TD 1 - Révisions, localisations

Exercice 1. Let A be a commutative ring and M, L, K be A-modules. We denote by $\mathcal{C}$ the category of A-modules.

(1) Show that $\mathcal{C} \ni N \mapsto \mathrm{Hom}_A(M \otimes_A \mathrm{Hom}_A(N, L), K)$ defines a (covariant) functor $F : \mathcal{C} \to \mathcal{C}$.

(2) **i)** Give conditions on L, M for F to be left exact.

ii) Give conditions on K, M for F to be right exact.

iii) Give conditions on K, L, M for F to be exact.

(3) Assume $A = \mathbb{Z}$, $M = \mathbb{Z}$ and $K = \mathbb{Q}/\mathbb{Z}$.

i) Show that F is right exact.

ii) Let $m \geq 1$ and $L = \mathbb{Z}$. Compute $L^i(F)(\mathbb{Z}/m\mathbb{Z})$ for all $i \in \mathbb{Z}$.

iii) Let $m, n \geq 1$ and $L = \mathbb{Z}/n\mathbb{Z}$. Compute $L^i(F)(\mathbb{Z}/m\mathbb{Z})$ for all $i \in \mathbb{Z}$.

Solution. **(1)** Rappelons que la composée de deux foncteurs covariants ou de deux foncteurs contravariant est un foncteur covariant. La composée d'un foncteur covariant et d'un foncteur contravariant (ou le contraire) est un foncteur contravariant. Or pour tout objet N de $\mathcal{C}$, $F(N) = \mathrm{Hom}_A(M \otimes_A \mathrm{Hom}_A(N, L), K)$ est la composée des foncteurs $\mathrm{Hom}_A(-, K) : \mathcal{C}^{op} \to \mathcal{C}$ (donc contravariant), $M \otimes_A - : \mathcal{C} \to \mathcal{C}$ (covariant) et $\mathrm{Hom}_A(-, L) : \mathcal{C}^{op} \to \mathcal{C}$ (donc contravariant). C'est donc un foncteur covariant.

(2) **i)** Par définition F est exact à gauche s'il transforme toute suite exacte $0 \to X' \to X \to X"$ en une suite exacte $0 \to F(X') \to F(X) \to F(X")$. En appliquant le foncteur $\mathrm{Hom}_A(-, L)$ à la suite exacte $0 \to X' \to X \to X"$, on obtient le complexe

$$\mathrm{Hom}_A(X", L) \to \mathrm{Hom}_A(X, L) \to \mathrm{Hom}_A(X', L) \to 0.$$

On sait que ce complexe est une suite exacte lorsque $\mathrm{Hom}_A(-, L)$ est exact à droite, c'est à dire si L est injectif. En ce cas, comme $M \otimes_A -$ est exact à droite, et $\mathrm{Hom}_A(-, K)$ exact à gauche, on obtient que $0 \to F(X') \to F(X) \to F(X")$ est exacte. Conclusion : Il suffit que L soit injectif pour que F soit exact à droite.

ii) Un raisonnement analogue assure que F est exact à gauche si M est plat (auquel cas $M \otimes_A -$ est exact à gauche) et K injectif.

iii) De **i), ii)**, on déduit que F est exact si K, L sont injectifs et M plat.

(3) On remarque que $M = \mathbb{Z}$ est libre sur $\mathbb{Z}$, donc plat et $K = \mathbb{Q}/\mathbb{Z}$ est injectif (voir le lemme de Baer plus bas).

i) Les hypothèses de **(2).ii)** sont satisfaites, donc F est exact à droite. En particulier il admet des foncteurs dérivés à gauche (puisque la catégorie $\mathrm{Mod}(\mathbb{Z})$ admet assez d'objets projectifs).

ii) Il faut trouver une résolution projective de $\mathbb{Z}/m\mathbb{Z}$. On utilise la résolution libre (en particulier projective) déjà vue en TD : $\cdots \to 0 \to 0 \to \mathbb{Z} \overset{\times m}{\to} \mathbb{Z}$. On alors que, pour tout $i \in \mathbb{Z}$, les groupes de cohomologie $L_i(F)(\mathbb{Z}/m\mathbb{Z})$ sont donnés par la formule :

$$L_i(F)(\mathbb{Z}/m\mathbb{Z}) = H^{-i}\big(\ldots \longrightarrow 0 \longrightarrow F(\mathbb{Z}) \overset{F(\times m)}{\longrightarrow} F(\mathbb{Z})\big).$$

Ceci donne immédiatement $L^{i>1}(F)(\mathbb{Z}/m\mathbb{Z}) = 0$. Comme $\mathbb{Z} \otimes_{\mathbb{Z}} X \cong X$ (l'isomorphisme étant donné par $k \otimes_{\mathbb{Z}} x \mapsto kx$) et $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, X) \cong X$ (l'isomorphisme étant donné par $\varphi \mapsto \varphi(1)$), on obtient $F(\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ et $F(\times m) = \mathbb{Q}/\mathbb{Z} \overset{\times m}{\longrightarrow} \mathbb{Q}/\mathbb{Z}$. Comme $\mathbb{Q}/\mathbb{Z}$ est divisible (c'est à dire que pour tout $x, n \in \mathbb{Z} - \{0\}$, il existe $y \in \mathbb{Q}/\mathbb{Z}$ avec $ny = x$), on en déduit que

$$L^0(F)(\mathbb{Z}/m\mathbb{Z}) \cong \mathrm{coker}(\mathbb{Q}/\mathbb{Z} \overset{\times m}{\longrightarrow} \mathbb{Q}/\mathbb{Z}) = 0.$$

Il reste à calculer $L^1(F)(\mathbb{Z}/m\mathbb{Z}) \cong \ker(\mathbb{Q}/\mathbb{Z} \overset{\times m}{\longrightarrow} \mathbb{Q}/\mathbb{Z})$. Tout élément de $\mathbb{Q}/\mathbb{Z}$ est représenté par la classe d'un élément $p/q \in \mathbb{Q}$ avec $p \wedge q = 1$ (en notant $p \wedge q$ le pgcd de p et q). On obtient que $mp/q = 0 \in \mathbb{Q}/\mathbb{Z}$ si q/m. La réciproque est immédiate. Il en découle que

$$L^1(F)(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}.$$

ii) Evidemment, on utilise la même résolution projective de $\mathbb{Z}/m\mathbb{Z}$. Avec $L = \mathbb{Z}/n\mathbb{Z}$, on a $F(\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. Un morphisme $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ est uniquement déterminé par l'image $\varphi(1)$ qui en outre doit vérifier $0 = \varphi(n) = n\varphi(1)$. On en déduit que $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \ker(\mathbb{Q}/\mathbb{Z} \overset{\times n}{\to} \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Par conséquent on doit calcler la cohomologie du complexe

$$\ldots 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \overset{\times m}{\longrightarrow} \mathbb{Z}/n\mathbb{Z}.$$

On obtient alors par des calculs similaires à ceux fait en TD

$$L^0(F)(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/n \wedge m\mathbb{Z}, \qquad L^1(F)(\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/n \wedge m\mathbb{Z}, \qquad L^{i>1}(F)(\mathbb{Z}/m\mathbb{Z}) = 0.$$

Exercice 2 (**Baer's Lemma**). **(1)** Let E be an injective A-module. Show that E satisfies the following condition :

$$\text{for every ideal } I \text{ of } A, \text{ the map } \quad \mathrm{Hom}_A(A, E) \longrightarrow \mathrm{Hom}_A(I, E) \quad \text{is surjective.} \tag{0.1}$$

(2) Let E be an A-module satisfying condition (0.1). We are given a diagram $\begin{array}{ccc} 0 \to & N' & \overset{f}{\to} N \\ & g\downarrow & \\ & E & \end{array}$. Let X denote the set of pairs (P, h_P) where P is a submodule of N satisfying $f(N') \subset P \subset N$ and $h_P : P \to E$ is an extension of g, that is $g = h_P \circ f$. We say that $(P, h_P) \leq (Q, h_Q)$ if $P \subset Q$ and $h_Q/P = h_P$. Show that $\leq$ is a partial order relation.

(3) Show that an A-module E is injective if and only if it satisfies condition (0.1) (one may use **(2)** and apply Zorn's lemma).

Solution. **(1)** Si E est injectif, alors comme I s'injecte dans A, on a que tout morphisme $I \to E$ se prolonge en un morphisme $A \to E$ par définition des module injectifs. Ce qui donne la surjectivité de $\mathrm{Hom}_A(A, E) \longrightarrow \mathrm{Hom}_A(I, E)$.

(2) On considère le diagramme $\begin{array}{ccc} 0 \to & N' & \overset{f}{\to} N \\ & g\downarrow & \\ & E & \end{array}$. L'ensemble des couples (P, h_P) où P est un sous-module de N vérifiant $f(N') \subset P \subset N$ et $h_P : P \to E$ est une extension de g est muni de la relation d'ordre $(P, h_P) \leq (Q, h_Q)$ si $P \subset Q$ et $h_Q/P = h_P$. On a bien $(P, h_P) \leq (P, h_p)$, si $P \subset Q \subset R$ est une suite croissante d'extension, alors, R est une extension de P. De plus si $(P, h_P) \leq (Q, h_Q)$ et $(Q, h_Q) \leq (P, h_P)$ alors $P = Q$. La relation est donc bien une relation d'ordre.

(3) Montrons que $\leq$ vérifie les hypothèses du Lemme de Zorn; c'est à dire que toute sous-famille totalement ordonnée admet un élément maximal. Soit $(P_i, h_{P_i})_{i\in\Im}$ une famille totalement ordonnée d'éléments de X. (Pour tout $i, j \in \Im$ on a $(P_i, h_{P_i}) \preceq (P_j, h_{P_j})$ ou $(P_j, h_{P_j}) \preceq (P_i, h_{P_i})$.) On pose $P = \bigcup_{i\in\Im} P_i$ et on définit $h : P \to E$ par $h(x) = h_{P_i}(x)$ si $x \in P_i$ de sorte que $(P, h_P) \in X$. D'après le lemme de Zorn, on a donc que X admet un

Supposons, par l'absurde, que $M \subsetneq N$. Il existe donc $x \in N \setminus M$. Alors $P = M + A.x$ est un sous module de N qui contient strictement M. On définit $I = (M : x) = \{a \in A,\ a.x \in M\}$ qui est un idéal. Soit $\gamma : I \to E$, l'application $\gamma(a) = h_M(a.x)$. D'après l'hypothèse de l'énoncé il existe alors $\varphi : A \to E$ tel que $\varphi_{|I} = \gamma$.

On définit alors $h_P : P \to E$ par $h_P(y + a.x) = h_M(y) + \varphi(a)$ (où $y \in M$). Cette définition est consistante car si $y + a.x = y' + a'.x$ alors $(a' - a).x = y - y' \in M$ donc $a' - a \in I$ et $\varphi(a' - a) = \gamma(a' - a) = h_M((a' - a).x) = h_M(y - y')$ d'où $h_M(y) + \varphi(a) = h_M(y') + \varphi(a')$.

On a ainsi obtenu un élément (P, h_P) tel que $(M, h_M) \prec (P, h_P)$ ce qui est impossible car (M, h_M) est maximal. On a donc $M = N$ et un morphisme $h : N \to E$ tel que $h \circ f = g$. Ceci étant vrai pour tout diagramme
$$\begin{array}{ccc} 0 \to N' & \xrightarrow{f} & N \\ g\downarrow & & \\ E & & \end{array}$$
on a bien montré que E est injectif.

Exercice 3 **(Cone of a morphism).** Let $X^\bullet, Y^\bullet \in Ch(\mathcal{C})$ be two complexes and $f : X^\bullet \to Y^\bullet$ a morphism of complexes. The cone of f is defined by $M^n(f) = X^{n+1} \oplus Y^n$. Let $d_f : M^\bullet(f) \to M^{\bullet+1}(f)$ be defined by the matrix $\begin{bmatrix} -d_X & 0 \\ f^\bullet & d_Y \end{bmatrix}$.

(1) Show that $(M(f), d_f)$ is an object of $Ch(\mathcal{C})$, i.e., a complex.

(2) Show that $M(f)$ is unique (up to isomorphism) depending only on the class of f in $K(\mathcal{C})$, the homotopy category of $\mathcal{C}$.

(3) Construct an exact sequence of complexes
$$0 \to Y^\bullet \to M^\bullet(f) \to X^\bullet[1] \to 0.$$

(4) Identify the morphisms $H^\bullet(X) \to H^\bullet(Y)$ in the long exact sequence associated with the short exact sequence from question **(3)**. Deduce that f is a quasi-isomorphism if and only if $H^\bullet(M(f)) = 0$.

Solution. **(1)** On a $(d_f \circ d_f)(x, y) = d_f(-d_X(x), d_Y(y) + f(x)) = d_X^2(x), d_Y^2(y) - f(d_X(x)) + d_Y f(x) = (0, 0)$ car d_X, d_Y sont de carrés nuls et f est un morphisme de complexes.

(2) Soit $f - f' = s d_X + d_Y s$ avec $s : X^\bullet \to Y^{\bullet-1}$. On vérifie aisément que le morphisme $h = \begin{bmatrix} 1 & 0 \\ s^\bullet & 1 \end{bmatrix}$ est un morphisme de $C(f) \to C(f')$. Il est de plus inversible; d'inverse $\begin{bmatrix} 1 & 0 \\ -s^\bullet & 1 \end{bmatrix}$.

(3) On vérifie que $i_Y : Y^\bullet \to X^{\bullet+1} \oplus Y^\bullet$ est un morphisme de complexes, de même que $p_X : X^{\bullet+1} \oplus Y^\bullet \to X^{\bullet+1} = X[1]^\bullet$ (rappelons que la différentielle sur le complexe $X[1]$ est $-d_X$). De plus $p_X i_Y = 0$, $p_X i_X = \mathrm{Id}$ et le noyau de p_X est $\mathrm{im}(i_Y)$ d'après l'équation $i_X p_X + i_Y p_Y = \mathrm{Id}$.On en conclut que la suite
$$0 \to Y^\bullet \xrightarrow{i_Y} M^\bullet(f) \xrightarrow{p_X} X^\bullet[1] \to 0$$
est exacte.

(4) Il suffit de reprendre la construction du morphisme de connexion δ dans le lemme du serpent. En effet, d'après le cours, la longue suite exacte en homologie s'écrit

$$\dots H^n(Y) \to H^n(M(f)) \to H^n(X[1]) \xrightarrow{\delta} H^{n+1}(Y) \to \dots$$

avec $H^n(X[1]) = H^{n+1}(X)$. Le morphisme δ est le morphisme $\ker(X^{n+1} \xrightarrow{d_X} X^{n+2}) \to \mathrm{coker}(Y^n \xrightarrow{d_Y} Y^{n+1})$ donné par le lemme du serpent, voir le cours et le TD 2. Précisément, étant donné $x \in X^{n+1}$ avec $d_X(x) = 0$, $\delta(x)$ est obtenu en prenant un antécédent (quelconque) de x par p_X, c'est à dire $h \in M(f)^n$ avec $p_X(h) = x$. Puis on prend $z \in Y^{n+1}$ tel que $i_Y(y) = d_f(h)$. On a alors $\delta(x) = [y]$, où $[y]$ est l'image de $y$par l'application canonique vers le conoyau. Dans le cas présent, on peut choisir $h = (x, 0)$. On obtient alors $d_f(h) = (0, f(y))$. On conclut que l'application $H^n(X[1]) \xrightarrow{\delta} H^{n+1}(Y)$ induite dans la longue suite exacte est l'application $H^{n+1}(f)$ induite par f via l'isomorphisme $H^n(X[1]) = H^{n+1}(X)$. Mais f est un quasi-isomorphisme si et seulement si $H^\bullet(f)$ est un isomorphisme. D'après la suite exacte longue, si $H^\bullet(f)$ est un isomorphisme alors, $H^\bullet(M(f)) \to H^{\bullet+1}(X)$ est l'application nulle de même que $H^\bullet(Y) \to H^\bullet(M(f))$ ce qui par exactitude force $H^\bullet(M(f)) = 0$. réciproquement, si $H^\bullet(M(f)) = 0$, la suite exacte implique immédiatement que $H^\bullet(f)$ est un isomorphisme puisqu'elle se réduit à des suites exactes $0 \to H^\bullet(X) \to H^\bullet(Y) \to 0$.

Exercice 4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor having a right adjoint $G : \mathcal{D} \to \mathcal{C}$ (we say that $\mathcal{D}$ is a reflexive subcategory of $\mathcal{C}$). Let $\mathcal{W}$ denote the collection of morphisms f in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$. Show that the following are equivalent:

1. G is fully faithful;
2. The natural transformation $F \circ G \to Id_{\mathcal{D}}$ is an isomorphism;
3. The natural functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ is an equivalence of categories.

Solution. That 1 implies 2 is standard for any adjunction:

$$\mathcal{D}(x, y) \cong \mathcal{C}(G(x), G(y)) \cong \mathcal{D}(F \circ G(x), y).$$

One can also see that 2 implies 1 by going in the other direction.
Let us prove 2 implies 3: The composition $\mathcal{D} \xrightarrow{F} \mathcal{C} \to \mathcal{C}[W^{-1}] \to \mathcal{D}$ is $G \circ F$, so it is isomorphic to the identity. It suffices to show that $\mathcal{C}[W^{-1}] \to \mathcal{D} \xrightarrow{G} \mathcal{C} \to \mathcal{C}[W^{-1}]$ is isomorphic to the identity. By the universality of $\mathcal{C} \to \mathcal{C}[W^{-1}]$, it suffices to show that $\mathcal{C} \to \mathcal{C}[W^{-1}] \to \mathcal{D} \xrightarrow{G} \mathcal{C} \to \mathcal{C}[W^{-1}]$ is isomorphic to $\mathcal{C} \to \mathcal{C}[W^{-1}]$. From the composition $F \to FGF \to F = id_F$ given by the unit and counit, we deduce that the transformation $F \to FGF$ is an isomorphism since the second part is. This shows that $\mathcal{C} \to \mathcal{C}[W^{-1}] \to \mathcal{D} \xrightarrow{G} \mathcal{C} \to \mathcal{C}[W^{-1}]$ is indeed what we want.
Now let's prove that 3 implies 1. We have that $Hom(\mathcal{C}[W^{-1}], \mathcal{D}) \cong Hom(\mathcal{C}, \mathcal{D})$ is fully faithful, and by composition, we get that $F^* : Hom(\mathcal{D}, \mathcal{D}) \cong Hom(\mathcal{C}, \mathcal{D})$ is fully faithful. We want to show that $F \circ G \to Id$ is an isomorphism, which reduces to showing that $FGF \to F$ is an isomorphism, a consequence of the adjunction.

Exercice 5. Let $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}$ be the category of abelian groups.

1. (Localization at a single prime) Let p be a prime. Show that the base change functor $- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}$ is a localization functor along the class $\mathcal{W}$ of all maps of abelian groups $f : X \to Y$ such that both $\ker f$ and $\mathrm{coker} f$ are p-torsion groups. *Hint: Use the pr flatness of* $\mathbb{Z}[\frac{1}{p}]$ *over* $\mathbb{Z}$*, and the previous exercise.*
2. Show that the map $\mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ sending $q \mapsto q \otimes 1$ is an isomorphism. Use this and the Exercice 3 to show that the category of $\mathbb{Q}$ vector spaces is a localization of the category of abelian groups.

Solution. We use exercise 4. We will check that the inclusion functor $\mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]} \to \mathbb{Z}\mathrm{Mod}$ is fully faithful. This amounts to check that $Hom_{\mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}}(A,B) = Hom_{\mathrm{Mod}_{\mathbb{Z}}}(A,B)$ for any two $\mathbb{Z}[\dfrac{1}{p}]-$modules A and B. The inclusion of the left term into the right term is automatic. Now take $f \in Hom_{\mathrm{Mod}_{\mathbb{Z}}}(A,B)$; We need to show it is automatically a morphism of $\mathbb{Z}[\dfrac{1}{p}]$-modules. But $pf(\dfrac{a}{p}) = f(p\cdot\dfrac{a}{p}) = f(a)$ so $f(\dfrac{a}{p}) = \dfrac{f(a)}{p}$ for all a and we are done. By Exercise 3, the base change functor

$$- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}$$

is a localization along maps inducing isomorphism after tensoring by $\mathbb{Z}[\dfrac{1}{p}]$. Let's describe these maps. A map $A \to B$ is such if and only if $\ker(A\otimes\mathbb{Z}[\dfrac{1}{p}] \to B\otimes\mathbb{Z}[\dfrac{1}{p}]) = 0$ and $\mathrm{coker}(A\otimes\mathbb{Z}[\dfrac{1}{p}] \to B\otimes\mathbb{Z}[\dfrac{1}{p}]) = 0$. By flatness of $\mathbb{Z}[\dfrac{1}{p}]) = 0$ taking ker and coker commute with tensoring, so this amounts to ask that $\mathbb{Z}[1/p] \otimes \ker(A \to B)$ and $\mathbb{Z}[1/p] \otimes \mathrm{coker}(A \to B)$ are zero. This is true if and only if $\ker A \to B$ and $\mathrm{coker}\, A \to B$ are torsion p-groups.
2. The same arguments work by flatness of $\mathbb{Q}$ over $\mathbb{Z}$. The fact that a morphism of abelian groups between two $\mathbb{Q}$-vector spaces is automatically a $\mathbb{Q}$ linear map works with the same proof as before.

Exercice 6 (Model structure on slice categories, by Victor Saunier)**.** Let $X \in \mathcal{A}$ and $(\mathcal{C},\mathcal{F},\mathcal{W})$ be a model structure on $\mathcal{A}$. We denote by $\mathcal{A}/X$ the category whose objects are maps $\alpha : Y \to X$ of $\mathcal{A}$ and whose morphisms are commutative triangles:

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "f"'] & X \\ Y' \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

Similarly, we denote $\mathcal{C}/X$ (resp. $\mathcal{F}/X, \mathcal{W}/X$) the morphisms of $\mathcal{A}/X$ as above where $f \in \mathcal{C}$ (resp. $\mathcal{F}, \mathcal{W}$).

Show that $(\mathcal{C}/X, \mathcal{F}/X, \mathcal{W}/X)$ determines a model structure on $\mathcal{A}/X$. We call it the *slice model structure*.

What are the fibrant objects in the above described model structure? The cofibrant objects?

Solution. Let us check the axioms of a model category.

1. MC1. Colimits in $\mathcal{C}/X$ can be computed as colimits in $\mathcal{C}$ (for example, because the forgetful functor $\mathcal{C}/X \to \mathcal{C}$ has a right adjoint given by $A \mapsto A \times X$). For limits : if I is a small category and $D : I \to \mathcal{C}/X$, consider I^{∇} the cocone of I (I to which we added a terminal object $*$). We have by construction a diagram $D^{\nabla} : I^{\nabla} \to \mathcal{C}$ where $*$ is sent to X. Then one can check that $\lim_{\mathcal{C}} D^{\nabla}$ is an object over X and is a limit of D in $\mathcal{C}/X$. So $\mathcal{C}/X$ is complete and cocomplete.

2. MC2. The 2 out of 3 from $\mathcal{C}$ gives the 2 out of 3 in $\mathcal{C}/X$ because weak equivalences are the same.

3. MC3. Automatic since retracts in $\mathcal{C}/X$ give retracts in $\mathcal{C}$.

4. MC4. Take a lifting problem in $\mathcal{C}/X$, for example :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "\wr i"'] & B \arrow[d, two heads, "p"] \\ C \arrow[r, "g"'] \arrow[ur, dashed, "\exists"] & D \end{tikzcd}$$

where all maps are maps over X.As $\mathcal{W}$, $\mathcal{C}$, $\mathcal{F}$ are the same as in $\mathcal{C}$, this gives a lifting problem in $\mathcal{C}$, so there exist a lifting (dotted arrow) φ in $\mathcal{C}$. There remains to check that this is a map over X. Call q_M the map $M \to X$ for any object M. We want to check that $q_B\varphi = q_A$. But $q_B\varphi = q_D p\varphi = q_D g = q_A$ so we are done.

5. MC5. We already have functorial factorizations in $\mathcal{C}$. We want to check they are maps over X. Let $f : A \to B$ be a map in $\mathcal{C}/X$. The factorization $A \rightarrowtail C_f \twoheadrightarrow B$ is automatically over X using the map $C_f \twoheadrightarrow B \xrightarrow{q_B} X$. The same holds for the second kind of factorizations.

Exercice 7 (Universal property of localization)**.** Let $\mathcal{C}$ be a small category and $\mathcal{W}$ a subset of the set of morphisms in of $\mathcal{C}$. A localization of $\mathcal{C}$ with respect to $\mathcal{W}$ is a category $C[\mathcal{W}^{-1}]$ together with a functor

$$l : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

satisfying the following universal property: For any category $\mathcal{D}$, composition with l:

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, D)$$

is a fully faithful functor and its essential image consists of those functors $\mathcal{C} \to \mathcal{D}$ sending $\mathcal{W}$ to isomorphisms. In other words, l, if it exists if the universal functor sending $\mathcal{W}$ to isomorphisms.

1. Check that $\mathcal{C}[\mathcal{W}^{-1}]$, if it exists, is unique up to canonical equivalences of categories.

2. Show that when $\mathcal{C}$ is the category with a single object $*$ and a monoid M of endomorphisms, and $\mathcal{W} = M$ then $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the category with one object $*$ and M^+ as endomorphisms, with M^+ the group completion of M.

Solution. Note that a functor $l : \mathcal{C} \to \mathcal{D}$ is a localization if and only if it verifies the following two properties:

- For any functor $F : \mathcal{C} \to \mathcal{E}$ that sends $\mathcal{W}$ to isomorphisms, there exists $G : \mathcal{D} \to \mathcal{E}$ such that $F \cong G \circ l$.
- The map $- \circ l : \mathrm{Nat}(G_1, G_2) \to \mathrm{Nat}(G_1 \circ l, G_2 \circ l)$ is a bijection for all functors $G_1, G_2 : \mathcal{D} \to \mathcal{E}$.

1. Suppose $l : \mathcal{C} \to \mathcal{E}$ and $l' : \mathcal{C} \to \mathcal{E}'$ are two localizations along $\mathcal{W}$. By the universal property of these localizations, both l and l' are isomorphic to functors that invert $\mathcal{W}$. In particular, l' belongs to the essential image of the functor $- \circ l : \mathrm{Fun}(\mathcal{E}, \mathcal{E}') \to \mathrm{Fun}^{\mathcal{W}}(\mathcal{C}, \mathcal{E}')$, so that there exists a functor $G : \mathcal{E} \to \mathcal{E}'$ such that $l' \cong G \circ l$.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "l'"] \arrow[dr, "l"'] & & \mathcal{E}' \\ & \mathcal{E} \arrow[ur, dashed, "G"'] & \end{tikzcd}$$

Similarly, $l \cong H \circ l'$ for some functor $H : \mathcal{E}' \to \mathcal{E}$. Using the bijection

$$\mathrm{Nat}(\mathrm{Id}_{\mathcal{E}}, H \circ G) \cong \mathrm{Nat}(l, H \circ G \circ l) \cong \mathrm{Nat}(l, l),$$

one finds a natural transformation $\alpha : \mathrm{Id}_{\mathcal{E}} \Rightarrow H \circ G$ (given by the image of Id_l). Similarly, there is a natural transformation $\beta : H \circ G \Rightarrow \mathrm{Id}_{\mathcal{E}}$. Finally, using the bijection $\mathrm{Nat}(\mathrm{Id}_{\mathcal{E}}, \mathrm{Id}_{\mathcal{E}}) \cong \mathrm{Nat}(l, l)$, one gets $\beta \circ \alpha = \mathrm{Id}_{\mathrm{Id}_{\mathcal{E}}}$ and similarly for the other composition. Hence $H \circ G \cong \mathrm{Id}_{\mathcal{E}}$. Reasoning with $\mathcal{E}'$ in a similar manner shows that G and H realize an equivalence of categories $\mathcal{E} \simeq \mathcal{E}'$.

2. Recall that the group completion of a monoid is a group M^+ together with a monoid map

$$\iota : M \to M^+$$

such that for any monoid map $\phi : M \to G$, there is a unique group morphism $\tilde{\phi} : M^+ \to G$ factorizing ϕ, that is $\phi = \tilde{\phi} \circ \iota$. It is usual abstract nonsense to prove it is unique up to isomorphism (and there is a unique such isomorphism that commutes with the structure maps from M to the completion). To prove the existence of M^+, it is enough to construct it, which can be done by defining it as a quotient of the free group on the generating set M by the obvious equivalence relation identifying $m \star m'$ with $m \cdot m'$, where $\star$ is the product in the free group and $\cdot$ is the product in M.

We now come to the proof. The key fact to note is the following: the full subcategory of Cat spanned by the categories with a unique object is isomorphic to the category of monoids. Therefore, given a monoid A, we will also write A for the category with one object $*$ and $\mathrm{End}(*) = A$ as morphisms. Let M be a monoid. We now show that $\iota : M \to M^+$, viewed as a functor, satisfies the universal property of the localization of M along all morphisms. Let $\mathcal{D}$ be a category. Then any functor $F : M^+ \to \mathcal{D}$ factors through the full subcategory $\mathcal{D}_{F(*)}$ spanned by the image of $*$. Thus

$$\mathrm{Fun}(M^+, \mathcal{D}) \cong \coprod_{x \in \mathrm{ob}(\mathcal{D})} \mathrm{Fun}(M^+, \mathcal{D}_x)$$

as categories. Observe that any morphism of monoid $f : M^+ \to \mathrm{End}_{\mathcal{D}}(x)$ factors through the subgroup of units $\mathrm{Aut}_{\mathcal{D}}(x)$. Hence precomposing with ι inverts all morphisms in M^+. Now the universal property of the group completion gives that the functor

$$- \circ \iota : \mathrm{Fun}(M^+, \mathrm{Aut}_{\mathcal{D}}(x)) \longrightarrow \mathrm{Fun}^{\mathcal{W}}(M, \mathrm{Aut}_{\mathcal{D}}(x))$$

is bijective on objects, hence essential surjective. The category $\mathrm{Fun}(M^+, \mathrm{Aut}_{\mathcal{D}}(x))$ has objects given by the group morphisms $M^+ \to \mathrm{Aut}_{\mathcal{D}}(x)$; the morphisms between f and g are the elements $\alpha \in \mathrm{Aut}_{\mathcal{D}}(x)$ such that $g = \alpha f \alpha^{-1}$. Using this description, one easily sees that $- \circ \iota$ is fully faithful, hence an equivalence of categories.