Kolyvagin's conjecture and Iwasawa theory

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(joint work with Ashay Burungale, Francesc Castella, and Christopher Skinner)

Let $E/\mathbb{Q}$ be an elliptic curve and p be an odd prime of good ordinary reduction for E. In 1991 Kolyvagin conjectured that the system of cohomology classes derived from Heegner points on the p-adic Tate module of E over an imaginary quadratic field K is non-trivial (see [10]). We report on a joint work with A. Burungale, F. Castella and C. Skinner, in which we prove Kolyvagin's conjecture in the cases where an anticyclotomic Iwasawa Main Conjecture for E/K is known. In particular, we provide the first known cases when p is an Eisenstein prime.

Let K be a quadratic imaginary field of odd conductor $D_K \neq -3$ and such that p and all primes dividing the conductor N of E split in K. Assume also that $E(K)[p] = \{0\}$. For any integer $n \geq 1$, using the modular parametrisation of the elliptic curve, one can construct the Heegner points

$$P_n \in E(K[n])$$

defined over $K[n]$, the ring class field of K of conductor n. The Kummer map yields a class in $H^1(K[n], T/p^M)$, where $T = T_p(E)$ is the p-adic Tate module of E and $M \geq 0$. Applying the *Kolyvagin derivative* and using the assumption $E(K)[p] = \{0\}$ (which yields the surjectivity of the restriction map), one can build a collection of classes

$$\{\kappa_n \in H^1(K, T/p^{M(n)})\}_{n\in\mathcal{N}},$$

where $\mathcal{N}$ is the set of square-free products of inert primes ℓ coprime to p and N such that $M(\ell) = \min\{\mathrm{ord}_p(\ell+1), \mathrm{ord}_p(a_\ell)\} > 0$, $M(n) = \min\{M(\ell) : \ell \mid n\}$, and by convention $1 \in \mathcal{N}$ with $M(1) = \infty$.

Such collection of classes forms a *Kolyvagin system*; in particular if $\ell n, n \in \mathcal{N}$, one can show that:

- the restriction to the cohomology group $H^1(K_\ell, T/p^{M(n)})$ of the completion K_ℓ of K at the unique prime of K above ℓ of the class κ_n, denoted by $\mathrm{loc}_\ell(\kappa_n)$, lies in the unramified subspace $H^1_f(K_\ell, T/p^{M(n)}) \subset H^1(K_\ell, T/p^{M(n)})$;
- the image $\mathrm{loc}^s_\ell(\kappa_{n\ell})$ of $\mathrm{loc}_\ell(\kappa_{n\ell})$ in the singular quotient $H^1_s(K_\ell, T/p^{M(n\ell)})$ of $H^1(K_\ell, T/p^{M(n\ell)})$ is non trivial (unless $\mathrm{loc}_\ell(\kappa_{n\ell}) = 0$);
- the elements $\mathrm{loc}^s_\ell(\kappa_{n\ell})$ and $\mathrm{loc}_\ell(\kappa_n)$ have the same order (in the corresponding local cohomology groups with coefficients T/p^k for $k \leq M(n), M(n\ell)$.

These properties follow from the definition of the classes and the so called *norm relations* between the point $P_n, P_{n\ell}$. One also has an explicit description of the bottom class κ_1 which is simply obtained as the image via the Kummer map in $H^1(K,T)$ of the Heegner point $P_K := \mathrm{Tr}_{K[1]/K} P_1 \in E(K)$. The celebrated work of Gross–Zagier [6] gives the following:

$$\text{(GZ)} \qquad \kappa_1 \neq 0 \quad \Leftrightarrow \quad L'(E/K, 1) \neq 0.$$

In [10] Kolyvagin conjectured that even when the analytic rank of E/K is not one, the *system* $\{\kappa_n\}_n$ is non-trivial, namely:

Conjecture A (Kolyvagin)**.** *There exists $n \in \mathcal{N}$ such that $\kappa_n \neq 0$.*

The first major progress towards this conjecture is due to W. Zhang [16], who proved Kolyvagin's conjecture using level raising techniques when $p \neq 2, 3$ is a prime of good ordinary reduction for E under the assumption that

$$\text{(sur)} \qquad \bar{\rho}_E : G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{\mathbb{F}_p}(E[p]) \text{ is surjective}$$

and $\bar{\rho}_E$ satisfies certain ramification hypotheses. More recently, some of the hypotheses have been relaxed by N. Sweeting [13] using an ultrapatching method for bipartite Euler systems.

The main result of [1] shows that Iwasawa theory can be used to prove new results about Conjecture A:

Theorem 1. *Let $E/\mathbb{Q}$ be an elliptic curve with good ordinary reduction at $p > 2$ and K a quadratic imaginary field as above. Conjecture A holds true if the rational anticyclotomic main conjecture for E/K holds.*

In particular, Conjecture A holds true if

(i) *$E[p]^{ss} \simeq \mathbb{F}_p(\phi) \oplus \mathbb{F}_p(\phi^{-1}\omega)$ as $G_{\mathbb{Q}}$-module, where $\phi : G_{\mathbb{Q}} \to \mathbb{F}_p^\times$ is a character such that $\phi|_{G_p} \neq 1, \omega$, with G_p a decomposition group at p and ω the Teichmüller character;*

(ii) *$p > 3$ is such that $E[p]$ is an irreducible $G_{\mathbb{Q}}$-module.*

In case (i), we rely on the work [4] where we prove the (integral) anticyclotomic main conjecture at Eisenstein primes, strengthening our previous work [3] which applied only in the co-rank one case (when $\kappa_1 \neq 0$). Case (ii) follows from the main conjecture proved in [2]. Previous results interlacing Iwasawa theory and Kolyvagin's conjecture (in the case where (sur) holds) appeared in [9].

Strategy of the proof. We briefly mention which are the main ingredients in the proof. Firstly, using Heegner points of p-power conductor, one can construct a Λ-adic Kolyvagin system, where $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and Γ is the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension K_∞ of K. It is a collection of classes

$$\{\kappa_{\Lambda,n} \in H^1(K, T\hat{\otimes}\Lambda/p^{M(n)})\}_{n\in\mathcal{N}}.$$

For any character $\alpha : \Gamma \to \mathbb{Z}_p^\times$, we can consider the *specialization* of the Λ-adic system at α, giving classes $\kappa_n(\alpha) \in H^1(K, T\otimes\mathbb{Z}_p(\alpha)/p^{M(n)})$. The main ingredients of the proof will then be:

(a) The existence of a character $\alpha_m : \Gamma \to \mathbb{Z}_p^\times$ with $\alpha_m \equiv 1 \bmod p^m$ for $m \gg 0$ such that

$$\kappa_1(\alpha_m) \neq 0.$$

This follows from Mazur's conjecture, proved by Cornut and Vatsal.

(b) A suitable control theorem/explicit reciprocity law characterizing the objects appearing in the Iwasawa main conjecture *specialized* at α_m. Combining such results with the (integral) anticyclotomic main conjecture, we will get:

$$\text{length}_{\mathbb{Z}_p} Ш(E \otimes \alpha_m/K)[p^\infty] + \text{ord}_p \text{Tam}(E/K) = 2 \cdot \text{ind}(\kappa_1(\alpha_m)),$$

where $Ш(E \otimes \alpha_m/K)[p^\infty]$ denotes the Shafarevic-Tate group of $E[p^\infty] \otimes \alpha_m$, $\text{Tam}(E/K)$ denotes the product of the Tamagawa factors of E/K and $\text{ind}(\kappa_1(\alpha_m))$ denotes the maximal power of p dividing the class $\kappa_1(\alpha_m)$. If one only has the rational main conjecture, an extra term (independent on α_m) appears in the equality, causing no harm for the proof of the result.

(c) A Kolyvagin system bound (with an *error term* $\mathcal{E}$, which is non-zero only when (sur) does not hold, but crucially not depending on α_m) for the Selmer groups of $E[p^\infty] \otimes \alpha_m$ given any weak Kolyvagin system $\{\kappa'_n\}$ with $\kappa'_1 \neq 0$:

$$\text{length}_{\mathbb{Z}_p} Ш(E \otimes \alpha_m/K)[p^\infty] \leq 2 \cdot \text{ind}(\kappa'_1) + \mathcal{E}.$$

The idea of the proof is then to bound the *index of divisibility* of the Kolyvagin system showing that if p^t divides κ_n for every n, then, choosing a suitable α_m as in (a), p^t also divides $\kappa_n(\alpha_m)$ for every n and we can consider the weak Kolyvagin system $\{\kappa'_n := p^{-t}\kappa_n(\alpha_m)\}$. Applying the bound in (c) and the equality in (b), we get a bound on t.

Further results. The above strategy also yields a bound on the index of divisibility μ_∞ of the Kolyvagin system in terms of the Tamagawa factors of E. Let $m_r := \min\{\text{ind}(\kappa_n) : n \text{ is divisible by exactly } r \text{ primes}\}$. One shows that $m_r \geq m_{r+1}$ for every $r \geq 0$. Let

$$\mu_\infty = \lim_{r\to\infty} m_r.$$

Note that Conjecture A is equivalent to $\mu_\infty < \infty$. In the case where (sur) holds, we are working on upgrading the bound $\mu_\infty \leq \text{ord}_p \text{Tam}(E/\mathbb{Q})$ obtained by the above argument to an exact equality, replacing the Kolyvagin system bound in (c) (which was proved in [4]) with the exact structure theorem

$$\text{length}_{\mathbb{Z}_p} Ш(E \otimes \alpha_m/K)[p^\infty] = 2 \cdot (\text{ind}(\kappa_1(\alpha_m)) - \mu_\infty).$$

That leads to the *refined* form of Conjecture A, as formulated in [17, Conjecture 4.5]:

$$\mu_\infty = \text{ord}_p \text{Tam}(E/\mathbb{Q}).$$

One could also ask about the non-vanishing of other Kolyvagin systems. For example we can consider the Kolyvagin system κ_n^{Kato}, for n in some set $\mathcal{N}_{\text{Kato}}$, obtained from the Euler system constructed by Kato [7] over cyclotomic extensions of $\mathbb{Q}$. A similar strategy applies: the non-vanishing results in (a) above are provided by Rohrlich's work [11] and Kato's explicit reciprocity law in [7] and the Kolyvagin system bound in (c) proved by Mazur–Rubin is generalized in our work to the case where (sur) does not hold. This allows us to prove the following:

Theorem 2. *Let $E/\mathbb{Q}$ be an elliptic curve without CM, and let p be an odd prime of good ordinary reduction for E such that $E(\mathbb{Q})[p] = 0$. Assume that the rational cyclotomic Iwasawa main conjecture holds, then*

$$\text{there exists } n \in \mathcal{N}_{\mathrm{Kato}} \text{ such that } \kappa_n^{\mathrm{Kato}} \neq 0.$$

In particular, $\{\kappa_n^{\mathrm{Kato}}\} \neq 0$ in the cases (i) and (ii) in Theorem 1.

For Eisenstein primes, we apply the new results on Mazur's cyclotomic main conjecture proved in [4]. Case (ii) follows from the main conjecture proved in [12, 7, 15]. Previous results on the non-vanishing of Kato's Kolyvagin system (in the case where (sur) holds) were proved in [8], where also an analogue of the refined Kolyvagin conjecture is formulated. We also prove such conjecture using Mazur–Rubin's structure theorem and the strategy outlined above in the Heegner point case.

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