TD 2 - Model structures on chain complexes

Let R be a commutative unital ring (or a commutative unital k-algebra). Recall that $Ch(R)$ is the category of (unbounded) chain complexes $(C_{i\in\mathbb{Z}}, d)$ of R-modules, and $Ch_{\geq 0}(R)$ its full subcategory of chain complexes concentrated in nonnegative degrees ($C_i = 0$ if $i < 0$). Finally, we denote by $Ch_{\leq 0}(R)$ the full subcategory of chain complexes concentrated in nonpositive degrees ($C_i = 0$ if $i > 0$); this subcategory is isomorphic (via $C^i = C_{-i}$) to the category of cochain complexes concentrated in nonnegative degrees.

Définition 0.1 (*Projective model structure*)**.** Let $\mathbf{C} = Ch(R)$ or $Ch_{\geq 0}(R)$. We define the so-called projective structure on $\mathbf{C}$ by setting

Weak equivalences $\mathcal{W}$ these are the quasi-isomorphisms (that is, morphisms of complexes inducing isomorphisms in homology).

Fibrations $\mathcal{F}$ these are the surjective morphisms of complexes (in every degree) in $Ch(R)$, and the morphisms of complexes that are surjective in every degree > 0 in $Ch_{\geq 0}(R)$.

Cofibrations $\mathcal{C}$ these are the morphisms of complexes having the left lifting property with respect to acyclic fibrations.

Dually, we define the so-called injective structure on $\mathbf{C}$ by setting $\mathcal{W}$ as the quasi-isomorphisms. Cofibrations are the injective morphisms of complexes (in every degree) in $Ch(R)$, and the morphisms of complexes that are injective in every degree < 0 in $Ch_{\leq 0}(R)$. Fibrations are the morphisms of complexes having the right lifting property with respect to acyclic cofibrations.

We admit the following theorem, and we will illustrate the structure thus defined in what follows.

Theorem 0.2. The above projective structures endow $Ch(R)$ and $Ch_{\geq 0}(R)$ with the structure of a model category. Moreover,

1. The cofibrations of $Ch_{\geq 0}(R)$ are exactly the degreewise inclusions whose cokernel is projective in every degree.
2. The cofibrations of $Ch(R)$ are the morphisms of complexes which are injective and have projective cokernel in every degree, and whose cokernel is cofibrant.
3. Any morphism of complexes in $Ch(R)$, injective in every degree, whose cokernel is a bounded below complex[1] of projective modules, is a cofibration.

Exercice 1.

1. State the analogue of the previous theorem for the injective structure.
2. Prove that in the projective structure, all objects are fibrant, and give the analogue in the injective structure.
3. Prove that if $f : A \rightarrowtail B$ is a cofibration, then $\mathrm{coker}(f)$ is cofibrant (hint: use a pushout).

Solution 1.

1. Simply replace cokernel by kernel, injective by surjective, projective by injective, and bounded below by bounded above.
2. All objects are fibrant in the projective structure because the terminal map $M \to 0$ is surjective.

[1] that is, $C_i = 0$ for $i \ll 0$

3. $\operatorname{coker} f$ is exactly the pushout of $A \to B$ along the zero map $A \to 0$. As cofibrations are preserved under pushouts, $0 \to \operatorname{coker} f$ is a cofibration and hence $\operatorname{coker} f$ is cofibrant.

In the following exercise, we will **not** use the characterization of cofibrations given by the theorem.

Exercice 2. Let R be a ring, and equip $Ch(R)$ with the projective structure.

1. We want to show that if a complex (C_*, d) is cofibrant, then C_n is projective for all n.
   (a) Show that morphisms of complexes $C_* \to D^{n+1}(M)$ are in bijection with linear maps $C_n \to M$.
   (b) Check, for all n, that $M \mapsto D^{n+1}(M)$ is a functor from R-modules to acyclic chain complexes.
   (c) Prove that C_* cofibrant implies that C_n is projective (hint: use the previous two questions to produce an acyclic fibration).
2. Prove that any cofibration $i : A_* \rightarrowtail B_*$ is degreewise injective. (Hint: check that $D^{n+1}(A_n) \to 0$ is an acyclic fibration and consider a morphism $A_* \to D^{n+1}(A_n) \to 0$.)
3. Deduce that for any cofibration $i : A_* \rightarrowtail B_*$, writing $P_* = \operatorname{coker}(i)$, one has $B_n \cong A_n \oplus P_n$ where P_n is projective.
4. The goal of this question is to show that a chain complex in $Ch(R)$ may be made of projective modules but be non-cofibrant.
   (a) Consider the quotient ring $R = K[x]/(x^2)$ where K is a field. For $n \in \mathbb{Z}$, set $P_n = R$ and consider the map $d_n : P_n \to P_{n-1}$ given by multiplication by x in R: $r \mapsto x \cdot r$. Prove that $P_* = (P_n, d_n)$ is a chain complex with zero homology. Deduce that if P_* is cofibrant, then $0 \to P_*$ is an acyclic cofibration.
   (b) Consider $p : R \to K = R/(x)$ the canonical projection, and equip K with the induced R-module structure. Prove that $S^0(p) : S^0(R) \to S^0(K)$ is a fibration in $Ch(R)$.
   (c) Prove that there exists a morphism of chain complexes $P_* \to S^0(K)$ which is equal to p in degree 0, and that, for any R-module M, if $u : P_* \to S^0(M)$ is a morphism of chain complexes, then $u(d_1(P_1)) = 0$.
   (d) Assuming that P_* is cofibrant, deduce a contradiction from the previous questions.
5. We want to prove that any chain complex $P_* \in Ch_{\geq 0}(R)$ consisting of projective modules is cofibrant. Let $f : X_* \overset{\sim}{\twoheadrightarrow} Y_*$ be an acyclic fibration in $Ch_{\geq 0}(R)$ and $\begin{array}{ccc} 0 & \longrightarrow & X_* \\ \downarrow & & \downarrow f \\ P_* & \underset{q}{\longrightarrow} & Y_* \end{array}$ a commutative diagram. We will construct a lift $\tilde{q}_* : P_* \to X_*$ by induction.
   (a) Prove that $f_0 : X_0 \to Y_0$ is surjective and that $\ker(f)$ is a chain complex with zero homology in every degree.
   (b) Using that P_0 is projective, construct a $\tilde{q}_0 : P_0 \to X_0$ which works.
   (c) Suppose now that $\tilde{q}_0, \dots, \tilde{q}_{n-1}$ have been constructed so that the diagrams commute and they commute with the differentials.
      i. Using that P_n is projective, construct an $h_n : P_n \to X_n$ making the diagram commute in degree n.
      ii. Prove that $d \circ h_n - \tilde{q}_{n-1} \circ d : P_n \to X_{n-1}$ takes values in $Z_{n-1}(\ker(f))$ (that is, the cycles of $\ker(f)$).

iii. Using that P_n is projective, show that there exists a map $\psi : P_n \to \ker(f)_n$ such that

$$d \circ \psi = d \circ h_n - \tilde{q}_{n-1} \circ d.$$

iv. Check that $\tilde{q}_n = h_n - \psi$ gives the desired lift.

Exercice 3 (Strong Cylinders)**.** Fix a model category $\mathcal{C}$ where every object is fibrant. A *strong* cylinder is a cylinder $X \sqcup X \rightarrowtail C \overset{\sim}{\twoheadrightarrow} X$ where the last arrow is acyclic fibration (for a general cylinder, we only ask for a weak equivalence). The goal of the exercise is to show that if C_X is a strong cylinder for X and two maps $f, g : X \to Y$ are left homotopic, they are left homotopic with C_X as a choice of cylinder.

1. Suppose f, g are left homotopic through $H : C \to Y$ where $C \overset{\sim}{\to} X$ is an arbitrary cylinder. Explain how we can reduce to the case where C is a strong cylinder (Hint : use the factorization axiom).

2. Conclude by finding a lift $C_X \to C$.

Solution 2. By $MC5$, factorize $C \overset{\sim}{\to} X$ by $C \overset{\sim}{\rightarrowtail} C' \overset{\sim}{\twoheadrightarrow} X$ (last fibration is a weak equivalence by two-out-of-three). We want to lift $C \to Y$ to $C' \to Y$. This can be done:

$$\begin{array}{ccc} C & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow\!\!\downarrow \\ C' & \longrightarrow & \bullet \end{array}$$

because $C \to C'$ is an acyclic cofibration and Y is fibrant. Thus, we lifted the left homotopy through a strong cylinder C'. Now let us show that we can get a homotopy through C_X. It suffices to find a map $C_X \to C$ which commutes with the structure maps $X \sqcup X \to C \to X$ and $X \sqcup X \to C_X \to X$. This is possible by finding a lift :

$$\begin{array}{ccc} X \sqcup X & \longrightarrow & C' \\ \downarrow & \nearrow & \downarrow\!\!\downarrow \\ C_X & \longrightarrow & X \end{array}$$

because $C' \overset{\sim}{\twoheadrightarrow} X$ is an acyclic fibration and $X \sqcup X \rightarrowtail C_X$ a cofibration.

Exercice 4. (*Homotopy equivalence in chain complexes*) We recall that two morphisms of chain complexes $f, g : X \to Y$ are *chain homotopic* if there exists $h : X_* \to Y_{*+1}$ such that $dh + hd = f - g$. Let I be the chain complex concentraed in degree 0 et 1 given by $I_0 = R \oplus R$, $I_1 = R$ with differential $\partial(r) = (r, -r)$.

1. Give for any chain complex X, a factorisation $id \coprod id : X \coprod X \to X$ of the form

$$X \sqcup X \overset{i_0 \coprod i_1}{\hookrightarrow} X \otimes I \overset{\sim}{\twoheadrightarrow} X$$

with $i_0(r) = (r, 0, 0)$ et $i_1(r) = (0, r, 0)$.

2. Prove that two chain complexes morphisms $f, g : X \to Y$ are chain homotopic if and only if there exists a chain complex morphism $H : X \otimes I \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

3. Prove that if X is cofibrant, f and g are chain homotopic if and only if they are homotopic in the sense of the model structure. (Hint: use that every chain complex is fibrant and that $X \otimes I$ is a strong cylinder object).

4. Deduce that for any complex A, we have a natural (in A) isomorphism

$$H_n(A) \cong \mathrm{Hom}_{\mathbf{Ho}(Ch(R))}(S^n(R), A).$$

Solution 3. 1.

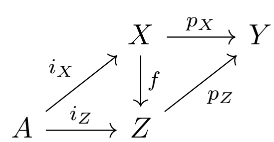
2.

3. Suppose f and g are chain homotopic. By the previous question, there is a map $H : X \otimes I \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. We need to check that $X \otimes I$ is a cylinder to deduce that they are left homotopic. For this it must be checked that $i : X \sqcup X \to X \otimes I$ is a cofibration. By the characterization of cofibrations of theorem 0.2, it is the case if and only if i is injective and $coker i$ is cofibrant. But i is obviously injective and a short computation shows that $coker i \simeq X[1]$ the shift of X. As X is cofibrant, $X[1]$ is degreewise projective and hence cofibrant. Finally, as X is cofibrant and Y projective, being left homotopic is equivalent to being homotopic.
Let's do the converse. Suppose f and g are homotopic. Hence they are left homotopic through an arbitrary cylinder C of X. By the previous exercise, they are homotopic through $X \otimes I$ which is a strong cylinder as the map $X \otimes I \to X$ is a fibration by question 1.

4. By the previous questions, $Hom_{Ho(Ch(R))}(S^nR, A) \simeq Hom_{Ch(R)}(S^nR, A)/(\text{chain homotopy})$. A chain map $S^nR \to A$ amounts to the choice of an element $a \in A_n$ such that $da = 0$, i.e. $Hom(S^nR, A) \simeq Z_n(A)$. But a and b in $Z_n(A)$ seen as maps $S^nR \to A$ are chain homotopic if and only if there exists $h \in A_{n+1}$ such that $a - b = dh$, i.e. $a - b \in B_nA$. Hence $Hom_{\mathbf{Ho}(Ch(R))}(S^nR, A) \simeq H_nA$.

Exercice 5. The goal of the exercise is to prove that liftings associated to a model structure are actually unique up to homotopy.

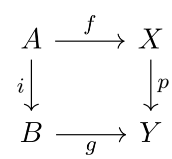
Let $(\mathcal{C}, \mathcal{W}, \mathrm{Cof}, \mathrm{Fib})$ be a model category, and $A \xrightarrow{\varphi} Y$ be a morphism. We denote $\mathcal{C}_\varphi^{A/}$ the category of objects of $\mathcal{C}$ under A and over φ. That is, an object of $\mathcal{C}_\varphi^{A/}$ is an object $X \in \mathcal{C}$ together with two morphisms $i_X : A \to X$ and $p_X : X \to Y$ satisfying $\varphi = p_X \circ i_X$. A morphism $(X, i_X, p_X) \to (Z, i_Z, p_Z)$ is a morphism $f : X \to Z$ such that the following diagram commutes:



1. Prove that $\mathcal{C}_\varphi^{A/}$ is complete and cocomplete and that it has a model structure such that the weak equivalences, cofibrations and fibrations are the morphisms $f : X \to Z$ that are respectively weak equivalences, cofibrations and fibrations in $\mathcal{C}$.

2. Determine a necessary and sufficient condition on $A \xrightarrow{i_X} X$ for (X, i_X, p_X) to be cofibrant in $\mathcal{C}_\varphi^{A/}$.

3. Consider a commutative square



in which i is a cofibration and p an acyclic fibration. Prove that if $h, h' : B \to X$ are two lifts in the diagram, then they are homotopic in $\mathcal{C}_\varphi^{A/}$ for a well chosen φ (hint: consider the induced morphism $B \bigcup_A B \to B$). Are they homotopic in $\mathcal{C}$?

Solution 4. 1. Let $X : \mathcal{D} \to C_\varphi^{A/}$ be a diagram, then in particular we get a diagram in the cone category $c(\mathcal{D})$ which has an additional initial object, and the data of the maps $X(i) \to X(j)$ and the canonical maps $i_{X(i)} : A \to X(i)$ determines a $c(\mathcal{D})$-diagram (still denoted X) for which we can take the colimit. This colimit has a canonical map from A. The collection of maps $p_{X(i)} : X(i) \to Y$ assembles to give a map from $\operatorname{colim}_{c(\mathcal{D})} X$ to Y whose composition with the map coming from A is indeed φ. We can proceed dually for limits.

The (co)completeness being proved, all other axioms of a model category follow from their analogues in $\mathcal{C}$, since they only require the data of maps $X \to Y$ in $\mathcal{C}$ sitting in commutative diagrams with respect to A and Y. These diagrams do not affect composition, retracts, lifting, or factorization properties.

2. The above construction shows that the initial object of $C_\varphi^{A/}$ is $A \xrightarrow{\text{id}} A \xrightarrow{\varphi} Y$. Therefore a cofibrant object is given by (X, i_X, p_X) and a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle p_X} \\ X & \xrightarrow[i_X]{} & X \end{array}$$

for which the vertical map, which is i_X, is a cofibration. Hence a cofibrant object is (X, i_X, p_X) with $i_X : A \to X$ a cofibration. And similarly (X, i_X, p_X) is fibrant iff $p_X : X \to Y$ is a fibration.

3. Let $h, h' : B \to X$ be two lifts. Choose $\varphi = p \circ f = g \circ i$. In the $C_\varphi^{A/}$ model structure, X is thus fibrant and B is cofibrant. Therefore h and h' being homotopic is equivalent to them being left homotopic. The coproduct of B with itself in this category is just $B \cup_A B$ (by question 1) with canonical structure maps $A \to B \cup_A B$ and $g \cup_A g \to Y$. By definition of lifts we have a well-defined map $h \cup_A h' : B \cup_A B \to X$, and we consider the canonical map $\text{id} \cup_A \text{id} : B \cup_A B \to B$. Factoring this as $B \cup_A B \to C \xrightarrow{g} B$ gives a cylinder object for $(B, i, g) \in C_\varphi^{A/}$. Therefore we obtain a commutative diagram

$$\begin{array}{ccc} B \cup_A B & \xrightarrow{h \cup_A h'} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[g \circ q]{} & Y \end{array}$$

Exercice 6. Let $\mathcal{C}$ be a category equipped with two model structures: $(\mathcal{W}_1, \text{Cof}_1, \text{Fib}_1)$ and $(\mathcal{W}_2, \text{Cof}_2, \text{Fib}_2)$. Assume that $\mathcal{W}_1 \subseteq \mathcal{W}_2$ and $\text{Fib}_1 \subseteq \text{Fib}_2$. The *mixed structure* $(W_m, \text{Cof}_m, \text{Fib}_m)$ is defined by setting $W_m = W_2$ and $\text{Fib}_m = \text{Fib}_1$. The mixed cofibrations Cof_m are defined via the lifting property.

1. Show that $\text{Cof}_2 \subseteq \text{Cof}_m \subseteq \text{Cof}_1$.

2. Prove that $\text{Cof}_m \cap \mathcal{W}_m = \text{Cof}_1 \cap \mathcal{W}_1$. *(Hint: Use MC3 and MC5. Start with $\supset$)*

3. Prove that the mixed structure $(\mathcal{W}_m, \text{Cof}_m, \text{Fib}_m)$ is a model category structure.

4. A map f is called a *special mixed cofibration* if there exist $i \in \text{Cof}_2$ and $j \in \text{Cof}_1 \cap \mathcal{W}_1$ such that $f = j \circ i$. Show that:
   - Every special mixed cofibration is a mixed cofibration,
   - Every mixed cofibration is a retract of a special mixed cofibration (Hint : MC5 twice).

5. A model category is called *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. Deduce that if $(\mathcal{W}_2, \mathrm{Cof}_2, \mathrm{Fib}_2)$ is left proper, then the mixed structure is also left proper.

Solution 5. 1. Note that if $C \subset D$, then the class of maps with left lifting properties satisfy the inverse inclusion $\mathrm{LLP}(C) \supset \mathrm{LLP}(D)$. In a model category we have that $\mathrm{Cof} = \mathrm{LLP}(\mathrm{Fib} \cap W)$. Since $\mathrm{Fib}_m \subset \mathrm{Fib}_2$ and $W_m = W_2$, we have $\mathrm{Fib}_m \cap W_m \subset \mathrm{Fib}_2 \cap W_2$, hence $\mathrm{Cof}_m \supset \mathrm{Cof}_2$. Using $\mathrm{Fib}_m = \mathrm{Fib}_1$ and $W_1 \subset W_2$, we obtain the other inclusion.

2. Let us first prove $\mathrm{Cof}_m \cap W_m \supset \mathrm{Cof}_1 \cap W_1$. Note that $\mathrm{Cof}_1 \cap W_1 = \mathrm{LLP}(\mathrm{Fib}_1)$ since $(W_1, \mathrm{Cof}_1, \mathrm{Fib}_1)$ is a model structure by assumption. In particular a map in Cof_1 has LLP with respect to any map in $\mathrm{Fib}_1 \cap W_2 = \mathrm{Fib}_m \cap W_m$. Therefore it lies in Cof_m. Further $W_1 \subset W_m = W_2$, hence $\mathrm{Cof}_1 \cap W_1 \subset W_m$ and we get the claimed inclusion.

   The reverse inclusion requires more work. Let $f \in \mathrm{Cof}_m \cap W_m$. We factor it as $f = A \xrightarrow{\sim} C \to B$ with respect to $(W_1, \mathrm{Cof}_1, \mathrm{Fib}_1)$ (which is a model structure). Since $f \in W_2 = W_m$ and $W_1 \subset W_2$, the 2-out-of-3 property guarantees that $C \to B$ is also in W_2. And since $\mathrm{Fib}_1 \subset \mathrm{Fib}_2$, we get that $C \to B$ belongs to $\mathrm{Fib}_1 \cap W_2 = \mathrm{Fib}_m \cap W_m$. Hence f (which lies in Cof_m) has LLP with respect to $C \to B$. This yields a lift

$$\begin{array}{ccc} A & \xrightarrow{\sim} & C \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow \\ B & = & B \end{array}$$

   which exhibits $f : A \to B$ as a retract of $A \xrightarrow{\sim} C$, hence it is indeed in $\mathrm{Cof}_1 \cap W_1$ by retract stability of acyclic cofibrations.

3. The (co)completeness axioms and 2-out-of-3 property are immediate since they hold for the second model structure. The retract stability of $\mathrm{Fib}_m = \mathrm{Fib}_1$ and $W_m = W_2$ follows from those of the first two model structures. The retract stability for cofibrations follows from cofibrations being defined as solutions to a LLP property, which is automatic, as is the lifting property of mixed cofibrations with respect to acyclic mixed fibrations by definition. Question 2 shows that $\mathrm{Cof}_m \cap W_m = \mathrm{Cof}_1 \cap W_1 = \mathrm{LLP}(\mathrm{Fib}_1)$, and since $\mathrm{Fib}_m = \mathrm{Fib}_1$, the second lifting axiom is satisfied.

   Only the factorization axiom remains. Given any morphism $f : X \to Y$, the first model structure gives a factorization $f : X \to A \to Y$ where the first map is in $\mathrm{Cof}_1 \cap W_1 = \mathrm{Cof}_m \cap W_m$ and the second is in $\mathrm{Fib}_1 = \mathrm{Fib}_m$. Hence the first factorization is trivial.

   For the second factorization, we factor f using the second model structure as $f = X \xrightarrow{2} B \xrightarrow{2} C$, where the subscript 2 indicates (co)fibrations in the second model structure. Now we also factor $B \xrightarrow{2} C$ using the first model structure as $B \xrightarrow{1} B' \xrightarrow{1} C$, so that f is factored as

$$f = (X \xrightarrow{2} B \xrightarrow{1} B') \xrightarrow{1} C.$$

   But $\mathrm{Cof}_2 \subset \mathrm{Cof}_m$ and $\mathrm{Cof}_1 \cap W_1 \subset \mathrm{Cof}_m$, hence the first part of the factorization is a mixed cofibration. It remains to prove that $B' \xrightarrow{1} C$ is also in $W_m = W_2$ (since $\mathrm{Fib}_m = \mathrm{Fib}_1$). By the 2-out-of-3 property, since $B \xrightarrow{2} C$ is in W_2 and $B \xrightarrow{1} B'$ is in $W_1 \subset W_2$, we conclude that indeed $B' \xrightarrow{1} C \in W_2$, completing the proof (all factorizations being given by the functors coming from both model structures in the above construction).

4. Let $f = j \circ i$ with $i \in \mathrm{Cof}_2$ and $j \in \mathrm{Cof}_1 \cap W_1$. By question 2, j is an (acyclic) mixed cofibration, and by question 1, i is a mixed cofibration. Hence their composition is a mixed cofibration.

Now let f be a mixed cofibration. We apply the same idea as in the proof of the factorization axiom. Factor $f : A \to C$ as $f = A \xrightarrow{2} B \xrightarrow{2} C$, and factor $B \xrightarrow{2} C$ as $B \xrightarrow{1} B' \xrightarrow{1} C$, so that f is factored as

$$f = (A \xrightarrow{2} B \xrightarrow{1} B') \xrightarrow{1} C.$$

But $\mathrm{Cof}_2 \subset \mathrm{Cof}_m$ and $\mathrm{Cof}_1 \cap W_1 \subset \mathrm{Cof}_m$, hence the first part of the factorization is a special mixed cofibration. By the 2-out-of-3 property, we again have that $B' \xrightarrow{1} C \in W_2$. Hence it is an acyclic mixed fibration (since $\mathrm{Fib}_m = \mathrm{Fib}_1$) and has the lifting property with respect to f. We deduce that f is a retract of $A \xrightarrow{2} B \xrightarrow{1} B'$.

5. Since the pushout of a retract is a retract, it is enough to prove the result for pushout along a special mixed cofibration. Let $A \xrightarrow{2} B \underset{\sim}{\xrightarrow{1}} C$ be such and $f : A \xrightarrow{m} Y$ be a mixed weak-equivalence (which is thus in $W_2 = W_m$). The pushout $B \cup_A Y$ splits as the composition of two pushout squares:

$$\begin{array}{ccc} A & \xhookrightarrow{\;2\;} & B \\ {\scriptstyle \sim m}\big\downarrow & & \big\downarrow{\scriptstyle \sim m} \\ Y & \longrightarrow & Y \cup_A B \\ & & \big\downarrow{\scriptstyle \sim 1} \\ & & Y \cup_A C \end{array}$$

where the middle vertical arrow is in $W_m = W_2$ by the left properness of the second model structure and the right horizontal arrow is an acyclic cofibration for the first structure as a pushout of such. Then by the 2-out-of-3 property (using that $W_1 \subset W_2$), the right vertical map is in $W_2 = W_m$, which concludes.

Exercice 7 (Model category of equivalence relations)**.** Let $\mathcal{E}q$ be the category whose objects are pairs $(X, \sim)$ where X is a set and $\sim$ is an equivalence relation on X, and whose morphisms are maps which preserve equivalence, i.e.:

$$\mathrm{Hom}_{\mathcal{E}q}((X, \sim_X), (Y, \sim_Y)) := \{f : X \to Y \mid \forall x, x' \in X,\, x \sim_X x' \implies f(x) \sim_Y f(x')\}.$$

We will often allow ourselves the notational shortcut $X = (X, \sim_X)$, $Y = (Y, \sim_Y)$, etc.

1. Prove that the categorical product is given by $(X, \sim_X) \times (Y, \sim_Y) = (X \times Y, \sim_{X\times Y})$, where:

$$(x, y) \sim_{X\times Y} (x', y') \iff (x \sim_X x' \text{ and } y \sim_Y y').$$

2. Let $A = \{a, b, c\}$ with $a \sim b \neq c$; $B = \{x, y\}$ with $x \sim y$; and $C = \{u, v\}$ with $u \neq v$. Let $f : C \to A$ be given by $f(u) = b$, $f(v) = c$, and $g : C \to B$ be given by $g(u) = x$ and $g(v) = y$. Prove that in the pushout $A \cup_C B$, one has $a \sim c$. (A picture can help.)

   For $X \in \mathcal{E}q$ and $x \in X$, we let $[x] = \{x' \in X \mid x' \sim_X x\}$ and $(X/\sim) := \{[x] \mid x \in X\}$. For any $X, Y \in \mathcal{E}q$, a morphism $f : X \to Y$ in $\mathcal{E}q$ is called a:

   - **Cofibration** if $f : X \to Y$ is injective as a map of sets.
   - **Fibration** if, for all $x \in X$, the restriction $f|_{[x]} : [x] \to [f(x)]$ is surjective.
   - **Weak equivalence** if the induced map on the quotient $f_* : (X/\sim) \to (Y/\sim)$ is bijective.

3. Let $j : \{0\} \to (\{0, 1\}, \sim)$ with $0 \sim 1$. Prove that a morphism is a fibration if, and only if, it has the right lifting property against j. (You may not yet assume that $\mathcal{E}q$ is a model category.)

4. Let $i_0 : \emptyset \to \{0\}$ and let $i_1 : (\{0,1\}, \sim_1) \to (\{0,1\}, \sim_2)$ where $0 \not\sim_1 1$ and $0 \sim_2 1$. Prove that a morphism is an acyclic fibration if, and only if, it has the right lifting property against i_0 and i_1.

5. Prove that two morphisms f, g are homotopic in $\mathcal{E}q$ if and only if $f \approx g$.

6. Prove that the functor $\pi : \mathcal{E}q \to \mathbf{Set}$, given on objects by $X \mapsto X/\sim_X$, induces an equivalence of categories $\mathrm{Ho}(\mathcal{E}q) \simeq \mathbf{Set}$.

7. Prove that the pullback of a weak equivalence along a fibration is a weak equivalence.