TD 2 - Localizations and model structures

Exercice 1 (Model structures on the category of sets)**.** Let Sets denote the category of sets.

1. Show that $(\mathrm{Sets}, \mathcal{W} = \text{bijections}, \mathcal{F} = All, \mathcal{C} = All)$ determines a model structure.

2. Do the same for $(\mathrm{Sets}, \mathcal{W} = All, \mathcal{F} = \text{surjections}, \mathcal{C} = \text{injections})$

In fact, there are precisely nine model structures in the category of sets. See link.

Exercice 2 (Universal property of localization)**.** Let $\mathcal{C}$ be a small category and $\mathcal{W}$ a subset of the set of morphisms in of $\mathcal{C}$. A localization of $\mathcal{C}$ with respect to $\mathcal{W}$ is a category $C[W^-1]$ together with a functor

$$l : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

satisfying the following universal property: For any category $\mathcal{D}$, composition with l:

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, D)$$

is a fully faithful functor and its essential image consists of those functors $\mathcal{C} \to \mathcal{D}$ sending $\mathcal{W}$ to isomorphisms. In other words, l, if it exists if the universal functor sending $\mathcal{W}$ to isomorphisms.

1. Check that $\mathcal{C}[\mathcal{W}^{-1}]$, if it exists, is unique up to canonical equivalences of categories.

2. Show that when $\mathcal{C}$ is the category with a single object $*$ and a monoid M of endomorphisms, and $\mathcal{W} = M$ then $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the category with one object $*$ and M^+ as endomorphisms, with M^+ the group completion of M.

Indication: we recall that the group completion M^+ of a monoid is a group M^+ together with a monoid map $can : M \to M^+$ such that for any monoid map $\phi : M \to G$, there is a unique group morphism $\tilde{\phi} : M^+ \to G$ factorizing ϕ, that is $\phi = \tilde{\phi} \circ can$. It is usual abstract nonsense to prove it is unique up (to unique if one requires that the isomorphisms commutes with the structure maps from M to the completion) isomorphism. To prove the existence of M^+, it is enough to construct it which can be obtained by defining as a quotient of the free group on the generating set M by the obvious equivalence relation identifiying $m \star m'$ with $m \cdot m'$ if $\star$ is the product in the free group and $\cdot$ is the product in M.

Exercice 3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor having a right adjoint $G : \mathcal{D} \to \mathcal{C}$[1]. Let $\mathcal{W}$ denote the collection of morphisms f in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$. Show that the following are equivalent:

1. G is fully faithful;

2. The natural transformation $F \circ G \to Id_{\mathcal{D}}$ is an isomorphism;

3. The natural functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ is an equivalence of categories.

[1] $\mathcal{D}$ is said to be a reflexive subcategory of $\mathcal{C}$.

Exercice 4 (Model structure on slice categories, by Victor Saunier)**.** Let $X \in \mathcal{A}$ and $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on $\mathcal{A}$. We denote by $\mathcal{A}/X$ the category whose objects are maps $\alpha : Y \to X$ of $\mathcal{A}$ and whose morphisms are commutative triangles:

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & X \\ {\scriptstyle f}\downarrow & \nearrow_{\alpha'} & \\ Y' & & \end{array}$$

Similarly, we denote $\mathcal{C}/X$ (resp. $\mathcal{F}/X, \mathcal{W}/X$) the morphisms of $\mathcal{A}/X$ as above where $f \in \mathcal{C}$ (resp. $\mathcal{F}, \mathcal{W}$).

Show that $(\mathcal{C}/X, \mathcal{F}/X, \mathcal{W}/X)$ determines a model structure on $\mathcal{A}/X$. We call it the *slice model structure.*

What are the fibrant objects in the above described model structure? The cofibrant objects?

Exercice 5 (Model structures on vector spaces, after Najib Idrissi)**.** Let k be a field and denote $\mathrm{Vect}(k)$ the category of vector spaces over k. We will use that every vector space, even the infinite-dimensional ones, has a basis (a.k.a. the axiom of choice).

1. Let there be a commutative square :

$$\begin{array}{ccc} E & \xrightarrow{u} & V \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ F & \xrightarrow[v]{} & W \end{array}$$

   (a) Show that u factors through i if and only if $\ker i \subseteq \ker u$.

   (b) Show that v factors through p if and only if $\operatorname{im} v \subseteq \operatorname{im} p$.

   (c) Show that there exists a lift $F \to V$ if and only if both conditions are met.

2. (a) Show that $i \perp p$ if and only if at least one of i, p is surjective and at least one is injective.

   (b) Deduce what are the possibilities for $\mathrm{LLP}(\mathcal{W})$, when $\mathcal{W}$ is any class of arrows.

3. Suppose $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is a model structure on $\mathrm{Vect}(k)$.

   (a) Show that $\mathcal{W} = \mathrm{RLP}(\mathcal{F}) \circ \mathrm{LLP}(\mathcal{C})$. Using the above, what are the possibilities for $\mathcal{W}, \mathcal{C}, \mathcal{C} \cap \mathcal{W}$ and $\mathcal{F} \cap \mathcal{W}$?

   (b) Using that a model structure is fully determined by the data of $\mathcal{W}$ and $\mathcal{F}$, make a list of all the model structures on $\mathrm{Vect}(k)$.

Exercice 6. Let $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}$ be the category of abelian groups.

1. (Localization at a single prime) Let p be a prime. Show that the base change functor $- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}$ is a localization functor along the class $\mathcal{W}$ of all maps of abelian groups $f : X \to Y$ such that both $\ker f$ and $\operatorname{coker} f$ are p-torsion groups. (Hint: Use the flatness of $\mathbb{Z}[\frac{1}{p}]$ over $\mathbb{Z}$.)

2. Show that the map $\mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ sending $q \mapsto q \otimes 1$ is an isomorphism. Use this and the Exercice 3 to show that the category of $\mathbb{Q}$ vector spaces is a localization of the category of abelian groups.