

TD 2 - MODEL STRUCTURES ON CHAIN COMPLEXES

Let R be a commutative unital ring (or a commutative unital k -algebra). Recall that $Ch(R)$ is the category of (unbounded) chain complexes $(C_{i \in \mathbb{Z}}, d)$ of R -modules, and $Ch_{\geq 0}(R)$ its full subcategory of chain complexes concentrated in nonnegative degrees ($C_i = 0$ if $i < 0$). Finally, we denote by $Ch_{\leq 0}(R)$ the full subcategory of chain complexes concentrated in nonpositive degrees ($C_i = 0$ if $i > 0$); this subcategory is isomorphic (via $C^i = C_{-i}$) to the category of cochain complexes concentrated in nonnegative degrees.

Définition 0.1 (*Projective model structure*). Let $\mathbf{C} = Ch(R)$ or $Ch_{\geq 0}(R)$. We define the so-called projective structure on \mathbf{C} by setting

Weak equivalences \mathcal{W} these are the quasi-isomorphisms (that is, morphisms of complexes inducing isomorphisms in homology).

Fibrations \mathcal{F} these are the surjective morphisms of complexes (in every degree) in $Ch(R)$, and the morphisms of complexes that are surjective in every degree > 0 in $Ch_{\geq 0}(R)$.

Cofibrations \mathcal{C} these are the morphisms of complexes having the left lifting property with respect to acyclic fibrations.

Dually, we define the so-called injective structure on \mathbf{C} by setting \mathcal{W} as the quasi-isomorphisms. Cofibrations are the injective morphisms of complexes (in every degree) in $Ch(R)$, and the morphisms of complexes that are injective in every degree < 0 in $Ch_{\leq 0}(R)$. Fibrations are the morphisms of complexes having the right lifting property with respect to acyclic cofibrations.

We admit the following theorem, and we will illustrate the structure thus defined in what follows.

Theorem 0.2. The above projective structures endow $Ch(R)$ and $Ch_{\geq 0}(R)$ with the structure of a model category. Moreover,

1. The cofibrations of $Ch_{\geq 0}(R)$ are exactly the degreewise inclusions whose cokernel is projective in every degree.
2. The cofibrations of $Ch(R)$ are the morphisms of complexes which are injective and have projective cokernel in every degree, and whose cokernel is cofibrant.
3. Any morphism of complexes in $Ch(R)$, injective in every degree, whose cokernel is a bounded below complex¹ of projective modules, is a cofibration.

Exercice 1. 1. State the analogue of the previous theorem for the injective structure.

2. Prove that in the projective structure, all objects are fibrant, and give the analogue in the injective structure.
3. Prove that if $f : A \rightarrow B$ is a cofibration, then $\text{coker}(f)$ is cofibrant (hint: use a pushout).

In the following exercise, we will **not** use the characterization of cofibrations given by the theorem.

Exercice 2. Let R be a ring, and equip $Ch(R)$ with the projective structure.

1. We want to show that if a complex (C_*, d) is cofibrant, then C_n is projective for all n .

¹that is, $C_i = 0$ for $i \ll 0$

- (a) Show that morphisms of complexes $C_* \rightarrow D^{n+1}(M)$ are in bijection with linear maps $C_n \rightarrow M$.
 - (b) Check, for all n , that $M \mapsto D^{n+1}(M)$ is a functor from R -modules to acyclic chain complexes.
 - (c) Prove that C_* cofibrant implies that C_n is projective (hint: use the previous two questions to produce an acyclic fibration).
2. Prove that any cofibration $i : A_* \rightarrow B_*$ is degreewise injective. (Hint: check that $D^{n+1}(A_n) \rightarrow 0$ is an acyclic fibration and consider a morphism $A_* \rightarrow D^{n+1}(A_n) \rightarrow 0$.)
 3. Deduce that for any cofibration $i : A_* \rightarrow B_*$, writing $P_* = \text{coker}(i)$, one has $B_n \cong A_n \oplus P_n$ where P_n is projective.
 4. The goal of this question is to show that a chain complex in $Ch(R)$ may be made of projective modules but be non-cofibrant.
 - (a) Consider the quotient ring $R = K[x]/(x^2)$ where K is a field. For $n \in \mathbb{Z}$, set $P_n = R$ and consider the map $d_n : P_n \rightarrow P_{n-1}$ given by multiplication by x in R : $r \mapsto x \cdot r$. Prove that $P_* = (P_n, d_n)$ is a chain complex with zero homology. Deduce that if P_* is cofibrant, then $0 \rightarrow P_*$ is an acyclic cofibration.
 - (b) Consider $p : R \rightarrow K = R/(x)$ the canonical projection, and equip K with the induced R -module structure. Prove that $S^0(p) : S^0(R) \rightarrow S^0(K)$ is a fibration in $Ch(R)$.
 - (c) Prove that there exists a morphism of chain complexes $P_* \rightarrow S^0(K)$ which is equal to p in degree 0, and that, for any R -module M , if $u : P_* \rightarrow S^0(M)$ is a morphism of chain complexes, then $u(d_1(P_1)) = 0$.
 - (d) Assuming that P_* is cofibrant, deduce a contradiction from the previous questions.
 5. We want to prove that any chain complex $P_* \in Ch_{\geq 0}(R)$ consisting of projective modules is cofibrant. Let $f : X_* \xrightarrow{\sim} Y_*$ be an acyclic fibration in $Ch_{\geq 0}(R)$ and
$$\begin{array}{ccc} 0 & \longrightarrow & X_* \\ \downarrow & & \downarrow f \\ P_* & \xrightarrow{q} & Y_* \end{array}$$
a commutative

diagram. We will construct a lift $\tilde{q}_* : P_* \rightarrow X_*$ by induction.

- (a) Prove that $f_0 : X_0 \rightarrow Y_0$ is surjective and that $\ker(f)$ is a chain complex with zero homology in every degree.
- (b) Using that P_0 is projective, construct a $\tilde{q}_0 : P_0 \rightarrow X_0$ which works.
- (c) Suppose now that $\tilde{q}_0, \dots, \tilde{q}_{n-1}$ have been constructed so that the diagrams commute and they commute with the differentials.
 - i. Using that P_n is projective, construct an $h_n : P_n \rightarrow X_n$ making the diagram commute in degree n .
 - ii. Prove that $d \circ h_n - \tilde{q}_{n-1} \circ d : P_n \rightarrow X_{n-1}$ takes values in $Z_{n-1}(\ker(f))$ (that is, the cycles of $\ker(f)$).
 - iii. Using that P_n is projective, show that there exists a map $\psi : P_n \rightarrow \ker(f)_n$ such that
$$d \circ \psi = d \circ h_n - \tilde{q}_{n-1} \circ d.$$
 - iv. Check that $\tilde{q}_n = h_n - \psi$ gives the desired lift.

Exercise 3 (Strong Cylinders). Fix a model category \mathcal{C} where every object is fibrant. A *strong* cylinder is a cylinder $X \sqcup X \rightarrow C \xrightarrow{\sim} X$ where the last arrow is an acyclic fibration (for a general cylinder, we only ask for a weak equivalence). The goal of the exercise is to show that if C_X is a strong cylinder for X and two maps $f, g : X \rightarrow Y$ are left homotopic, they are left homotopic with C_X as a choice of cylinder.

1. Suppose f, g are left homotopic through $H : C \rightarrow Y$ where $C \xrightarrow{\sim} X$ is an arbitrary cylinder. Explain how we can reduce to the case where C is a strong cylinder (Hint : use the factorization axiom).
2. Conclude by finding a lift $C_X \rightarrow C$.

Exercise 4. (*Homotopy equivalence in chain complexes*) We recall that two morphisms of chain complexes $f, g : X \rightarrow Y$ are *chain homotopic* if there exists $h : X_* \rightarrow Y_{*+1}$ such that $dh + hd = f - g$. Let I be the chain complex concentrated in degree 0 et 1 given by $I_0 = R \oplus R$, $I_1 = R$ with differential $\partial(r) = (r, -r)$.

1. Give for any chain complex X , a factorisation $id \coprod id : X \coprod X \rightarrow X$ of the form

$$X \sqcup X \xrightarrow{i_0 \coprod i_1} X \otimes I \xrightarrow{\sim} X$$

with $i_0(r) = (r, 0, 0)$ et $i_1(r) = (0, r, 0)$.

2. Prove that two chain complexes morphisms $f, g : X \rightarrow Y$ are chain homotopic if and only if there exists a chain complex morphism $H : X \otimes I \rightarrow Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.
3. Prove that if X is cofibrant, f and g are chain homotopic if and only if they are homotopic in the sense of the model structure. (Hint: use that every chain complex is fibrant and that $X \otimes I$ is a strong cylinder object).
4. Deduce that for any complex A , we have a natural (in A) isomorphism

$$H_n(A) \cong \text{Hom}_{\mathbf{Ho}(Ch(R))}(S^n(R), A).$$

Exercise 5. The goal of the exercise is to prove that liftings associated to a model structure are actually unique up to homotopy.

Let $(\mathcal{C}, \mathcal{W}, \text{Cof}, \text{Fib})$ be a model category, and $A \xrightarrow{\varphi} Y$ be a morphism. We denote $\mathcal{C}_\varphi^{A/}$ the category of objects of \mathcal{C} under A and over φ . That is, an object of $\mathcal{C}_\varphi^{A/}$ is an object $X \in \mathcal{C}$ together with two morphisms $i_X : A \rightarrow X$ and $p_X : X \rightarrow Y$ satisfying $\varphi = p_X \circ i_X$. A morphism $(X, i_X, p_X) \rightarrow (Z, i_Z, p_Z)$ is a morphism $f : X \rightarrow Z$ such that the following diagram commutes:

$$\begin{array}{ccc} & X & \xrightarrow{p_X} Y \\ i_X \nearrow & \downarrow f & \nwarrow p_Z \\ A & \xrightarrow{i_Z} & Z \end{array}$$

1. Prove that $\mathcal{C}_\varphi^{A/}$ is complete and cocomplete and that it has a model structure such that the weak equivalences, cofibrations and fibrations are the morphisms $f : X \rightarrow Z$ that are respectively weak equivalences, cofibrations and fibrations in \mathcal{C} .
2. Determine a necessary and sufficient condition on $A \xrightarrow{i_X} X$ for (X, i_X, p_X) to be cofibrant in $\mathcal{C}_\varphi^{A/}$.
3. Consider a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

in which i is a cofibration and p an acyclic fibration. Prove that if $h, h' : B \rightarrow X$ are two lifts in the diagram, then they are homotopic in $\mathcal{C}_\varphi^{A/}$ for a well chosen φ (hint: consider the induced morphism $B \cup_A B \rightarrow B$). Are they homotopic in \mathcal{C} ?

Exercise 6. Let \mathcal{C} be a category equipped with two model structures: $(\mathcal{W}_1, \text{Cof}_1, \text{Fib}_1)$ and $(\mathcal{W}_2, \text{Cof}_2, \text{Fib}_2)$. Assume that $\mathcal{W}_1 \subseteq \mathcal{W}_2$ and $\text{Fib}_1 \subseteq \text{Fib}_2$. The *mixed structure* $(\mathcal{W}_m, \text{Cof}_m, \text{Fib}_m)$ is defined by setting $\mathcal{W}_m = \mathcal{W}_2$ and $\text{Fib}_m = \text{Fib}_1$. The mixed cofibrations Cof_m are defined via the lifting property.

1. Show that $\text{Cof}_2 \subseteq \text{Cof}_m \subseteq \text{Cof}_1$.
2. Prove that $\text{Cof}_m \cap \mathcal{W}_m = \text{Cof}_1 \cap \mathcal{W}_1$. (*Hint: Use MC3 and MC5. Start with \supseteq*)
3. Prove that the mixed structure $(\mathcal{W}_m, \text{Cof}_m, \text{Fib}_m)$ is a model category structure.
4. A map f is called a *special mixed cofibration* if there exist $i \in \text{Cof}_2$ and $j \in \text{Cof}_1 \cap \mathcal{W}_1$ such that $f = j \circ i$. Show that:
 - Every special mixed cofibration is a mixed cofibration,
 - Every mixed cofibration is a retract of a special mixed cofibration (*Hint : MC5 twice*).
5. A model category is called *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. Deduce that if $(\mathcal{W}_2, \text{Cof}_2, \text{Fib}_2)$ is left proper, then the mixed structure is also left proper.

Exercise 7 (Model category of equivalence relations). Let $\mathcal{E}q$ be the category whose objects are pairs (X, \sim) where X is a set and \sim is an equivalence relation on X , and whose morphisms are maps which preserve equivalence, i.e.:

$$\text{Hom}_{\mathcal{E}q}((X, \sim_X), (Y, \sim_Y)) := \{f : X \rightarrow Y \mid \forall x, x' \in X, x \sim_X x' \implies f(x) \sim_Y f(x')\}.$$

We will often allow ourselves the notational shortcut $X = (X, \sim_X)$, $Y = (Y, \sim_Y)$, etc.

1. Prove that the categorical product is given by $(X, \sim_X) \times (Y, \sim_Y) = (X \times Y, \sim_{X \times Y})$, where:

$$(x, y) \sim_{X \times Y} (x', y') \iff (x \sim_X x' \text{ and } y \sim_Y y').$$

2. Let $A = \{a, b, c\}$ with $a \sim b \neq c$; $B = \{x, y\}$ with $x \sim y$; and $C = \{u, v\}$ with $u \neq v$. Let $f : C \rightarrow A$ be given by $f(u) = b$, $f(v) = c$, and $g : C \rightarrow B$ be given by $g(u) = x$ and $g(v) = y$. Prove that in the pushout $A \cup_C B$, one has $a \sim c$. (A picture can help.)

For $X \in \mathcal{E}q$ and $x \in X$, we let $[x] = \{x' \in X \mid x' \sim_X x\}$ and $(X/\sim) := \{[x] \mid x \in X\}$. For any $X, Y \in \mathcal{E}q$, a morphism $f : X \rightarrow Y$ in $\mathcal{E}q$ is called a:

- **Cofibration** if $f : X \rightarrow Y$ is injective as a map of sets.
 - **Fibration** if, for all $x \in X$, the restriction $f|_{[x]} : [x] \rightarrow [f(x)]$ is surjective.
 - **Weak equivalence** if the induced map on the quotient $f_* : (X/\sim) \rightarrow (Y/\sim)$ is bijective.
3. Let $j : \{0\} \rightarrow (\{0, 1\}, \sim)$ with $0 \sim 1$. Prove that a morphism is a fibration if, and only if, it has the right lifting property against j . (You may not yet assume that $\mathcal{E}q$ is a model category.)
 4. Let $i_0 : \emptyset \rightarrow \{0\}$ and let $i_1 : (\{0, 1\}, \sim_1) \rightarrow (\{0, 1\}, \sim_2)$ where $0 \not\sim_1 1$ and $0 \sim_2 1$. Prove that a morphism is an acyclic fibration if, and only if, it has the right lifting property against i_0 and i_1 .
 5. Prove that two morphisms f, g are homotopic in $\mathcal{E}q$ if and only if $f \approx g$.
 6. Prove that the functor $\pi : \mathcal{E}q \rightarrow \mathbf{Set}$, given on objects by $X \mapsto X/\sim_X$, induces an equivalence of categories $\text{Ho}(\mathcal{E}q) \simeq \mathbf{Set}$.
 7. Prove that the pullback of a weak equivalence along a fibration is a weak equivalence.