TD 3 - Quillen functors, Derived functors and (a bit of) homotopy colimits

Exercice 1. A model structure on Cat (Charles Rezk)

Denote Cat the category of small categories. We assume that it is complete and cocomplete. We let $\mathcal{W}$ denote equivalences of categories and $\mathcal{C}$ denote functors that are injective on objects. We let $\mathcal{F}$ denote functors $F : \mathcal{A} \to \mathcal{B}$ such that for every isomorphism $g : F(a) \to b$ of $\mathcal{B}$, there is a map $f : a \to a'$ with $g = F(f)$; such functors are called isofibrations.

1. Denote $*$ the category with one object and no non-trivial arrows, and I the category with two objects $0, 1$ and exactly one isomorphism in each direction. Let $i : * \to I$ be the inclusion at 0. Show that $\mathcal{F} = RLP(\{i\})$.
2. Show that $\mathcal{W}$ verifies 2-out-of-3, and that $\mathcal{W}, \mathcal{C}$ and $\mathcal{F}$ are stable under retracts.
3. (a) Show that every functor F of $\mathcal{C} \cap \mathcal{W}$ has a left inverse G which is also a quasi-inverse and such that the natural transformation $FG \simeq \mathrm{id}$ is equal to the identity on the image of F.
   (b) Deduce that $\mathcal{F} \subset RLP(\mathcal{C} \cap \mathcal{W})$.
4. Show that $\mathcal{C} \subset LLP(\mathcal{F} \cap \mathcal{W})$.
5. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Denote $\mathrm{Path}(F) := \mathcal{A} \times_{\mathcal{B}} \mathcal{B}^I$ where the map $\mathcal{B}^I \to \mathcal{B}$ is the source map, and $\mathrm{Cyl}(F) := \mathcal{A} \times (I \coprod_{\mathcal{A}} \mathcal{B})$. Show that F factors through $\mathrm{Path}(F)$ and $\mathrm{Cyl}(F)$; deduce that $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ is a model structure on Cat. What are the fibrant objects, the cofibrant objects?

Exercice 2 (Derived functors in homological algebra vs model categories)**.** The goal of this exercise is to understand how the model-categorical notion of derived functor generalizes what you have seen in homological algebra. Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be an additive functor between abelian categories. We suppose that F is right exact.

1. Consider the projective model structure on $Ch_{\geq 0}(\mathcal{A})$. Show that F sends quasi-isos between cofibrant objects to quasi-isos in $Ch_{\geq 0}\mathcal{B}$. Deduce that it has a total left derived functor in the model categorical sense.
2. Show that $\mathbb{L}F(V) \cong F(P)$ where P is a projective resolution of V.
3. What is the link between the homological-algebraic derived functors $L^iF(V)$ and $\mathbb{L}F(V)$?
4. Apply this to prove the existence and identify the total derived functors of $\mathrm{Hom}_R(-, M)$:
$$Ch_{\geq 0}(A)^{op} \to Ch_{\geq 0}(A).$$
   Identify them with the derived functors $\mathrm{Ext}^j(-,-)$ from the homological algebra course. Distinguish between the cases of projective and injective structures, and explain how this affects the computations.
5. Take R a commutative ring. Do the functors $- \otimes - : Ch_{\geq 0}(R) \times Ch_{\geq 0}(R) \to Ch_{\geq 0}(\mathcal{R})$ and $\mathrm{Hom}(-,-) : Ch_{\geq 0}(R)^{op} \times Ch_{\geq 0}(R) \to Ch_{\geq 0}(R)$ have total derived functors?

Exercice 3 (Composition of Derived Functors)**.** Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_3$ be functors and let $\mathcal{W}_i$ be a class of morphisms in $\mathcal{C}_i$.

1. Assuming all the relevant total left derived functors exist, use their universal properties to construct a natural transformation $\mathbb{L}G \circ \mathbb{L}F \to \mathbb{L}(G \circ F)$.

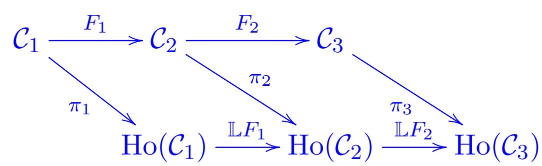
2. Suppose now that $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ are model categories and that F and G are left Quillen functors. Show that $G \circ F$ is a left Quillen functor.

3. Show that the arrow of 1) induces a natural equivalence $\mathbb{L}(G \circ F) \simeq (\mathbb{L}G) \circ (\mathbb{L}F)$.

4. Let (L, R) be an adjoint pair. Show that L is left Quillen if and only if R is right Quillen.

5. Suppose the restriction of a functor F to cofibrant objects sends acyclic cofibrations to weak equivalences, show that F is left derivable. (Hint: Ken Brown's lemma).

Solution 1. 1. Denote $\pi_i : \mathcal{C}_i \to \mathrm{Ho}(\mathcal{C}_i)$ the canonical functors. Let us recall that the total left derived functor $\mathbb{L}F_1 : \mathrm{Ho}(\mathcal{C}_1) \to \mathrm{Ho}(\mathcal{C}_2)$ come equipped with a natural transformation $\mathbb{L}F_1 \circ \pi_1 \to \pi_2 \circ F_1$ which is universal among such (it is a right Kan extension). In particular we have a commutative diagram



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and natural transformations, given forn any $X \in \mathcal{C}_1$ and $Y \in \mathcal{C}_2$ by $\mathbb{L}F_1(\pi_1(X)) \to \pi_2(F_1(X))$ and $\mathbb{L}F_2(\pi_2(Y) \to \pi_3(F_2(Y))$. Taking $Y = F_1(X)$, the commutativity of the diagram gives a natural transformation

$$\mathbb{L}F_2 \circ \mathbb{L}F_1(\pi_1(X)) \to \pi_3(F_2 \circ F_1(X))$$

hence by universal property we get a unique natural transformation $\mathbb{L}L_2 \circ \mathbb{L}F_1 \to \mathbb{L}(F_2 \circ F_1)$.

2. Let us now address the second question: first we remark that the composition of left Quillen functors is again a left Quillen functor. Indeed, by definition if F_1 preserves both cofibrations and acyclic cofibrations and F_2 also, clearly so does the composition $F_2 \circ F_1$. Therefore, by the theorem given in class, the model structures garantee the existence of $\mathbb{L}F_1$, $\mathbb{L}F_2$ and $\mathbb{L}(F_2 \circ F_1)$, given on objects, respectively by $\mathbb{L}F_1(X) = F_1(Q_1(X))$, $\mathbb{L}F_2(Y) = F_2(Q_2(Y))$ and $\mathbb{L}(F_2 \circ F_1)(X) = F_2(F_1(Q_1(X))$ where Q_1 is a cofibrant replacement functor in $\mathcal{C}_1$ and Q_2 is a cofibrant replacement in $\mathcal{C}_2$.

3. In this case the natural transformation $\mathbb{L}L_2 \circ \mathbb{L}F_1 \to \mathbb{L}(F_2 \circ F_1)$ is given on each object $X \in \mathcal{C}$ by a morphism

$$F_2(Q_2(F_1(Q_1(X))) \to F_2(F_1(Q_1(X))$$

We only have to notice that by construction (in fact, we have to unfold the proof given in class that the formula $\mathbb{L}F = F \circ Q$ has the universal property of total left derived functor) this morphism is the image under F_2 of the cofibrant-replacement

$$Q_2(F_1(Q_1(X)) \to F_1(Q_1(X)$$

which by definition is a weak-equivalence with both source and target cofibrant: the source is cofibrant by definition. The target is cofibrant because F_1 is a left Quillen functor so sends cofibrant objects to cofibrant objects. Therefore by Brown's lemma[1] its image under F_2 is a weak-equivalence and therefore an isomorphism in the homotopy category.

4. Assume L is left Quillen. We show R preserves fibrations and trivial fibrations.

Let $p : X \to Y$ be a fibration in the target model category. To prove $R(p)$ is a fibration, we verify it has the *right lifting property (RLP)* against all trivial cofibrations in the source category.

[1] it is always worth recalling that this lemma does imply that all left Quillen functors send all weak equivalences between cofibrant to weak equivalences and right Quillen functors send weak equivalences between fibrant to weak equivalences

Consider a lifting problem for a trivial cofibration $i : A \to B$:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & R(X) \\ i \downarrow & & \downarrow R(p) \\ B & \xrightarrow{\beta} & R(Y) \end{array}$$

By adjointness, this corresponds to a lifting problem in the target category:

$$\begin{array}{ccc} L(A) & \xrightarrow{\tilde{\alpha}} & X \\ L(i) \downarrow & & \downarrow p \\ L(B) & \xrightarrow{\tilde{\beta}} & Y \end{array}$$

Since L is left Quillen, $L(i)$ is a trivial cofibration. As p is a fibration, there exists a lift $h : L(B) \to X$. By adjunction, h induces a lift $h^\sharp : B \to R(X)$ in the source category, proving $R(p)$ is a fibration. The proof that R preserves trivial fibrations is exactly the same.

5. By assumption, the restriction of F to cofibrant objects sends acyclic cofibrations to weak equivalences. Applying Ken Brown's lemma, which states that a functor preserving acyclic cofibrations between cofibrant objects must preserve all weak equivalences between cofibrant objects, we conclude that F preserves all weak equivalences between cofibrant objects. To construct the left derived functor $\mathbb{L}F$, we define $\mathbb{L}F(X) = F(QX)$, where QX is a cofibrant replacement of X equipped with a weak equivalence $QX \xrightarrow{\sim} X$. Since F preserves weak equivalences between cofibrant objects, $\mathbb{L}F(X)$ is well-defined up to weak equivalence and does not depend on the choice of cofibrant replacement, because they are all equivalent. The universal property of $\mathbb{L}F$ follows because any functor G that preserves weak equivalences admits a natural transformation $\mathbb{L}G \to G$, induced by the weak equivalence $QX \xrightarrow{\sim} X$ and the fact that G preserves weak equivalences. This shows that $\mathbb{L}F$ is the left derived functor of F, and thus F is left derivable.

Exercice 4 (Slice categories II, by Victor Saunier)**.** Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on $\mathcal{A}$. Let $f : X \to Y$ be a morphism. Recall that we defined in the first exercise sheet a model structure on every slice category $\mathcal{A}/X$.

1. Show that the functor $f_! : \mathcal{A}/X \to \mathcal{A}/Y$ which postcomposes by f admits a right adjoint f^* and describe it.

2. Show that the pair $(f_!, f^*)$ is a Quillen pair of adjoints.

3. Suppose $\mathcal{A}$ is right proper, i.e. weak equivalences are stable under pullback by fibrations, and that $f \in \mathcal{W}$. Show that the pair $(f_!, f^*)$ is a Quillen equivalence.

4. (Rezk) Suppose that for every weak equivalence f, the pair $(f_!, f^*)$ is a Quillen equivalence. Show that $\mathcal{A}$ is right proper.

On homotopy (co)limits

Exercice 5 (Homotopy colimits)**.** In this exercise, we first deal with generalities on homotopy pushouts and then specialize to chain complexes with the projective model structure. Let $\mathcal{C}$ be a model category and let I be the category given by the diagram-shape

$$\begin{array}{ccc} b & \longrightarrow & c \\ \downarrow & & \\ a & & \end{array}$$

1. Let $f : X \to Y$ be a natural transformation of diagrams $X, Y \in \mathrm{Fun}(I, \mathcal{C})$. Show that f has the left lifting property with respect to all projective acyclic fibrations if and only if the the natural maps

$$X_a \coprod_{X_b} Y_b \to Y_a, \quad X_b \to Y_b, \quad X_c \coprod_{X_b} Y_b \to Y_c$$

are cofibrations in $\mathcal{C}$. (Here we mean the usual pushouts in $\mathcal{C}$.)
Deduce that a diagram $Y : I \to \mathcal{C}$ is cofibrant if and only if Y_b is cofibrant in $\mathcal{C}$ and the maps $Y_a \to Y_b$ and $Y_a \to Y_c$ are cofibrations. Moreover, show that $X \to Y$ has the left lifting property with respect to projective fibrations if and only the above three maps are acyclic cofibrations.

2. Show that the category of diagrams $\mathrm{Fun}(I, \mathcal{C})$ admits the projective model structure (without using the result seen in class that such a structure exists since I is very small).

3. Show that the colimit functor $\mathrm{colim}\colon \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$ is a left Quillen functor.

4. Assume that $\mathcal{C}$ is left proper (i.e. weak equivalences are stable under pushouts along cofibrations). Show that any pushout diagram

$$\begin{array}{ccc} B & \overset{f}{\rightarrowtail} & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \coprod_B C \end{array}$$

where $f\colon B \longrightarrow C$ a cofibration, is also a homotopy pushout diagram.

5. **Case of Topological spaces.** Assume now that $\mathcal{C} = \mathbf{Top}$.

   (a) Using that Top is proper (as seen in exercise 3. from the sheet on Quillen model structure), show that there is a canonical isomorphism

   $$\mathbb{L}\,\mathrm{colim}(X \leftarrow A \to Y) \cong X \coprod_A^{\mathrm{h}} Y = X \coprod_{A \times \{0\}} \mathrm{Cyl}(A \to Y)$$

   in Ho(Top) between the homotopy pushout computed by the projective model structure and the formula given by the mapping cylinder.

   (b) Give a formula for computing the homotopy colimit of a tower $(X_0 \to X_1 \to X_2 \to \dots)$ as well as the homotopy limit of a tower $(\dots \to Y_2 \to Y_1 \to Y_0)$.

6. **Case of chain complexes.** Assume now that $\mathcal{C}$ is the model category of chain complexes over a ring R.

   (a) Show that $\mathcal{C}$ is left proper.

   (b) Let $g : A \to B$ be a map of chain complexes. Recall that the *mapping cone* of g, denoted $C(g)$, is the chain complex given in level n by $B_n \oplus A_{n-1}$ and whose differential $B_{n+1} \oplus A_n \to B_n \oplus A_{n-1}$ is given $(b, a) \mapsto (\partial_B(b) + g(a), -\partial_A(a))$. Let I denote the chain complex given by $R \oplus R$ in degree 0 and R in degree 1 with differential given by $\partial_R : R \to R \oplus R$ given by $r \mapsto (-r, r)$. We define the *mapping cylinder* of g, denoted $\mathrm{Cyl}(g)$, as the pushout in chain complexes of

   $$\begin{array}{ccc} A & \overset{g}{\longrightarrow} & B \\ \big\downarrow{\scriptstyle i_0} & & \big\downarrow \\ I \otimes A & \longrightarrow & \mathrm{Cyl}(g) \end{array}$$

   where the vertical arrow $A \to I \otimes A$ is induced by the inclusion $i_0 : R \to I$ corresponding to the inclusion of the second factor $R \hookrightarrow R \oplus R$ in degree 0. The differential on $I \otimes A$ is given by $r \otimes a \mapsto \partial_R(r) \otimes a + (-1)^{\deg(r)} r \otimes \partial_A(a)$. Show that the mapping cone of g is the pushout of

$$\begin{array}{ccc} I \otimes A & \longrightarrow & \mathrm{Cyl}(g) \\ \downarrow & & \downarrow \\ C(\mathrm{Id}_A) & \longrightarrow & C(g). \end{array}$$

(c) Let Δ^1 be the category with two objects and one non trivial morphism in between them. Show that the construction of the mapping cone defines a functor $C : \mathrm{Fun}(\Delta^1, \mathrm{Ch}(R)) \to \mathrm{Ch}(R)$ sending natural transformations objectwise given by quasi-isomorphisms to quasi-isomorphisms.

(d) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be a diagram in $\mathcal{C}$. Show that there exists a diagram of the form $Y' := (0 \longleftarrow A' \xrightarrow{g'} B')$ with g' a cofibration and A' and B' cofibrant, together with a natural transformation $u : Y' \to Y$ which is objectwise a weak equivalence. Notice that by the previous question the induced map $C(g') \to C(g)$ is a weak equivalence.

(e) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be a diagram in $\mathcal{C}$ with A and B cofibrant and g a cofibration. Show that $A \to I \otimes A$ is a weak equivalence and show that we can construct a zigzag of diagrams $Y \leftarrow Y' \to Y''$ of the form

$$\begin{array}{ccccc} 0 & \longleftarrow & A & \xrightarrow{g} & B \\ \uparrow & & \uparrow & & \uparrow \\ C(A) & \longleftarrow & A & \xrightarrow{g} & B \\ \downarrow & & \downarrow & & \downarrow \\ C(A) & \longleftarrow & I \otimes A & \xrightarrow{g} & \mathrm{Cyl}(g) \end{array}$$

where each vertical arrow is a weak equivalence and the map $I \otimes A \to \mathrm{Cyl}(g)$ is a cofibration.

(f) Let $Y := (0 \longleftarrow A \xrightarrow{g} B)$ be any diagram. Conclude that the mapping cone $C(g)$ is a model for the homotopy colimit of the diagram Y.

Solution 2. First we advise the reader to write down a commutative square of functors in $Fun(I, \mathcal{C})$, which are given by glueing two commutative cubes on their common face, and in which each face is commutative, as well as to write down what a lifting mean (which is a family of three maps divding parallel faces into two commutative triangles). A key feature of the diagram we are considering is that the object b has only outgoing non-identity arrows and the other two objects have only incoming non-identity arrows. The object b and its image by a functor will play a specific role. **1.** Suppose a morphim $X \to Y$ in $Fun(I, \mathcal{C})$ has the left lifting property with respect to projective acyclic fibrations. We first show that the the map $X(b) \to Y(b)$ has the left lifting property. Thus, we need to see that for any $U \to V$ a acyclic fibration in $\mathcal{C}$ the dotted lifting arrow exists in the diagram

$$\begin{array}{ccc} X(b) & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow \\ Y(b) & \longrightarrow & V \end{array} .$$

For this, we notice that the data of such a diagram is equivalent to the data of a morphism of diagrams

$$\begin{array}{ccc} X & \longrightarrow & (*, U, *) \\ \downarrow & \nearrow & \downarrow \\ Y & \longrightarrow & (*, V, *) \end{array}$$

where $(*, U, *)$ is a notation for the diagram $* \leftarrow U \rightarrow *$ (and $*$ is the terminal object). The lifting exists by the assumption that $X \rightarrowtail Y$. This shows that $X(b) \to Y(b)$ is a cofibration. Let us now use this to show that the map $X(a) \coprod_{X(b)} Y(b) \to Y(a)$ has the left lifting property

$$\begin{array}{ccc} X(a) \coprod_{X(b)} Y(b) & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow \\ Y(a) & \longrightarrow & V \end{array}$$

with respect to any acyclic fibration $U \to V$ in $\mathcal{C}$. We do this using the remark that the data of such a commutative square is equivalent to the data of a commutative square of diagrams

$$\begin{array}{ccc} X & \longrightarrow & (U, U, *) \\ \downarrow & \nearrow & \downarrow \\ Y & \longrightarrow & (V, U, *) \end{array}$$

The case of the remaining map is completely analogous.

We now have to check the converse, that if $X \to Y$ is of the form given in the exercise then it has the left lifting property with respect to projective acyclic fibrations. The idea is again to use first the fact that $X(b) \to Y(b)$ is a cofibration in $\mathcal{C}$ to construct the lifting in the middle. This is possible since each arrow $U(a) \to V(a)$, $U(b) \to V(b)$ and $U(c) \to V(c)$ are acyclic fibrations if $U \to V$ is an acyclic fibration in $Fun(I, \mathcal{C})$.

This being done, we see that the lifing $Y(b) \to U(b)$ gives a commutative diagram

$$\begin{array}{ccccc} & & X(b) & \longrightarrow & X(a) \\ & \swarrow & \downarrow & & \downarrow \\ Y(b) & \longrightarrow & U(b) & \longrightarrow & U(a) \end{array}$$

from which we get a canonical map $X(a) \coprod_{X(b)} Y(b) \to U(a)$ which, by the commutativity of the diagram of squares fits into the commuative square

$$\begin{array}{ccc} X(a) \coprod_{X(b)} Y(b) & \longrightarrow & U(a) \\ \downarrow & \nearrow & \downarrow \wr \\ Y(a) & \longrightarrow & V(a) \end{array}$$

for which the dotted arrow exists since the left hand vertical map is assumed to be a cofibration. The remaining lifts is the same. This proves the first equivalence.

The case of acyclic cofibrations is similar, using fibration on the right hand side instead of acyclic ones.

2. One has to check that all the axioms are satisfied. First one checks that $\mathrm{Fun}(I, \mathcal{C})$ admits all limits and colimits: this is true as long as they exist in $\mathcal{C}$ because colimits and limits in $\mathrm{Fun}(I, \mathcal{C})$ are computed objectwise in $\mathcal{C}$. Then one has to check the two-out-of-three property of weak-equivalences. But again this follows by definition of the weak-equivalences as objectwise weak-equivalences in $\mathcal{C}$ which verifies this property. Then we have to check that fibrations, cofibrations and weak-equivalences are stable under retracts. For fibrations and weak-equivalences this follows again from the definitions, so we only have to say something about cofibrations: but since cofibrations are maps defined by a left lifting property, and the latter are stable under retracts, this is also OK (see the proof of the closedness of a model category in Class).

The lifting properties were already checked in the previous question so all we have to check is the factorization property: we explain the case $X \to Y$ factored as acyclic cofibration + fibration. Here is the idea: again, first we factor the middle term $X(b) \to Y(b)$ as a acyclic cofibration followed by a fibration $X(b) \to A \to Y(b)$ in $\mathcal{C}$. Then we complete this into a diagram by taking pushouts

$X(c) \to X(c) \coprod_{X(b)} A \to Y(b)$ and $X(a) \to X(a) \coprod_{X(b)} A \to Y(a)$. Now we factor the last two maps $X(c) \coprod_{X(b)} A \to H_c \to Y(b)$ and $X(a) \coprod_{X(b)} A \to H_a \to Y(a)$ again in $\mathcal{C}$. The resulting factorization $X \to H \to Y$ has the required properties.

3.This follows because by definition its right adjoint is the constant diagram functor which is right Quillen as by definition it preserves fibrations and acyclic fibrations.

Remark on computations of homotopy pushouts. As we have seen in class, the last point implies in particular that homotopy pushouts exists for any model category $\mathcal{C}$ and are computed as the left total derived functor of the pushout functor $Fun(I, \mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ where the diagram category is given the projective model structure. This means that it is computed by taking the pushout of a cofibrant replacement of $X(a) \leftarrow X(b) \to X(c)$ in $Fun(I, \mathcal{C})$, that is

$$\mathbb{L}\,\mathrm{colim}\,\big(X(a) \leftarrow X(b) \to X(c)\big) = \mathrm{colim}\,\big(L_X(a) \leftarrow L_X(b) \to L_X(c)\big) = L_X(a) \coprod_{L_X(b)} L_X(c)$$

where $L_X \overset{\sim}{\twoheadrightarrow} X$ is the cofibrant replacement.

Note that by question 2., we have that a diagram Z is cofibrant if $Z(b)$ is cofibrant and (since $0 \coprod\limits_{O} Z(b) = Z(b)$) the maps $Z(b) \to Z(c)$ and $Z(b) \to Z(a)$ are cofibrations. Thus:

a cofibrant replacement of a diagram X is a diagram $L_X(a) \leftarrowtail L_X(b) \rightarrowtail L_X(c)$, with $L_X(b)$ cofibrant, and a commutative diagram:

$$\begin{array}{ccccc} L_X(a) & \leftarrowtail & L_X(b) & \rightarrowtail & L_X(c) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ X(a) & \longleftarrow & X(b) & \longrightarrow & X(c). \end{array}$$

The next question and the proposition below shows that in model categories where weak equivalences are preserved by pushouts, there is an easier formula to compute it.

4. Indeed, let

$$\begin{array}{ccccc} C' & \leftarrowtail & A' & \rightarrowtail & B' \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ C & \longleftarrow & A & \overset{f}{\rightarrowtail} & B \end{array}$$

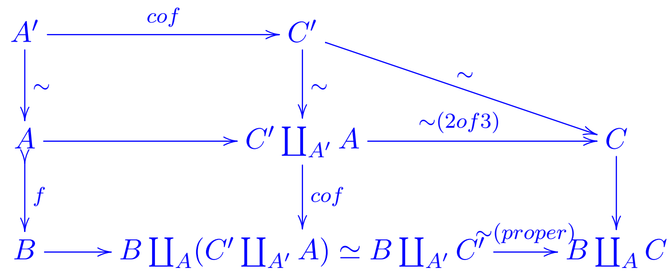
be a cofibrant resolution of the diagram $C \leftarrow A \to B$ (as explained in the remark above). We have to show that the natural map

$$C' \coprod_{A'} B' \to C \coprod_{A} B$$

is a weak-equivalence. But this map can be obtained as a composition of two maps : $C' \coprod_{A'} B' \to C' \coprod_{A'} B$ followed by $C' \coprod_{A'} B \to C \coprod_A B$. The first map can be obtained as a pushout

$$\begin{array}{ccc} B' & \longrightarrow & C' \coprod_{A'} B' \\ \downarrow & & \downarrow \\ B & \longrightarrow & C' \coprod_{A'} B \end{array}$$

The top horizontal arrow is a cofibration (because cofibrations are stable under pushout and $A' \rightarrowtail C'$ is a cofibration) and as $B' \to B$ is a weak-equivalence , left properness implies that the left vertical arrow is a weak-equivalence. The second map can be obtained as a composition of pushout diagrams.



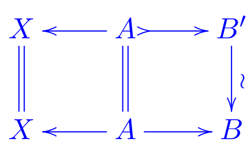
where the middle vertical arrow is a cofibration as a pushout of the cofibration f and the lower right horizontal arrow is a weak-equivalence thanks to the properness assumption.

5. Noticing that the factorisation $A \hookrightarrow A \times [0,1] \coprod\limits_{A\times\{1\}} Y \xrightarrow{\sim} Y$ given by the mapping cylinder is a relative cell complex followed by a weak equivalence, we wee that the result will follow from the following general fact: *If $\mathcal{C}$ is a left proper model category, and $A \rightarrowtail B' \xrightarrow{\sim} B$ is a replacement of a morphism $i : A \to B$ by a cofibration, then there is a natural isomorphism $\mathbb{L}\,\mathrm{colim}(X \leftarrow A \to B) \cong X \coprod\limits_A B'$.* Strictly speaking the proposition asserts that there is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ between the homotopy pushout and the pushout induced by the cofibrant replacement of $A \to B$ and that in fact, this isomorphism is inducec by a *natural* zigzag of weak equivalence

$$L_X \coprod_{L_A} L_B \xleftarrow{\sim} ? \xrightarrow{\sim} X \coprod_A B'$$

where the $L_X \leftarrow L_A \to L_B$ is a cofibrant replacement of $X \leftarrow A \to B$ (and thus the source of the weak equivalence is precisely the homotopy pushout) and the question mark ? depends functorially on the diagram.

We now prove the proposition. By question (3.), the target $X \coprod\limits_A B'$ is the homotopy pushout $\mathbb{L}\,\mathrm{colim}(X \leftarrow A \rightarrowtail B')$. The map $B' \to B$ induces a map of diagrams



for which all vertical maps are weak equivalences. Hence this is a weak equivalence of diagrams, and thus, the induced map on homotopy colimits is an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

We thus have a natural isomorphism $\mathbb{L}\,\mathrm{colim}(X \leftarrow A \to B) \xleftarrow{\simeq} \mathbb{L}\,\mathrm{colim}(X \leftarrow A \rightarrowtail B') \xrightarrow{\simeq} X \coprod\limits_A B'$ in $\mathrm{Ho}(\mathcal{C})$ as claimed and the question mark ? above is just the pushout $L_X \coprod\limits_{L_A} L_{B'}$ where $L_X \leftarrow L_A \to L_{B'}$ is the cofibrant replacement of $X \leftarrow A \to B'$.The category N depicting the colimit of tower is simply $0 \to 1 \to 2 \to 3 \to \cdots$ (that is the category associated to the ordinal $\mathbb{N}$, or said otherwise to the ordered set $\mathbb{N}$) the category with exactly one arrow in between two consecutive non-negative integers. It is not a very small category so that the theorem seen in class does not guarantee the existence of homotopy colimit.

However, we can apply the same ideas as in the study of the homotopy pushout. Proceeding exactly as in question 1. we see that, for any model category $\mathcal{C}$, a morphism $X \to Y$ in $Fun(N,\mathcal{C})$ is a projective cofibration (resp. acyclic cofibration) if and only if $X(0) \to Y(0)$ is a cofibration (resp. acyclic cofibration) and for every $i > 0$, the natural map $X_i \coprod\limits_{Y_{i-1}} X_{i-1} \to Y_i$ is a cofibration (resp. acyclic cofibration). Then one can prove as in 2. that the projective structure on $Fun(N,\mathcal{C})$ makes the category of towers a model category so that the homotopy colimit of the tower exists.

Further a cofibrant replacement of a diagram $X : N \to \mathcal{C}$ is thus given by a cofibrant object $L_X(0)$ and cofibrations $L_X(\rightarrowtail L_X(i+1)$ (for any $i \in \mathbb{N}$) together with acyclic fibrations $L_X(i) \overset{\sim}{\twoheadrightarrow} X_i$ making the obvious squares commutative. In the specific case where $X(0)$ is cofibrant and all the maps $X(i) \to X(i+1)$ are cofibrations, we thus have that X is cofibrant and therefore as seen in class, the canonical map from the homotopy pushout of the tower X to its pushout $\operatorname{colim} X(i)$ is a weak equivalence. It follows that if we have a commutative diagram

$$\begin{array}{ccccccc} Y(0) & \rightarrowtail & Y(1) & \rightarrowtail & Y(2) & \rightarrowtail & \cdots \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ X(0) & \longrightarrow & Y(1) & \longrightarrow & Y(2) & \longrightarrow & \cdots \end{array}$$

with $Y(0)$ cofibrant, then the diagram is a weak equivalence of diagram and by above we thus have a zigzag of weak equivalences

$$\operatorname*{colim}_{\mathbb{N}} Y(i) \overset{\sim}{\leftarrow} \operatorname*{colim}_{\mathbb{N}} L_Y(i) \overset{\sim}{\rightarrow} \operatorname*{colim}_{\mathbb{N}} L_X(i).$$

This proves that to compute the homotopy colimit of a tower it is enough to replace it by a weakly equivalent tower consisting of cofibrations whose first object is cofibrant.

A completely dual analysis shows that the injective model structure is also a model category for $Fun(N, \mathcal{C})$ and thus that homotopy limit of tower exists and can be computed by replacing a tower by a weakly equivalent tower such that all maps are fibrations and the last object Y_0 is fibrant.

Now, recall from class that in Top, every object is fibrant and that for every object X_0 there is a CW-complex $\tilde{X}_0$ weakly equivalent to it: $\tilde{X}_0 \overset{\sim}{\to} X_0$ (and by composition we have an induced map $\tilde{X}_0 \to X_1$).

Hence in Top , the homotopy colimit of a tower is given by the "telescope"

$$\mathbb{L}colim X_i \cong \tilde{X}_0 \times [0,1] \coprod_{\tilde{X}_0 \times \{1\}} X_1 \times [1,2] \coprod_{X_1 \times \{2\}} X_2 \times [2,3] \coprod_{X_2 \times \{3\}} X_3 \times [3,4] \coprod \cdots$$

that is a tower of glued cylinders. Now consider the colimit of almost the same telescope but for which we start at X_0. Then we have a pushout diagram

$$\begin{array}{ccc} \tilde{X}_0 \times [0,1[& \rightarrowtail & \tilde{X}_0 \times [0,1] \coprod_{\tilde{X}_0} \big(\coprod_{X_{i-1}} X_i \times [i, i+1] \big) \\ \downarrow \wr & & \downarrow \\ X_0 \times [0,1[& \rightarrowtail & X_0 \times [0,1] \coprod_{\tilde{X}_0} \big(\coprod_{X_{i-1}} X_i \times [i, i+1] \big) \end{array}$$

in which the right vertical arrow is a weak equivalence by left properness. Hence the homotopy colimit of a tower $X_0 \to X_1 \to \cdots$ is given by the telescope

$$\mathbb{L}colim X_i \cong X_0 \times [0,1] \coprod_{X_0 \times \{1\}} X_1 \times [1,2] \coprod_{X_1 \times \{2\}} X_2 \times [2,3] \coprod_{X_2 \times \{3\}} X_3 \times [3,4] \coprod \cdots .$$

By a similar argument and induction one can prove that if all the maps in the sequence $X_0 \to X_1 \to \cdots$ are cofibration then the colimit of the sequence $\operatorname{colim}(X_i)$ is weakly equivalent to its homotopy colimit $\mathbb{L}colim X_i$.

Similarly a homotopy limit of $\ldots Y_2 \to Y_1 \to Y_0$ by replacing each map by a fibration and taking the limit hence as a limit of path spaces.

6. Let

$$\begin{array}{ccc} M & \xrightarrow{g} & M' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ N & \xrightarrow{g'} & N' \end{array}$$

be a pushout diagram in $Ch(A)$ where g is assumed to be a cofibration and f is weak-equivalence. We must show that f' is a weak-equivalence. But notice that as g is a cofibration and therefore injective, we have a short exact sequence of chain complexes and therefore long exact sequence of homology groups, and finally we have maps of exact sequences

$$\begin{array}{ccccccccc} H_{n+1}(M'/M) & \longrightarrow & H_n(M) & \longrightarrow & H_n(M') & \longrightarrow & H_n(M'/M) & \longrightarrow & H_{n-1}(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(N'/N) & \longrightarrow & H_n(N) & \longrightarrow & H_n(N') & \longrightarrow & H_n(N'/N) & \longrightarrow & H_{n-1}(N) \end{array}$$

where the first and fourth vertical maps are isomorphisms because the diagram is a pushout and the second and last vertical maps are isomorphisms because f is a weak-equivalence. So f' is also a weak-equivalence. Note that in degree n, one has $(I^1 \otimes A)_n = A_n \oplus A_n \oplus A_{n-1}$. The formula given for the differential gives $d(x, y, w) = \big(\partial_A(x) - z, \partial_A(y) + z, -\partial_A(z)\big)$. Hence the pushout $Cyl(g) := B \coprod_A I^1 \otimes A$ is given in degree n by $B_n \oplus A_n \oplus A_{n-1}$ and the map $I^1 \otimes A \to B \coprod_A I^1 \otimes A$ is given in degree n by $(x, y, w) \mapsto (g(y), x, w)$. Thus the differential on the pushout $Cyl(g)$ is given by

$$(b, x, w) \mapsto \big(\partial_B(b) + g(w), \partial_A(x) - w, -\partial_A(z)\big).$$

The formula for the differential of $I^1 \otimes A$ above shows that their linear maps $(I^1 \otimes A)_n = A_n \oplus A_n \oplus A_{n-1} \to A_n \oplus A_{n-1}$ given by $(x, y, z) \mapsto (y, z)$ defines a chain map $t : I^1 \otimes A \to C(Id_A)$.

Now we compute the pushout $Cyl(g) \coprod\limits_{I^1 \otimes A} C(Id_A)$. In degree n, we have

$$\Big(Cyl(g) \coprod_{I^1 \otimes A} C(Id_A)\Big)_n = (B_n \oplus A_n \oplus A_{n-1}) \oplus (A_n \oplus A_{n-1}) / (g(y), x, w, 0, 0) \sim (0, 0, 0, y, w)$$

and hence it is isomorphic to $B_n \oplus A_{n-1}$ (the terms corresponding to x being killed off in the quotient). The differential then reads $(b, w) \mapsto (\partial_B(b) + g(w), -\partial_A(w))$ which proves that the pushout is indeed the cone $C(g)$. A functor from Δ^1 to any category is simply the data of two objects and one morphism between them, that is the data of an arrow $A \overset{g}{\to} B$. A map between functors is simply a natural transformation thus a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ A' & \xrightarrow{g'} & B' \end{array}$$

. Now if we are in chain complexes, the linear maps $\beta \oplus \alpha : B_n \oplus A_{n-1} \to B'_n \oplus A'_{n-1}$ are a map $C(g) \to C(g')$ of chain complexes (because α and β commutes with differential and the diagram is commutative). And it is easy to check that this assignement does make $g \mapsto C(g)$ into a functor $Fun(\Delta^1, Ch(R)) \to Ch(R)$. It reamains to prove it send objectwise weak equivalences to weak equivalence. To see this, we note that given $f : A \to B$ one has an exact sequence of complexes $0 \to B \to C(f) \to A[1] \to 0$. Hence a map of morphisms produces a map of exact sequences and if the maps are quasi-isomorphisms, by the five-lemma, the middle terms will also be.

Take $u_a : A' \overset{\sim}{\twoheadrightarrow} A$ a cofibrant replacement of A. Then choose a factorization of $A' \to A \to B$ as a cofibration $g' : A' \rightarrowtail B'$ followed by a acyclic fibration $u_b : B' \overset{\sim}{\twoheadrightarrow} B$ and set $u_c : 0 \to 0$ as the identity. This gives us the required natural transformation with g' a cofibration as we have a commutative diagram

$$\begin{array}{ccccc} 0 & \longleftarrow & A' & \overset{g'}{\rightarrowtail} & B' \\ \| & & {\scriptstyle u_a}\downarrow\wr & & \wr\downarrow{\scriptstyle u_b} \\ 0 & \overset{g}{\longleftarrow} & A & \longrightarrow & B \end{array}$$

First note that the composition $A \overset{i_0 1}{I} \otimes A \to C(Id_A)$ is given $y \mapsto (0, y, 0) \mapsto$ Since g is assumed to be a cofibration and cofibrations are stable under pushouts, by definition of the mapping cylinder, the map $I^1 \otimes A \to Cyl(g)$ is a cofibration as well. Now we are only left to prove the vertical arrows in the diagram

$$\begin{array}{ccccc} 0 & \longleftarrow & A & \overset{g}{\rightarrowtail} & B \\ \uparrow & & \| & & \| \\ C(Id_A) & \overset{t \circ i_0}{\longleftarrow} & A & \overset{g}{\rightarrowtail} & B \\ \| & & \downarrow{\scriptstyle i_0} & & \downarrow \\ C(Id_A) & \overset{t}{\longleftarrow} & I^1 \otimes A & \rightarrowtail & Cyl(g) \end{array}$$

are weak equivalences. For the lower right one, it follows by left properness once we prove that i_0 is. Note that the linear maps $s : I^1 \otimes A \to A$ given in degree n by $s(x, y, w) = x + y$ are a chain complex morphism. Further $s \circ i_0 = Id_A$. To prove that i_0 is a quasi-isomorphism it is thus enough to prove that $i_0 \circ s$ induces the identity in homology and thus it is enough to prove it is homotopic to the identity of $I^1 \otimes A$. Let $h : I^1 \otimes A \to I^1 \otimes A[-1]$ be given by $h(x, y, w) = (0, 0, x)$. Then $dh + hd(x, y, w) = (-x, x, -w) = -(x, y, w) + i_0 \circ s(x, y, w)$ which proves that h is indeed a chain homotopy in between Id and $i_0 \circ s$. Finally, since $C(Id_A)$ is acyclic[2] the upper left vertical map is a quasi-isomorphism. Given Y we can apply (d) to find $Y' \overset{\sim}{\to} Y$ where Y' is of the form $Y' = (0 \longleftarrow A' \overset{g'}{\rightarrowtail} B'$

with A' and B' cofibrant. The natural transformation $Y' \overset{\sim}{\to} Y$ being a weak equivalence, it induces a quasi-isomorphism of the homotopy colimits of Y' to the one of Y. Further, by (c) the mapping cone of Y' and Y are weak-equivalent. Thus it is enough to prove that the mapping cone of $g' : A' \to B'$ is quasi-isomorphic to the homotopy pushout of the diagram Y'. In other words, we are left to prove (f) in the case where $g; A \to B$ is a cofibration and A, B are cofibrant, which are exactly the assumptions of (e). Now because of left properness we know that the the pushout of such a Y is a homotopy pushout. For the same reason, this is also the case for each horizontal diagram in the string of weak equivalences $Y \overset{\sim}{\leftarrow} Y' \overset{\sim}{\to} Y$ given by (e). The vertical maps being all weak equivalences we thus have that the homotopy pushout of the top horizontal diagram is equivalent to the one of the lower horizontal diagram. The later is thus the same (again by left properness) as the pushout $C(Id_A) \overset{t}{\longleftarrow} I^1 \otimes A \rightarrowtail Cyl(g)$, that is by definition the mapping cone $C(g)$ of the original map g while the first was by above quasi-isomorphic to the homotopy pushout.

Exercice 6. We recall the definition of a coend : Let $F : I^{\mathrm{op}} \to \mathbf{sSet}$ be a functor and $G : I \to \mathbf{sSet}$ be another functor. We define their coend denoted $F \otimes_I G$ as the following coequalizer

$$\coprod_{f:i \to j} F(j) \times G(i) \rightrightarrows \coprod_{i \in I} F(i) \times G(i) \to F \otimes_I G$$

which is a simplicial set.

1. We admit that $* \otimes_I G$ computes the colimit of the functor G. Let X be a simplicial set, that we see as a functor $sSet^{op} \to sSet$ by seeing $X(n)$ as a discrete simplicial set. The *geometric realization* of X is defined as the coend $\Delta[-] \otimes X \in Fun(sSet^{op}, sSet)$. Convince yourself that it is well-named and compute it.

2. Fix a small category I. Recall why $Fun(I, sSet)$ has the projective model structure. Consider the coend pairing defined by
$$sSet^{D^{op}} \times sSet^D \to sSet$$
which sends (F, G) to $F \otimes G$. Our goal is to see how this pairing interacts with the model structure. Show that if we fix a cofibrant G,it preserves weak equivalences in the F variabl (hint : show it on the generating cofibrations + small object argument).

[2] as follow from the long exact sequence since id is a quasi-isomorphism

3. Deduce that if X is a simplicial set which is cofibrant for the projective model structure, then the geometric realization $\Delta[-] \otimes_{sSet^{op}} X$ is a model for the homotopy colimit of X seen as a functor $\Delta^{op} \to sSet$.

Actually, this fact is true even if X is not projectively cofibrant, but we need more work : the Reedy model structure on simplicial objects.

Exercice 7. We assume that there is a model structure on $\mathbf{sSet}^{\Delta^{op}}$, called the *Reedy model structure*, such that

(a) The weak equivalences are the objectwise weak equivalences.

(b) The cofibrations are the maps $f : F \to G$ such that for all n, the map

$$F[n] \sqcup_{L_n F} L_n G \to G[n]$$

is a cofibration. Here $L_n F$ denotes the n-th latching object defined by $L_n(F) = \mathrm{colim}_{f:k\to n, k<n} F(k)$

(c) The fibrations are the maps such that for all n, the map

$$F[n] \to G[n] \times_{M_n G} M_n F$$

is a fibration. Here $M_n F$ denotes the n-th matching object defined by $M_n(G) = \lim_{f:k\to n, k>n} G(k)$.

1. Show that the geometric realization functor $\mathbf{sSet}^{\Delta^op} \to \mathbf{sSet}$ is left Quillen for the Reedy structure. Hint : show that its right adjoint is right Quillen.

2. Deduce from the previous exercise that $|X| \simeq \mathrm{hocolim}\, X$ for any simplicial set X.

Exercice 8 (The fundamental theorem of homotopy theory, after Geoffroy Horel)**.** We assume the result of the previous exercise. Let $F : \mathbf{sSet} \to \mathbf{sSet}$ be a homotopical functor that preserves homotopy colimits. Then F is naturally weakly equivalent to the functor

$$X \mapsto X \otimes^{\mathbb{L}} F(*)$$

If you are familiar with simplicial cofibrantly generated model categories, you can try to do the exercise replacing the target of F by any such model category M.

Solution 3. For exercices 6,7,8, see section 7 and 8 of Geoffroy Horel's course `https://www.math.univ-paris13.fr/%7Eginot/HomotopieII/Horel_HomotopieII.pdf`.