TD 2 - LOCALIZATIONS AND MODEL STRUCTURES

Exercice 1 (Model structures on the category of sets)**.** Let Sets denote the category of sets.

1. Show that $(\text{Sets}, \mathcal{W} = \text{bijections}, \mathcal{F} = All, \mathcal{C} = All)$ determines a model structure.
2. Do the same for $(\text{Sets}, \mathcal{W} = All, \mathcal{F} = \text{surjections}, \mathcal{C} = \text{injections})$

In fact, there are precisely nine model structures in the category of sets. See link.

Solution. 1. Let's check the axioms of a model structure :

1. MC1. It's a well-known fact that the category of sets is complete and cocomplete.
2. MC2. Obvious for bijections.
3. MC3. Stability under retracts : Obvious for $\mathcal{C}$ and $\mathcal{F}$. For $\mathcal{W}$, Consider a retract diagram
$$\begin{tikzcd} A \arrow[r, "i"'] \arrow[d, "q"] & B \arrow[r, "r"'] \arrow[d, "f"] & A \arrow[d, "q"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "r'"] & A' \end{tikzcd}$$
Suppose f is a bijection. Then $rq^{-1}i'$ and $iq^{-1}r$ are respectively right and left inverses of f, so f is a bijection.
4. MC4. The lifting condition is easy in both cases since it suffices to compose with the inverse i^{-1} (in the case $i \in \mathcal{C} \cap \mathcal{W}$)) and with p^{-1} in the case $p \in \mathcal{F} \cap \mathcal{W}$.
5. MC5. The factorization $X \to X \to Y$ of $X \to Y$ is an acyclic cofibration followed by a fibration; whereas $X \to Y \to Y$ is a cofibration followed by an acyclic fibration.

Let us check the axioms for the second model structure :

1. MC1. Same as 1.
2. MC2. Obvious since $\mathcal{W} = \text{All}$.
3. MC3. Obvious for $\mathcal{W} = \text{All}$. If q is an injection, it has a left inverse and thus one can construct a left inverse of f following the proof of 1. If q is a surjection we can construct similarly a right inverse for f.
4. MC4. The lifting property amounts to check that for any square diagram
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r, "g"'] \arrow[ur, dashed, "\exists"] & D \end{tikzcd}$$
where i is an injection and p a surjection, there always exists a dotted lift. This amounts to find exactly a lift of the map $C - i(A) \to D$ to B.We can construct one cpostcomposing with a right inverse of the surjection $B \to D$ (which exists by axiom of choice).
5. MC5. $X \xrightarrow{f} Y$ factors as $X \xrightarrow{id} X \sqcup Y \xrightarrow{f \sqcup id_Y} Y$ which works for the two types of functorial factorizations we ask for.

Exercice 2 (Universal property of localization)**.** Let $\mathcal{C}$ be a small category and $\mathcal{W}$ a subset of the set of morphisms in of $\mathcal{C}$. A localization of $\mathcal{C}$ with respect to $\mathcal{W}$ is a category $C[W^{-}1]$ together with a functor

$$l : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

satisfying the following universal property: For any category $\mathcal{D}$, composition with l:

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, D)$$

is a fully faithful functor and its essential image consists of those functors $\mathcal{C} \to \mathcal{D}$ sending $\mathcal{W}$ to isomorphisms. In other words, l, if it exists if the universal functor sending $\mathcal{W}$ to isomorphisms.

1. Check that $\mathcal{C}[\mathcal{W}^{-1}]$, if it exists, is unique up to canonical equivalences of categories.

2. Show that when $\mathcal{C}$ is the category with a single object $*$ and a monoid M of endomorphisms, and $\mathcal{W} = M$ then $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the category with one object $*$ and M^+ as endomorphisms, with M^+ the group completion of M.

Indication: we recall that the group completion M^+ of a monoid is a group M^+ together with a monoid map $can : M \to M^+$ such that for any monoid map $\phi : M \to G$, there is a unique group morphism $\tilde{\phi} : M^+ \to G$ factorizing ϕ, that is $\phi = \tilde{\phi} \circ can$. It is usual abstract nonsense to prove it is unique up (to unique if one requires that the isomorphisms commutes with the structure maps from M to the completion) isomorphism. To prove the existence of M^+, it is enough to construct it which can be obtained by defining as a quotient of the free group on the generating set M by the obvious equivalence relation identifiying $m \star m'$ with $m \cdot m'$ if $\star$ is the product in the free group and $\cdot$ is the product in M.

Solution. Note that a functor $l : \mathcal{C} \to \mathcal{D}$ is a localization if and only if it verifies the following two properties:

- For any functor $F : \mathcal{C} \to \mathcal{E}$ that sends $\mathcal{W}$ to isomorphisms, there exists $G : \mathcal{D} \to \mathcal{E}$ such that $F \cong G \circ l$.
- The map $- \circ l : \mathrm{Nat}(G_1, G_2) \to \mathrm{Nat}(G_1 \circ l, G_2 \circ l)$ is a bijection for all functors $G_1, G_2 : \mathcal{D} \to \mathcal{E}$.

1. Suppose $l : \mathcal{C} \to \mathcal{E}$ and $l' : \mathcal{C} \to \mathcal{E}'$ are two localizations along $\mathcal{W}$. By the universal property of these localizations, both l and l' are isomorphic to functors that invert $\mathcal{W}$. In particular, l' belongs to the essential image of the functor $- \circ l : \mathrm{Fun}(\mathcal{E}, \mathcal{E}') \to \mathrm{Fun}^{\mathcal{W}}(\mathcal{C}, \mathcal{E}')$, so that there exists a functor $G : \mathcal{E} \to \mathcal{E}'$ such that $l' \cong G \circ l$.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "l'"] \arrow[dr, "l"'] & & \mathcal{E}' \\ & \mathcal{E} \arrow[ur, dashed, "G"'] & \end{tikzcd}$$

Similarly, $l \cong H \circ l'$ for some functor $H : \mathcal{E}' \to \mathcal{E}$. Using the bijection

$$\mathrm{Nat}(\mathrm{Id}_{\mathcal{E}}, H \circ G) \cong \mathrm{Nat}(l, H \circ G \circ l) \cong \mathrm{Nat}(l, l),$$

one finds a natural transformation $\alpha : \mathrm{Id}_{\mathcal{E}} \Rightarrow H \circ G$ (given by the image of Id_l). Similarly, there is a natural transformation $\beta : H \circ G \Rightarrow \mathrm{Id}_{\mathcal{E}}$. Finally, using the bijection $\mathrm{Nat}(\mathrm{Id}_{\mathcal{E}}, \mathrm{Id}_{\mathcal{E}}) \cong \mathrm{Nat}(l, l)$, one gets $\beta \circ \alpha = \mathrm{Id}_{\mathrm{Id}_{\mathcal{E}}}$ and similarly for the other composition. Hence $H \circ G \cong \mathrm{Id}_{\mathcal{E}}$. Reasoning with $\mathcal{E}'$ in a similar manner shows that G and H realize an equivalence of categories $\mathcal{E} \simeq \mathcal{E}'$.

2. Recall that the group completion of a monoid is a group M^+ together with a monoid map

$$\iota : M \to M^+$$

such that for any monoid map $\phi : M \to G$, there is a unique group morphism $\tilde{\phi} : M^+ \to G$ factorizing ϕ, that is $\phi = \tilde{\phi} \circ \iota$. It is usual abstract nonsense to prove it is unique up to isomorphism (and there is a unique such isomorphism that commutes with the structure maps from M to the completion). To prove the existence of M^+, it is enough to construct it, which can be done by defining it as a quotient of the free group on the generating set M by the obvious equivalence relation identifying $m \star m'$ with $m \cdot m'$, where $\star$ is the product in the free group and $\cdot$ is the product in M.

We now come to the proof. The key fact to note is the following: the full subcategory of Cat spanned by the categories with a unique object is isomorphic to the category of monoids. Therefore, given a monoid A, we will also write A for the category with one object $*$ and $\mathrm{End}(*) = A$ as morphisms. Let M be a monoid. We now show that $\iota : M \to M^+$, viewed as a functor, satisfies the universal property of the localization of M along all morphisms. Let $\mathcal{D}$ be a category. Then any functor $F : M^+ \to \mathcal{D}$ factors through the full subcategory $\mathcal{D}_{F(*)}$ spanned by the image of $*$. Thus

$$\mathrm{Fun}(M^+, \mathcal{D}) \cong \coprod_{x \in \mathrm{ob}(\mathcal{D})} \mathrm{Fun}(M^+, \mathcal{D}_x)$$

as categories. Observe that any morphism of monoid $f : M^+ \to \mathrm{End}_{\mathcal{D}}(x)$ factors through the subgroup of units $\mathrm{Aut}_{\mathcal{D}}(x)$. Hence precomposing with ι inverts all morphisms in M^+. Now the universal property of the group completion gives that the functor

$$- \circ \iota : \mathrm{Fun}(M^+, \mathrm{Aut}_{\mathcal{D}}(x)) \longrightarrow \mathrm{Fun}^{\mathcal{W}}(M, \mathrm{Aut}_{\mathcal{D}}(x))$$

is bijective on objects, hence essential surjective. The category $\mathrm{Fun}(M^+, \mathrm{Aut}_{\mathcal{D}}(x))$ has objects given by the group morphisms $M^+ \to \mathrm{Aut}_{\mathcal{D}}(x)$; the morphisms between f and g are the elements $\alpha \in \mathrm{Aut}_{\mathcal{D}}(x)$ such that $g = \alpha f \alpha^{-1}$. Using this description, one easily sees that $- \circ \iota$ is fully faithful, hence an equivalence of categories.

Exercice 3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor having a right adjoint $G : \mathcal{D} \to \mathcal{C}$[1]. Let $\mathcal{W}$ denote the collection of morphisms f in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$. Show that the following are equivalent:

1. G is fully faithful;
2. The natural transformation $F \circ G \to Id_{\mathcal{D}}$ is an isomorphism;
3. The natural functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ is an equivalence of categories.

(Solution: That 1 implies 2 is standard for any adjunction:

$$\mathcal{D}(x, y) \cong \mathcal{C}(G(x), G(y)) \cong \mathcal{D}(F \circ G(x), y).$$

One can also see that 2 implies 1 by going in the other direction.
Let's prove 2 implies 3: The composed map $\mathcal{D} \xrightarrow{F} \mathcal{C} \to \mathcal{C}[W^{-1}] \to \mathcal{D}$ is $G \circ F$, so it is isomorphic to the identity. It suffices to show that $\mathcal{C}[W^{-1}] \to \mathcal{D} \xrightarrow{G} \mathcal{C} \to \mathcal{C}[W^{-1}]$ is isomorphic to the identity. By the universality of $\mathcal{C} \to \mathcal{C}[W^{-1}]$, it suffices to show that $\mathcal{C} \to \mathcal{C}[W^{-1}] \to \mathcal{D} \xrightarrow{G} \mathcal{C} \to \mathcal{C}[W^{-1}]$ is isomorphic to $\mathcal{C} \to \mathcal{C}[W^{-1}]$. From the composition $F \to FGF \to F = id_F$ given by the unit and counit, we deduce that the transformation $F \to FGF$ is an isomorphism since the second part is. This shows that $\mathcal{C} \to \mathcal{C}[W^{-1}] \to \mathcal{D} \xrightarrow{G} \mathcal{C} \to \mathcal{C}[W^{-1}]$ is indeed what we want.
Now let's prove that 3 implies 1. We have that $Hom(\mathcal{C}[W^{-1}], \mathcal{D}) \cong Hom(\mathcal{C}, \mathcal{D})$ is fully faithful, and by composition, we get that $F^* : Hom(\mathcal{D}, \mathcal{D}) \cong Hom(\mathcal{C}, \mathcal{D})$ is fully faithful. We want to show that $F \circ G \to Id$ is an isomorphism, which reduces to showing that $FGF \to F$ is an isomorphism, a consequence of the adjunction.)

[1] $\mathcal{D}$ is said to be a reflexive subcategory of $\mathcal{C}$.

Exercice 4 (Model structure on slice categories, by Victor Saunier)**.** Let $X \in \mathcal{A}$ and $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on $\mathcal{A}$. We denote by $\mathcal{A}/X$ the category whose objects are maps $\alpha : Y \to X$ of $\mathcal{A}$ and whose morphisms are commutative triangles:

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "f"'] & X \\ Y' \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

Similarly, we denote $\mathcal{C}/X$ (resp. $\mathcal{F}/X, \mathcal{W}/X$) the morphisms of $\mathcal{A}/X$ as above where $f \in \mathcal{C}$ (resp. $\mathcal{F}, \mathcal{W}$).

Show that $(\mathcal{C}/X, \mathcal{F}/X, \mathcal{W}/X)$ determines a model structure on $\mathcal{A}/X$. We call it the *slice model structure.*

What are the fibrant objects in the above described model structure? The cofibrant objects?

Let us check the axioms of a model category.

1. MC1. Colimits in $\mathcal{C}/X$ can be computed as colimits in $\mathcal{C}$ (for example, because the forgetful functor $\mathcal{C}/X \to \mathcal{C}$ has a right adjoint given by $A \mapsto A \times X$). For limits : if I is a small category and $D : I \to \mathcal{C}/X$, consider I^∇ the cocone of I (I to which we added a terminal object $*$). We have by construction a diagram $D^\nabla : I^\nabla \to \mathcal{C}$ where $*$ is sent to X. Then one can check that $\lim_{\mathcal{C}} D^\nabla$ is an object over X and is a limit of D in $\mathcal{C}/X$. So $\mathcal{C}/X$ is complete and cocomplete.

2. MC2. The 2 out of 3 from $\mathcal{C}$ gives the 2 out of 3 in $\mathcal{C}/X$ because weak equivalences are the same.

3. MC3. Automatic since retracts in $\mathcal{C}/X$ give retracts in $\mathcal{C}$.

4. MC4. Take a lifting problem in $\mathcal{C}/X$, for example :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "\wr i"] & B \arrow[d, two heads, "p"] \\ C \arrow[r, "g"'] \arrow[ur, dashed, "\exists"] & D \end{tikzcd}$$

   where all maps are maps over X.As $\mathcal{W}$, $\mathcal{C}$, $\mathcal{F}$ are the same as in $\mathcal{C}$, this gives a lifting problem in $\mathcal{C}$, so there exist a lifting (dotted arrow) φ in $\mathcal{C}$. There remains to check that this is a map over X. Call q_M the map $M \to X$ for any object M. We want to check that $q_B\varphi = q_A$. But $q_B\varphi = q_D p\varphi = q_D g = q_A$ so we are done.

5. MC5. We already have functorial factorizations in $\mathcal{C}$. We want to check they are maps over X. Let $f : A \to B$ be a map in $\mathcal{C}/X$. The factorization $A \overset{\sim}{\rightarrowtail} C_f \twoheadrightarrow B$ is automatically over X using the map $C_f \twoheadrightarrow B \xrightarrow{q_B} X$. The same holds for the second kind of factorizations.

Exercice 5 (Model structures on vector spaces, after Najib Idrissi)**.** Let k be a field and denote $\mathrm{Vect}(k)$ the category of vector spaces over k. We will use that every vector space, even the infinite-dimensional ones, has a basis (a.k.a. the axiom of choice).

1. Let there be a commutative square :

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "i"'] & V \arrow[d, "p"] \\ F \arrow[r, "v"'] & W \end{tikzcd}$$

   (a) Show that u factors through i if and only if $\ker i \subseteq \ker u$.

(b) Show that v factors through p if and only if $\operatorname{im} v \subseteq \operatorname{im} p$.

(c) Show that there exists a lift $F \to V$ if and only if both conditions are met.

2. (a) Show that $i \perp p$ if and only if at least one of i, p is surjective and at least one is injective.

   (b) Deduce what are the possibilities for $\mathrm{LLP}(\mathcal{W})$, when $\mathcal{W}$ is any class of arrows.

3. Suppose $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is a model structure on $\mathrm{Vect}(k)$.

   (a) Show that $\mathcal{W} = \mathrm{RLP}(\mathcal{F}) \circ \mathrm{LLP}(\mathcal{C})$. Using the above, what are the possibilities for $\mathcal{W}, \mathcal{C}, \mathcal{C} \cap \mathcal{W}$ and $\mathcal{F} \cap \mathcal{W}$?

   (b) Using that a model structure is fully determined by the data of $\mathcal{W}$ and $\mathcal{F}$, make a list of all the model structures on $\mathrm{Vect}(k)$.

Exercice 6. Let $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}$ be the category of abelian groups.

1. (Localization at a single prime) Let p be a prime. Show that the base change functor $- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}$ is a localization functor along the class $\mathcal{W}$ of all maps of abelian groups $f : X \to Y$ such that both $\ker f$ and $\operatorname{coker} f$ are p-torsion groups. (Hint: Use the flatness of $\mathbb{Z}[\frac{1}{p}]$ over $\mathbb{Z}$.)

2. Show that the map $\mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ sending $q \mapsto q \otimes 1$ is an isomorphism. Use this and the Exercice 3 to show that the category of $\mathbb{Q}$ vector spaces is a localization of the category of abelian groups.

Solution. We use exercise 3. We will check that the inclusion functor $\mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]} \to \mathbb{Z}\mathrm{Mod}$ is fully faithful. This amounts to check that $Hom_{\mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}}(A, B) = Hom_{\mathrm{Mod}_{\mathbb{Z}}}(A, B)$ for any two $\mathbb{Z}[\frac{1}{p}]-$modules A and B. The inclusion of the left term into the right term is automatic. Now take $f \in Hom_{\mathrm{Mod}_{\mathbb{Z}}}(A, B)$; We need to show it is automatically a morphism of $\mathbb{Z}[\frac{1}{p}]$-modules. But $pf(\frac{a}{p}) = f(p \cdot \frac{a}{p}) = f(a)$ so $f(\frac{a}{p}) = \frac{f(a)}{p}$ for all a and we are done. By Exercise 3, the base change functor

$$- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}] : \mathrm{Mod}_{\mathbb{Z}} \to \mathrm{Mod}_{\mathbb{Z}[\frac{1}{p}]}$$

is a localization along maps inducing isomorphism after tensoring by $\mathbb{Z}[\frac{1}{p}]$. Let's describe these maps. A map $A \to B$ is such if and only if $\ker(A \otimes \mathbb{Z}[\frac{1}{p}] \to B \otimes \mathbb{Z}[\frac{1}{p}]) = 0$ and $\operatorname{coker}(A \otimes \mathbb{Z}[\frac{1}{p}] \to B \otimes \mathbb{Z}[\frac{1}{p}]) = 0$. By flatness of $\mathbb{Z}[\frac{1}{p}]) = 0$ taking ker and coker commute with tensoring, so this amounts to ask that $\mathbb{Z}[1/p] \otimes \ker(A \to B)$ and $\mathbb{Z}[1/p] \otimes \operatorname{coker}(A \to B)$ are zero. This is true if and only if $\ker A \to B$ and $\operatorname{coker} A \to B$ are torsion p-groups.
2. The same arguments work by flatness of $\mathbb{Q}$ over $\mathbb{Z}$. The fact that a morphism of abelian groups between two $\mathbb{Q}$-vector spaces is automatically a $\mathbb{Q}$ linear map works with the same proof as before.