Cofibrantly generated model structures and simplicial sets

Exercice 1 (The Kan model structure on **sSet**)**.** We let Δ^n (or $\Delta[n]$ in the course notes) be the standard n-simplex, $\partial\Delta^n$ its interior, S^n the quotient of Δ^n by its interior and Λ^n_k the k^{th}-horn.

(1) We write $\mathcal{I}$ for the set of maps $\partial\Delta^n \to \Delta^n$.
   (a) Show that a map of simplicial sets $f : X \to S$ is injective in all degrees if and only if it belongs to $LLP(RLP(\mathcal{I}))$, i.e. it is a cofibration.

(2) Suppose X is a Kan complex, show that for any simplicial set S, $\mathrm{Map}(S, X)$ is also a Kan complex.
   Let X be a simplicial set and $x \in X$. Recall that if X is a Kan complex, we have denoted $\pi_n(X, x)$ the quotient of $\mathrm{Map}(S^n, X)$ by the equivalence relation $\sim$ generated by $f \sim g$ if there is a map $\varphi : \Delta^1 \to \mathrm{Map}(S^n, X)$ whose source and target are f and g.

(3) Let $f : K \to L$ be a map of Kan complexes. Show that f is a weak equivalence if and only if f induces an isomorphism on every homotopy group as defined above.

(4) Let X be a topological space and denote $\mathrm{Sing}_\bullet X$ the simplicial set $\mathrm{Hom}(\Delta^\bullet, X)$. Show that $\mathrm{Sing}_\bullet X$ is a Kan complex and $\pi_n(X, x) \simeq \pi_n(\mathrm{Sing}_\bullet X, x)$.

(5) Let X be a Kan complex. Show that $\pi_0(X) \simeq \pi_0(\mathrm{Sing}_\bullet |X|)$. Deduce inductively that $\pi_n(X, x) \simeq \pi_n(\mathrm{Sing}_\bullet |X|, x)$.

(6) Conclude to show that the Kan model structure on **sSet** is Quillen-equivalent to the classical model structure on **Top**.

Exercice 2 (*Modules over cdgas*)**.** Let A be a cdga (commutative differential graded algebra) over $\mathbb{Q}$. Let $\mathbf{Mod}(A)$ denote the category of dg modules over A. An object in $\mathbf{Mod}(A)$ is thus a cochain complex M together with a morphism of complexes $A \otimes_{\mathbb{Q}} M \to M$ satisfying the module axioms (i.e. $(a \cdot b) \cdot m = a \cdot (b \cdot m)$, $1 \cdot m = m$).

(1) Show that the forgetful functor $U : \mathbf{Mod}(A) \to \mathrm{Ch}(\mathbb{Q})$ is a right adjoint and describe its left adjoint F.

(2) Show that there is a model structure on $\mathbf{Mod}(A)$ where
   - weak equivalences are the morphisms f such that $U(f)$ is a quasi-isomorphism,
   - fibrations are the morphisms f such that $U(f)$ is surjective.

(3) Show that the functor $- \underset{A}{\otimes} - : \mathbf{Mod}(A) \times \mathbf{Mod}(A) \longrightarrow \mathbf{Mod}(A)$ admits a total left derived functor $- \overset{\mathbb{L}}{\underset{A}{\otimes}} - : \mathbf{Ho}(\mathbf{Mod}(A) \times \mathbf{Mod}(A)) \cong \mathbf{Ho}(\mathbf{Mod}(A)) \times \mathbf{Ho}(\mathbf{Mod}(A)) \longrightarrow \mathbf{Ho}(\mathbf{Mod}(A))$.

(4) Let $f : A \to B$ be a morphism of cdgas. Show that the functor $f_* : \mathbf{Mod}(B) \to \mathbf{Mod}(A)$, given by $A \otimes_{\mathbb{Q}} M \overset{f \otimes id}{\to} B \otimes_{\mathbb{Q}} M \to M$, is a right Quillen functor.

(5) Assume $f : A \to B$ is a quasi-isomorphism of cdgas. Show that f_* is a Quillen equivalence.

Solution 1. One possible reference is the book *Modules over operads and functors*, by Benoît Fresse (sections 11.2.5 – 11.2.10).

Exercice 3 (Loops and Suspensions *(by Victor Saunier)*)**.** Let $\mathcal{C}$ be a model category with a zero object. For $X \in \mathcal{C}$, we denote ΣX the homotopy colimit of the following diagram $0 \longleftarrow X \longrightarrow 0$ and ΩX the homotopy limit of $0 \longrightarrow X \longleftarrow 0$.

(1) Compute ΩX in $\mathbf{sSet}_*$, $\mathbf{Top}_*$, $\mathbf{Ch}(\mathbb{Z})$.

(2) Compute ΣX in $\mathbf{sSet}_*$, $\mathbf{Top}_*$, $\mathbf{Ch}(\mathbb{Z})$.

(3) Show that $\Sigma : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ is adjoint to $\Omega : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$. In which of the previous cases is this adjunction an equivalence?

Exercice 4 (Detailled construction of the Nerve of a category)**.** In this exercise, we establish a link between the theory of categories and the theory of simplicial sets. More precisely, we check that we can translate the information provided by a category $\mathcal{C}$ into a simplicial set, called the nerve of $\mathcal{C}$ and denoted by $N(\mathcal{C})$. We will see that this translation does not lose any information and that in fact the theory of categories can be seen as a sub-theory of that of simplicial sets.

(1) The category of simplexes Δ can be canonically identified with a full subcategory of Cat, spanned by the categories of the form $[n] := [0 \to 1 \to \cdots \to n]$. Use this inclusion and the previous exercise to produce an adjunction sending $\tau(\Delta[n]) = [n]$.

(2) Let $\mathcal{C}$ be a small category. Check that the functor N is characterized as follows: $N(\mathcal{C})_n$ consists of composable strings of morphims in $\mathcal{C}$ of lenght n: $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} X_n$. In particular, the 0-simplexes of $N(\mathcal{C})$ are the objects of $\mathcal{C}$ and the 1-cells are morphisms in $\mathcal{C}$. Describe the face and degeneracy maps in terms of compositions and identity morphims.

(3) Show that the canonical morphism induced by the inclusion $\tau(\partial\Delta[n]) \to \tau(\Delta[n]) = [n]$ is an isomorphism of categories for $n \geq 3$. Describe both $\tau(\partial\Delta[1])$ and $\tau(\partial\Delta[2])$.
(*Hint:* use the construction of $\partial\Delta[n]$ as a cokernel).

(4) Deduce that the canonical map $\tau(\mathrm{sk}_2(X)) \to \tau(X)$ is an isomorphism of categories for every simplicial set X. In other words, the category $\tau(X)$ only depends on the 2-skeleton of X.

(5) Let X be a simplicial set. Check that the category $\tau(\mathrm{sk}_2(X))$ is isomorphic to the quotient of the free category with X_0 as objects and X_1 as morphisms under the following relation on morphisms:
- for every 2-simplex $\sigma : \Delta[2] \to X$, we identify $\partial_1(\sigma)$ with the composition $\partial_0(\sigma) \circ \partial_2(\sigma)$.
- for every $x \in X_0$, identify $\epsilon_0(x)$ with Id_x

(6) Let $\mathcal{C}$ be a category and describe the category $\tau(\mathrm{sk}_2(N(\mathcal{C}))$. Conclude that the adjunction map $\tau(N(\mathcal{C})) \to \mathcal{C}$ is an isomorphism of categories and that N is fully faithful.

(7) Let I_n denote the sub-simplicial set (subfunctor) of $\Delta[n]$ given by $\bigcup_i^n \mathrm{im}\ \alpha_i \subseteq \Delta[n]$ where $\alpha_i : \Delta[1] \to \Delta[n]$ is the map sending $0 \to i$ and $1 \mapsto i+1$. Show that I_n is the colimit of the diagram

$$\begin{tikzcd}
\Delta[1] & & \Delta[1] & & \cdots & & \Delta[1] \\
& \Delta[0] \arrow[ul, "\partial_1"] \arrow[ur, "\partial_0"'] & & \Delta[0] \arrow[ul, "\partial_1"] \arrow[ur, "\partial_0"'] & \cdots & \Delta[0] \arrow[ul, "\partial_1"] \arrow[ur, "\partial_0"'] &
\end{tikzcd}$$

where $\Delta[1]$ appears n times.

(8) Let $\mathcal{C}$ be a category and let $N(\mathcal{C})$ denote its nerve. Show that the composition with the inclusion $I_n \subseteq \Delta[n]$ produces a bijection

$$\mathrm{Hom}_{\mathbf{sEns}}(\Delta[n], N(\mathcal{C})) \cong \mathrm{Hom}_{\mathbf{sEns}}(I_n, N(\mathcal{C}))$$

for all $n \geq 2$. Conclude that the canonical map $\tau(I_n) \to \tau(\Delta[n]) = [n]$ is an isomorphism of categories for $n \geq 2$.