R}_*)^n}$ is constant, that is in the essential image of the functor $\mathbf{Fac}^{lc}_{\mathbb{R}^n}\to\mathbf{Fac}^{lc}_{(\mathbb{R}_*)^n}$. In that case, $\mathscr{F}$ belongs to the essential image of $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*}\to\mathbf{Fac}^{lc}_{(\mathbb{R}_*)^n}$. Thus Corollary 9 and Corollary 11 imply

Corollary 12. *There is a natural equivalence*

$$E_n\text{-}\boldsymbol{Mod} \cong E_1\text{-}\boldsymbol{Mod}\Big(E_1\text{-}\boldsymbol{Mod}\big(\dots(E_1\text{-}\boldsymbol{Mod})\dots\big)\Big)$$

inducing an equivalence

$$E_n\text{-}\boldsymbol{Mod}_A \cong E_1\text{-}\boldsymbol{Mod}_A\Big(E_1\text{-}\boldsymbol{Mod}_A\big(\dots(E_1\text{-}\boldsymbol{Mod}_A)\dots\big)\Big)$$

between the relevant ∞-subcategories[62].

Let us describe now informally $\mathbf{Fac}^{lc}_{(\mathbb{R}_*)^2}$. A basis of (stratified) disks is given by the convex open subsets. Such a subset is a good disk of index 0 if it is a neigborhood of the origin, of index 1 if it is in $\mathbb{R}^2\setminus\{0\}$ and intersects one and only one half open coordinate axis. It is a good disk of index 2 if it lies in the complement of the coordinate axis. We can construct a factorization algebra on $(\mathbb{R}_*)^2$ as follows. Let E_i, $i=1\dots4$ be four E_2-algebras (labelled by the cyclically ordered four quadrants of $\mathbb{R}^2$). Let $A_{1,2}$, $A_{2,3}$, $A_{3,4}$ and $A_{4,1}$ be four E_1-algebras endowed, for each $i\in\mathbb{Z}/4\mathbb{Z}$, with a compatible (E_i,E_{i+1})-bimodule structure[63] on $A_{i,i+1}$. Also let M be a E_2-module over each E_i in a compatible way. Precisely, this means that M is endowed with a right action[64] of the E_1-algebra $\big(A_{4,1}\overset{\mathbb{L}}{\underset{E_1}{\otimes}}A_{1,2}\big)\overset{\mathbb{L}}{\underset{E_2\otimes(E_4)^{op}}{\otimes}}\big(A_{2,3}\overset{\mathbb{L}}{\underset{E_3}{\otimes}}A_{3,4}\big)$.

[62] the A-E_1-module structure on the right hand side are taken along the various underlying E_1-structures of A obtained by projecting on the various component $\mathbb{R}$ of $\mathbb{R}^n$

[63] that is a map of E_2-algebras $E_i\otimes(E_{i+1})^{op}\to\mathbb{R}\mathrm{Hom}^{E_1}_{A_{i,i+1}}(A_{i,i+1},A_{i,i+1})$

[64] In particular, it implies that the tuple $(M,A_{i-1,i},A_{i,i+1},E_i)$ defines a stratified locally constant factorization algebra on $\tilde{H}$ by Proposition 32

Similarly to previous examples (in particular example 40), we see that we obtain a locally constant factorization algebra $\mathscr{M}$ on $(\mathbb{R}_*)^2$ whose value on an open rectangle is given by

$$\mathscr{M}(R) := \begin{cases} M & \text{if } R \text{ is a good neighborhood of the origin;} \\ A_{i,i+1} & \text{if } R \text{ is a good neighborhood of index 1} \\ & \quad \text{intersecting the quadrant labelled } i \text{ and } i+1; \\ E_i & \text{if } R \text{ lies in the interior of the } i^{\text{th}}\text{-quadrant.} \end{cases} \tag{35}$$

The structure maps are given by the various module and algebras structures. As in example 40, we get

Proposition 38. *Any stratified locally constant factorization algebra on* $(\mathbb{R}_*)^2$ *is quasi-isomorphic to a factorization algebra associated to a tuple* $(M, A_{i,i+1}, E_i, i = 1 \dots 4)$ *as in the rule* (35) *above.*

Example 42 (Butterfly). Let us give one of the most simple singular stratified example. Consider the "(semi-open) butterfly" that is the subspace $B := \{(x,y) \in \mathbb{R}^2 \,|\, |y| < |x|\} \cup \{(0,0)\}$ of $\mathbb{R}^2$. B has a dimension 0 strata given by the origin and two open strata B_+, B_- of dimension 2 given respectively by restricting to those points $(x,y) \in B$ such that $x > 0$, resp. $x < 0$.

The restriction to B_+ of a stratified locally constant factorization algebra on B is locally constant factorization over $B_+ \cong \mathbb{R}^2$ (hence is determined by an E_2-algebra). This way, we get the restriction functor $\mathrm{res}^* : \mathbf{Fac}^{lc}_B \to \mathbf{Fac}^{lc}_{\mathbb{R}^2} \times \mathbf{Fac}^{lc}_{\mathbb{R}^2}$.

Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection $(x,y) \mapsto x$ on the first coordinate. Then, by Corollary 6, we get a functor $\pi_* : \mathbf{Fac}^{lc}_B \to \mathbf{Fac}^{lc}_{\mathbb{R}_*}$ where $\mathbb{R}_*$ is the pointed line (example 34). The restriction of π_* to $B_+ \cup B_-$ thus yields a the functor $\mathbf{Fac}^{lc}_{\mathbb{R}^2} \times \mathbf{Fac}^{lc}_{\mathbb{R}^2} \to \mathbf{Fac}^{lc}_{\mathbb{R}\setminus\{0\}}$.

A proof similar (and slightly easier) to the one of Proposition 45 shows that the induced functor

$$(\pi_*, \mathrm{res}^*) : \mathbf{Fac}^{lc}_B \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}\setminus\{0\}}} (\mathbf{Fac}^{lc}_{\mathbb{R}^2} \times \mathbf{Fac}^{lc}_{\mathbb{R}^2})$$

is an equivalence. From Proposition 29, we then deduce

Proposition 39. *There is an equivalence*

$$\boldsymbol{Fac}^{lc}_B \cong \boldsymbol{BiMod} \times^h_{(E_1\text{-}\boldsymbol{Alg} \times E_1\text{-}\boldsymbol{Alg})} \big(E_2\text{-}\boldsymbol{Alg} \times E_2\text{-}\boldsymbol{Alg}\big).$$

In other words, locally constant factorization algebras on the (semi-open) butterfly are equivalent to the ∞*-category of triples* (A,B,M) *where* A*,* B *are* E_2*-algebras and* M *is a left* $A \otimes B^{op}$*-module (for the underlying* E_1*-algebras structures of* A *and* B*).*

Let $\overline{B} = \{(x,y) \in \mathbb{R}^2 \,|\, |y| \le |x|\}$ be the closure of the butterfly. It has four additional dimension 1 strata, given by the boundary of B_+ and B_-. The above argument yields an equivalence

$$\mathbf{Fac}^{lc}_{\overline{B}} \xrightarrow{\simeq} \mathbf{Fac}^{lc}_{\mathbb{R}_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}\setminus\{0\}}} \big(\mathbf{Fac}^{lc}_{\tilde{H}} \times \mathbf{Fac}^{lc}_{\tilde{H}}\big)$$

where $\tilde{H}$ is the pointed half-plane, see example 37. From Propositions 32, 27 and 29 we see that a locally constant factorization algebra on the closed butterfly $\overline{B}$ is equivalent to the data of two E_2-algebras E_+, E_-, four E_1-algebras A_+, B_+, A_-, B_-, equipped respectively with left or right modules structures over E_+ or E_-, and a left $\left(A_+ \otimes^{\mathbb{L}}_{E_+} B_+\right) \otimes \left(A_- \otimes^{\mathbb{L}}_{E_-} B_-\right)^{op}$-module M.

Example 43 (Homotopy calculus). The canonical action of $S^1 = SO(2)$ on $\mathbb{R}^2$ has the origin for fixed point. It follows that S^1 acts canonically on $\mathbf{Fac}_{\mathbb{R}^2_*}$, the category of factorization algebras over the pointed disk. If $\mathscr{F} \in \mathbf{Fac}^{lc}_{\mathbb{R}^2_*}$, its restriction to $\mathbb{R}^2 \setminus \{0\}$ determines an E_2-algebra A with monodromy by Corollary 4, Proposition 18 and Theorem 11. If $\mathscr{F}$ is S^1-equivariant, then its monodromy is trivial and it follows that the global section $\mathscr{F}(\mathbb{R}^2)$ is an E_2-module over A. The S^1-action yields an S^1-action on $\mathscr{F}(\mathbb{R}^2)$ which, algebraically, boils down to an additional differential of homological degree 1 on $\mathscr{F}(\mathbb{R}^2)$. We believe that the techniques in this section suitably extended to the case of compact group actions on factorization algebras allow to prove

Claim. The category $\left(\mathbf{Fac}^{lc}_{\mathbb{R}^2_*}\right)^{S^1}$ *of* S^1*-equivariant locally constant factorization algebras on the pointed disk* $\mathbb{R}^2_*$ *is equivalent to the category of homotopy calculus of Tamarkin-Tsygan [TT] describing homotopy Gerstenhaber algebras acting on a BV-module.*

There are nice examples of homotopy calculus arising in algebraic geometry [BF].

Example 44. Let $K : S^1 \to \mathbb{R}^3$ be a knot, that is a smooth embedding of S^1 inside $\mathbb{R}^3$. Then we can consider the stratified manifold $\mathbb{R}^3_K$ with a 1 dimensional open strata given by the image of K, and another 3-dimensional open strata given by the knot complement $\mathbb{R}^3 \setminus K(S^1)$. Then the category of locally constant factorization algebra on $\mathbb{R}^3_K$ is equivalent to the category of quadruples (A, B, f, ρ) where A is an E_3-algebras, B is an E_1-algebra, $f : B \to B$ is a monodromy and $\rho : A \otimes B \to B$ is an action of A onto B (compatible with all the structures (that is making B an object of E_2-$\mathbf{Mod}_A(E_1\text{-}\mathbf{Alg})$. We refer to [AFT] and [Ta] for details on the invariant of knots produced this way.

7 Applications of factorization algebras and homology

7.1 Enveloping algebras of E_n-algebras and Hochschild cohomology of E_n-algebras

In this section we describe the universal enveloping algebra of an E_n-algebra in terms of factorization algebras, and apply it to describe E_n-Hochschild cohomology.

Given an E_n-algebra A, we get a factorization algebra on $\mathbb{R}^n$ and thus on its submanifold $S^{n-1} \times \mathbb{R}$ (equipped with the induced framing); see Example 19. We can use the results of § 6 to study the category of E_n-modules over A.

In particular, from Corollary 8, and Proposition 30, we obtain the following corollary which was first proved by J. Francis [F1].

Corollary 13. *Let A be an E_n-algebra. The functor $N_* : E_n\text{-}\boldsymbol{Mod}_A \to E_1\text{-}\boldsymbol{RMod}_{\int_{S^{n-1}\times\mathbb{R}} A}$ is an equivalence.*

Similarly, the functor $(-N)_* : E_n\text{-}\boldsymbol{Mod}_A \to E_1\text{-}\boldsymbol{LMod}_{\int_{S^{n-1}\times\mathbb{R}} A}$ is an equivalence.

We call $\int_{S^{n-1}\times\mathbb{R}} A$ *the universal (E_1-)enveloping algebra of the E_n-algebra A.*

A virtue of Corollary 13 is that it reduces the homological algebra aspects in the category of E_n-modules to standard homological algebra in the category of modules over a differential graded algebra (given by any strict model of $\int_{S^{n-1}\times\mathbb{R}} A$).

Remark 29. Corollary 8 remains true for $n = \infty$ (and follows from the above study, see [GTZ3]), in which case, since S^∞ is contractible, it boils down to the following result:

Proposition 40 ([L3], [KM]). *Let A be an E_∞-algebra. There is an natural equivalence of ∞-categories $E_\infty\text{-}\boldsymbol{Mod}_A \cong E_1\text{-}\boldsymbol{RMod}_A$ (where in the right hand side, A is identified with its underlying E_1-algebra).*

Example 45. Let A be a smooth commutative algebra (or the sheaf of functions of a smooth scheme or manifold) viewed as an E_n-algebra. Then, by Theorem 3 and Theorem 10, we have

Proposition 41. *For $n \geq 2$, there is an equivalence*

$$E_n\text{-}\boldsymbol{Mod}_A \;\cong\; E_1\text{-}\boldsymbol{RMod}_{S^\bullet_A(\Omega^1_A[n-1])}.$$

The right hand side is just a category of modules over a graded commutative algebra. If $A = \mathcal{O}_X$, then one thus has an equivalence between E_n-$\mathbf{Mod}_{\mathcal{O}_X}$ and right graded modules over $\mathcal{O}_{T_X[1-n]}$, the functions on the graded tangent space of X.

Example 46. Let A be an E_n-algebra. It is canonically a E_n-module over itself; thus by Corollary 13 it has a structure of right module over $\int_{S^{n-1}\times\mathbb{R}} A$. The later has an easy geometrict description. Indeed, by the dimension axiom, $A \cong \int_{\mathbb{R}^n} A$. The euclidean norm gives the trivialization $S^{n-1}\times(0,+\infty)$ of the end(s) of $\mathbb{R}^n$ so that, by Lemma 5, $\int_{\mathbb{R}^n} A$ has a canonical structure of right module over $\int_{S^{n-1}\times(0,+\infty)} A$.

Let us consider the example of an n-fold loop space. Let Y be an n-connective pointed space ($n \geq 0$) and let $A = C_*(\Omega^n(Y))$ be the associated E_n-algebra. By non-abelian Poincaré duality (Theorem 7) we have an equivalence

$$\int_{S^{n-1}\times\mathbb{R}} A \cong C_*\big(\mathrm{Map}_c(S^{n-1}\times\mathbb{R}, Y)\big) \cong C_*\Big(\Omega\Big(Y^{S^{n-1}}\Big)\Big).$$

By Corollary 13 we get

Corollary 14. *The category of E_n-modules over $C_*(\Omega^n(Y))$ is equivalent to the category of right modules over $C_*\Big(\Omega\Big(Y^{S^{n-1}}\Big)\Big)$.*

The algebra $C_*\left(\Omega\left(Y^{S^{n-1}}\right)\right)$ in Corollary 14 is computed by the cobar construction of the differential graded coalgebra $C_*\big(\mathrm{Map}(S^{n-1},Y)\big)$. If Y is of finite type, $n-1$-connected and the ground ring k is a field of characteristic zero, the latter is the linear dual of the commutative differential graded algebra $CH_{S^{n-1}}(\Omega^*_{dR}(Y))$ where $\Omega^*_{dR}(Y)$ is (by Theorem 5) the differential graded algebra of Sullivan polynomial forms on Y. In that case, the structure can be computed using rational homotopy techniques.

Example 47. Assume $Y = S^{2m+1}$, with $2m \geq n$. Then, Y has a Sullivan model given by the CDGA $S(y)$, with $|y| = 2m+1$. By Theorem 3 and Theorem 5, $C_*(\mathrm{Map}(S^{n-1},S^{2m+1}))$ is equivalent to the cofree cocommutative coalgebra $S(u,v)$ with $|u| = -1-2m$ and $|v| = n-2m-2$. By Corollary 14 and Corollary 19 we find that the category of E_n-modules over $C_*(\Omega^n(S^{2m+1})$ is equivalent to the category of right modules over the graded commutative algebra $S(a,b)$ where $|u| = -2m-2$ and $|v| = n-2m-3$.

There is an natural notion of cohomology for E_n-algebras which generalizes Hochschild cohomology of associative algebras. It plays the same role with respect to deformations of E_n-algebras as Hochschild cohomology plays with respect to deformations of associatives algebras.

Definition 23. Let M be an E_n-module over an E_n-algebra A. The E_n-Hochschild cohomology[65] of A with values in M, denoted by $HH_{E_n}(A,M)$, is by definition (see [F1]) $\mathbb{R}\mathrm{Hom}^{E_n}_A(A,M)$ (Definition 35).

Corollary 15. *Let A be an E_n-algebra, and M, N be E_n-modules over A.*

1. *There is a canonical equivalence*
$$\mathbb{R}Hom^{E_n}_A(M,N) \cong \mathbb{R}Hom^{left}_{\int_{S^{n-1}\times\mathbb{R}} A}(M,N)$$
where the right hand side are homomorphisms of left modules (Definition 36).
2. *In particular $HH_{E_n}(A,M) \cong \mathbb{R}Hom^{left}_{\int_{S^{n-1}\times\mathbb{R}} A}(A,M)$.*
3. *If A is a CDGA (or E_∞-algebra) and M is a left module over A, then*
$$HH_{E_n}(A,M) \cong \mathbb{R}Hom^{E_n}_A(A,M) \cong CH^{S^n}(A,M).$$

Proof. The first two points follows from from Corollary 13. The last one follows from Theorem 10 which yields equivalences

$$\begin{aligned}\mathbb{R}\mathrm{Hom}^{\mathrm{left}}_{\int_{S^{n-1}\times\mathbb{R}} A}(A,M) &\cong \mathbb{R}\mathrm{Hom}^{left}_{CH_{S^{n-1}}(A)}(CH_{\mathbb{R}^n}(A),M)\\ &\cong \mathbb{R}\mathrm{Hom}^{left}_A\left(CH_{\mathbb{R}^n}(A)\otimes^{\mathbb{L}}_{CH_{S^{n-1}}(A)} A, M\right)\\ &\cong \mathbb{R}\mathrm{Hom}^{left}_A(CH_{S^n}(A),M) \cong CH^{S^n}(A,M) \qquad (36)\end{aligned}$$

when A is an E_∞-algebra. □ □

[65] which is an object of $Chain(k)$

Example 48. Let A be a smooth commutative algebra and M a symmetric A-bimodule. By the HKR Theorem (see Theorem 3), one has $CH_{S^d}(A) \cong S^\bullet_A(\Omega^1_A[d])$ which is a projective A-module since A is smooth. Thus, Corollary 15 implies

$$HH_{E_d}(A,M) \cong S^\bullet_A(\mathrm{Der}(A,M)[-d]).$$

We now explain the relationship in between E_n-Hochschild cohomology and deformation of E_n-algebras. Denote $E_n\textbf{-Alg}_{|A}$ the ∞-category of E_n-algebras over A. The bifunctor of E_n-derivations $\mathrm{Der} : (E_n\textbf{-Alg}_{|A})^{op} \times E_n\textbf{-Mod}_A \to Chain(k)$ is defined as

$$\mathrm{Der}(R,N) := \mathrm{Map}_{E_n\textbf{-Alg}_{|A}}(R, A \oplus N).$$

The (absolute) *cotangent complex* of A (as an E_n-algebra) is the value on A of the left adjoint of the split square zero extension functor $E_n\textbf{-Mod}_A \ni M \mapsto A \oplus M \in E_n\textbf{-Alg}_{|A}$. In other words, there is an natural equivalence

$$\mathrm{Map}_{E_n\textbf{-Mod}_A}(L_A, -) \xrightarrow{\simeq} \mathrm{Der}(A, -)$$

of functors. The (absolute) *tangent complex of A* (as an E_n-algebra) is the dual of L_A (as an E_n-module):

$$T_A := \mathbb{R}\mathrm{Hom}_A^{E_n}(L_A, A) \cong \mathbb{R}\mathrm{Hom}^{\mathrm{left}}_{\int_{S^{n-1}\times\mathbb{R}} A}(L_A, A).$$

The tangent complex has a structure of an (homotopy) Lie algebra which controls the deformation of A as an E_n-algebra (that is, its deformations are precisely given by the solutions of Maurer-Cartan equations in T_A). Indeed, Francis [F1] has proved the following beautiful result which solve (and generalize) a conjecture of Kontsevich [Ko]. His proof relies heavily on factorization homology and in particular on the excision property to identify $E_n\textbf{-Mod}_A$ with the E_{n-1}-Hochschild homology of $E_1\textbf{-LMod}_A$ which is a E_{n-1}-monoidal category.

Theorem 13 ([F1])**.** *Let A be an E_n-algebra and T_A be its tangent complex. There is a fiber sequence of non-unital E_{n+1}-algebras*

$$A[-1] \longrightarrow T_A[-n] \longrightarrow HH_{E_n}(A)$$

inducing a fiber sequence of (homotopy) Lie algebras

$$A[n-1] \longrightarrow T_A \longrightarrow HH_{E_n}(A)[n]$$

after suspension.

7.2 Centralizers and (higher) Deligne conjectures

We will here sketch applications of factorization algebras to study *centralizers* and solve the (relative and higher) Deligne conjecture.

The following definition is due to Lurie [L3] (and generalize the notion of center of a category due to Drinfeld).

Definition 24. The (derived) centralizer of an E_n-algebra map $f : A \to B$ is the *universal* E_n-algebra $\mathfrak{z}_n(f)$ equipped with a map of E_n-algebras $e_{\mathfrak{z}_n(f)} : A \otimes \mathfrak{z}_n(f) \to B$ making the following diagram

$$\begin{tikzcd} & A \otimes \mathfrak{z}_n(f) \arrow[dr, dotted, "e_{\mathfrak{z}_n(f)}"] & \\ A \arrow[ur, "id\otimes 1_{\mathfrak{z}_n(f)}"] \arrow[rr, "f"] & & B \end{tikzcd} \tag{37}$$

commutative in E_n-Alg. The *(derived) center* of an E_n-algebra A is the centralizer $\mathfrak{z}_n(A) := \mathfrak{z}_n(A \xrightarrow{id} A)$ of the identity map.

The existence of the derived centralizer $\mathfrak{z}_n(f)$ of an E_n-algebra map $f : A \to B$ is a *non-trivial* result of Lurie [L3]. The universal property means that if $C \xrightarrow{\varphi} B$ is an E_n-algebra map fitting inside a commutative diagram

$$\begin{tikzcd} & A \otimes C \arrow[dr, "\varphi"] & \\ A \arrow[ur, "id\otimes 1_C"] \arrow[rr, "f"] & & B \end{tikzcd} \tag{38}$$

then there is a unique[66] factorization $\varphi : A \otimes C \overset{id\otimes\kappa}{\dashrightarrow} A \otimes \mathfrak{z}_n(f) \xrightarrow{e_{\mathfrak{z}_n(f)}} B$ of φ by an E_n-algebra map $\kappa : C \to \mathfrak{z}_n(f)$. In particular, the commutative diagram

$$\begin{tikzcd} & & A \otimes \mathfrak{z}_n(f) \otimes \mathfrak{z}_n(g) \arrow[dr, "e_{\mathfrak{z}_n(f)}\otimes id"] & & \\ & A \otimes \mathfrak{z}_n(f) \arrow[ur, "id\otimes 1_{\mathfrak{z}_n(g)}"] \arrow[dr, "e_{\mathfrak{z}_n(f)}"] & & B \otimes \mathfrak{z}_n(g) \arrow[dr, "e_{\mathfrak{z}_n(g)}"] & \\ A \arrow[ur, "id\otimes 1_{\mathfrak{z}_n(f)}"] \arrow[rr, "f"] & & B \arrow[ur, "id\otimes 1_{\mathfrak{z}_n(g)}"] \arrow[rr, "g"] & & C \end{tikzcd}$$

induces natural maps of E_n-algebras

$$\mathfrak{z}_n(\circ) : \mathfrak{z}_n(f) \otimes \mathfrak{z}_n(g) \longrightarrow \mathfrak{z}_n(g \circ f). \tag{39}$$

Example 49. Let M be a monoid (for instance a group). That is an E_1-algebra in the (discrete) category of sets with cartesian product for monoidal structure. Then $\mathfrak{z}_1(M) = Z(M)$ is the usual center $\{m \in M, \forall n \in M, n \cdot m = m \cdot n\}$ of M. Let $f : H \hookrightarrow G$ be the inclusion of a subgroup in a group G. Then $\mathfrak{z}_1(f)$ is the usual centralizer of the subgroup H in G. This examples explain the name centralizer.

Similarly, let *k-Mod* be the (discrete) category of k-vector spaces over a field k. Then an E_1-algebra in *k-Mod* is an associative algebra and $\mathfrak{z}_1(A) = Z(A)$ is its usual (non-derived) center. However, if one sees A as an E_1-algebra in the ∞-category *Chain*(k) of chain complexes, then $\mathfrak{z}_1(A) \cong \mathbb{R}\mathrm{Hom}_{A\otimes A^{op}}(A,A)$ is (computed by) the usual Hochschild cochain complex ([L]) in which the usual center embeds naturally, but is different from it even when A is commutative.

[66] up to a contractible space of choices

Let $A \xrightarrow{f} B$ be an E_n-algebra map. Then B *inherits a* canonical *structure of E_n-A-module, denoted B_f*, which is the pullback along f of the tautological E_n-B-module structure on B.

The relative Deligne conjecture claims that the centralizer is computed by E_n-Hochschild cohomology.

Theorem 14 (Relative Deligne conjecture)**.** *Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be maps of E_n-algebras.*

1. *There is an E_n-algebra structure on $HH_{E_n}(A,B_f) \cong \mathbb{R}Hom_A^{E_n}(A,B_f)$ which makes $HH_{E_n}(A,B_f)$ the centralizer $\mathfrak{z}_n(f)$ of f (in particular, $\mathfrak{z}(f)$ exists);*
2. *the diagram*

$$\begin{tikzcd} \mathfrak{z}_n(f)\otimes\mathfrak{z}_n(g) \arrow[rr, "\mathfrak{z}_n(\circ)"] & & \mathfrak{z}_n(g\circ f) \\ \mathbb{R}Hom_A^{E_n}(A,B_f)\otimes\mathbb{R}Hom_B^{E_n}(B,C_g) \arrow[u, "\cong"] \arrow[rr, "\circ"] & & \mathbb{R}Hom_A^{E_n}(A,C_{g\circ f}) \arrow[u, "\cong"'] \end{tikzcd}$$

   *is commutative in E_n-**Alg** (where the lower arrow is induced by composition of maps in E_n-**Mod**).*

3. *If $A \xrightarrow{f} B$ is a map of E_∞-algebras, then there is an equivalence of E_n-algebras $\mathfrak{z}_n \cong CH^{S^n}(A,B_f)$ where $CH^{S^n}(A,B)$ is endowed with the structure given by Proposition 7.*

SKETCH OF PROOF. This result is proved in [GTZ3] (the techniques of [F1] shall also give an independent proof) and we only briefly sketch the main point of the argument. We first define an E_n-algebra structure on $\mathbb{R}\mathrm{Hom}_A^{E_n}(A,B_f)$. By Corollary 8 and Theorem 9, we can assume that A, B are factorization algebras on $\mathbb{R}^n$ and that a morphism of modules is a map of the underlying (stratified) factorization algebras. We are left to prove that there is a locally constant factorization algebra structure on $\mathbb{R}^n$ whose global sections are $HH_{E_n}(A,B_f) \cong \mathbb{R}\mathrm{Hom}_A^{E_n}(A,B_f)$. It is enough to define it on the basis of convex open subsets $\mathscr{CV}$ of $\mathbb{R}^n$ (by Proposition 17). To any convex open set U (with central point x_U), we associate the chain complex $\mathbb{R}\mathrm{Hom}_{A_{|U}}^{\mathbf{Fac}_U}(A_{|U},B_{f\,|U})$ of factorization algebras morphisms from the restrictions $A_{|U}$ to $B_{|U}$ which, on the restriction to $U\setminus\{x_U\}$ are given by f. Note that since U is convex, there is a quasi-isomorphism

$$\mathbb{R}\mathrm{Hom}_{A_{|U}}^{\mathbf{Fac}_U}(A_{|U},B_{f\,|U}) \cong \mathbb{R}\mathrm{Hom}_A^{E_n}(A,B_f) = \mathbb{R}\mathrm{Hom}_{A_{|\mathbb{R}^n}}^{\mathbf{Fac}_{\mathbb{R}^n}}(A_{|\mathbb{R}^n},B_{f\,|\mathbb{R}^n}). \tag{40}$$

Now, given convex sets $U_1,\dots,U_r$ which are pairwise disjoint inside a bigger convex V, we define a map

$$\rho_{U_1,\dots,U_r,V} : \bigotimes_{i=1\dots r}\mathbb{R}\mathrm{Hom}_{A_{|U_i}}^{\mathbf{Fac}_{U_i}}(A_{|U_i},B_{f\,|U_i}) \longrightarrow \mathbb{R}\mathrm{Hom}_{A_{|V}}^{\mathbf{Fac}_V}(A_{|V},B_{f\,|V})$$

as follows. To define $\rho_{U_1,\dots,U_r,V}(g_1,\dots,g_r)$, we need to define a factorization algebra map on V and for this, it is enough to do it on the open set consisting of convex subsets of V which either are included in one of the U_i and contains x_{U_i} or else does not contains any x_{U_i}. To each open set $x_{U_i} \in D_i \subset U_i$ of the first kind, we define

$$\rho_{U_1,\dots,U_r,V}(g_1,\dots,g_r)(D_i) : A(D_i) \xrightarrow{g_i} B_f(D_i)$$

to be given by g_i, while for any open set $D \subset V \setminus \{x_{U_1},\dots,x_{|U_r}\}$, we define

$$\rho_{U_1,\dots,U_r,V}(g_1,\dots,g_r)(D) : A(D) \xrightarrow{f} B_f(D)$$

to be given by f. The conditions that $g_1,\dots,g_r$ are maps of E_n-modules over $A(U_i)$ ensures that $\rho_{U_1,\dots,U_r,V}$ define the structure maps of a factorization algebra which is further locally constant since $\rho_{U,\mathbb{R}^n}$ is the equivalence (40).

The construction is roughly described in Figure 4. One can check that the natural evaluation map $eval : A \otimes \mathbb{R}\mathrm{Hom}_A^{E_n}(A,B_f) \to B$ is a map of E_n-algebras.

Now let C be an E_n-algebra fitting in the commutative diagram (38), which we again identify with a factorization algebra map (over $\mathbb{R}^n$). By adjunction (in $Chain(k)$), the map $\varphi : A \otimes C \to B$ has a (derived) adjoint $\theta_\varphi : C \to RHom(A,B)$. Since φ is a map of factorization algebras and diagram (37) is commutative, one check that θ_φ factors through a map

$$\tilde{\theta_\varphi} : C \longrightarrow RHom_A^{E_n}(A,B) \cong CH^{S^n}(A,B). \tag{41}$$

which can be proved to be a map of factorization algebras. Further, by definition of θ_φ, the identity

$$eval \circ \big(id_A \otimes \theta_\varphi\big) \;=\; \varphi$$

holds. The uniqueness of the map $\tilde{\theta_\varphi}$ follows from the fact that the composition

$$RHom_A^{E_n}(A,B) \xrightarrow{1_{RHom_A^{E_n}(A,A)} \otimes id} RHom_A^{E_n}\Big(A, A \otimes RHom_A^{E_n}(A,B)\Big) \xrightarrow{ev_*} RHom_A^{E_n}(A,B)$$

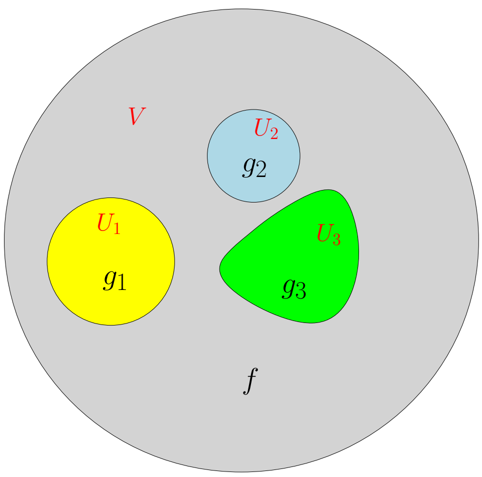

Figure 4: The factorization algebra map $A_{|V} \to B_{|V}$ obtained by applying the relevant maps of modules g_1, g_2, g_3 (viewed as maps of factorizations algebras) and the E_n-algebra map $f : A \to B$ on the respective regions

is the identity map.

Finally the equivalence between $\mathfrak{z}_n(f)$ and $CH^{S^n}(A,B_f)$ in the commutative case follows from the string of equivalences (36) which can be checked to be an equivalence of E_n-algebras using diagram (24) connecting algebras over the operads of little rectangles of dimension n and factorization algebras. $\square$

$\square$

Example 50. Let $A \xrightarrow{f} B$ be a map of CDGAs. then by Proposition 15 we have an equivalence $HH_{E_n}(A,B) \cong \mathbb{R}\mathrm{Hom}^{left}_{CH_{\partial I^n}(A)}(CH_{I^n}(A), CH_{I^n}(B))$ where $I=[0,1]$ and $\partial I^n \cong S^{n-1}$ is the boundary of the unit cube I^n. We have a simplicial model $I^n_\bullet$ of I^n where $I_\bullet$ is the standard model of Example 1; its boundary $\partial I^n_\bullet$ is a simplicial model for S^{n-1}. Then Theorem 14 identifies the derived composition (39) as the usual composition (of left dg-modules)

$$Hom^{left}_{CH_{\partial I^n_\bullet}(A)}\big(CH_{I^n_\bullet}(A), CH_{I^n_\bullet}(B)\big) \otimes Hom^{left}_{CH_{\partial I^n_\bullet}(B)}\big(CH_{I^n_\bullet}(B), CH_{I^n_\bullet}(C)\big) \\ \xrightarrow{\circ} Hom^{left}_{CH_{\partial I^n_\bullet}(A)}\big(CH_{I^n_\bullet}(A), CH_{I^n_\bullet}(C)\big).$$

The relative Deligne conjecture implies easily the standard one and also the Swiss cheese conjecture. Indeed, Theorem 14 implies that the multiplication $\mathfrak{z}_n(A)\otimes \mathfrak{z}_n(A) \xrightarrow{\mathfrak{z}_n(\circ)} \mathfrak{z}_n(A)$ makes $\mathfrak{z}_n(A)$ into an E_1-algebra in the ∞-category E_n-**Alg**. The ∞-category version of Dunn Theorem (Theorem 11) gives an equivalence E_1-**Alg**$\big(E_n$-**Alg**$\big) \cong E_{n+1}$-**Alg**. This yields the following solution to the higher Deligne conjecture, see [GTZ3, L3].

Corollary 16 (Higher Deligne Conjecture)**.** *Let A be an E_n-algebra. The E_n-Hochschild cohomology $HH_{E_n}(A,A)$ has an E_{n+1}-algebra structure lifting the Yoneda product which further lifts the E_n-algebra structure of the centralizer $\mathfrak{z}(A \xrightarrow{id} A)$.*

In particular, if A is commutative, there is a natural E_{n+1}-algebra structure on $CH^{S^n}(A,A) \cong HH_{E_n}(A,A)$ whose underlying E_n-algebra structure is the one given by Theorem 7. Hence the underlying E_1-algebra structure is given by the cup-product (Example 9).

Example 51. Let $\mathscr{C}$ be a monoidal (ordinary) category. Then the center $\mathfrak{z}_1(\mathscr{C})$ is in E_2-**Alg**(Cat), that is a braided monoidal category. One can prove that $\mathfrak{z}_1(\mathscr{C})$ is actually the Drinfeld center of $\mathscr{C}$, see [L3].

Remark 30. Presumably, the E_{n+1}-structure on $HH_{E_n}(A)$ given by Corollary 16 shall be closely related to the one given by Theorem 13.

Example 52. Let $1_A : k \to A$ be the unit of an E_n-algebra A. Then $\mathfrak{z}_n(1_A) \cong A$ as an E_n-algebra. The derived composition (39) yields canonical map[67] of E_n-algebras $\mathfrak{z}_n(A) \otimes \mathfrak{z}_n(1_A) \longrightarrow \mathfrak{z}_n(1_A)$ which exhibits $A \cong \mathfrak{z}_n(1_A)$ as a right E_1-module over $\mathfrak{z}_n(A) \cong HH_{E_n}(A,A)$ in the category of E_n-algebras (by Theorem 16). Hence, in

[67] which is equivalent to $e_{\mathfrak{z}(1_A)}$

view of Example 37, we obtain, as an immediate corollary (see [Ca2]), a proof of the Swiss-Cheese version of Deligne conjecture[68]:

Corollary 17 (Deligne conjecture with action). *Let A be an E_n-algebra. Then the pair $(HH_{E_n}(A,A),A)$ is canonically an object of E_1-**RMod**$(E_n$-**Alg**$)$, that is A has an natural action of the E_{n+1}-algebra $HH_{E_n}(A,A)$.*

Example 53 ((Higher homotopy) calculus again [Ca2]). Let A be an E_n-algebra. Assume $n = 0,1,3,7$, so that A defines canonically a (locally constant) factorization algebra A_{S^n} on the framed manifold S^n (see Example 19). Similarly the E_{n+1}-algebra given by the higher Hochschild cohomology $HH_{E_n}(A,A)$ defines canonically a (locally constant) factorization algebra on the manifold $S^n \times (0,\infty)$ endowed with the product framing.

The Deligne conjecture with action (Corollary 17) shows that A is also a left module over $HH_{E_n}(A,A)$. Thus, according to Proposition 30, the pair $(HH_{E_n}(A,A),A)$ yields a stratified locally constant factorization algebra $\mathscr{H}$ on $D^{n+1} \setminus \{0\}$, the closed disk in which we have removed the origin.

By Theorem 8, we have that $(A_{S^n})(\partial D^{n+1}) \cong \int_{S^n} A$. Collapsing the boundary ∂D^{n+1} to a point yields an adequatly stratified map $\tau : D^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1}_*$ so that $\mathscr{A} := \tau_*(\mathscr{H})$ is stratified locally constant on $\mathbb{R}^{n+1}_*$. It is further $SO(n+1)$-equivariant. Together with example 43, the above paragraph thus sketches a proof of the following fact:

Corollary 18. *Let A be an E_n-algebra and $n = 0,1,3,7$. Then A gives rise to an $SO(n+1)$-equivariant stratified locally constant factorization algebra $\mathscr{A}$ on the pointed disk $\mathbb{R}^{n+1}_*$ such that $\mathscr{A}(\mathbb{R}^{n+1}) \cong \int_{S^n} A$ and for any sub-disk $D \subset \mathbb{R}^{n+1} \setminus \{0\}$, there is an natural (with respect to disk inclusions) equivalence of E_{n+1}-algebras $\mathscr{A}(D) \xrightarrow{\simeq} HH_{E_n}(A,A)$.*

In particular, for $n = 1$, we recover that the pair $(HH_{E_1}(A,A), CH_{S^1}(A))$, given by Hochschild cohomology and Hochschild homology of an associative or A_∞-algebra A, defines an homotopy calculus (see [KS, TT] or Example 43).

This corollary is proved in details (using indeed factorization homology techniques) in the interesting paper [Ho] along with many other examples in which D^n is replaced by other framed manifold.

7.3 Higher string topology

The formalism of factorization homology for CDGAs and higher Deligne conjecture was applied in [Gi, GTZ3] to higher string topology which we now explain briefly. We also refer to the work [Hu, HKV] for a related approach.

Let M be a closed oriented manifold, equipped with a Riemannian metric. String topology is about the algebraic structure of the chains and homology of the free loop space $LM := \mathrm{Map}(S^1,M)$ and its higher free sphere spaces $M^{S^n} :=$

[68] originally proved, for a slightly different variant of the swiss cheese operad, in [DTT, V]

$\mathrm{Map}(S^n, M)$. These spaces have Fréchet manifold structures and there is a submersion $ev : M^{S^n} \to M$ given by evaluating at a chosen base point in S^n. The canonical embedding $Map(S^n \vee S^n, M) \xrightarrow{\rho_{in}} Map(S^n, M) \times Map(S^n, M)$ has an oriented normal bundle[69]. It follows that there is a Gysin map $(\rho_{in})_! : H_*\left(M^{S^n \coprod S^n}\right) \longrightarrow H_{*-\dim(M)}\left(M^{S^n \vee S^n}\right)$. The pinching map $\delta_{S^n} : S^n \to S^n \vee S^n$ (obtained by collapsing the equator to a point) yields the map $\delta_{S^n}^* : \mathrm{Map}(S^n \vee S^n, M) \longrightarrow M^{S^n}$. The *sphere product* is the composition

$$\begin{aligned} \star_{S^n} : H_{*+\dim(M)}\left(M^{S^n}\right)^{\otimes 2} &\to H_{*+2\dim(M)}\left(M^{S^n \coprod S^n}\right) \\ &\xrightarrow{(\rho_{in})_!} H_{*+\dim(M)}\left(M^{S^n \vee S^n}\right) \xrightarrow{(\delta_{S^n}^*)_*} H_{*+\dim(M)}\left(M^{S^n}\right). \end{aligned} \tag{42}$$

The circle action on itself induces an action $\gamma : LM \times S^1 \to LM$.

Theorem 15. *1. **(Chas-Sullivan [CS])** Let $\Delta : H_*(LM) \xrightarrow{\times [S^1]} H_{*+1}(LM \times S^1) \xrightarrow{\gamma_*} H_{*+1}(LM)$ be induced by the S^1-action. Then $(H_{*+\dim(M)}(LM), \star_{S^1}, \Delta)$ is a Batalin Vilkoviski-algebra and in particular a P_2-algebra.*

*2. **(Sullivan-Voronov [CV])** $(H_{*+\dim(M)}(LM), \star_{S^n})$ is a graded commutative algebra.*

*3. **(Costello [C1], Lurie [L2])** If M is simply connected, the chains $C_*(LM)[\dim(M)]$ have a structure of E_2-algebra (and actually of Disk_2^{or}-algebra) which, in characteristic 0, induces Chas-Sullivan P_2-structure in homology (by [FT]).*

In [CV], Sullivan-Voronov also sketched a proof of the fact that $H_{\bullet+\dim(M)}(M^{S^n})$ is an algebra over the homology $H_*(\mathrm{Disk}_{n+1}^{or})$ of the operad Disk_{n+1}^{or} and in particular has a P_{n+1}-algebra structure (see Example 64). Their work and the aforementioned work for $n = 1$ (Theorem 15.3) rise the following

Question 1. Is there a natural E_{n+1}-algebra (or even Disk_{n+1}^{or}-algebra) on the *chains* $C_*\left(M^{S^n}\right)[\dim(M)]$ which induces Sullivan-Voronov product in homology ?

Using the solution to the higher Deligne conjecture and the relationship between factorization homology and mapping spaces, one obtains a positive solution to the above question for sufficiently connected manifolds.

Theorem 16 ([GTZ3])**.** *Let M be an n-connected[70] Poincaré duality space. The shifted chain complex $C_{\bullet+\dim(X)}(X^{S^n})$ has a natural[71] E_{n+1}-algebra structure which induces the sphere product $\star_{S^n}$ (given by the map (42)) of Sullivan-Voronov [CV]*

$$H_p\left(X^{S^n}\right) \otimes H_q\left(X^{S^n}\right) \to H_{p+q-\dim(X)}\left(X^{S^n}\right)$$

in homology when X is an oriented closed manifold.

[69] which can be obtained as a pullback along ev of the normal bundle of the diagonal $M \to M \times M$

[70] it is actually sufficient to assume that M is nilpotent, connected and has finite homotopy groups $\pi_i(M, m_0)$ for $1 \le i \le n$

[71] with respect to maps of Poincaré duality spaces

SKETCH OF PROOF. We only gives the key steps of the proof following [GTZ3, Gi].

- Let $[M]$ be the fundamental class of M. The Poincaré duality map $\chi_M : x \mapsto x \cap [M]$ is a map of left modules and thus, by Proposition 40 (and since, by assumption, the biduality homomorphism $C_*(X) \to (C^*(X))^\vee$ is a quasi-isomorphism),
$$\chi_M : C^*(X) \to C_*(X)[\dim(X)] \;\cong\; \big(C^*(X)\big)^\vee[\dim(X)]$$
has an natural lift as an E_∞-module. And thus as an E_n-module as well.

- From the previous point we deduce that there is an equivalence
$$\begin{aligned} HH_{E_n}(C^*(X), C^*(X)) &\cong \mathbb{R}\mathrm{Hom}^{E_n}_{C^*(X)}\big(C^*(X), C^*(X)\big) \\ &\xrightarrow{(\chi_M)_*} \mathbb{R}\mathrm{Hom}^{E_n}_{C^*(X)}\big(C^*(X), \big(C^*(X)\big)^\vee\big)[\dim(M)] \\ &\cong \mathbb{R}\mathrm{Hom}^{left}_{\int_{S^{n-1}\times\mathbb{R}} C^*(X)}\big(C^*(X), \big(C^*(X)\big)^\vee\big)[\dim(M)] \\ &\cong \mathbb{R}\mathrm{Hom}^{left}_{C^*(X)}\big(C^*(X) \otimes^{\mathbb{L}}_{\int_{S^{n-1}\times\mathbb{R}} C^*(X)} C^*(X), k\big)[\dim(M)] \\ &\cong \mathbb{R}\mathrm{Hom}^{left}_{C^*(X)}\big(CH_{S^n}(C^*(X)), k\big)[\dim(M)]. \end{aligned} \tag{43}$$
where the last equivalence follows from Theorem 10.

- By Theorem 5 relating Factorization homology of singular cochains with mapping spaces, the above equivalence (43) induces an equivalence
$$C_*\Big(M^{S^n}\Big)[\dim(M)] \xrightarrow{\simeq} HH_{E_n}(C^*(X), C^*(X)). \tag{44}$$

- Now, one uses the higher Deligne conjecture (Corollary 16) and the latter equivalence (44) to get an E_{n+1}-algebra structure on $C_*\Big(M^{S^n}\Big)[\dim(M)]$. The explicit definition of the cup-product given by Proposition 7 (and Theorem 14) allows to describe explicitly the E_1-algebra structure at the level of the cochains $C^*\Big(M^{S^n}\Big)$ through the equivalence (44), which, in turn allows to check it induces the product $\star_{S^n}$.

□ □

Example 54. Let $M = G$ be a Lie group and $A = S(V) \stackrel{\simeq}{\to} \Omega_{dR}(G)$ be its minimal model. The graded space V is concentrated in positive odd degrees. If G is n-connected, by the generalized HKR Theorem 3, there is an equivalence
$$S(V \oplus V^*[-n]) \;\cong\; CH^{S^n}(A,A) \;\cong\; C_*\Big(G^{S^n}\Big)[\dim(G)] \tag{45}$$
in *Chain*(k). The *higher formality conjecture* shows that the equivalence (45) is an equivalence of E_{n+1}-algebras. Here the left hand side is viewed as an E_{n+1}-algebra obtained by the formality of the E_{n+1}-operad from the P_{n+1}-structure on $S(V \oplus V^*[-n])$ whose multiplicative structure is the one given by the symmetric algebra and the bracket is given by the pairing between V and V^*.

7.4 Iterated loop spaces and Bar constructions

In this section we apply the formalism of factorization homology to describe iterated Bar constructions *equipped with* their algebraic structure and relate them to the E_n-algebra structure of n^{th}-iterated loop spaces. We follow the approach of [F1, AF, GTZ3].

Bar constructions have been introduced in topology as a model for the coalgebra structure of the cochains on $\Omega(X)$, the based loop space of a pointed space X. Similarly the cobar construction of coaugmented coalgebra has been studied originally as a model for the (E_1-)algebra structure of the chains on $\Omega(X)$. The Bar and coBar constructions also induce equivalences between algebras and coalgebras under sufficient nilpotence and degree assumptions [HMS, FHT, FG].

Let (A,d) be a differential graded unital associative algebra which is *augmented*, that is equipped with an algebra homomorphism $\varepsilon : A \to k$.

Definition 25. The standard *Bar functor* of the augmented algebra (A,d,ε) is

$$Bar(A) := \ k \overset{\mathbb{L}}{\underset{A}{\otimes}} k.$$

If A is flat over k, it is computed by the standard chain complex $Bar^{std}(A) = \bigoplus_{n\geq 0} \overline{A}^{\otimes n}$ (where $\overline{A} = ker(A \xrightarrow{\varepsilon} k)$ is the augmentation ideal of A) endowed with the differential

$$\begin{aligned} b(a_1 \otimes \cdots a_n) &= \sum_{i=1}^{n} \pm a_1 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_n \\ &\quad + \sum_{i=1}^{n-1} \pm a_1 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_n \end{aligned}$$

see [FHT, Fr2] for details (and signs).

The Bar construction has a standard coalgebra structure. It is well-known that if A is a commutative differential graded algebra, then the shuffle product makes the Bar construction $Bar^{std}(A)$ a CDGA and a bialgebra as well. It was proved by Fresse [Fr2] that Bar constructions of E_∞-algebras have an (augmented) E_∞-structure, allowing to consider iterated Bar constructions and further that there is a canonical n^{th}-iterated Bar construction functor for augmented E_n-algebras as well [Fr3].

Let us now describe the factorization homology/algebra point of view on Bar constructions.

An *augmented E_n-algebra* is an E_n-algebra A equipped with an E_n-algebra map $\varepsilon : A \to k$, called the augmentation. We denote ***E_n-Alg**^{aug} the ∞-category of augmented E_n-algebras* (see Definition 31). The augmentation makes k an E_n-module over A.

By Proposition 35, an augmented E_n-algebra defines naturally a locally constant factorization algebra on the closed unit disk (with its stratification given by its boundary) of dimension less than n. Indeed, we obtain functors

$$\omega_{D^i} : E_n\text{-}\mathbf{Alg}^{aug} \longrightarrow E_i\text{-}\mathbf{Alg}^{aug}(E_{n-i}\text{-}\mathbf{Alg}^{aug}) \longrightarrow \mathbf{Fac}^{lc}_{D^i}(E_{n-i}\text{-}\mathbf{Alg}^{aug}). \tag{46}$$

Definition 26. Let A be an augmented E_n-algebra. Its Bar construction is

$$Bar(A) := \int_{I\times\mathbb{R}^{n-1}} A \overset{\mathbb{L}}{\underset{\int_{S^0\times\mathbb{R}^{n-1}} A}{\otimes}} k.$$

This definition agrees with Definition 25 for differential graded associative algebras and further we have equivalences

$$Bar(A) := \int_{I\times\mathbb{R}^{n-1}} A \overset{\mathbb{L}}{\underset{\int_{S^0\times\mathbb{R}^{n-1}} A}{\otimes}} k \;\cong\; k \overset{\mathbb{L}}{\underset{A}{\otimes}} k \;\cong\; p_*(\omega_{D^1}(A)) \tag{47}$$

where I is the closed interval $[0,1]$ and $p : I \to pt$ is the unique map; in particular the right hand side of (47) is just the factorization homology of the associated factorization algebra on D^1.

The functor (46) shows that $Bar(A) \cong p_*(\omega_{D^1}(A))$ has an natural structure of augmented E_{n-1}-algebra (which can also be deduced from Lemma 5.2).

We can thus iterate (up to n-times) the Bar constructions of an augmented E_n-algebra.

Definition 27. Let $0 \le i \le n$. The i^{th}-iterated Bar construction of an augmented E_n-algebra A is the augmented E_{n-i}-algebra

$$Bar^{(i)}(A) := Bar(\cdots(Bar(A))\cdots).$$

Using the excision axiom of factorization homology, one finds

Lemma 8 (Francis [F1], [GTZ3])**.** *Let A be an E_n-algebra and $0 \le i \le n$. There is a natural equivalence of E_{n-i}-algebras*

$$Bar^{(i)}(A) \cong \int_{D^i\times\mathbb{R}^{n-i}} A \overset{\mathbb{L}}{\underset{\int_{S^{i-1}\times\mathbb{R}^{n-i+1}} A}{\otimes}} k \;\cong\; p_*(\omega_{D^i}(A))$$

In particular, taking $n = \infty$, we recover an E_∞-structure on the iterated Bar construction $Bar^{(i)}(A)$ of an augmented E_∞-algebra.

We now describes the expected coalgebras structures. We start with the E_∞-case, for which we can use the derived Hochschild chains from § 2. Then, Lemma 8, Theorem 10 and the excision axiom give natural equivalences of E_∞-algebras ([GTZ3]):

$$CH_{S^i}(A,k) \;\cong\; Bar^{(i)}(A). \tag{48}$$

Recall the continuous map (13) $pinch : \mathrm{Cube}_d(r) \times S^d \longrightarrow \bigvee_{i=1\dots r} S^d$. Similarly to the definition of the map (15), applying the singular set functor to the map (13) we get a morphism

$$\begin{aligned} pinch_*^{S^n,r} : C_*\big(\mathrm{Cube}_d(r)\big) \otimes CH_{S^d}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \\ \overset{pinch_* \otimes_A^{\mathbb{L}} id}{\longrightarrow} CH_{\bigvee_{i=1}^r S^d}(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \cong \Big(CH_{\coprod_{i=1}^r S^d}(A) \overset{\mathbb{L}}{\underset{A^{\otimes r}}{\otimes}} A\Big) \overset{\mathbb{L}}{\underset{A}{\otimes}} k \\ \cong \Big(CH_{\coprod_{i=1}^r S^d}(A)\Big) \overset{\mathbb{L}}{\underset{A^{\otimes r}}{\otimes}} k \cong \Big(CH_{S^d}(A,k)\Big)^{\otimes r} \end{aligned} \quad (49)$$

Proposition 42 ([GTZ3])**.** *Let A be an augmented E_∞-algebra. The maps* (49) *$pinch_*^{S^d,r}$ makes the iterated Bar construction $Bar^{(d)}(A) \cong CH_{S^d}(A,k)$ a natural E_n-coalgebra in the ∞-category of E_∞-algebras.*

If Y is a pointed space, its E_∞-algebra of cochains $C^*(Y)$ has a canonical augmentation $C^*(Y) \to C^*(pt) \cong k$ induced by the base point $pt \to Y$. By Theorem 4, we have an E_∞-algebra morphism

$$\begin{aligned} \mathscr{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \cong CH_{S^n}(C^*(Y),k) \\ \overset{\mathscr{I}t \otimes_{C^*(Y)}^{\mathbb{L}} k}{\longrightarrow} C^*\big(Y^{S^n}\big) \otimes_{C^*(Y)}^{\mathbb{L}} k \longrightarrow C^*\big(\Omega^n(Y)\big). \end{aligned} \quad (50)$$

We now have a nice application of factorization homology in algebraic topology.

Corollary 19. 1. *The map* (50) *$\mathscr{I}t^{\Omega^n} : Bar^{(n)}(C^*(Y)) \to C^*\big(\Omega^n(Y)\big)$ is an E_n-coalgebra morphism in the category of E_∞-algebras. It is further an equivalence if Y is connected, nilpotent and has finite homotopy groups $\pi_i(Y)$ in degree $i \leq n$.*

2. *The dual of* (50) *$C_*\big(\Omega^n(Y)\big) \longrightarrow \big(C_*\big(\Omega^n(Y)\big)\big)^{\vee\vee} \overset{\mathscr{I}t_{\Omega^n}}{\longrightarrow} \Big(Bar^{(n)}(C^*(Y))\Big)^{\vee}$ is a morphism in E_n-**Alg**. If Y is n-connected, it is an equivalence.*

We now sketch the construction of the E_i-coalgebra structure of the i^{th}-iterated Bar construction of an E_n-algebra. To do so, we only need to define a locally constant *cofactorization algebra* structure[72] whose global section is $Bar^{(i)}(A)$. By (the dual of) Proposition 17, it is enough to define such a structure on the basis of convex open disks of $\mathbb{R}^i$. Let $\mathscr{A} \in \mathbf{Fac}_{\mathbb{R}^i}^{lc}(E_{n-i}\text{-}\mathbf{Alg}^{aug})$ be the factorization algebra associated to A (by Theorem 9 and Theorem 11).

Let V be a convex open subset. By Corollary 10 (and Theorem 9), the augmentation gives us a stratified locally constant factorization algebra $\widehat{\omega}(\mathscr{A}_{|V})$ on $\widehat{V} = V \cup \{\infty\}$ (with values in E_{n-i}-$\mathbf{Alg}^{aug}$).

[72] that is a locally constant coalgebra over the ∞-operad $\mathrm{Disk}(\mathbb{R}^i)$ see remark 24

If $U \subset V$ is a convex open subset, we have a continuous map $\pi_U : \widehat{V} \to \widehat{U}$ which maps the complement of U to a single point. Further, the augmentation defines maps of factorization algebras (on $\widehat{U}$)

$$\varepsilon_U : \pi_{U*}(\widehat{\omega}(\mathscr{A}_{|V})) \longrightarrow \widehat{\omega}(\mathscr{A}_{|U})$$

which, on every open convex subset of U is the identity, and, on every open convex neighborhood of ∞ is given by the augmentation.

Define

$$Bar^{(i)}(A)(U) := \int_V \widehat{\omega}(\mathscr{A}_{|V}) \cong \int_U \pi_{U*}(\widehat{\omega}(\mathscr{A}_{|V}))$$

to be the factorization homology of $\widehat{\omega}(\mathscr{A}_{|V})$. We finally get, for $U_1,\dots,U_s$ pairwise disjoint convex subsets of a convex open subset V, a structure map

$$\begin{aligned}\nabla_{U_1,\dots,U_s,V} : Bar^{(i)}(A)(V) = \int_V (\widehat{\omega}(\mathscr{A}_{|V}) \\ \xrightarrow{\otimes \int_{U_i} \varepsilon_{U_i}} \bigotimes_{i=1\dots s} \int_{U_i} \widehat{\omega}(\mathscr{A}_{|U_i}) \xrightarrow{\simeq} Bar^{(i)}(A)(U_1) \otimes \cdots \otimes Bar^{(i)}(A)(U_s). \quad (51)\end{aligned}$$

The maps $\nabla_{U_1,\dots,U_s,V}$ are the structure maps of a locally constant factorization coalgebras (see [GTZ3]) hence they make $Bar^{(i)}(A)$ into an E_i-coalgebra (with values in the category of E_{n-i}-algebras), naturally in A:

Theorem 17 ([F1, GTZ3, AF])**.** *The iterated Bar construction lifts into an ∞-functor*

$$Bar^{(i)} : E_n\text{-}Alg^{aug} \longrightarrow E_i\text{-}coAlg\Big(E_{n-i}\text{-}Alg^{aug}\Big).$$

One has an natural equivalence[73] $Bar^{(i)}(A) \cong k \overset{\mathbb{L}}{\underset{A}{\otimes}} k$ *in* E_{n-i}***-Alg***.

Further this functor is equivalent to the one given by Proposition 42 when restricted to augmented E_∞-algebras.

Example 55. Let Free_n be the free E_n-algebra on k as in Example 16. By Definition 26, we have equivalences of E_{n-1}-algebras.

$$\begin{aligned} Bar(\mathrm{Free}_n) &= \int_{D^1\times\mathbb{R}^{n-1}} \mathrm{Free}_n \overset{\mathbb{L}}{\underset{\int_{S^0\times\mathbb{R}^{n-1}}\mathrm{Free}_n}{\otimes}} k \\ &\cong \int_{S^1\times\mathbb{R}^{n-1}} \mathrm{Free}_n \overset{\mathbb{L}}{\underset{\int_{\mathbb{R}^{n-1}}\mathrm{Free}_n}{\otimes}} k \\ &\cong \Big(\mathrm{Free}_n \otimes \mathrm{Free}_{n-1}(k[1])\Big) \overset{\mathbb{L}}{\underset{\int_{\mathbb{R}^{n-1}}\mathrm{Free}_n}{\otimes}} k \qquad \text{(by Proposition 8)} \\ &\cong \mathrm{Free}_{n-1}(k[1]). \end{aligned}$$

The result also holds for $\mathrm{Free}_n(V)$ instead of $Free_n$, see [F2]. Iterating, one finds

[73] the relative tensor product being the tensor product of E_i-modules over A

Proposition 43 ([F2]). *There is a natural equivalence of E_{n-i}-algebras*

$$Bar^{(i)}(Free_n(V)) \cong Free_{n-i}(V[i]).$$

If one works in Top_* instead of $Chain(k)$, then $\mathrm{Free}_n(X) = \Omega^n\Sigma^n X$ and the above proposition boils down to $Bar^{(i)}(\Omega^n\Sigma^n X) \cong B^i\Omega^i(\Omega^{n-i}\Sigma^n X) \xleftarrow{\simeq} \Omega^{n-i}\Sigma^{n-i}(\Sigma^i X)$.

7.5 E_n-Koszul duality and Lie algebras homology

Let $A \xrightarrow{\varepsilon} k$ be an augmented E_n-algebra. The linear dual $RHom(Bar^{(n)}(A), k)$ of the n^{th}-iterated Bar construction inherits an E_n-algebra structure (Theorem 17).

Definition 28 ([L3, F1])**.** The E_n-algebra $A^{(n)!} := RHom(Bar^{(n)}(A), k)$ dual to the iterated Bar construction is called the (derived) *E_n-Koszul dual of A*.

The terminology is chosen because it agrees with the usual notion of Koszul duality for quadratic associative algebras but it is really more like a *E_n-Bar*-duality.

Direct inspection of the E_n-algebra structures show that the dual of the iterated Bar construction is equivalent to the centralizer of the augmentation. (see § 7.2):

Lemma 9 ([L3, GTZ3])**.** *Let $A \xrightarrow{\varepsilon} k$ be an augmented E_n-algebra. There is an natural equivalence of E_n-algebras $A^{(n)!} \cong \mathfrak{z}_n(A \xrightarrow{\varepsilon} k)$.*

Let M be a dimension m manifold endowed with a framing of $M \times \mathbb{R}^n$. By Proposition 35, Theorem 11 and Proposition 18, we have the functor

$$\omega_{M\times D^n} : E_{n+m}\text{-}\mathbf{Alg}^{aug} \to E_n\text{-}\mathbf{Alg}^{aug}(E_m\text{-}\mathbf{Alg}) \xrightarrow{\omega_{D^n}} \mathbf{Fac}^{lc}_{D^n}(E_m\text{-}\mathbf{Alg}) \cong \mathbf{Fac}^{lc}_{\mathbb{R}^m\times D^n} \longrightarrow \mathbf{Fac}^{lc}_{M\times D^n}$$

where the last map is induced by the framing of $M \times \mathbb{R}^n$ as in Example 19. Let p be the map $p : M \times D^n \to pt$. We can compute the factorization homology $p_*(\omega_{M\times D^n})$ by first pushing forward along the projection on D^n and then applying p_* or first pushing forward on M and then pushing forward to the point. By Theorem 17, we thus obtain an equivalence

$$p_*(\omega_{M\times D^n}) \cong Bar^{(n)}\Big(\int_{M\times\mathbb{R}^n} A\Big) \cong \int_M \big(Bar^{(n)}(A)\big) \tag{52}$$

where the right equivalence is an equivalence of E_n-coalgebras. When M is further *closed*, this result can be extended to obtain :

Proposition 44 (Francis [F2, AF])**.** *Let A be an augmented E_{n+m}-algebra, $M \times \mathbb{R}^n$ be a framed closed manifold. There is an natural equivalence of E_n-algebras*

$$\int_{M\times\mathbb{R}^n} A^{(n+m)!} \cong \Big(\int_{M\times\mathbb{R}^n} A\Big)^{(n)!}$$

if $Bar^{(n)}\Big(\int_{M\times\mathbb{R}^n} A\Big)$ has projective finite type homology groups in each degree. In particular, $\int_M A^{(m)!} \cong \big(\int_M A\big)^{\vee}$ when M is framed (and the above condition is satisfied).

In plain english, we can say that the factorization homology over a closed framed manifold of an algebra and its E_n-Koszul dual are the same (up to finiteness issues).

Example 56 (Lie algebras and their E_n-enveloping algebras). Let **Lie-Alg** be the ∞-category of Lie algebras[74]. The forgetful functor E_n-**Alg** $\to$ **Lie-Alg**, induced by $A \mapsto A[n-1]$, has a left adjoint $U^{(n)}$: **Lie-Alg** $\to E_n$-**Alg**, the E_n-enveloping algebra functor (see [Fr4, FG] for a construction). For $n = 1$, this functor agrees with the standard universal enveloping algebra.

Proposition 45 (Francis [F2]). *Let $\mathfrak{g}$ be a (differential graded) Lie algebra. There is an natural equivalence of E_n-coalgebras*

$$(U^{(n)}(\mathfrak{g}))^{(n)!} \cong C^{\bullet}_{Lie}(\mathfrak{g})$$

where $C^{\bullet}_{Lie}(\mathfrak{g})$ is the usual Chevalley-Eilenberg cochain complex (endowed with its differential graded commutative algebra structure).

Then using Proposition 44 and Proposition 45, we obtain for $n = 1,3,7$ that

$$\int_{S^n} U^{(n)}(\mathfrak{g}) \cong \left(\int_{S^n} C^{\bullet}_{Lie}(\mathfrak{g})\right)^{\vee}$$

which for $n = 1$ gives the following standard result computing the Hochschild homology groups of an universal enveloping algebra: $HH_*(U(\mathfrak{g})) \cong HH_*(C^{\bullet}_{Lie}(\mathfrak{g}))^{\vee}$. Applying the Fubini formula, we also find

$$\int_{S^1\times S^1} U^{(2)}(\mathfrak{g}) \cong CH_*\big(CH_*(C^{\bullet}_{Lie}(\mathfrak{g}))^{(1)!}\big).$$

7.6 Extended topological quantum field theories

In [L2], Lurie introduced factorization homology as a (generalization of) an invariant of an extended topological field theory and offshoot of the cobordism hypothesis. We wish now to reverse this construction and explain very roughly how factorization homology can be used to produce an extended topological field theory.

Following [L2], there is an ∞-category[75] of *extended topological field theories* with values in a symmetric monoidal (∞,n)-category with duals $(\mathscr{C},\otimes)$. It is the category of symmetric monoidal functors $\mathbf{Fun}^{\otimes}\big((\mathrm{Bord}_n^{fr},\coprod),(\mathscr{C},\otimes)\big)$ where Bord_n^{fr} is the (∞,n)-category of bordisms of framed manifolds with monoidal structure given by disjoint union. In [L2], Bord_n^{fr} is defined as an n-fold Segal space which precisely models the following *intuitive* notion of an (∞,n)-category whose objects are framed compact 0-dimensional manifolds. The morphisms between objects are framed 1-bordism, that is $\mathrm{Hom}_{\mathrm{Bord}_n^{fr}}(X,Y)$ consists of 1-dimensional

[74] which is equivalent to the category of L_∞-algebras

[75] the cobordism hypothesis actually ensures that it is an ∞-groupoid

framed manifolds T with boundary $\partial T = Y \coprod X^{op}$ (where X^{op} has the opposite framing to the one of the object X). The 2-morphisms in Bord_n^{fr} are framed 2-bordisms between 1-dimensional framed manifolds (with corners) and so on. The n-morphisms are n-framed bordisms between $n-1$-dimensional framed manifolds with corners, its $n+1$-morphisms diffeomorphisms and the higher morphisms are isotopies. Note that in the precise model of Bord_n^{fr}, the boundary component $N_1,\dots,N_r$ of a manifold M are represented by an open manifold with boundary components $N_1\times\mathbb{R},\dots,N_r\times\mathbb{R}$ (in other words are replaced by open collars).

There is an $(\infty,n+1)$-category $E_{\leq n}$**-Alg** whose construction is only sketched in [L2] and detailled in [Sc] using a model based on factorization algebras. The category $E_{\leq n}\text{-}Alg$ can be described informally as the ∞-category with objects the E_n-algebras, 1-morphisms $\mathrm{Hom}_{E_{\leq n}\text{-}\mathbf{Alg}}(A,B)$ is the space of all (A,B)-bimodules in E_{n-1}**-Alg** and so on. The (∞-)category $n\text{-}\mathrm{Hom}_{E_{\leq n}\text{-}\mathbf{Alg}}(P,Q))$ of n-morphisms is the ∞-category of (P,Q)-bimodules where P,Q are E_1-algebras (with additional structure). In other words we have

$$\begin{aligned} n\text{-}\mathrm{Hom}_{E_{\leq n}\text{-}\mathbf{Alg}}(P,Q)) &\cong \{P\} \underset{\mathbf{Fac}^{lc}_{(-\infty,0)}}{\times} \mathbf{Fac}^{lc}_{\mathbb{R}_*} \underset{\mathbf{Fac}^{lc}_{(0,+\infty)}}{\times} \{R\} \\ &\cong \{P\} \underset{E_1\text{-}\mathbf{Alg}}{\times} \mathbf{BiMod} \underset{E_1\text{-}\mathbf{Alg}}{\times} \{R\} \end{aligned}$$

see Example 34. The composition

$$n\text{-}\mathrm{Hom}_{E_{\leq n}\text{-}\mathbf{Alg}}(P,Q)) \times n\text{-}\mathrm{Hom}_{E_{\leq n}\text{-}\mathbf{Alg}}(Q,R)) \longrightarrow n\text{-}\mathrm{Hom}_{E_{\leq n}\text{-}\mathbf{Alg}}(P,R))$$

is given by tensor products of bimodules: ${}_PM_Q \otimes^{\mathbb{L}}_Q {}_QN_R$ which in terms of factorization algebras is induced by the pushforward along the map $q:\mathbb{R}_*\times_{\mathbb{R}}\mathbb{R}_*\to\mathbb{R}_*$ where $\mathbb{R}_*\times_{\mathbb{R}}\mathbb{R}_*$ is identified with $\mathbb{R}$ stratified in two points -1 and 1 and q is the quotient map identifying the interval $[-1,1]$ with the stratified point $0\in\mathbb{R}_*$.

Let $E_{\leq n}\text{-}\mathbf{Alg}_{(0)}$ be the (∞,n)-category obtained from $E_{\leq n}$**-Alg** by discarding non-invertible $n+1$-morphisms, that is $E_{\leq n}\text{-}\mathbf{Alg}_{(0)} = Gr^{(n)}(E_{\leq n}\text{-}\mathbf{Alg})$ where $Gr^{(n)}$ is the right adjoint of the forgetful functor $(\infty,n+1)\text{-}\mathbf{Cat}\to(\infty,n)\text{-}\mathbf{Cat}$.

The (∞,n)-category $E_{\leq n}\text{-}\mathbf{Alg}_{(0)}$ is fully dualizable (the dual of an algebra is its opposite algebra) hence every E_n-algebra determines in an unique way an extended topological field theory by the cobordism hypothesis.

In fact, this extended field theory can be constructed by factorization homology. Let M be a m-dimensional manifold. We say that M is *stably n-framed* if $M\times\mathbb{R}^{n-m}$ is framed. Assume that M has two ends which are trivialized as $L\times\mathbb{R}^{op}\subset M$ and $R\times\mathbb{R}\subset M$, where L,R are stably framed codimension 1 closed sub-manifolds; here $\mathbb{R}^{op}$ means $\mathbb{R}$ endowed with the opposite framing to the standard one. For instance, M can be the interior of compact manifold $\overline{M}$ with two boundary component L, R and trivializations $L\times[0,\infty)\hookrightarrow\overline{M}$ and $R\times(-\infty,0]\hookrightarrow\overline{M}$ where the trivialization on $L\times[0,\infty)$ has the opposite orientation as the one induced by M.

In that case, Lemma 5 (and Proposition 29) imply that the factorization homology $\int_M A$ is an E_{n-m}-algebra which is also a bimodule over the E_{n-m+1}-algebras

$\big(\int_{L\times\mathbb{R}^{n-m+1}} A, \int_{R\times\mathbb{R}^{n-m+1}} A\big)$:

$$\int_{M\times\mathbb{R}^{n-m}} A \in \Big\{\int_{L\times\mathbb{R}^{n-m+1}} A\Big\} \underset{E_1-\textbf{-Alg}}{\times} \textbf{BiMod}\big(E_{n-m}\textbf{-Alg}\big) \underset{E_1-\textbf{-Alg}}{\times} \Big\{\int_{R\times\mathbb{R}^{n-m+1}} A\Big\}.$$

Thus $\int_M A$ is a m-morphism in $E_{\leq n}\textbf{-Alg}_{(0)}$ from R to L. In fact, one can prove

Theorem 18 ([Sc])**.** *Let A be an E_n-algebra. The rule which, to a stably n-framed manifold M of dimension m, associates*

$$Z_A(M) := \int_{M\times\mathbb{R}^{n-m}} A$$

extends as an extended field theory $Z_A \in \boldsymbol{Fun}^{\otimes}\big(Bord_n^{fr}, E_{\leq n}\textbf{-}\boldsymbol{Alg}_{(0)}\big)$.

8 Commutative factorization algebras

In this Section, we explain in details the relationship in between classical homology theory *à la* Eilenberg-Steenrod with factorization homology and more generally between (co)sheaves and factorization algebras. The main point is that when $\mathscr{C}$ is endowed with the monoidal structure given by the coproduct, then Factorization algebras boils down to the usual theory of cosheaves. This is in particular the case of factorization algebras with values in a category of commutative algebras $E_\infty\textbf{-Alg}(\mathscr{C},\otimes)$ in $(\mathscr{C},\otimes)$.

8.1 Classical homology as factorization homology

In this section we explain the relationship between factorization homology and singular homology (as well as generalized (co)homology theories for spaces).

Let $Chain(\mathbb{Z})$ be the (∞-)category of differential graded abelian groups (i.e. chain complexes of $\mathbb{Z}$-modules). It has a symmetric monoidal structure given by the direct sum of chain complexes, which is the coproduct in $Chain(\mathbb{Z})$. We can thus define homology theory for manifolds with values in $\big(Chain(\mathbb{Z}),\oplus\big)$. These are precisely (restrictions of) the (generalized) cohomology theories for spaces and nothing more. Recall that Mfld_n^{or} is the (∞-)category of oriented manifolds, Example 11.

Corollary 20. *Let G be an abelian group[76]. There is an unique homology theory for oriented manifolds with coefficient in G (Definition 11), that is (continuous) functor $\mathscr{H}_G : \mathrm{Mfld}_n^{or} \to Chain(\mathbb{Z})$ satisfying the axioms*

- ***(dimension)*** *$\mathscr{H}_G(\mathbb{R}^n) \cong G$;*
- ***(monoidal)*** *the canonical map $\bigoplus_{i\in I} \mathscr{H}_G(M_i) \xrightarrow{\simeq} \mathscr{H}_G(\coprod_{i\in I} M_i)$ is a quasi-isomorphism;*

[76] or even a graded abelian group or chain complex of abelian groups

- ***(excision)*** *If M is an oriented manifold obtained as the gluing $M = R\cup_{N\times\mathbb{R}} L$ of two submanifolds along a a codimension 1 submanifold N of M with a trivialization $N\times\mathbb{R}$ of its tubular neighborhood in M, there is an natural equivalence*

$$\mathscr{H}_G(M) \overset{\simeq}{\longleftarrow} cone\Big(\mathscr{H}_G(N\times\mathbb{R}) \overset{i_L-i_N}{\longrightarrow} \mathscr{H}_G(L)\oplus\mathscr{H}_G(R)\Big).$$

  Here $\mathscr{H}_G(N\times\mathbb{R}) \overset{i_L}{\longrightarrow} \mathscr{H}_G(L)$ and $\mathscr{H}_G(N\times\mathbb{R}) \overset{i_R}{\longrightarrow} \mathscr{H}_G(R)$ are the maps induced by functoriality by the inclusions of $N\times\mathbb{R}$ in L and R.

Then, this homology theory is singular homology[77] with coefficient in G. In particular, it extends as an homology theory for spaces.

The uniqueness means of course up to a contractible choice, meaning that any two homology theory with coefficient in G will be naturally equivalent and any two choices of equivalences will also be naturally equivalent and so on.

Proof. This is a consequence of Proposition 46 below applied to $\mathscr{C} = Chain(\mathbb{Z})$ and the fact that the homotopy colimit

$$hocolim\Big(\ \mathscr{H}_G(R) \xleftarrow{i_R} \mathscr{H}_G(N\times\mathbb{R}) \xrightarrow{i_L} \mathscr{H}_G(L)\ \Big)$$

is precisely computed by the cone[78] of the map $i_L - i_R$. □ □

Remark 31. Let H be a topological group, $f : H \to \mathrm{Homeo}(\mathbb{R}^n)$ a map of topological groups and $Bf : BH \to B\mathrm{Homeo}(\mathbb{R}^n)$ the induced map, so that we have the category $\mathrm{Mfld}_n^{(BH,Bf)}$ of manifolds with H-structure, see Example 11. As shown by its proof, Corollary 20 also holds with oriented manifolds replaced by manifold with a H-structure; in particular for *all manifolds or* a contrario for *framed manifolds*.

Remark 32. Corollary 20 and Theorem 10 implies that $\mathscr{H}_G(M)$ is computed by (derived) Hochschild homology $CH_M(G)$ (in $(Chain(\mathbb{Z}),\oplus)$). If $M_\bullet$ is a simplicial set model of M, then $\mathscr{H}_G(M) \cong CH_{M_\bullet}(G)$ which is exactly (by § 2.3) the chain complex of the simplicial abelian group $G[M_\bullet]$. In particular, for $M_\bullet = \Delta_\bullet(M) = \mathrm{Hom}(\Delta^\bullet, M)$, one recovers *exactly* the singular chain complex $C_*(M)$ of M.

Corollary 20 is a particular case of a more general result which we now describe. Let $\mathscr{C}$ be a category with coproducts. Then $(\mathscr{C},\coprod)$ is symmetric monoidal, with unit given by its initial object $\emptyset$. As we have seen in § 2.1, any object X of $\mathscr{C}$ carries a canonical (thus natural in X) structure of commutative algebra in $(\mathscr{C},\coprod)$ which is given by the "multiplication" $X\coprod X \overset{\coprod id_X}{\longrightarrow} X$ induced by the identity map

[77] or generalized exceptional homology when G is graded or a chain complex

[78] note that in this case, we know *a posteriori* that we can choose $\mathscr{H}_G$ to be singular chains, so that $\mathscr{H}_G(i_L)\oplus\mathscr{H}_G(i_R)$ is injective and the cone is equivalent to a quotient of chain complexes

$id_X : X \to X$ on each component. This algebra structure is further unital, with unit given by the unique map $\emptyset \to X$. This defines a functor $triv : \mathscr{C} \to E_\infty\text{-}\mathbf{Alg}(\mathscr{C})$ which to an object associates its trivial commutative algebra structure. In fact, the latter algebras are the only only possible commutative and even associative ones in $(\mathscr{C}, \coprod)$.

Lemma 10 (Eckman-Hilton principle)**.** *Let $\mathscr{C}$ be a category with coproducts and $(\mathscr{C}, \coprod)$ the associated symmetric monoidal category. Let $H \xrightarrow{f} \mathrm{Homeo}(\mathbb{R}^n)$ be a topological group morphism and $\iota : \mathrm{Disk}_n^{(BH,Bf)}\text{-}\boldsymbol{Alg}(\mathscr{C}) \to \mathscr{C}$ be the underlying object functor* (16) *(Definition 9). We have a commutative diagram of equivalences*

$$\begin{tikzcd} E_\infty\text{-}\boldsymbol{Alg}(\mathscr{C}) \arrow[rr, "\simeq"] & & \mathrm{Disk}_n^{(BH,Bf)}\text{-}\boldsymbol{Alg}(\mathscr{C}) \arrow[dl, "\simeq", "\iota"'] \\ & \mathscr{C} \arrow[ul, "\simeq"', "triv"] & \end{tikzcd}$$

where the horizontal arrow is the canonical functor of Example 13.

In particular, any E_n-algebra ($n \geq 1$) in $(\mathscr{C}, \coprod)$ is a (trivial) commutative algebra.

Proof. Let $I = \{1, \dots, n\}$ be a finite set and $m_I : \coprod_{i\in I} C \to C$ be any map. The universal property of the coproduct yields a commutative diagram

$$\begin{tikzcd} \left(\coprod_{i\in I} C\right) \coprod \left(\coprod_{i\in I} C\right) \arrow[r, "m_I \coprod m_I"] \arrow[d, "\left(\coprod_{i\in I}(\coprod id_C)\right)\circ\sigma_I"'] & C \coprod C \arrow[d, "\coprod id_C"] \\ \coprod_{i\in I} C \arrow[r, "m_I"'] & C \end{tikzcd}$$

where σ_I is the permutation induced by the bijection $(1, n+1)(2, n+2)\cdots(n, 2n)$ of $I \coprod I = \{1, \dots, 2n\}$ on itself. Hence, if C is an E_n-algebra, it is naturally an object of $E_n\text{-}\mathbf{Alg}(E_\infty\text{-}\mathbf{Alg})$ where the commutative algebra structure is the trivial one $C \coprod C \overset{\coprod id_C}{\to} C$. By Dunn Theorem 11 (or Eckman-Hilton principle), the forgetful map (induced by the pushforward of factorization algebras) $E_n\text{-}\mathbf{Alg}((E_\infty\text{-}\mathbf{Alg})) \longrightarrow E_0\text{-}\mathbf{Alg}(E_\infty\text{-}\mathbf{Alg})$ is an equivalence. It follows that the E_n-algebra C is an E_∞-algebra whose structure is equivalent to $triv(C)$.

The group map $\{1\} \to H$ induces a canonical functor $\mathrm{Disk}_n^{(BH,Bf)}\text{-}\mathbf{Alg} \to E_n\text{-}\mathbf{Alg}$ so that the above result implies that such a $\mathrm{Disk}^{(BH,Bf)}$-algebra with underlying object C is necessarily of the form $triv(C)$. □ □

Proposition 46. *Let $(\mathscr{C}, \coprod)$ be a ∞-category whose monoidal structure is given by the coproduct and $f : H \to \mathrm{Homeo}(\mathbb{R}^n)$ be a topological group morphism .*

- *Any homology theory for (BH, Bf)-structured manifolds (Definition 10)* extends uniquely *into an homology theory for spaces (Definition 1).*

- *Any object $C \in \mathscr{C}$ determines a unique homology theory for (BH, Bf)-manifolds with values in C (Definition 11); further the evaluation map $\mathscr{H} \mapsto \mathscr{H}(\mathbb{R}^n)$ is an equivalence between the category of homology theories for (BH, Bf)-manifolds in $(\mathscr{C}, \coprod)$ and $\mathscr{C}$.*

Proof. By Theorem 6, homology theories for (BH, Bf)-structured manifolds are equivalent to $\mathrm{Disk}_n^{(BH,Bf)}$-algebras which, by Lemma 10, are equivalent to $\mathscr{C}$. In particular, any $\mathrm{Disk}_n^{(BH,Bf)}$-algebra is given by the commutative algebra associated to an object of $\mathscr{C}$ so that by Theorem 10, it extends to an homology theory for spaces. ☐

8.2 Cosheaves as factorization algebras

In this Section we identify (pre-)cosheaves and (pre-)factorization algebras when the monoidal structure is given by the coproduct.

Let $(\mathscr{C}, \otimes)$ be symmetric monoidal and let $\mathscr{F}$ be in $\mathbf{PFac}_X(\mathscr{C})$. Then the structure maps $\mathscr{F}(U) \to \mathscr{F}(V)$ for any open U inside an open V induces a functor $\gamma_{\mathscr{C}} : \mathbf{PFac}_X(\mathscr{C}) \longrightarrow \mathbf{PcoShv}_X(\mathscr{C})$ where $\mathbf{PcoShv}_X(\mathscr{C}) := \mathbf{Fun}(\mathrm{Open}(X), \mathscr{C})$ is the ∞-category of *precosheaves on X with values in $\mathscr{C}$*.

Lemma 11. *Let $(\mathscr{C}, \coprod)$ be an ∞-category with coproducts whose monoidal structure is given by the coproduct and X be a topological space.*

1. *The functor $\gamma_{\mathscr{C}} : \boldsymbol{PFac}_X(\mathscr{C}) \longrightarrow \boldsymbol{PcoShv}_X(\mathscr{C})$ is an equivalence.*

2. *If X has a factorizing basis of opens[79], then the functor $\gamma_{\mathscr{C}} : \boldsymbol{PFac}_X(\mathscr{C}) \longrightarrow \boldsymbol{PcoShv}_X(\mathscr{C})$ restricts to an equivalence*

$$\boldsymbol{Fac}_X(\mathscr{C}) \xrightarrow{\simeq} \boldsymbol{coShv}_X(\mathscr{C})$$

   between factorization algebras on X and the ∞-category $\boldsymbol{coShv}_X(\mathscr{C})$ of (homotopy) cosheaves on X with values in $\mathscr{C}$.

3. *If X is a manifold, then the above equivalence also induces an equivalence $\boldsymbol{Fac}_X^{lc}(\mathscr{C}) \xrightarrow{\simeq} \boldsymbol{coShv}_X^{lc}(\mathscr{C})$ between locally constant factorization algebras and locally constant cosheaves.*

Proof. Let $\mathscr{F}$ be in $\mathbf{PFac}_X$ and $U_1, \dots, U_r$ be open subsets of $V \in Open(X)$, which are pairwise disjoint. Let $\rho_{U_1,\dots,U_i,V} : \mathscr{F}(U_1) \coprod \dots \coprod \mathscr{F}(U_i) \to \mathscr{F}(V)$ be the structure map of $\mathscr{F}$. The associativity of the structure maps (diagram (19)) shows that the structure map $\rho_{U_j,V} : \mathscr{F}(U_j) \to \mathscr{F}(V)$ factors as

$$\begin{tikzcd}
\mathscr{F}(U_j) \arrow[rr, "\rho_{U_j,V}"] \arrow[d, "\cong"'] & & \mathscr{F}(V) \\
\emptyset \coprod \dots \emptyset \coprod \mathscr{F}(U_j) \coprod \emptyset \dots \coprod \emptyset \arrow[rr, "\left(\coprod_{k=1}^{j-1} \rho_{\emptyset,U_k}\right) \coprod id \coprod \left(\coprod_{k=j+1}^{i} \rho_{\emptyset,U_k}\right)"] & & \mathscr{F}(U_1) \coprod \dots \coprod \mathscr{F}(U_i). \arrow[u, "\rho_{U_1,\dots,U_i,V}"']
\end{tikzcd}$$

[79] for instance when X is Hausdorff

The universal property of the coproduct implies that the following diagram

$$\begin{tikzcd} \mathscr{F}(U_1)\coprod\dots\coprod\mathscr{F}(U_i) \arrow[rr, "\rho_{U_1,\dots,U_i,V}"] \arrow[d, "\rho_{U_1,V}\coprod\dots\coprod\rho_{U_i,V}"'] & & \mathscr{F}(V) \\ \mathscr{F}(V)\coprod\dots\coprod\mathscr{F}(V) \arrow[urr, "\coprod id_{\mathscr{F}(V)}"'] & & \end{tikzcd} \tag{53}$$

is commutative. It follows that the structure maps are completely determined by the precosheaf structure. Conversely, any precosheaf on $\mathscr{C}$ gives rise functorially to a prefactorization algebra with structure maps given by the composition

$$\mathscr{F}(U_1)\coprod\dots\coprod\mathscr{F}(U_i) \longrightarrow \mathscr{F}(V)\coprod\dots\coprod\mathscr{F}(V) \stackrel{\coprod id_{\mathscr{F}(V)}}{\longrightarrow} \mathscr{F}(V)$$

yielding a functor $\theta_{\mathscr{C}} : \mathbf{PcoShv}_X(\mathscr{C}) \to \mathbf{PFac}_X(\mathscr{C})$. The commutativity of diagram (53) implies that this functor $\theta_{\mathscr{C}}$ is inverse to $\gamma_{\mathscr{C}} : \mathbf{PFac}_X(\mathscr{C}) \to \mathbf{PcoShv}_X(\mathscr{C})$.

Note that both the cosheaf condition and the factorization algebra conditions implies that the canonical map $\mathscr{F}(U_1)\coprod\dots\coprod\mathscr{F}(U_i) \to \mathscr{F}(U_1\coprod\dots\coprod U_i)$ is an equivalence (of constant simplicial objects). Now, we can identify the two gluing conditions. Since $\mathscr{U}$ is a (factorizing) cover of V, then

$$\mathscr{P}(U) := \{\{U_1,\dots,U_k\},\ \text{which are pairwise disjoint}\}$$

is a cover of V. Further, if $\mathscr{F} \in \mathbf{Fac}_X$, the Čech complex $\check{C}(\mathscr{U},\mathscr{F})$ is precisely the (standard) Čech complex $\check{C}^{\mathrm{cosheaf}}(P\mathscr{U},\gamma_{\mathscr{C}}(\mathscr{F}))$ of the cosheaf $\gamma_{\mathscr{C}}(\mathscr{F})$ computed on the cover $P\mathscr{U}$ so that the map $\check{C}(\mathscr{U},\mathscr{F}) \to \mathscr{F}(V)$ is an equivalence if and only if $\check{C}^{\mathrm{cosheaf}}(P\mathscr{U},\gamma_{\mathscr{C}}(\mathscr{F})) \to \mathscr{F}(V)$ is an equivalence.

Assume X has a factorizing basis of opens. Both factorization algebras and cosheaf are determined by their restriction on a basis of opens. It follows that $\gamma_{\mathscr{C}}$ sends factorization algebras to cosheaves and $\theta_{\mathscr{C}}$ sends cosheaves to factorization algebras.

It remains to consider the locally constant condition when X is a manifold, thus has a basis of euclidean neighborhood. On each euclidean neighborhood D, by Theorem 9, the restriction of $\mathscr{F} \in \mathbf{Fac}_X^{lc}$ to D is the factorization algebra given by an E_n-algebra $A \in E_n\text{-}\mathbf{Alg}((\mathscr{C},\coprod))$ in $\mathscr{C}$. Lemma 10 implies that A is given by the trivial commutative algebra $triv(C)$ associated to an object $C \in \mathscr{C}$. It follows from the identification of Čech complexes above, that $\mathscr{F}_{|D}$ is thus equivalent to the constant cosheaf on D associated to the object C. The converse follows from the fact if $\mathscr{G} \in \mathbf{coShv}_X^{lc}(\mathscr{C})$, then for any point $x \in X$, there is an euclidean neighborhood $D_x \cong \mathbb{R}^n$ on which $\mathscr{G}_{|D_x}$ is constant. The identification of the Čech complexes above implies that $\theta_{\mathscr{C}}(\mathscr{G}_{|D_x})$ is locally constant on D_x. Proposition 13 implies that the factorization algebra $\theta_{\mathscr{C}}(\mathscr{G})$ is locally constant on X, hence finishes the proof. $\square$ $\square$

From the identification between cosheaves and factorization algebras, we deduce that factorization homology in $(\mathscr{C},\coprod)$ agrees with homology with local coefficient:

Proposition 47. *Let $(\mathscr{C}, \coprod)$ be a ∞-category whose monoidal structure is given by the coproduct and X a manifold.*

- *There is an equivalence between homology theories for (X, TX)-structured manifolds (Definition 10) and $\boldsymbol{coShv}_X^{lc}(\mathscr{C})$, the ($\infty$-)category of locally constant cosheaves on X with values in $\mathscr{C}$.*
- *The above equivalence is given, for any (X, TX)-structured manifold M and (homotopy) cosheaf $\mathscr{G} \in \boldsymbol{coShv}_X^{lc}(\mathscr{C})$, by*

$$\int_M \mathscr{G} := \mathbb{R}\Gamma(M, \mathscr{G})$$

  the cosheaf homology of M with values in the cosheaf $p^*(\mathscr{G})$ where $p : M \to X$ is the map defining the (X, TX)-structure.

Proof. The first claim is an immediate application of Lemma 11.3 and Theorem 8. The latter result implies that factorization homology is computed by the Čech complex of the locally constant factorization algebra associated to a $\mathrm{Disk}_n^{(M,TM)}$-algebra given by the pullback along $p : M \to X$ of some $A \in \mathrm{Disk}_n^{(X,TX)}$-**Alg**. Now the second claims follows from Lemma 11.2. □ □

Remark 33. Factorization homology on a (X, e)-structured manifold M depends only on its value on open sub sets of M. Thus Proposition 47 implies that, for any manifold M and $A \in \mathrm{Disk}_n^{(X,e)}$-**Alg**$(\mathscr{C})$, factorization homology $\int_M A$ is given by cosheaf homology of the locally constant cosheaf $\mathscr{G}$ given by the image of A under the functor $\mathrm{Disk}_n^{(X,e)}$-**Alg** $\to \mathrm{Disk}_n^{(M,TM)}$-**Alg** (of Example 12) and the equivalence $\mathbf{Fac}_X^{lc}(\mathscr{C}) \xrightarrow{\simeq} \mathbf{coShv}_X^{lc}(\mathscr{C})$ of Lemma 11.

Let $(\mathscr{C}, \otimes)$ be a symmetric monoidal (∞-)category. We say that a factorization algebra $\mathscr{F}$ on X is *commutative* if each $\mathscr{F}(U)$ is given a structure of differential graded commutative (or E_∞-) algebra and the structure maps are maps of algebras. In other words, the category of *commutative factorization algebras* is $\mathbf{Fac}_X(E_\infty\text{-}\mathbf{Alg})$.

A peculiar property of (differential graded) commutative algebras is that their coproduct is given by their tensor product (that is the underlying tensor product in $\mathscr{C}$ endowed with its canonical algebra structure). From Lemma 11, we obtain the following:

Proposition 48. *Let $(\mathscr{C}, \otimes)$ be a symmetric monoidal (∞-)category. The functor*

$$\gamma_{E_\infty\text{-}\boldsymbol{Alg}(\mathscr{C})} : \boldsymbol{Fac}_X(E_\infty\text{-}\boldsymbol{Alg}(\mathscr{C})) \longrightarrow \boldsymbol{coShv}_X(E_\infty\text{-}\boldsymbol{Alg}(\mathscr{C}))$$

is an equivalence.

In other words, commutative factorization algebras are cosheaves (in E_∞-**Alg**).

Remark 34. In view of Proposition 48, one can think general factorization algebras as non-commutative cosheaves.

Combining Proposition 48, Proposition 47 and Theorem 10, we obtain:

Corollary 21. *Let $\mathscr{F} \in \boldsymbol{Fac}_X^{lc}(E_\infty\text{-}\boldsymbol{Alg})$ be a locally constant commutative factorization algebra on X. Then*

$$\int_X A \cong \mathscr{C}\mathscr{H}_X(\mathscr{F})$$

where $\mathscr{C}\mathscr{H}_X(\mathscr{F})$ is the (derived) global section of the cosheaf which to any open U included in an euclidean Disk D associates the derived Hochschild homology $CH_U(\mathscr{F}(D))$.

9 Complements on factorization algebras

In this Section, we give several proofs of results, some of them probably known by the experts, about factorization algebras that we have postponed and for which we do not know any reference in the literature.

9.1 Some proofs related to the locally constant condition and the push-forward

Proof of Propositions 13 and 24. Let $U \subset D \subset M$ be an inclusion of open disks; we need to prove that $\mathscr{F}(U) \to \mathscr{F}(D)$ is a quasi-isomorphism. We can assume $D = \mathbb{R}^n$ (by composing with a homeomorphism); the proof in the stratified case is similar to the non-stratified one by replacing $\mathbb{R}^n$ with $\mathbb{R}^i \times [0,+\infty)^{n-i}$. We first consider the case where $\mathscr{F}_{|U}$ is locally constant and further, that U is an euclidean disk (with center x and radius r_0). Denote $D(y,r)$ an euclidean open disk of center y and radius $r > 0$ and let

$$T_+ := \sup(t \in \mathbb{R}, \text{ such that } \forall\, \tfrac{r_0}{2} \le s < t,\ \mathscr{F}(D(x,\tfrac{r_0}{2})) \to \mathscr{F}(D(x,s)) \text{ is an equivalence}).$$

By assumption $T_+ \ge r_0$. We claim that $T = +\infty$. Indeed, let T be finite and such that , $\mathscr{F}(D(x,\frac{r_0}{2})) \to \mathscr{F}(D(x,s))$ is an equivalence for all $s < T$. We will prove that T can not be equal to T_+. Every point y on the sphere of center x and radius T has a neighborhood in which $\mathscr{F}$ is locally constant. In particular, there is a number $\varepsilon_y > 0$, an open angular sector $S_{[0,T+\varepsilon_y)}$ of length $T+\varepsilon_y$ and angle θ_y containing y such that $\mathscr{F}_{|S_{(T-\varepsilon_y,T+\varepsilon_y)}}$ is locally constant. Here, $S_{(\tau,\gamma)}$ denotes the restriction of the angular sector to the band containing numbers of radius lying in (τ,γ).

We first note that $S_{[0,T+\varepsilon_y)}$ has a factorizing cover $\mathscr{A}_y$ consisting of open angular sectors of the form $S_{(T-\tau,T+\varepsilon_y)}$ ($0 < \tau \le \varepsilon_y$) and $S_{[0,\kappa)}$ ($0 < \kappa < T$); there is an induced similar cover $\mathscr{A}_y \cap U$ of $S_{[0,T)}$ given by the angular sectors of the form $S_{(T-\tau,T)}$ ($0 < \tau \le \varepsilon_y$) and $S_{[0,\kappa)}$ ($0 < \kappa < T$). The structure maps $\mathscr{F}(S_{(T-\tau,T)}) \to \mathscr{F}(S_{(T-\tau,T+\varepsilon_y)})$ induce a map of Čech complex $\psi_y : \check{C}(\mathscr{A}_y \cap U, \mathscr{F}) \to \check{C}(\mathscr{A}_y, \mathscr{F})$ so that the following diagram is commutative:

$$\begin{tikzcd} \check{C}(\mathscr{A}_y \cap U, \mathscr{F}) \arrow[r, "\psi_y"] \arrow[d, "\simeq"'] & \check{C}(\mathscr{A}_y, \mathscr{F}) \arrow[d, "\simeq"] \\ \mathscr{F}(S_{[0,T)}) \arrow[r] & \mathscr{F}(S_{[0,T+\varepsilon_y)}) \end{tikzcd} \quad . \tag{54}$$

Since the map $S_{(T-\tau,T+\varepsilon_y)} \to S_{(T-\tau,T+\varepsilon_y)}$ is the inclusion of a sub-disk inside a disk in $S_{(T-\varepsilon_y,T+\varepsilon_y)}$, it is a quasi-isomorphism and, thus, so is the map $\psi_y : \check{C}(\mathscr{A}_y \cap U, \mathscr{F}) \to \check{C}(\mathscr{A}_y, \mathscr{F})$. It follows from diagram (54) that the structure map $\mathscr{F}(S_{[0,T)}) \to \mathscr{F}(S_{[0,T+\varepsilon_y)})$ is a quasi-isomorphism. In the above proof, we could also have taken any angle $\theta \le \theta_y$ or replaced ε_y by any $\varepsilon \le \varepsilon_y$ without changing the result.

By compactness of the sphere of radius T, we can thus find an $\varepsilon > 0$ and a $\theta > 0$ such that the structure map $\mathscr{F}(S \cap U) \to \mathscr{F}(S)$ is a quasi-isomorphism for any angular sector S around x of radius $r = T + \varepsilon' < T + \varepsilon$ and arc length $\phi_S < \theta$. The collection of such angular sectors S is a (stable by intersection) factorizing basis of the disk $D(x, T+\varepsilon')$ while the collection of sectors $S \cap U$ is a (stable by intersection) factorizing basis of the disk $D(x,T)$. Further, we have proved that the structure maps $\mathscr{F}(S \cap U) \to \mathscr{F}(S)$ is a quasi-isomorphism for any such S. It follows that the map $\mathscr{F}(D(x,T)) \to \mathscr{F}(D(x,T+\varepsilon')$ is a quasi-isomorphism (since again the induced map in between the Čech complexes associated to this two covers is a quasi-isomorphism). It follows that $T_+ > T$ for any finite T hence is infinite as claimed above. In particular, the canonical map $\mathscr{F}(D(x,T)) \to \mathscr{F}(D(x,T+r))$ is a quasi-isomorphism for any $r \ge 0$.

Now, since the collection of disks of radius $T > 0$ centered at x is a factorizing cover of $\mathbb{R}^n$, we deduce that $\mathscr{F}(D(x,T)) \to \mathscr{F}(\mathbb{R}^n)$ is a quasi-isomorphism. Indeed, fix some $R > 0$ and let $j_R : x + y \mapsto x + R/(R - |y|)y$ be the homothety centered at x mapping $D(x,R)$ homeomorphically onto $\mathbb{R}^n$. The map j_R is a bijection between the set $\mathscr{D}_R$ of (euclidean) sub-disks of $D(x,R)$ centered at x and the set $\mathscr{D}$ of all (euclidean) disks of $\mathbb{R}^n$ centered at x. For any disk centered at x, the inclusion $D(x,T) \hookrightarrow D(x, j_R(T))$ yields a quasi-isomorphism $\mathscr{F}(D(x,T)) \to \mathscr{F}(D(x, j_R(T))$. If $\alpha = \{D(x,r_0), \dots, D(x,r_i)\} \in (P\mathscr{D}_R)^{i+1}$, we thus get a quasi-isomorphism

$$\mathscr{F}(\alpha) \cong \mathscr{F}\Big(D\big(x, \min(r_j, j = 0 \dots i)\big)\Big) \xrightarrow{\simeq} \mathscr{F}\Big(D\big(x, \min(j_R(r_j), j = 0 \dots i)\big)\Big) \cong \mathscr{F}(j_R(\alpha)).$$

Assembling those for all α's yields a quasi-isomorphism $\check{C}(\mathscr{D}_R, \mathscr{F}) \xrightarrow{\simeq} \check{C}(\mathscr{D}, \mathscr{F})$ which fit into a commutative diagram

$$\begin{tikzcd} \check{C}(\mathscr{D}_R, \mathscr{F}) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \check{C}(\mathscr{D}, \mathscr{F}) \arrow[d, "\simeq"] \\ \mathscr{F}(D(x,R)) \arrow[r] & \mathscr{F}(\mathbb{R}^n) \end{tikzcd}$$

whose vertical arrows are quasi-isomorphisms since $\mathscr{F}$ is a factorization algebra. It follows that the lower horizontal arrow is a quasi-isomorphism as claimed.

We are left to prove the result for $U \hookrightarrow D = \mathbb{R}^n$ when U is not necessarily an euclidean disk. Choose an euclidean open disk $\tilde{D}$ inside U small enough so that $\mathscr{F}_{|\tilde{D}}$ is locally constant. Let $h : \tilde{D} \cong \mathbb{R}^n$ be an homothety (with the same center as $\tilde{D}$) identifying $\tilde{D}$ and $\mathbb{R}^n$. Then $\tilde{U} := h^{-1}(U) \subset D \subset U$ is an open disk homothetic to U. So that by the above reasoning (after using an homeomorphism between U and an euclidean disk $\mathbb{R}^n$) we have that the structure map $\mathscr{F}(\tilde{U}) \to \mathscr{F}(U)$ is a quasi-isomorphism as well. Since $\mathscr{F}_{|\tilde{D}}$ is locally constant, the structure map $\mathscr{F}(\tilde{U}) \to \mathscr{F}(\tilde{D})$ is a quasi-isomorphism. Now, Proposition 13 follows from the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(\tilde{U}) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathscr{F}(U) \arrow[dr] & \\
\mathscr{F}(\tilde{D}) \arrow[ur] \arrow[rr, "\simeq"] & & \mathscr{F}(D)
\end{tikzcd}$$

which implies that the structure map $\mathscr{F}(U) \to \mathscr{F}(D)$ is a quasi-isomorphism. $\square$

Proof of Proposition 18. First we check that if $\mathscr{F}$ is a factorization algebra on $X \times Y$ and $U \subset X$ is open, then $\underline{\pi_1}_*(\mathscr{F}(U))$ is a factorization algebra over Y. If $\mathscr{V}$ is a factorizing cover of an open set $V \subset Y$, then $\{U\} \times \mathscr{V}$ is a factorizing cover of $U \times V$ and the Čech complex $\check{C}\big(\mathscr{V}, \underline{\pi_1}_*\mathscr{F}(U)\big)$ is equal to $\check{C}(\{U\} \times \mathscr{V}, \mathscr{F})$. Hence the natural map $\check{C}\big(\mathscr{V}, \underline{\pi_1}_*\mathscr{F}(U)\big) \to \underline{\pi_1}_*(\mathscr{F})(U,V)$ factors as

$$\check{C}\big(\mathscr{V}, \underline{\pi_1}_*\mathscr{F}(U)\big) = \check{C}(\{U\} \times \mathscr{V}, \mathscr{F}) \to \mathscr{F}(U \times V) = \underline{\pi_1}_*(\mathscr{F})(U,V).$$

It is a quasi-isomorphism since $\mathscr{F}$ is a factorization algebra. We have proved that $\underline{\pi_1}_*(\mathscr{F}) \in \mathbf{PFac}_X(\mathbf{Fac}_Y)$. To show that $\underline{\pi_1}_*(\mathscr{F}) \in \mathbf{Fac}_X(\mathbf{Fac}_Y)$, we only need to check that for every open $V \subset Y$, and any factorizing cover $\mathscr{U}$ of U, the natural map $\check{C}\big(\mathscr{U}, \underline{\pi_1}_*(\mathscr{F})(-,V)\big) \to \underline{\pi_1}_*(\mathscr{F})(U,V)$ is a quasi-isomorphism, which follows by the same argument. Hence $\underline{\pi_1}_*$ factors as a functor $\underline{\pi_1}_* : \mathbf{Fac}_{X\times Y} \longrightarrow \mathbf{Fac}_X(\mathbf{Fac}_Y)$.

When $\mathscr{F}$ is locally constant, Proposition 15 applied to the first and second projection implies that $\underline{\pi_1}_*(\mathscr{F}) \in \mathbf{Fac}_X^{lc}(\mathbf{Fac}_Y^{lc})$.

Now we build an inverse of $\underline{\pi_1}_*$ in the locally constant case. Let $\mathscr{B}$ be in $\mathbf{Fac}_X(\mathbf{Fac}_Y)$. A (stable by finite intersection) basis of neighborhood of $X \times Y$ is given by the products $U \times V$, with $(U,V) \in \mathscr{CV}(X) \times \mathscr{CV}(Y)$ where $\mathscr{CV}(X)$, $\mathscr{CV}(Y)$ are bounded geodesically convex neighborhoods (for some choice of Riemannian metric on X and Y). Thus by § 5.2, in order to extend $\mathscr{B}$ to a factorization algebra on $X \times Y$, it is enough to prove that the rule $(U \times V) \mapsto \mathscr{B}(U)(V)$ (where $U \subset X$, $V \subset Y$) defines an $\mathscr{CV}(X) \times \mathscr{CV}(Y)$-factorization algebra. If $U \times V \in \mathscr{CV}(X) \times \mathscr{CV}(Y)$, then U and V are canonically homeomorphic to $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Now, the construction of the structure maps for opens in

$\mathscr{CV}(U)\times\mathscr{CV}(V)$ restricts to proving the result for $\mathscr{B}_{|U\times V}\in \mathbf{Fac}^{lc}_{\mathbb{R}^n}(\mathbf{Fac}^{lc}_{\mathbb{R}^m})$. By Theorem 9, $\mathbf{Fac}_{\mathbb{R}^d}\cong E_d$-**Alg**, hence by Dunn Theorem 11 below, we have that $\mathbf{Fac}^{lc}_{\mathbb{R}^{n+m}}\xrightarrow{\pi_{1*}}\mathbf{Fac}^{lc}_{\mathbb{R}^n}(\mathbf{Fac}^{lc}_{\mathbb{R}^m})$ is an equivalence which allows to define a $\mathscr{CV}(X)\times\mathscr{CV}(Y)$-factorization algebra structure associated to $\mathscr{B}$. We denote $j(\mathscr{B})\in\mathbf{Fac}_{X\times Y}$ the induced factorization algebra on $X\times Y$. Note that $j(\mathscr{B})$ is locally constant, since, again, the question reduces to Dunn Theorem.

It remains to prove that $j:\mathbf{Fac}^{lc}_X(\mathbf{Fac}^{lc}_Y)\to\mathbf{Fac}_{X\times Y}$ is a natural inverse to $\underline{\pi_1}_*$. This follows by uniqueness of the factorization algebra extending a factorization algebra on a factorizing basis, that is, Proposition 17. □

9.2 Complements on § 6.2

Here we collect the proofs of statements relating factorization algebras and intervals.

Proof of Proposition 27. The (sketch of) proof is extracted from the excision property for factorization algebras in [GTZ2]. By definition of a Disk_1^{fr}-algebra, a factorization algebra $\mathscr{G}$ on $\mathbb{R}$ carries a structure of Disk_1^{fr}-algebra (by simply restricting the value of $\mathscr{G}$ to open sub-intervals, just as in Remark 23).

Similarly, if $\mathscr{G}$ is a factorization algebra on $[0,+\infty)$, it carries a structure of a Disk_1^{fr}-algebra and a (pointed) right module over it, while a factorization algebra on $(-\infty,0]$ carries the structure of a left (pointed) module over a Disk_1^{fr}-algebra (see Definition 36). It follows that a factorization algebra over the closed interval $[0,1]$ determines an E_1-algebra $\mathscr{A}$ and pointed left module $\mathscr{M}^\ell$ and pointed right module $\mathscr{M}^r$ over $\mathscr{A}$.

By strictification we can replace the E_1-algebra and modules by differential graded associative ones so that we are left to the case of a factorization algebra $\mathscr{F}$ on $[-1,1]$ which, on the factorizing basis $\mathscr{I}$ of $[0,1]$, is precisely the $\mathscr{I}$-prefactorization algebra $\mathscr{F}$ defined before the Proposition 27.

Now, we are left to prove that, for any A,M^ℓ,M^r,m^r,m^ℓ, $\mathscr{F}$ is a $\mathscr{I}$-factorization algebra, and then to compute its global section $\mathscr{F}([0,1])$. Theorem 9 implies that the restriction $\mathscr{F}_A$ of $\mathscr{F}$ to $(0,1)$ is a factorization algebra. In order to conclude we only need to prove that the canonical maps

$$\check{C}\big(\mathscr{U}_{[0,1)},\mathscr{F})\big)\longrightarrow\mathscr{F}([0,1))=M^r,\quad \check{C}\big(\mathscr{U}_{(0,1]},\mathscr{F})\big)\longrightarrow\mathscr{F}((0,1])=M^\ell$$

$$\text{and}\quad \check{C}\big(\mathscr{U}_{[0,1]},\mathscr{F})\big)\longrightarrow\mathscr{F}([0,1])\cong M^r\overset{\mathbb{L}}{\underset{A}{\otimes}}M^\ell$$

are quasi-isomorphisms. Here, $\mathscr{U}_{[0,1]}$ is the factorizing covers given by all opens $U_t:=[0,1]\setminus\{t\}$ where $t\in[0,1]$ (in other words by the complement of a singleton). Similarly, $\mathscr{U}_{[0,1)}$, $\mathscr{U}_{(0,1]}$ are respectively covers given by all opens $U_t^\ell:=([0,1)\setminus\{t\}$ where $t\in[0,1)$ and all opens $U_t^r:=(0,1]\setminus\{t\}$ where $t\in(0,1]$.

The proof in the 3 cases are essentially the same so we only consider the case of the opens U_t. Since $M^r \overset{\mathbb{L}}{\underset{A}{\otimes}} M^\ell \cong M^r \underset{A}{\otimes} B(A,A,A) \otimes_A M^\ell$ where $B(A,A,A)$ is the two-sided Bar construction of A, it is enough to prove the result for $M^r = M^\ell = A$ in which case we are left to prove that the canonical map

$$\check{C}\big(\mathscr{U}_{[0,1]}, \mathscr{F})\big) \longrightarrow A \overset{\mathbb{L}}{\underset{A}{\otimes}} A \cong B(A,A,A) \xrightarrow{\simeq} A$$

is an equivalence.

Any two open sets in $\mathscr{U}_{[0,1]}$ intersect non-trivially so that the set $P\mathscr{U}$ are singletons. We have $\mathscr{F}(U_t) \cong \mathscr{F}([-1,t)) \otimes \mathscr{F}((t,1])$ which is $A \otimes A$ if $t \neq \pm 1$ and is $A \otimes k$ or $k \otimes A$ if $t = 1$ or $t = -1$. More generally,

$$\mathscr{F}(U_{t_0}, \dots, U_{t_n}, U_{\pm 1}) \cong \mathscr{F}(U_{t_0}, \dots, U_{t_n}) \otimes k.$$

Further, if $0 < t_0 < \cdots < t_n < 1$, then $\mathscr{F}(U_{t_0}, \dots, U_{t_n}) \cong A \otimes A^{\otimes n} \otimes A$ and the structure map $\mathscr{F}(U_{t_0}, \dots, U_{t_n}) \to \mathscr{F}(U_{t_0}, \dots, \widehat{U_{t_i}}, \dots, U_{t_n})$ is given by the multiplication

$$a_0 \otimes \cdots \otimes a_{n+1} \mapsto a_0 \otimes \cdots (a_i a_{i+1}) \otimes \cdots \otimes a_{n+1}.$$

This identifies the Čech complex $\check{C}\big(\mathscr{U}, \mathscr{F}\big)$ with a kind of parametrized analogue of the standard two sided Bar construction with coefficients in A. We have canonical maps

$$\phi_t : \mathscr{F}(U_t) \cong \mathscr{F}([-1,t)) \otimes \mathscr{F}((t,1]) \to \mathscr{F}([-1,1)) \otimes \mathscr{F}((-1,1]) \cong A \otimes A \to A$$

induced by the multiplication in A. The composition $\bigoplus_{U_r, U_s \in P\mathscr{U}} \mathscr{F}(U_r, U_s)[1] \to \bigoplus_{U_t \in P\mathscr{U}} \mathscr{F}(U_t)[0] \to A$ is the zero map so that we have a map of (total) chain complexes: $\eta : \check{C}\big(\mathscr{U}, \mathscr{F}\big) \to A$. In order to prove that η is an equivalence, we consider the retract $\kappa : A \cong \mathscr{F}(U_1) \hookrightarrow \bigoplus_{U_t \in P\mathscr{U}} \mathscr{F}(U_t)[0] \hookrightarrow \check{C}\big(\mathscr{U}, \mathscr{F}\big)$ which satisfies $\eta \circ \kappa = id_A$. Let h be the homotopy operator on $\check{C}\big(\mathscr{U}, \mathscr{F}\big)$ defined, on $\mathscr{F}(U_{t_0}, \dots, U_{t_n})[n]$, by

$$\sum_{i=0}^{n} (-1)^i s_i^{t_0, \dots, t_n} : \mathscr{F}(U_{t_0}, \dots, U_{t_n})[n] \longrightarrow \bigoplus_{U_{r_0}, \dots U_{r_{n+1}} \in P\mathscr{U}} \mathscr{F}(U_{r_0}, \dots, U_{r_{n+1}})[n+1]$$

where, for $0 \le i \le n-1$, $s_i^{t_0, \dots, t_n}$ is defined as the suspension of the identity map

$$\mathscr{F}(U_{t_0}, \dots, U_{t_n})[n] \to \mathscr{F}(U_{t_0}, \dots, U_{t_n})[n+1] \cong \mathscr{F}(U_{t_0}, \dots, U_{t_i}, U_{t_i}, \dots, U_{t_n})[n+1]$$

followed by the inclusion in the Čech complex.

Similarly, the map $s_n^{t_0, \dots, t_n}$ is defined as the suspension of the identity map $\mathscr{F}(U_{t_0}, \dots, U_{t_n})[n] \to \mathscr{F}(U_{t_0}, \dots, U_{t_n})[n+1] \cong \mathscr{F}(U_{t_0}, \dots, U_{t_n}, U_1)[n+1]$ (followed by the inclusion in the Čech complex). Note that $dh + hd = id - \kappa \circ \eta$ where d is the total differential on $\check{C}\big(\mathscr{U}, \mathscr{F}\big)$. It follows that $\eta : \check{C}\big(\mathscr{U}, \mathscr{F}\big) \to A$ is an equivalence. $\square$

Proof of Corollary 7. The functor $\underline{\pi_1}_* : \mathbf{Fac}^{lc}_{X\times[0,+\infty)} \to E_1\text{-}\mathbf{RMod}(\mathbf{Fac}^{lc}_X)$ is well-defined by Corollary 6 and Proposition 28. In order to check it is an equivalence, as in the proof of Proposition 18, we only need to prove it when $X = \mathbb{R}^n$, that is that, if $\mathscr{F} \in \mathbf{Fac}^{lc}_{[0,+\infty)}(\mathbf{Fac}^{lc}_X)$, then it is in the essential image of $\underline{\pi_1}_*$. By Proposition 18, we can also assume that the restriction $\mathscr{F}_{|(0,+\infty)}$ is in $\underline{\pi_1}_*\big(\mathbf{Fac}^{lc}_{\mathbb{R}^n\times(0,+\infty)}\big)$.

Let $\mathscr{I}_\varepsilon$ be the factorizing cover of $[0,+\infty)$ consisting of all intervals with the restriction that intervals containing 0 are included in $[0,\varepsilon)$ note that $I_\tau \subset I_\varepsilon$ if $\varepsilon > \tau$. We can replace $\mathscr{F}$ by its Čech complex on $\mathscr{I}_\varepsilon$ (for any ε) and thus by its limit over all $\varepsilon > 0$, which we still denote $\mathscr{F}$. As in the proof of Proposition 18, we only need to prove that $(U,V) \mapsto \big(\mathscr{F}(V)\big)(U)$ extends as a factorization algebra relative to the factorizing basis of $\mathbb{R}^n \times [0,+\infty)$ consisting of cubes (with sides parallel to the axes). The only difficulty is to define the prefactorization algebra structure on this basis (since we already know it is locally constant, and thus will extend into a factorization algebra). As noticed above, we already have such structure when no cubes intersect $\mathbb{R}^n \times \{0\}$. Given a finite family of pairwise disjoint cubes lying in a bigger cube $K \times [0,R)$ intersecting $\{0\}$, we can find $\varepsilon > 0$ such that no cubes of the family lying in $\mathbb{R}^n \times (0,+\infty)$ lies in the band $\mathbb{R}^n \times [0,\varepsilon)$. The value of $\mathscr{F}$ on each square containing $\mathbb{R}^n \times \{0\}$ can be computed using the Čech complex associated to I_ε. This left us, in every such cube, with one term containing a summand $[0,\tau)$ ($\tau \le \varepsilon$) and cubes in the complement. Now choosing the maximum of the possible τ allows to first maps $(\mathscr{F}(c))(d)$ to $\big(\mathscr{F}(\tau,R)\big)(K)$ for every cube $c\times d$ in $\mathbb{R}^n \times (\tau,R)$ (since we already have a factorization algebra on $\mathbb{R}^n \times (0,+\infty)$). Then to maps all other summands to terms of the form $\mathscr{F}([0,\tau))(d)$, then all of them in $\mathscr{F}([0,\tau))(K)$ and finally to evaluate the last two remaining summand in $\mathscr{F}([0,R))(K)$ using the prefactorization algebra structure of $\mathscr{F}$ with respect to intervals in $[0,\infty)$. This is essentially the same argument as in the proof of Corollary 8. □

9.3 Complements on § 6.3

Here we collect proofs of statements relating factorization algebras and E_n-modules.

Proof of Theorem 12 The functoriality is immediate from the construction. Let $\mathbf{Fin}_*$ be the ∞-category associated to the category Fin_* of pointed finite sets. If $\mathscr{O}$ is an operad, the ∞-category $\mathscr{O}\text{-}\mathbf{Mod}_A$ of $\mathscr{O}$-modules[80] over an $\mathscr{O}$-algebra A is the category of $\mathscr{O}$-linear functors $\mathscr{O}\text{-}\mathbf{Mod}_A := \mathrm{Map}_{\mathbf{O}}(\mathbf{O}_*, Chain(k))$. Here, following the notations of § 10.2, $\mathbf{O}$ is the ∞-categorical enveloppe of $\mathscr{O}$ as in [L3] and $\mathbf{O}_* := \mathbf{O} \times_{\mathbf{Fin}} \mathbf{Fin}_*$. There is an natural fibration $\pi_{\mathscr{O}} : \mathscr{O}\text{-}\mathbf{Mod} \longrightarrow \mathscr{O}\text{-}\mathbf{Alg}$ whose fiber at $A \in \mathscr{O}\text{-}\mathbf{Alg}$ is $\mathscr{O}\text{-}\mathbf{Mod}_A$.

Let $\mathscr{D}_{isk}$ be the set of all open disks in $\mathbb{R}^n$. Recall from Remark 24 that $\mathscr{D}_{isk}$-prefactorization algebras are exactly algebras over the operad $\mathrm{Disk}(\mathbb{R}^n)$ and that

[80] in $Chain(k)$

locally constant $\mathscr{D}_{isk}$-prefactorization algebras are the same as locally constant factorization algebras on $\mathbb{R}^n$ (Proposition 12). The map of operad $\mathrm{Disk}(\mathbb{R}^n) \to \mathbb{E}_{\mathbb{R}^n}$ of [L3, §5.2.4] induces a fully faithful functor E_n-**Alg** $\to \mathrm{Disk}(\mathbb{R}^n)$-**Alg** and thus a functor

$$\tilde{\psi} : E_n\text{-}\mathbf{Mod} \longrightarrow \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod} \longrightarrow \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}.$$

The map $\tilde{\psi}$ satisfies that, for every convex subset $C \subset \mathbb{R}^n$, one has

$$\tilde{\psi}(M)(C) = M(C) = \mathscr{F}_M(C).$$

By definition, $\tilde{\psi} \circ can : E_n\text{-}\mathbf{Alg} \to \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}$ is the composition

$$E_n\text{-}\mathbf{Alg} \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}^n} \longrightarrow \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}.$$

Hence, the commutativity of the diagram in the Theorem will follow automatically once we have proved that $\tilde{\psi}$ factors as a composition of functors

$$\tilde{\psi} : E_n\text{-}\mathbf{Mod} \xrightarrow{\psi} \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \longrightarrow \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}. \tag{55}$$

Assuming for the moment that we have proved that $\tilde{\psi}$ factors through $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*}$, let us show that ψ is fully faithful. By definition of categories of modules, we have a commutative diagram

$$\begin{tikzcd} E_n\text{-}\mathbf{Mod} \arrow[r] \arrow[d, "\pi_{E_n}"'] & \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod} \arrow[d, "\pi_{\mathrm{Disk}(\mathbb{R}^n)}"] \\ E_n\text{-}\mathbf{Alg} \arrow[r] & \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg} \end{tikzcd}$$

whose bottom arrow is a fully faithful embedding by [L3, §5.2.4]. Since the mapping spaces of $\mathscr{F} \in \mathbf{Fac}^{lc}_X$ are the mapping spaces of the underlying prefactorization algebra, the map $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \to \mathrm{Disk}(\mathbb{R}^n)$-**Alg** is fully faithful, and we are left to prove that

$$\tilde{\psi} : \mathrm{Map}_{E_n\text{-}\mathbf{Mod}}(M,N) \to \mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(\tilde{\psi}(M), \tilde{\psi}(N))$$

is an equivalence for all $M \in E_n\text{-}\mathbf{Mod}_A$ and $N \in E_n\text{-}\mathbf{Mod}_B$. The fiber at (the image of) an E_n-algebra A of $\mathrm{Disk}(\mathbb{R}^n)$-**Mod** $\to \mathrm{Disk}(\mathbb{R}^n)$-**Alg** is the (homotopy) pullback

$$\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod}_A := \mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}^{/A} \times^h_{\mathrm{Disk}(\mathbb{R}^n\setminus\{0\})\text{-}\mathbf{Alg}^{/A}} \mathrm{Iso}_{\mathrm{Disk}(\mathbb{R}^n\setminus\{0\})\text{-}\mathbf{Alg}}(A).$$

Here we write $\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}^{/A}$ for the ∞-category of $\mathrm{Disk}(\mathbb{R}^n)$-algebras under A and $\mathrm{Iso}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(A)$ its subcategory of objects $A \xrightarrow{f} B$ such that f is an equivalence. In plain english, $\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod}_A$ is the ∞-category of maps $A \xrightarrow{f} B$ (where B runs through $\mathrm{Disk}(\mathbb{R}^n)$-**Alg**) whose restriction to $\mathbb{R}^n \setminus \{0\}$ is an equivalence.

It is now sufficient to prove, given E_n-algebras A and B (identified with objects of $\mathbf{Fac}^{lc}_{\mathbb{R}^n}$) and two locally constant factorization algebras $\tilde{\psi}(M)$, $\tilde{\psi}(N)$ on $\mathbb{R}^n_*$

together with two maps of factorizations algebras $f: A \to \tilde{\psi}(M)$, $g: B \to \tilde{\psi}(N)$ whose restrictions to $\mathbb{R}^n \setminus \{0\}$ are quasi-isomorphisms, that the canonical map

$$\mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(A,B) \times^h_{\mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(A,\tilde{\psi}(N))} \mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(\tilde{\psi}(M),\tilde{\psi}(N)) \\ \longrightarrow \mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(\tilde{\psi}(M),\tilde{\psi}(N)) \quad (56)$$

is an equivalence. This pullback is the mapping space $\mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Mod}}(\tilde{\psi}(M),\tilde{\psi}(N))$ and the maps to $\mathrm{Map}_{\mathrm{Disk}(\mathbb{R}^n)\text{-}\mathbf{Alg}}(A,\tilde{\psi}(N))$ are induced by post-composition by g and precomposition by f.

The fiber of the map (56) at $\Theta: \tilde{\psi}(M) \to \tilde{\psi}(N))$ is the mapping space of $\mathrm{Disk}(\mathbb{R}^n)$-algebras $A \xrightarrow{\tau} B$ such that, for any disk $U \subset \mathbb{R}^n \setminus \{0\}$, which is a sub-disk of a disk D containing 0, the following diagram is commutative:

$$\begin{tikzcd} A(D) \arrow[d, "\tau(D)"'] & A(U) \arrow[l, "\rho^A_{U,D}"'] \arrow[r, "\simeq", "f"'] \arrow[d, "\tau(U)"'] & \tilde{\psi}(M)(U) \arrow[d, "\Theta(U)"] \\ B(D) & B(U) \arrow[l, "\rho^B_{U,D}"'] \arrow[r, "\simeq", "g"'] & \tilde{\psi}(N)(U). \end{tikzcd} \quad (57)$$

Here $\rho^A_{U,D}$ and $\rho^B_{U,D}$ are the structure maps of the factorization algebras associated to A and B. The right hand square of Diagram (57) shows that τ is uniquely determined by Θ on every open disk in $\mathbb{R}^n \setminus \{0\}$.

Since A and B are locally constant factorization algebras on $\mathbb{R}^n$, the maps $\rho^A_{U,D}$ and $\rho^B_{U,D}$ are natural quasi-isomorphisms. It follows from the left hand square in Diagram (57) that the restriction of τ to $\mathbb{R}^n \setminus \{0\}$ also determines the map τ on $\mathbb{R}^n$. Hence the map (56) is an equivalence which concludes the proof that ψ is fully faithful.

It remains to prove that $\tilde{\psi}$ factors through a functor ψ, that is that we have a composition as written in (55). This amounts to prove that for any $M \in E_n\text{-}\mathbf{Mod}_A$ (that is M is an E_n-module over A), $\tilde{\psi}(M)$ is a locally constant factorization algebra on the stratified manifold given by the pointed disk $\mathbb{R}^n_*$. Since the convex subsets are a factorizing basis stable by finite intersection, we only have to prove this result on the cover $\mathscr{CV}(\mathbb{R}^n)$ (by Proposition 17 and Proposition 13).

Note that if $V \in \mathscr{CV}(\mathbb{R}^n)$ is a subset of $\mathbb{R}^n \setminus \{0\}$, then $\psi(M)_{|V}$ lies in the essential image of $\psi \circ can(M)_{|V}$ where $\psi \circ can(M)$ is the functor inducing the equivalence between E_n-algebras and locally constant factorization algebras on $\mathbb{R}^n$ (Theorem 9). We denote $\mathscr{F}_A := \psi \circ can(A)$ the locally constant factorization algebra on $\mathbb{R}^n$ induced by A. In particular, the canonical map

$$\check{C}(\mathscr{CV}(V),\mathscr{F}_M) = \check{C}(\mathscr{CV}(V),\mathscr{F}_A) \to \mathscr{F}_A(V) \cong \mathscr{F}_M(V)$$

is a quasi-isomorphism and further, if $U \subset V$ is a sub disk, then $\mathscr{F}_M(U) \to \mathscr{F}_M(V)$ is a quasi-isomorphism.

We are left to consider the case where V is a convex set containing 0. Let $\mathscr{U}_V$ be the cover of V consisting of all open sets which contains 0 and are a finite union of disjoint convex subsets of V. It is a factorizing cover, and, by construction, two open sets in $\mathscr{U}_V$ intersects non-trivially since they contain 0. Hence $P\mathscr{U}_V = \mathscr{U}_V$. Since $\mathscr{U}_V \subset P\mathscr{C}\mathscr{V}(V)$, we have a diagram of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & \check{C}(\mathscr{U}_V, \mathscr{F}_M) \arrow[r, "i_M"] & \check{C}(\mathscr{C}\mathscr{V}(V), \mathscr{F}_M) \arrow[r] & \check{C}(\mathscr{C}\mathscr{V}(V), \mathscr{F}_M)/\check{C}(\mathscr{U}_V, \mathscr{F}_M) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \check{C}(\mathscr{U}_V, \mathscr{F}_A) \arrow[r, "i_A"] & \check{C}(\mathscr{C}\mathscr{V}(V), \mathscr{F}_A) \arrow[r] & \check{C}(\mathscr{C}\mathscr{V}(V), \mathscr{F}_A)/\check{C}(\mathscr{U}_V, \mathscr{F}_A) \arrow[r] & 0
\end{tikzcd}$$

where the vertical equivalence follows from the fact that $\mathscr{F}_M(U) \cong \mathscr{F}_A(U)$ if U is a convex set not containing 0. Moreover, since $\mathscr{U}_V$ is a factorizing cover of V and $\mathscr{F}_A$ a factorization algebra, i_A is a quasi-isomorphism, hence $\check{C}(\mathscr{C}\mathscr{V}(V), \mathscr{F}_A)/\check{C}(\mathscr{U}_V, \mathscr{F}_A)$ is acyclic. It follows that $i_M : \check{C}(\mathscr{U}_V, \mathscr{F}_M) \to \check{C}(\mathscr{C}\mathscr{V}(V), \mathscr{F}_M)$ is a quasi-isomorphism as well.

We are left to prove that the canonical map $\check{C}(\mathscr{U}_V, \mathscr{F}_M) \to \mathscr{F}(M) \cong M$ is a quasi-isomorphism. Note that for any $U \in \mathscr{U}_V$, we have $\mathscr{F}_M(U) \cong M \otimes_A^{\mathbb{L}} \mathscr{F}_A(U)$. We deduce that $\check{C}(\mathscr{U}_V, \mathscr{F}_M) \cong M \otimes_A^{\mathbb{L}} \check{C}(\mathscr{U}_V, \mathscr{F}_A)$ as well. The chain map

$$M \otimes_A^{\mathbb{L}} \check{C}(\mathscr{U}_V, \mathscr{F}_A) \cong \check{C}(\mathscr{U}_V, \mathscr{F}_M) \longrightarrow \mathscr{F}(M) \cong M \otimes_A^{\mathbb{L}} A$$

is an equivalence since it is obtained by tensoring (by M over A) the quasi-isomorphism $\check{C}(\mathscr{U}_V, \mathscr{F}_A) \to \mathscr{F}_A(V) \cong A$ (which follows from the fact that $\mathscr{F}_A$ is a factorization algebra).

It remains to prove that $\mathscr{F}_M(U) \to \mathscr{F}_M(V)$ is a quasi-isomorphism if both U, V are convex subsets containing 0. This is immediate since $M(U) \to M(V)$ is an equivalence by definition of an E_n-module over A. $\square$

Proof of Corollary 8. By Theorem 12, we have a commutative diagram

$$\begin{tikzcd}
E_n\text{-}\mathbf{Mod} \arrow[r] \arrow[d, "\pi_{E_n}"'] & \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times^h_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \mathbf{Fac}^{lc}_{\mathbb{R}^n} \arrow[d] \\
E_n\text{-}\mathbf{Alg} \arrow[r] & \mathbf{Fac}^{lc}_{\mathbb{R}^n}
\end{tikzcd}$$

with fully faithful horizontal arrows. Since $E_n\text{-}\mathbf{Alg} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n}$ is an equivalence, we only need to prove that, for any E_n-algebra A, the induced fully faithful functor $E_n\text{-}\mathbf{Mod}_A \longrightarrow \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{A\}$ between the fibers is essentially surjective[81].

Let $\mathscr{M}$ be a locally constant factorization algebra on $\mathbb{R}^n_*$ such that $\mathscr{M}_{|\mathbb{R}\setminus\{0\}}$ is equal to $\mathscr{A}_{|\mathbb{R}^n\setminus\{0\}}$ where $\mathscr{A}$ is the factorization algebra associated to A (by Theorem 9). Recall that $N : \mathbb{R}^n \to [0, +\infty)$ is the euclidean norm map. Lemma 7 implies

[81] that is we are left to prove Corollary 9

that $N_*(\mathscr{M})$ is is locally constant on the stratified half-line $[0,+\infty)$ and thus equivalent to a right module over the E_1-algebra $N_*(\mathscr{M})(\mathbb{R}^n\setminus\{0\}) \cong \mathscr{A}(\mathbb{R}^n\setminus\{0\}) \cong \int_{S^{n-1}\times\mathbb{R}} A$.

By homeomorphism invariance of (locally constant) factorization algebras, we can replace $\mathbb{R}^n$ by the unit open disk D^n of $\mathbb{R}^n$ in the above analysis. We also denote D^n_* the disk D^n viewed as a pointed space with base point 0. We now use this observation to define a structure of E_n-module over A on $M := \mathscr{M}(D^n) = N_*(\mathscr{M})\big([0,1)\big)$. It amount to define, for any finite set I, continuous maps (compatible with the structure of the operad of little disks of dimension n)

$$\begin{aligned}\mathrm{Rect}_*\Big(D^n_*\coprod\big(\coprod_{i\in I} D^n\big), D^n_*\Big) &\longrightarrow \mathrm{Map}_{Chain(k)}\big(M\otimes A^{\otimes I}, M\big)\\ &\xrightarrow{\simeq} \mathrm{Map}_{Chain(k)}\Big(M\otimes\Big(\int_{D^n} A\Big)^{\otimes I}, M\Big)\end{aligned}$$

where Rect_* is the space of rectilinear embeddings which maps the center of the first copy D^n_* to the center of D^n_* (i.e. preserves the base point of D^n). Let I_N be the map that sends an element $f\in \mathrm{Rect}_*\big(D^n\coprod\big(\coprod_{i\in I} D^n\big), D^n\big)$ to the smallest open sub-interval $I_N(f)\subset(0,1)$ which contains $N\big(f\big(\coprod_{i\in I} D^n\big)\big)$, that is the smallest interval that contains the image of the non-pointed disks. By definition I_N is continuous (meaning the lower and the upper bound of $I_N(f)$ depends continuously of f) and its image is disjoint from the image $N(D^n_*)$ of the pointed copy of D^n. Similarly we define $r(f)$ to be the radius of $f(D^n_*)$. We have a continuous map

$$\tilde{N}:\mathrm{Rect}_*\Big(D^n_*\coprod\big(\coprod_{i\in I} D^n\big), D^n_*\Big)\longrightarrow \mathrm{Rect}\Big([0,1)\coprod(0,1),[0,1)\Big)$$

given by $\tilde{N}(f)\big((0,1)\big) = I_N(f)$ and $\tilde{N}(f)\big([0,1)\big) = [0,r(f))$. Since $f(\coprod_{i\in I} D^n)\subset S^{n-1}\times(0,1)$, we have the composition

$$\begin{aligned}\Upsilon:\mathrm{Rect}_*\Big(D^n_*\coprod\big(\coprod_{i\in I} D^n\big), D^n_*\Big) &\longrightarrow \mathrm{Rect}\Big(\coprod_{i\in I} D^n, S^{n-1}\times(0,1)\Big)\\ &\longrightarrow \mathrm{Map}_{Chain(k)}\Big(\Big(\int_{D^n} A\Big)^{\otimes I}, \int_{S^{n-1}\times(0,1)} A\Big)\end{aligned}$$

where the first map is induced by the restriction to $\coprod_{i\in I} D^n$ and the last one by functoriality of factorization homology with respect to embeddings. We finally define

$$\mu : \mathrm{Rect}_*\Big(D^n_* \coprod \big(\coprod_{i\in I} D^n\big), D^n_*\Big) \xrightarrow{\tilde{N}\times\Upsilon}$$

$$\mathrm{Rect}\Big([0,1)\coprod(0,1),[0,1)\Big) \times \mathrm{Map}_{Chain(k)}\Big(\Big(\int_{D^n} A\Big)^{\otimes I}, \int_{S^{n-1}\times(0,1)} A\Big) \longrightarrow$$

$$\mathrm{Map}_{Chain(k)}\Big(M \otimes \int_{S^{n-1}\times(0,1)} A, M\Big) \times \mathrm{Map}_{Chain(k)}\Big(\Big(\int_{D^n} A\Big)^{\otimes I}, \int_{S^{n-1}\times(0,1)} A\Big)$$

$$\xrightarrow{id_M\otimes\circ} \mathrm{Map}_{Chain(k)}\Big(M \otimes \Big(\int_{D^n} A\Big)^{\otimes I}, M\Big)$$

where the second map is induced by the E_1-module structure of $M = N_*(\mathscr{M})([0,1))$ over $\int_{S^{n-1}\times(0,1)} A$ and the last one by composition. That μ is compatible with the action of the little disks operad follows from the fact that Υ is induced by the E_n-algebra structure of A and M is an E_1-module over $\int_{S^{n-1}\times(0,1)} A$. Hence, M is in E_n-$\mathbf{Mod}_A$.

We now prove that the factorization algebra $\psi(M)$ is $\mathscr{M}$. For all euclidean disks D centered at 0, one has $\psi(M)(D) = \mu(\{D \hookrightarrow D^n\})(M) = \mathscr{M}(D)$ and further $\psi(M)(U) = \mathscr{A}(U)$ if U is a disk that does not contain 0. The $\mathscr{D}$-prefactorization algebra structure of $\psi(M)$ (where $\mathscr{D}$ is the basis of opens consisting of all euclidean disks centered at 0 and all those who do not contain 0) is precisely given by μ according to the construction of ψ (see Theorem 12). Hence, by Proposition 17, $\psi(M) \cong \mathscr{M}$ and the essential surjectivity follows. $\square$

Proof of Proposition 29. By [L3, §4.3], we have two functors $i_\pm : \mathbf{BiMod} \to E_1\text{-}\mathbf{Alg}$ and the (homotopy) fiber of $\mathbf{BiMod} \xrightarrow{(i_-,i_+)} E_1\text{-}\mathbf{Alg} \times E_1\text{-}\mathbf{Alg}$ at a point (L,R) is the category of (L,R)-bimodules which is equivalent to the category $E_1\text{-}\mathbf{LMod}_{L\otimes R^{op}}$. We have a factorization

$$\begin{tikzcd}
\mathbf{Fac}^{lc}_{\mathbb{R}_*} \arrow[r, "{(j^*_\pm,(-N)_*)}"] \arrow[drr, "{(j^*_-,j^*_+)}"'] & \mathbf{BiMod} \arrow[r, "{(i_-,i_+)}"] & E_1\text{-}\mathbf{Alg} \times E_1\text{-}\mathbf{Alg} \\
& & \mathbf{Fac}^{lc}_{(-\infty,0)} \times \mathbf{Fac}^{lc}_{(0,+\infty)}. \arrow[u, "\simeq"']
\end{tikzcd} \tag{58}$$

We can assume that L,R are strict and consider the fiber

$$\big(\mathbf{Fac}^{lc}_{\mathbb{R}_*}\big)_{L,R} := \{\mathscr{F}_L, \mathscr{F}_R\} \times_{\mathbf{Fac}^{lc}_{(-\infty,0)}\times\mathbf{Fac}^{lc}_{(0,+\infty)}} \mathbf{Fac}^{lc}_{\mathbb{R}_*}$$

of (j^*_-, j^*_+) at the pair of factorization algebras $(\mathscr{F}_L, \mathscr{F}_R)$ on $(-\infty,0)$, $(0,+\infty)$ corresponding to L, R respectively (using Proposition 27). The pushforward along the opposite of the euclidean norm map gives the functor $(-N)_* : \big(\mathbf{Fac}^{lc}_{\mathbb{R}_*}\big)_{L,R} \to E_1\text{-}\mathbf{LMod}_{L\otimes R^{op}}$.

We further have a locally constant factorization algebra $\mathscr{G}^{L,R}_M$ on $\mathbb{R}_*$ which is defined on the basis of disks by the same rule as for the open interval (for disks included in a component $\mathscr{R}\setminus\{0\}$) together with $\mathscr{G}_M^{L,R}(\alpha,\beta)=M$ for $\alpha<0<\beta$. For $r<t_1<u_1\cdots<t_n<u_n<\alpha<0<\beta<x_1<y_1<\cdots<x_m<y_m<s$, the structure maps

$$\Big(\bigotimes_{i=1\dots n}\mathscr{G}_M^{L,R}\big((u_i,t_i)\big)\Big)\otimes\mathscr{G}_M^{L,R}\big((\alpha,\beta)\big)\otimes\Big(\bigotimes_{i=1\dots n}\mathscr{G}_M^{L,R}\big((u_i,t_i)\big)\Big)$$
$$\cong\ L^{\otimes n}\otimes M\otimes R^{\otimes m}\longrightarrow M\cong\mathscr{G}_M^{L,R}\big((r,s)\big)$$

are given by $\ell_1\otimes\cdots\otimes\ell_n\otimes a\otimes r_1\otimes\cdots\otimes r_n\mapsto(\ell_1\cdots\ell_n)\cdot a\cdot(r_1\cdots r_n)$.

One checks as in Proposition 27 that $\mathscr{G}_M^{L,R}$ is a locally constant factorization algebra on $\mathbb{R}_*$. The induced functor E_1-$\mathbf{LMod}_{L\otimes R^{op}}\to\mathbf{Fac}^{lc}_{\mathbb{R}_*}$ is an inverse of $(-N)_*$. Thus the fiber $\big(\mathbf{Fac}^{lc}_{\mathbb{R}_*}\big)_{L,R}$ of (j_-^*,j_+^*) is equivalent to E_1-$\mathbf{LMod}_{L\otimes R^{op}}$. It now follows from diagram (58) that the functor $(j_\pm^*,(-N)_*):\mathbf{Fac}^{lc}_{\mathbb{R}_*}\cong\mathbf{BiMod}$ is an equivalence. □

Proof of Proposition 30. We define a functor $G:E_1\text{-}\mathbf{RMod}_{\mathscr{A}(S^{n-1}\times\mathbb{R})}\to\mathbf{Fac}^{lc}_{\mathbb{R}^n_*}\times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}}\{\mathscr{A}\}$ (which will be an inverse of N_*) as follows. By Proposition 28 we have an equivalence

$$E_1\text{-}\mathbf{RMod}_{\mathscr{A}(S^{n-1}\times\mathbb{R})}\cong\mathbf{Fac}^{lc}_{[0,+\infty)}\times_{\mathbf{Fac}^{lc}_{(0,+\infty)}}\{N_*(\mathscr{A})\}.$$

It is enough to define G as a functor from $\mathbf{Fac}^{lc}_{[0,+\infty)}\times_{\mathbf{Fac}^{lc}_{(0,+\infty)}}\{N_*(\mathscr{A})\}$ to locally constant $\mathscr{U}$-factorization algebras, where $\mathscr{U}$ is a (stable by finite intersections) factorizing basis of $\mathbb{R}^n$ (by Proposition 17). We choose $\mathscr{U}$ to be the basis consisting of all euclidean disks centered at 0 and all convex open subsets not containing 0. Let $\mathscr{R}\in\mathbf{Fac}^{lc}_{[0,+\infty)}\times_{\mathbf{Fac}^{lc}_{(0,+\infty)}}\{N_*(\mathscr{A})\}$. If $U\in\mathscr{U}$ does not contains 0, then we set $G(\mathscr{R})(U)=\mathscr{A}(U)$ and structure maps on open sets in $\mathscr{U}$ not containing 0 to be the one of $\mathscr{A}$; this defines a locally constant factorization algebra on $\mathbb{R}^n\setminus\{0\}$ since $\mathscr{A}$ does.

We denote $D(0,r)$ the euclidean disk of radius $r>0$ and set $G(\mathscr{R})(D(0,r))=\mathscr{R}([0,\varepsilon))$. Let $D(0,r),U_1,\dots,U_i$ be pairwise subsets of $\mathscr{U}$ which are sub-sets of an euclidean disk $D(0,s)$. Then, $U_1,\dots,U_i$ lies in $S^{n-1}\times(r,s)$. Denoting respectively $\rho^{\mathscr{A}}$, $\rho^{\mathscr{R}}$ the structure maps of the factorization algebras $\mathscr{A}\in\mathbf{Fac}^{lc}_{S^{n-1}\times(0,+\infty)}$ and $\mathscr{R}\in\mathbf{Fac}^{lc}_{[0,+\infty)}$, we have the following composition

$$\begin{aligned}G(\mathscr{R})\big(D(0,r)\big)\otimes G(\mathscr{R})(U_1)\otimes\cdots\otimes G(\mathscr{R})(U_i)&\cong\mathscr{R}\big([0,r)\big)\otimes\mathscr{A}(U_1)\otimes\cdots\otimes\mathscr{A}(U_i)\\ \xrightarrow{id\otimes\rho^A_{U_1,\dots,U_i,S^{n-1}\times(r,s)}}\ \mathscr{R}\big([0,r)\big)\otimes\mathscr{A}\big(S^{n-1}\times(r,s)\big)&\cong\mathscr{R}\big([0,r)\big)\otimes N_*(\mathscr{A})\big((r,s)\big)\\ \xrightarrow{\rho^{\mathscr{R}}_{[0,r),(r,s),[0,s)}}\ \mathscr{R}\big([0,s)\big)&=G(\mathscr{R})\big(D(0,s)\big).\end{aligned}\tag{59}$$

The maps (59) together with the structure maps of $\mathscr{A}_{|\mathbb{R}^n\setminus\{0\}}\cong\mathscr{R}_{|\mathbb{R}^n\setminus\{0\}}$ define the structure of a $\mathscr{U}$-factorization algebra since $\mathscr{R}$ and $\mathscr{A}$ are factorization algebras.

The maps $G(\mathscr{R})\big(D(0,r)\big) \to G(\mathscr{R})\big(D(0,s)\big)$ are quasi-isomorphisms since $\mathscr{R}$ is locally constant. Since the maps (59) only depend on the structure maps of $\mathscr{R}$ and $\mathscr{A}$, the rule $\mathscr{R} \mapsto G(\mathscr{R})$ extends into a functor

$$G : E_1\text{-}\mathbf{RMod}_{\mathscr{A}(S^{n-1}\times\mathbb{R})} \cong \mathbf{Fac}^{lc}_{[0,+\infty)} \times_{\mathbf{Fac}^{lc}_{(0,+\infty)}} \{N_*(\mathscr{A})\} \to \mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{\mathscr{A}\}.$$

In order to check that $N_* \circ G$ is equivalent to the identity functor of $E_1\text{-}\mathbf{RMod}_{\mathscr{A}(S^{n-1}\times\mathbb{R})}$ it is sufficient to check it on the basis of opens of $[0,+\infty)$ given by the open intervals and the half-closed intervals $[0,s)$ for which the result follows from the definition of the maps (59). Similarly, one can check that $G \circ N_*$ is equivalent to the identity of $\mathbf{Fac}^{lc}_{\mathbb{R}^n_*} \times_{\mathbf{Fac}^{lc}_{\mathbb{R}^n\setminus\{0\}}} \{\mathscr{A}\}$ by checking it on the open cover $\mathscr{U}$. $\square$

10 Appendix

In this appendix, we briefly collect several notions and results about ∞-categories and (∞-)operads and in particular the E_n-operad and its algebras and their modules.

10.1 ∞-category overview

There are several equivalent (see [Be1]) notions of (symmetric monoidal) ∞-categories and the reader shall feel free to use its favorite ones in these notes though we choose

Definition 29. In this paper, an ∞-category means a complete Segal spaces [R, L2].

Other appropriate models[82] are given by Segal category [HS, TV1] or Joyal quasi-categories [L1]. Almost all ∞-categories in these notes arise as some (derived) topological (or simplicial or dg) categories or localization of a category with weak equivalences. They carry along derived functors (such as derived homomorphisms) lifting the usual derived functors of usual derived categories. We recall below (examples 59 and 58) how to go from a model or topological category to an ∞-category.

Following [R, L2], a ***Segal space*** is a functor $X_\bullet : \Delta^{op} \to Top$, that is a simplicial space[83], which is Reedy fibrant (see [H]) and satisfies the the condition that for every integers $n \geq 0$, the natural map (induced by the face maps)

$$X_n \longrightarrow X_1 \times_{X_0} \times X_1 \times_{X_0} \cdots \times_{X_0} X_1 \tag{60}$$

(where there is n copies of X_1) is a weak homotopy equivalence. [84]

[82] Depending on the context some models are more natural to use than others

[83] here a space can also mean a simplicial set and it is often technically easier to work in this setting

[84] Alternatively, one can work out an equivalent notion fo Segal spaces which forget about the Reedy fibrancy condition and replace condition (60) by the condition that the following natural map is is a weak homotopy equivalence:

$$X_n \longrightarrow holim\big(X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} \cdots \xleftarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1\big).$$

Associated to a Segal space $X_\bullet$ is a (discrete) category $ho(X_\bullet)$ with objects the points of X_0 and morphisms $ho(X_\bullet)(a,b) = \pi_0\big(\{a\} \times_{X_0} X_1 \times_{X_0} \{b\}\big)$. We *call $ho(X_\bullet)$ the homotopy category of $X_\bullet$.*

A Segal space $X_\bullet$ is *complete* if the canonical map $X_0 \to \mathrm{Iso}(X_1)$ is a weak equivalence, where $\mathrm{Iso}(X_1)$ is the subspace of X_1 consisting of maps f whose class $[f] \in ho(X_\bullet)$ is invertible.

There is a simplicial closed model category structure, denoted $\mathscr{S}e\mathscr{S}p$ on the category of simplicial spaces such that a fibrant object in $\mathscr{S}e\mathscr{S}p$ is precisely a Segal space. The category of simplicial spaces has another simplicial closed model structure, denoted $\mathscr{C}\mathscr{S}e\mathscr{S}p$, whose fibrant objects are precisely complete Segal spaces [R, Theorem 7.2]. Let $\mathbb{R} : \mathscr{S}e\mathscr{S}p \to \mathscr{S}e\mathscr{S}p$ be a fibrant replacement functor and $\widehat{\cdot}: SeSp \to \mathscr{C}\mathscr{S}e\mathscr{S}p$ be the completion functor that assigns to a Segal space $X_\bullet$ an equivalent complete Segal space $\widehat{X_\bullet}$. The composition $X_\bullet \mapsto \widehat{\mathbb{R}(X_\bullet)}$ gives a fibrant replacement functor $L_{\mathscr{C}\mathscr{S}e\mathscr{S}p}$ from simplicial spaces to complete Segal spaces.

Example 57 (Discrete categories). Let $\mathscr{C}$ be an ordinary category (which we also referred to as a discrete category since its Hom-spaces are discrete). Its nerve is a Segal space which is not complete in general. However, one can form its *classifying diagram*, abusively denoted $N(\mathscr{C})$ which is a complete Segal space [R]. This is the ∞-category associated to $\mathscr{C}$.

By definition, the classifying diagram is the simplicial space $n \mapsto \big(N(\mathscr{C})\big)_n := N_\bullet(\mathrm{Iso}(\mathscr{C}^{[n]}))$ given by the ordinary nerves (or classifying spaces) of $\mathrm{Iso}(\mathscr{C}^{[n]})$ the subcategories of isomorphisms of the categories of n-composables arrows in $\mathscr{C}$.

Example 58 (**Topological category**). Let T be a topological (or simplicial) category. Its nerve $N_\bullet(T)$ is a simplicial space. Applying the complete Segal Space replacement functor we get the *∞-category $T_\infty := L_{\mathscr{C}\mathscr{S}e\mathscr{S}p}(N_\bullet(T))$ associated to T*.

Note that there is a model category structure on topological category which is Quillen equivalent[85] to $\mathscr{C}\mathscr{S}e\mathscr{S}p$, ([Be1]). The functor $T \mapsto T_\infty$ realizes this equivalence. If T is a discrete topological category (in other words an usual category viewed as a topological category), then T_∞ is equivalent to the ∞-category $N(T)$ associated to T in Example 57 ([Be2]). It is worth mentioning that the functor $T \mapsto T_\infty$ is the analogue for complete Segal spaces of the *homotopy coherent nerve* ([L1]) for quasi-categories, see [Be3] for a comparison.

Example 59 (**The ∞-category of a model category**). Let $\mathscr{M}$ be a model category and $\mathscr{W}$ be its subcategory of weak-equivalences. We denote $L^H(\mathscr{M},\mathscr{W})$ its *hammock localization*, see [DK]. One of the main property of $L^H(\mathscr{M},\mathscr{W})$ is that it is a simplicial category and that the (usual) category $\pi_0(L^H(\mathscr{M},\mathscr{W}))$ is the homotopy category of $\mathscr{M}$. Further, every weak equivalence has a (weak) inverse in $L^H(\mathscr{M},\mathscr{W})$. If $\mathscr{M}$ is a simplicial model category, then for every pair (x,y) of

[85] more precisely there is a zigzag of Quillen equivalences in between them; zigzag which goes through the model category structure of Segal categories.

objects the simplicial set of morphisms $Hom_{L^H(\mathscr{M},\mathscr{W})}(x,y)$ is naturally homotopy equivalent to the function complex $\mathrm{Map}_{\mathscr{M}}(x,y)$.

By construction, the nerve $N_\bullet(L^H(\mathscr{M},\mathscr{W}))$ is a simplicial space. Applying the complete Segal Space replacement functor we get

Proposition 49. *[Be1]The simplicial space $\boldsymbol{L}_\infty(\mathscr{M}) := L_{\mathscr{C}\mathscr{S}e\mathscr{S}p}(N_\bullet(L^H(\mathscr{M},\mathscr{W})))$ is a complete Segal space, which is the ∞-category associated to $\mathscr{M}$.*

Note that the above construction extends to any category with weak equivalences.

Also, the *limit and colimit* in the ∞-category $\boldsymbol{L}_\infty(\mathscr{M})$ associated to a closed model category $\mathscr{M}$ can be computed by the *homotopy limit and homotopy colimit* in $\mathscr{M}$, that is by using fibrant and cofibrant resolutions. The same is true for derived functors. For instance a right Quillen functor $f:\mathscr{M}\to\mathscr{N}$ has a lift $\mathbb{L}f:\boldsymbol{L}_\infty(\mathscr{M})\to\boldsymbol{L}_\infty(\mathscr{N})$.

Remark 35. There are other functors that yields a complete Segal space out of a model category. For instance, one can generalize the construction of Example 57. For $\mathscr{M}$ a model category and any integer n, let $\mathscr{M}^{[n]}$ be the (model) category of n-composables morphisms, that is the category of functors from the poset $[n]$ to $\mathscr{M}$. The *classification diagram* of $\mathscr{M}$ is the simplicial space $n\mapsto N_\bullet(\mathscr{W}e(\mathscr{M}^{[n]}))$ where $\mathscr{W}e(\mathscr{M}^{[n]})$ is the subcategory of weak equivalences of $\mathscr{M}^{[n]}$. Then taking a *Reedy* fibrant replacement yields another complete Segal space $N_\bullet(\mathscr{W}e(\mathscr{M}^{[n]}))^f$ ([Be2, Theorem 6.2], [R, Theorem 8.3]). It is known that the Segal space $N_\bullet(\mathscr{W}e(\mathscr{M}^{[n]}))^f$ is equivalent to $\boldsymbol{L}_\infty(\mathscr{M}) = L_{\mathscr{C}\mathscr{S}e\mathscr{S}p}(N_\bullet(L^H(\mathscr{M},\mathscr{W})))$ ([Be2]).

Definition 30. The *objects* of an ∞-category $\boldsymbol{C}$ are the points of $\boldsymbol{C}_0$. By definition, an ∞-category has a *space* (and not just a set) of morphisms

$$\mathrm{Map}_{\boldsymbol{C}}(x,y) := \{x\}\times^h_{\boldsymbol{C}_0}\boldsymbol{C}_1\times^h_{\boldsymbol{C}_0}\{y\}$$

between two objects x and y. A morphism $f\in\mathrm{Map}_{\boldsymbol{C}}(x,y)$ *is called an equivalence* if its image $[f]\in\mathrm{Map}_{ho(\boldsymbol{C})}(x,y)$ is an isomorphism.

From Example 59, we get an ∞-*category of* ∞-*categories*, denoted **∞-Cat**, whose morphisms are called ∞-*functors* (or just functors for short). An equivalence of ∞-categories is an equivalence in **∞-Cat** in the sense of Definition 30.

The model category of complete Segal spaces is cartesian closed [R] hence so is the ∞-category **∞-Cat**. In particular, given two ∞-categories $\mathscr{C}$, $\mathscr{D}$ we have an ∞-category $\mathbf{Fun}(\mathscr{C},\mathscr{D})$ of functors[86] from $\mathscr{C}$ to $\mathscr{D}$. There is an natural weak equivalence of spaces:

$$\mathrm{Map}_{\infty\text{-}\mathbf{Cat}}\big(\mathscr{B},\mathbf{Fun}(\mathscr{C},\mathscr{D})\big)\xrightarrow{\simeq}\mathrm{Map}_{\infty\text{-}\mathbf{Cat}}\big(\mathscr{B}\times\mathscr{C},\mathscr{D}\big). \tag{61}$$

[86]computed from the Hom-space in the category of simplicial spaces using the fibrant replacement functor $L_{\mathscr{C}\mathscr{S}e\mathscr{S}p}$

Remark 36 (the case of simplicial model categories). When $\mathscr{M}$ is a simplicial closed model category, there are natural equivalences ([R]) of spaces

$$\mathrm{Map}_{\boldsymbol{L}_\infty(\mathscr{M})}(x,y) \cong \mathrm{Map}_{\left(L^H(\mathscr{M},\mathscr{W})\right)_\infty}(x,y) \cong \mathrm{Map}_{L^H(\mathscr{M},\mathscr{W})}(x,y) \cong \mathrm{Map}_{\mathscr{M}}(x,y)$$

where the right hand side is the function complex of $\mathscr{M}$ and x,y two objects. The first two equivalences also hold for general model categories ([Be1]). In particular, the two constructions of an ∞-category associated to a simplicial model category, either viewed as topological category as in Example 58, or as a model category as in Example 59, are equivalent:

Proposition 50. *Let $\mathscr{M}$ be a simplicial model category. Then $\mathscr{M}_\infty \cong \boldsymbol{L}_\infty(\mathscr{M})$.*

Let $\boldsymbol{I}$ be the ∞-category associated to the trivial category $\Delta^1 = \{0\} \to \{1\}$ which has two objects and only one non-trivial morphism. We have two maps $i_0, i_1 : \{pt\} \to \boldsymbol{I}$ from the trivial category to $\boldsymbol{I}$ which respectively maps the object pt to 0 and 1.

Definition 31. Let A be an object of an ∞-category $\mathscr{C}$. The ∞-*category* $\mathscr{C}_A$ of objects over A is the pullback

$$\begin{tikzcd} \mathscr{C}_A \arrow[r] \arrow[d] & \mathbf{Fun}(\boldsymbol{I},\mathscr{C}) \arrow[d, "i_1^*"] \\ \{A\} \arrow[r] & \mathscr{C}. \end{tikzcd}$$

The ∞-*category* ${}_A\mathscr{C}$ of objects under A is the pullback

$$\begin{tikzcd} \mathscr{C}_A \arrow[r] \arrow[d] & \mathbf{Fun}(\boldsymbol{I},\mathscr{C}) \arrow[d, "i_0^*"] \\ \{A\} \arrow[r] & \mathscr{C}. \end{tikzcd}$$

Informally, the ∞-category $\mathscr{C}_A$ is just the category of objects $B \in \mathscr{C}$ equipped with a map $f : B \to A$ in $\mathscr{C}$.

There is a notion of symmetric monoidal ∞-category generalizing the classical notion for discrete categories. There are several equivalent way to define this notion, see [L3, L1, TV3] for details. Let Γ be the skeleton of the category Fin_* of finite pointed sets, that is the subcategory spanned by the objects $n_+ := \{0,\ldots,n\}$, $n \in \mathbb{N}$. For $i = 1\ldots n$, let $s_i : n_+ \to 1_+$ be the map sending i to 1 and everything else to 0.

Definition 32. A symmetric monoidal ∞-category is a functor $T \in \mathbf{Fun}(\Gamma, \infty\text{-}\mathbf{Cat})$ such that the canonical map $T(n_+) \xrightarrow{\prod_{i=0}^n s_i} \left(T(1_+)\right)^n$ is an equivalence. The full sub-category of $\mathbf{Fun}(\Gamma, \infty\text{-}\mathbf{Cat})$ spanned by the symmetric monoidal categories is denoted ∞-$\mathbf{Cat}^{\otimes}$. Its morphisms are called *symmetric monoidal functors*. Further ∞-$\mathbf{Cat}^{\otimes}$ is enriched over ∞-$\mathbf{Cat}$.

A symmetric monoidal category $T : \Gamma \to \infty$-$\mathbf{Cat}$ will usually be denoted as $(\boldsymbol{T}, \otimes)$ where $\boldsymbol{T} := T(1)$. If $C : \Gamma \to \infty$-$\mathbf{Cat}$ and $D : \Gamma \to \infty$-$\mathbf{Cat}$ are symmetric monoidal categories, we will *denote* $\boldsymbol{Fun}^{\otimes}(\mathscr{C}, \mathscr{D})$ *the* ∞*-categories of symmetric monoidal functors*.

Equivalently, a symmetric monoidal category is an E_∞-algebra object in the ∞-category ∞-$\mathbf{Cat}$. An (∞-)category with finite coproducts has a canonical structure of symmetric monoidal ∞-category and so does a category with finite products.

Example 60 (The ∞-category Top). Applying the above procedure(Example 59) to the model category of simplicial sets, we obtain the ∞-category $sSet$. Similarly, the model category of topological spaces yields the ∞-category Top of topological spaces. By Remark 36, we can also apply Example 58 to the standard enrichment of these categories into topological (or simplicial) categories to construct (equivalent) models of $sSet$ and Top.

Since the model categories $sSet$ and Top are Quillen equivalent [GJ, H], their associated ∞-categories are equivalent. The left and right equivalences $|-| : sSet \underset{\sim}{\overset{\sim}{\rightleftarrows}} Top : \Delta_\bullet(-)$ are respectively induced by the singular set and geometric realization functors. The *disjoint union of simplicial sets and topological spaces make sSet and Top into symmetric monoidal ∞-categories*.

The above analysis also holds for the pointed versions $sSet_*$ and Top_* of the above ∞-categories (using the model categories of these pointed versions [H]).

Example 61 (Chain complexes). The model category of (unbounded) chain complexes over k (say with the projective model structure) [H] yields the ∞-category of chain complexes $Chain(k)$(Example 59). The mapping space between two chain complex P_*, Q_* is equivalent to the geometric realization of the simplicial set $n \mapsto \mathrm{Hom}_{Chain(k)}(P_* \otimes C_*(\Delta^n), Q_*)$ where $\mathrm{Hom}_{Chain(k)}$ stands for morphisms of chain complexes. It follows from Proposition 50, that one can also use Example 58 applied to the category of chain complexes endowed with the above topological space of morphisms to define $Chain(k)$. In particular a chain homotopy between two chain maps $f, g \in \mathrm{Map}_{Chain(k)}(P_*, Q_*)$ is a path in $\mathrm{Map}_{Chain(k)}(P_*, Q_*)$.

In fact, $ho(Chain(k)) \cong \mathscr{D}(k)$ is the usual *derived category of k-modules*. The (derived) tensor product over k yields a symmetric monoidal structure to $Chain(k)$ which will usually simply denote by $\otimes$. Note that $Chain(k)$ *is enriched over itself*, that is, for any $P_*, Q_* \in Chain(k)$, there is an object $\mathbb{R}\mathrm{Hom}_k(P_*, Q_*) \in Chain(k)$ together with an adjunction

$$\mathrm{Map}_{Chain(k)}\big(P_* \otimes Q_*, R_*\big) \cong \mathrm{Map}_{Chain(k)}\big(P_*, \mathbb{R}\mathrm{Hom}_k(Q_*, R_*)\big).$$

The interested reader can refer to [GH, L3] for details of ∞-categories enriched over ∞-categories and to [GM, DS] for model categories enriched over symmetric monoidal closed model categories (which is the case of the category of chain complexes).

Example 62. In characteristic zero, there is a standard closed model category structure on the category of commutative differential graded algebras (CDGA for short), see [Hi, Theorem 4.1.1]. Its fibrations are epimorphisms and (weak) equivalences are quasi-isomorphisms (of CDGAs). We thus get the ∞-category $CDGA$ of CDGAs. The category $CDGA$ also has a monoidal structure given by the (derived) tensor product (over k) of differential graded commutative algebras, which makes $CDGA$ a symmetric monoidal model category. Given $A, B \in CDGA$, the mapping space $\mathrm{Map}_{CDGA}(A,B)$ is the (geometric realization of the) simplicial set of maps $[n] \mapsto Hom_{\text{dg-Algebras}}(A, B\otimes \Omega^*(\Delta^n))$ (where $\Omega^*(\Delta^n)$ is the CDGA of forms on the n-dimensional standard simplex and $\mathrm{Hom}_{\text{dg-Algebras}}$ is the module of differential graded algebras maps). It has thus a canonical enrichment over chain complexes.

The model categories of left modules and commutative algebras over a CDGA A yield the ∞-categories E_1-$\mathbf{LMod}_A$ and $CDGA_A$. The base change functor lifts to a functor of ∞-categories. Further, if $f : A \to B$ is a weak equivalence, the natural functor $f_* : E_1\text{-}\mathbf{LMod}_B \to: E_1\text{-}\mathbf{LMod}_A$ induces an equivalence $E_1\text{-}\mathbf{LMod}_B \overset{\sim}{\to} E_1\text{-}\mathbf{LMod}_A$ of ∞-categories since it is induced by a Quillen equivalence.

Moreover, if $f : A \to B$ is a morphism of CDGAs, we get a natural functor $f^* : E_1\text{-}\mathbf{LMod}_A \to E_1\text{-}\mathbf{LMod}_B, M \mapsto M \otimes^{\mathbb{L}}_A B$, which is an equivalence of ∞-categories when f is a quasi-isomorphism, and is a (weak) inverse of f_* (see [TV2] or [KM]). The same results applies to monoids in E_1-$\mathbf{LMod}_A$ that is to the categories of commutative differential graded A-algebras.

10.2 E_n-algebras and E_n-modules

The classical definition of an E_n-algebra (in chain complexes) is an algebra over any E_n-operad in chain complexes, that is an operad weakly homotopy equivalent to the chains on the little (n-dimensional) cubes operad $(\mathrm{Cube}_n(r))_{r\geq 0}$ ([Ma]). Here

$$\mathrm{Cube}_n(r) := \mathrm{Rect}\Big(\coprod_{i=1}^{r}(0,1)^n, (0,1)^n\Big)$$

is the space of rectilinear embeddings of r-many disjoint copies of the unit open cube in itself. It is topologized as the subspace of the space of all continuous maps. By a *rectilinear embedding*, we mean a composition of a translation and dilatations in the direction given by a vector of the canonical basis of $\mathbb{R}^n$. In other words, $\mathrm{Cube}_n(r)$ is the configuration space of r-many disjoint open rectangles[87] parallel to the axes lying in the unit open cube. The operad structure $\mathrm{Cube}_n(r) \times \mathrm{Cube}_n(k_1)\times\cdots\times \mathrm{Cube}_n(k_r) \to \mathrm{Cube}_n(k_1+\cdots+k_r)$ is simply given by composition of embeddings.

An *E_n-algebra in chain complexes* is thus a chain complex A together with chain maps $\gamma_r : C_*(\mathrm{Cube}_n(r))\otimes A^{\otimes r} \longrightarrow A$ compatible with the composition of operads [BV, Ma, Fr5]. By definition of the operad Cube_n, we are only considering (weakly) unital versions of E_n-algebras.

[87] more precisely rectangular parallelepiped in dimension bigger than 2

The model category of E_n-algebras gives rises to *the ∞-category E_n-**Alg*** of E_n-algebras in the symmetric monoidal ∞-category of differential graded k-modules. The symmetric structure of $Chain(k)$ lifts to a a symmetric monoidal structure on $(E_n\textbf{-Alg},\otimes)$ given by the tensor product of the underlying chain complexes[88].

One can extend the above notion to define E_n-algebras with coefficient in any symmetric monoidal ∞-category following [L3]. One way is to rewrite it in terms of symmetric monoidal functor as follows. Any topological (resp. simplicial) operad $\mathscr{O}$ defines a symmetric monoidal category, denoted **O**, fibered over the category of pointed finite sets Fin_*. This category **O** has the finite sets for objects. For any sets $n_+ := \{0,\dots n\}$, $m_+ := \{0,\dots,m\}$ (with base point 0), its morphism space $\mathbf{O}(n_+,m_+)$ (from n_+ to m_+) is the disjoint union $\coprod_{f:n_+\to m_+}\prod_{i\in m_+}\mathscr{O}((f^{-1}(i))_+)$ and the composition is induced by the operadic structure. The rule $n_+\otimes m_+ = (n+m)_+$ makes canonically **O** into a symmetric monoidal topological (resp. simplicial) category. We abusively denote **O** its associated ∞-category. Note that this construction extends to colored operad and is a special case of an ∞-operad [89].

Then, if $(\mathscr{C},\otimes)$ is a symmetric monoidal ∞-category, *a $\mathscr{O}$-algebra in $\mathscr{C}$ is a symmetric monoidal functor $A\in \textbf{\textit{Fun}}^{\otimes}(\mathbf{O},\mathscr{C})$*. We call $A(1_+)$ the underlying algebra object of A and we usually denote it simply by A.

Definition 33. [L3, F1] Let $(\mathscr{C},\otimes)$ be symmetric monoidal (∞-)category. The ∞-category of *E_n-algebras with values in $\mathscr{C}$* is

$$E_n\textbf{-Alg}(\mathscr{C}) := \mathbf{Fun}^{\otimes}(\mathbf{Cube}_n,\mathscr{C}).$$

Similarly $E_n\textbf{-coAlg}(\mathscr{C}) := \mathbf{Fun}^{\otimes}(\mathbf{Cube}_n,\mathscr{C}^{op})$ is the category of *E_n-coalgebras in $\mathscr{C}$*. We denote $Map_{E_n\textbf{-}\textbf{\textit{Alg}}}(A,B)$ *the mapping space of E_n-algebras maps from A to B*.

Note that Definition 33 is a definition of categories of (weakly) *(co)unital E_n*-coalgebra objects.

One has an equivalence $E_n\textbf{-Alg}\cong E_n\textbf{-Alg}(Chain(k))$ of symmetric monoidal ∞-categories (see [L3, F2]) where E_n**-Alg** is the ∞-category associated to algebras over the operad Cube_n considered above. It is clear from the above definition that any (∞-)operad $\mathbb{E}_n$ weakly homotopy equivalent (as an operad) to Cube_n gives rise to an equivalent ∞-category of algebra. In particular, the inclusion of rectilinear embeddings into all framed embeddings gives us an alternative definition for E_n-algebras:

Proposition 51 ([L3])**.** *Let Disk_n^{fr} be the category with objects the integers and morphism the spaces $\mathrm{Disk}_n^{fr}(k,\ell) := \mathrm{Emb}^{fr}(\coprod_k \mathbb{R}^n,\coprod_\ell \mathbb{R}^n)$ of framed embeddings of k disjoint copies of a disk $\mathbb{R}^n$ into ℓ such copies (see Example 12). The natural map $\textbf{\textit{Fun}}^{\otimes}(\mathrm{Disk}_n^{fr}, Chain(k)) \xrightarrow{\simeq} E_n\textbf{-}\textbf{\textit{Alg}}$ is an equivalence.*

[88] other possible models for the symmetric monoidal ∞-category $(E_n\textbf{-Alg},\otimes)$ are given by algebraic Hopf operads such as those arising from the filtration of the Barratt-Eccles operad in [BF]

[89] *An ∞-operad $\mathscr{O}^{\otimes}$* is a ∞-category together with a functor $\mathscr{O}^{\otimes}\to N(Fin_*)$ satisfying a list of axioms, see [L3]. It is to colored topological operads what ∞-categories are to topological categories.

Example 63. (Iterated loop spaces) The standard examples[90] of E_n-algebras are given by iterated loop spaces. If X is a pointed space, we denote $\Omega^n(X) := \mathrm{Map}_*(S^n, X)$ the set of all pointed maps from $S^n \cong I^n/\partial I^n$ to X, equipped with the compact-open topology. The pinching map (13) $pinch : \mathrm{Cube}_n(r) \times S^n \longrightarrow \bigvee_{i=1\dots r} S^n$ induces an E_n-algebra structure (in $(Top, \times)$) given by

$$\mathrm{Cube}_n(r) \times \big(\Omega^n(X)\big)^r \cong \mathrm{Cube}_n(r) \times \mathrm{Map}_*(\bigvee_{i=1\dots r} S^n, X) \stackrel{pinch^*}{\longrightarrow} \Omega^n(X).$$

Since the construction is functorial in X, the singular chain complex $C_*(\Omega^n(X))$ is also an E_n-algebra in chain complexes, and further this structure is compatible with the E_∞-coalgebra structure of $C_*(\Omega^n(X))$ (from Example 65). Similarly, the singular cochain complex $C^*(\Omega^n(X))$ is an E_n-coalgebra in a way compatible with its E_∞-algebra structure; that is an object of E_n**-coAlg**$(E_\infty$**-Alg**$)$.

Example 64 (P_n-algebras). A standard result of Cohen [Co] shows that, for $n \geq 2$, the homology of an E_n-algebra is a P_n-algebra (also see [Si, Fr5]). *A P_n-algebra* is a graded vector space A endowed with a degree 0 multiplication with unit which makes A a graded commutative algebra, and a (cohomological) degree 1-n operation $[-,-]$ which makes $A[1-n]$ a graded Lie algebra. These operations are also required to satisfy the Leibniz rule $[a \cdot b, c] = a[b,c] + (-1)^{|b|(|c|+1-n)}[a,c] \cdot b$.

For $n = 1$, P_n-algebras are just usual Poisson algebras while for $n = 2$, they are Gerstenhaber algebras.

In characteristic 0, the operad Cube_n is formal, thus equivalent as an operad to the operad governing P_n-algebras (for $n \geq 2$). It follows that P_n-algebras gives rise to E_n-algebras in that case, that is there is a functor[91] P_n**-Alg** $\to E_n$**-Alg**.

There are natural maps (sometimes called the stabilization functors)

$$\mathrm{Cube}_0 \longrightarrow \mathrm{Cube}_1 \longrightarrow \mathrm{Cube}_2 \longrightarrow \cdots \tag{62}$$

(induced by taking products of cubes with the interval $(0,1)$). It is a fact ([Ma, L3]) that the colimit of this diagram, denoted by $\mathbb{E}_\infty$ is equivalent to the commutative operad *Com* (whose associated symmetric monoidal ∞-category is $\mathbf{Fin}_*$).

Definition 34. The(∞-) category of E_∞-algebras with value in $\mathscr{C}$ is $E_\infty\text{-}\mathbf{Alg}(\mathscr{C}) := \mathbf{Fun}^{\otimes}(\mathbf{Cube}_\infty, \mathscr{C})$. It is simply denoted E_∞**-Alg** if $(\mathscr{C}, \otimes) = (Chain(k), \otimes)$. Similarly, the category of E_∞-coalgebras. is $E_\infty\text{-coAlg} := \mathbf{Fun}^{\otimes}(\mathbf{Cube}_\infty, \mathscr{C}^{op})$.

Note that Definition 34 is a definition of (weakly) unital E_∞-algebras.

The category E_∞**-Alg** is (equivalent to) the ∞-category associated to the model category of $\mathbb{E}_\infty$-algebras for any E_∞-operad $\mathbb{E}_\infty$.

The natural map $\mathbf{Fun}^{\otimes}(\mathbf{Fin}_*, \mathscr{C}) \longrightarrow \mathbf{Fun}^{\otimes}(\mathbf{Cube}_\infty, \mathscr{C}) = E_\infty\text{-}\mathbf{Alg}$ is also an equivalence.

[90] May's recognition principle [Ma] actually asserts that any E_n-algebra in $(Top, \times)$ which is group-like is homotopy equivalent to such an iterated loop space

[91] which is *not* canonical, see [T]

For any $n \in \mathbb{N} - \{0\} \cup \{+\infty\}$, the map $\mathrm{Cube}_1 \to \mathrm{Cube}_n$ (from the nested sequence (62)) induces a *functor* E_n**-Alg** $\longrightarrow E_1$**-Alg** which associates to an E_n-algebra its underlying E_1-algebra structure.

Example 65 (Singular (co)chains). Let X be a topological space. Its *singular cochain complex* $C^*(X)$ *has a natural structure of* E_∞*-algebra*, whose underlying E_1-structure is given by the usual (strictly associative) cup-product (for instance see [M2]). The *singular chains* $C_*(X)$ *have a natural structure of* E_∞*-coalgebra* which is the predual of $(C^*(X), \cup)$. There are similar constructions for simplicial sets $X_\bullet$ instead of spaces, see [BF]. We recall that $C^*(X)$ is the linear dual of the singular chain complex $C_*(X)$ with coefficient in k which is the geometric realization (in the ordinary category of chain complexes) of the simplicial k-module $k[\Delta_\bullet(X)]$ spanned by the singular set $\Delta_\bullet(X) := \{\Delta^\bullet \xrightarrow{f} X, f \text{ continuous}\}$. Here Δ^n is the standard n-dimensional simplex.

Remark 37. The mapping space $Map_{E_\infty\text{-}\mathbf{Alg}}(A,B)$ of two E_∞-algebras A, B (in the model category of E_∞-algebras) is the (geometric realization of the) simplicial set $[n] \mapsto Hom_{E_\infty\text{-}\mathbf{Alg}}\big(A, B \otimes C^*(\Delta^n)\big)$.

The ∞-category E_∞-Alg is enriched over $sSet$ (hence Top as well by Example 60) and has all (∞-)colimits. In particular, it is *tensored* over $sSet$, see [L1, L3] for details on tensored ∞-categories or [EKMM, MCSV] in the context of topologically enriched model categories. We recall that it means that there is a functor E_∞-**Alg** $\times\, sSet \to E_\infty$-**Alg**, denoted $(A, X_\bullet) \mapsto A \boxtimes X_\bullet$, together with natural equivalences

$$Map_{E_\infty\text{-}\mathbf{Alg}}\big(A \boxtimes X_\bullet, B\big) \;\cong\; Map_{sSet}\big(X_\bullet, Map_{E_\infty\text{-}\mathbf{Alg}}(A,B)\big).$$

To compute explicitly this tensor, it is useful to know the following proposition.

Proposition 52. *Let $(\mathscr{C}, \otimes)$ be a symmetric monoidal ∞-category. In the symmetric monoidal ∞-category E_∞-**Alg**$(\mathscr{C})$, the tensor product is a coproduct.*

For a proof see Proposition 3.2.4.7 of [L3] (or [KM, Corollary 3.4]); for $\mathscr{C} = Chain(k)$, this essentially follows from the observation that an E_∞-algebra is a commutative monoid in $(Chain(k), \otimes)$, see [L3] or [KM, Section 5.3]. In particular, Proposition 52 implies that, for any finite set I, $A^{\otimes I}$ has a natural structure of E_∞-algebra.

Modules over E_n-algebras. In this paragraph, we give a brief account of various categories of modules over E_n-algebras. Note that by definition (see below), the categories we considered are categories of *pointed* modules. Roughly, an A-modules M being pointed means it is equipped with a map $A \to M$.

Let Fin, (resp. Fin_*) be the category of (resp. pointed) finite sets. There is a forgetful functor $Fin_* \to Fin$ forgetting which point is the base point. There is also a functor $Fin \to Fin_*$ which adds an extra point called the base point. We write **Fin**, $\mathbf{Fin}_*$ for the associated ∞-categories (see Example 57). Following [L3, F1], if $\mathscr{O}$ is a (coherent) operad, *the ∞-category $\mathscr{O}$-**Mod**$_A$ of $\mathscr{O}$-modules*[92] *over an $\mathscr{O}$-algebra A* is the category of $\mathscr{O}$-linear functors $\mathscr{O}\text{-}\mathbf{Mod}_A := \mathrm{Map}_{\mathbf{O}}(\mathbf{O}_*, Chain(k))$ where $\mathbf{O}$ is the (∞-)category associated[93] to the operad $\mathscr{O}$ and $\mathbf{O}_* := \mathbf{O} \times_{\mathbf{Fin}} \mathbf{Fin}_*$ (also see [Fr1] for similar constructions in the model category setting of topological operads).

The categories $\mathscr{O}\text{-}\mathbf{Mod}_A$ for $A \in \mathscr{O}\text{-}\mathbf{Alg}$ assemble to form an ∞-category $\mathscr{O}\text{-}\mathbf{Mod}$ describing pairs consisting of an $\mathscr{O}$-algebra and a module over it. More precisely, there is an natural fibration $\pi_{\mathscr{O}} : \mathscr{O}\text{-}\mathbf{Mod} \longrightarrow \mathscr{O}\text{-}\mathbf{Alg}$ whose fiber at $A \in \mathscr{O}\text{-}\mathbf{Alg}$ is $\mathscr{O}\text{-}\mathbf{Mod}_A$. When $\mathscr{O}$ is an $\mathbb{E}_n$-operad (that is an operad equivalent to Cube_n), we simply write E_n instead of $\mathscr{O}$:

Definition 35. Let A be an E_n-algebra (in $Chain(k)$). We denote $E_n\text{-}\mathbf{Mod}_A$ the ∞-category of (pointed) E_n-modules over A. Since $Chain(k)$ is bicomplete and enriched over itself, $E_n\text{-}\mathbf{Mod}_A$ is naturally enriched over $Chain(k)$ as well.

We *denote*[94] $\mathbb{R}Hom_A^{E_n}(M,N) \in Chain(k)$ *the enriched mapping space of morphisms of E_n-modules over A*. Note that if $\mathbb{E}_n$ is a cofibrant E_n-operad and further M, N are modules over an $\mathbb{E}_n$-algebra A, then $\mathbb{R}Hom_A^{E_n}(M,N)$ is computed by $Hom_{\mathrm{Mod}_A^{\mathbb{E}_n}}(Q(M),R(N))$. Here $Q(M)$ is a cofibrant replacement of M and $R(N)$ a fibrant replacement of N in the model category $\mathrm{Mod}_A^{\mathbb{E}_n}$ of modules over the $\mathbb{E}_n$-algebra A. In particular, $\mathbb{R}Hom_A^{E_n}(M,N) \cong Hom_{E_n\text{-}\mathbf{Mod}_A}(M,N)$. This follows from the fact that $E_n\text{-}\mathbf{Mod}_A$ is equivalent to the ∞-category associated to the model category $\mathrm{Mod}_A^{\mathbb{E}_n}$.

If $(\mathscr{C},\otimes)$ is a symmetric monoidal (∞-)category and $A \in E_n\text{-}\mathbf{Alg}$, then we *denote E_n-**Mod**$_A(\mathscr{C})$ the ∞-category of E_n-modules over A (in $\mathscr{C}$).*

We *denote respectively E_n-**Mod*** the ∞-category of all E_n-modules in $Chain(k)$ *and E_n-**Mod**$(\mathscr{C})$* the ∞-category of all E_n-modules in $(\mathscr{C},\otimes)$.

By definition, the canonical functor[95] $\pi_{E_n} : E_n\text{-}\mathbf{Mod}(\mathscr{C}) \to E_n\text{-}\mathbf{Alg}(\mathscr{C})$ gives rise, for any E_n-algebra A, to a (homotopy) pullback square:

$$\begin{tikzcd}
E_n\text{-}\mathbf{Mod}_A(\mathscr{C}) \arrow[r] \arrow[d] & E_n\text{-}\mathbf{Mod}(\mathscr{C}) \arrow[d, "\pi_{E_n}"] \\
\{A\} \arrow[r] & E_n\text{-}\mathbf{Alg}(\mathscr{C})
\end{tikzcd} \tag{63}$$

Note that the functor π_{E_n} is monoidal.

We also have a canonical functor $can : E_n\text{-}\mathbf{Alg} \to E_n\text{-}\mathbf{Mod}$ induced by the tautological module structure that any algebra has over itself.

[92] in $Chain(k)$. Of course, similar construction hold with $Chain(k)$ replaced by a symmetric monoidal ∞-category

[93] in the paragraph above Definition 33

[94] The $\mathbb{R}$ in the notation is here to recall that this corresponds to a functor that can be computed as a derived functor associated to ordinary model categories using standard techniques of homological/homotopical algebras

[95] which essentially forget the module in the pair (A,M)

Example 66. If A is a differential graded algebra, E_1**-Mod**$_A$ is equivalent to the ∞-category of (pointed) A-bimodules. If A is a CDGA, E_∞**-Mod**$_A$ is equivalent to the ∞-category of (pointed) left A-modules.

Example 67 (**left and right modules**). If $n = 1$, we also have naturally defined ∞-categories of left and right modules over an E_1-algebra A (as well as ∞-categories of all right modules and left modules). They are the immediate generalization of the (∞-categories associated to the model) categories of pointed left and right differential graded modules over a differential graded associative unital algebra. We refer to [L3] for details.

Definition 36. We write respectively $E_1\textbf{-LMod}_A(\mathscr{C})$, $E_1\textbf{-RMod}_A(\mathscr{C})$, $E_1\textbf{-LMod}(\mathscr{C})$ and $E_1\textbf{-RMod}(\mathscr{C})$ for the ∞-categories of left modules over a fixed A, right modules over A, and all left modules and all right modules (with values in $(\mathscr{C}, \otimes)$).

If $\mathscr{C} = Chain(k)$, we simply write $E_1\textbf{-LMod}_A$, $E_1\textbf{-RMod}_A$, $E_1\textbf{-LMod}$, $E_1\textbf{-RMod}$. Further, we will *denote* $\mathbb{R}Hom_A^{left}(M,N) \in Chain(k)$ *the enriched mapping space of morphisms of left modules over A* (induced by the enrichment of $Chain(k)$). In particular $\mathbb{R}Hom_A^{left}(M,N) \cong Hom_{E_1\textbf{-LMod}_A}(M,N)$.

There are standard models for these categories. For instance, the category of right modules over an E_1-algebra can be obtained by considering a colored operad Cube_1^{right} obtained from the little interval operad Cube_1 as follows. Denote c, i the two colors. We define $\mathrm{Cube}_1^{right}(\{X_j\}_{j=1}^r, i) := \mathrm{Cube}_1(r)$ if all $X_j = i$. If $X_1 = c$ and all others $X_j = i$, we set $\mathrm{Cube}_1^{right}(\{X_j\}_{j=1}^r, c) := \mathrm{Rect}\big([0,1) \coprod \big(\coprod_{i=1}^r (0,1)\big), [0,1)\big)$ where Rect is the space of rectilinear embeddings (mapping 0 to itself). All other spaces of maps are empty. Then the ∞-category associated to the category of Cube_1^{right}-algebras is equivalent to $E_1\textbf{-RMod}$.

Let A be an E_1-algebra, then the usual tensor product of right and left A-modules has a canonical lift

$$-\underset{A}{\overset{\mathbb{L}}{\otimes}}- : E_1\textbf{-RMod}_A \times E_1\textbf{-LMod}_A \longrightarrow Chain(k)$$

which, for a differential graded associative algebra over a field k is computed by the two-sided Bar construction. There is a similar derived functor $E_1\textbf{-RMod}_A(\mathscr{C}) \times E_1\textbf{-LMod}_A(\mathscr{C}) \longrightarrow \mathscr{C}$, still denoted $(R,L) \mapsto R \otimes_A^{\mathbb{L}} L$, whenever $(\mathscr{C}, \otimes)$ is a symmetric monoidal ∞-category with geometric realization and such that $\otimes$ preserves geometric realization in both variables, see [L3]. There are (derived) adjunction

$$\mathrm{Map}_{E_1\textbf{-LMod}_A}\big(P_* \otimes L, N\big) \;\cong\; \mathrm{Map}_{Chain(k)}\big(P_*, \mathbb{R}\mathrm{Hom}_A^{left}(L,N)\big),$$

$$\mathrm{Map}_{Chain(k)}\big(R \underset{A}{\overset{\mathbb{L}}{\otimes}} L, N\big) \;\cong\; \mathrm{Map}_{E_1\textbf{-LMod}_A}\big(L, \mathbb{R}\mathrm{Hom}_k(R,N)\big)$$

which relates the tensor product with the enriched mapping spaces of modules.

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