# TD 1 - Révisions, localisations

**Exercice 1.** Let $A$ be a commutative ring and $M, L, K$ be $A$-modules. We denote by $\mathcal{C}$ the category of $A$-modules.

**(1)** Show that $\mathcal{C} \ni N \mapsto \mathrm{Hom}_A(M \otimes_A \mathrm{Hom}_A(N, L), K)$ defines a (covariant) functor $F : \mathcal{C} \to \mathcal{C}$.

**(2)** **i)** Give conditions on $L, M$ for $F$ to be left exact.

**ii)** Give conditions on $K, M$ for $F$ to be right exact.

**iii)** Give conditions on $K, L, M$ for $F$ to be exact.

**(3)** Assume $A = \mathbb{Z}$, $M = \mathbb{Z}$ and $K = \mathbb{Q}/\mathbb{Z}$.

**i)** Show that $F$ is right exact.

**ii)** Let $m \geq 1$ and $L = \mathbb{Z}$. Compute $L^i(F)(\mathbb{Z}/m\mathbb{Z})$ for all $i \in \mathbb{Z}$.

**iii)** Let $m, n \geq 1$ and $L = \mathbb{Z}/n\mathbb{Z}$. Compute $L^i(F)(\mathbb{Z}/m\mathbb{Z})$ for all $i \in \mathbb{Z}$.

**Exercice 2 (Baer's Lemma).** **(1)** Let $E$ be an injective $A$-module. Show that $E$ satisfies the following condition :

$$\text{for every ideal } I \text{ of } A, \text{ the map } \quad \mathrm{Hom}_A(A, E) \longrightarrow \mathrm{Hom}_A(I, E) \quad \text{is surjective.} \tag{0.1}$$

**(2)** Let $E$ be an $A$-module satisfying condition (0.1). We are given a diagram $\begin{array}{ccc} 0 \to N' & \xrightarrow{f} & N \\ g \downarrow & & \\ E & & \end{array}$. Let $X$ denote the set of pairs $(P, h_P)$ where $P$ is a submodule of $N$ satisfying $f(N') \subset P \subset N$ and $h_P : P \to E$ is an extension of $g$, that is $g = h_P \circ f$. We say that $(P, h_P) \leq (Q, h_Q)$ if $P \subset Q$ and $h_Q/P = h_P$. Show that $\leq$ is a partial order relation.

**(3)** Show that an $A$-module $E$ is injective if and only if it satisfies condition (0.1) (one may use **(2)** and apply Zorn's lemma).

**Exercice 3 (Cone of a morphism).** Let $X^\bullet, Y^\bullet \in Ch(\mathcal{C})$ be two complexes and $f : X^\bullet \to Y^\bullet$ a morphism of complexes. The cone of $f$ is defined by $M^n(f) = X^{n+1} \oplus Y^n$. Let $d_f : M^\bullet(f) \to M^{\bullet+1}(f)$ be defined by the matrix $\begin{bmatrix} -d_X & 0 \\ f^\bullet & d_Y \end{bmatrix}$.

**(1)** Show that $(M(f), d_f)$ is an object of $Ch(\mathcal{C})$, i.e., a complex.

**(2)** Show that $M(f)$ is unique (up to isomorphism) depending only on the class of $f$ in $K(\mathcal{C})$, the homotopy category of $\mathcal{C}$.

**(3)** Construct an exact sequence of complexes

$$0 \to Y^\bullet \to M^\bullet(f) \to X^\bullet[1] \to 0.$$

**(4)** Identify the morphisms $H^\bullet(X) \to H^\bullet(Y)$ in the long exact sequence associated with the short exact sequence from question **(3)**. Deduce that $f$ is a quasi-isomorphism if and only if $H^\bullet(M(f)) = 0$.

**Exercice 4.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor having a right adjoint $G : \mathcal{D} \to \mathcal{C}$ (we say that $\mathcal{D}$ is a reflexive subcategory of $\mathcal{C}$). Let $\mathcal{W}$ denote the collection of morphisms $f$ in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$. Show that the following are equivalent:

1. $G$ is fully faithful;

2. The natural transformation $F \circ G \to Id_{\mathcal{D}}$ is an isomorphism;

3. The natural functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ is an equivalence of categories.

**Exercice 5.** Let $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}$ be the category of abelian groups.

1. (Localization at a single prime) Let $p$ be a prime. Show that the base change functor $- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] :$ $\mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}$ is a localization functor along the class $\mathcal{W}$ of all maps of abelian groups $f : X \to Y$ such that both $\ker f$ and $\mathrm{coker} f$ are $p$-torsion groups. *Hint: Use the pr flatness of* $\mathbb{Z}[\frac{1}{p}]$ *over* $\mathbb{Z}$*, and the previous exercise.*

2. Show that the map $\mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ sending $q \mapsto q \otimes 1$ is an isomorphism. Use this and the Exercice 3 to show that the category of $\mathbb{Q}$ vector spaces is a localization of the category of abelian groups.

**Exercice 6** (Model structure on slice categories, by Victor Saunier)**.** Let $X \in \mathcal{A}$ and $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on $\mathcal{A}$. We denote by $\mathcal{A}/X$ the category whose objects are maps $\alpha : Y \to X$ of $\mathcal{A}$ and whose morphisms are commutative triangles:

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & X \\ {\scriptstyle f}\downarrow & \nearrow_{\alpha'} & \\ Y' & & \end{array}$$

Similarly, we denote $\mathcal{C}/X$ (resp. $\mathcal{F}/X, \mathcal{W}/X$) the morphisms of $\mathcal{A}/X$ as above where $f \in \mathcal{C}$ (resp. $\mathcal{F}, \mathcal{W}$).

Show that $(\mathcal{C}/X, \mathcal{F}/X, \mathcal{W}/X)$ determines a model structure on $\mathcal{A}/X$. We call it the *slice model structure.*

What are the fibrant objects in the above described model structure? The cofibrant objects?

**Exercice 7** (Universal property of localization)**.** Let $\mathcal{C}$ be a small category and $\mathcal{W}$ a subset of the set of morphisms in of $\mathcal{C}$. A localization of $\mathcal{C}$ with respect to $\mathcal{W}$ is a category $C[\mathcal{W}^{-1}]$ together with a functor

$$l : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

satisfying the following universal property: For any category $\mathcal{D}$, composition with $l$:

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, D)$$

is a fully faithful functor and its essential image consists of those functors $\mathcal{C} \to \mathcal{D}$ sending $\mathcal{W}$ to isomorphisms. In other words, $l$, if it exists if the universal functor sending $\mathcal{W}$ to isomorphisms.

1. Check that $\mathcal{C}[\mathcal{W}^{-1}]$, if it exists, is unique up to canonical equivalences of categories.

2. Show that when $\mathcal{C}$ is the category with a single object $*$ and a monoid $M$ of endomorphisms, and $\mathcal{W} = M$ then $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the category with one object $*$ and $M^+$ as endomorphisms, with $M^+$ the group completion of $M$.