Cofibrantly generated model structures and simplicial sets

Exercice 1 (The Kan model structure on **sSet**)**.** We let Δ^n (or $\Delta[n]$ in the course notes) be the standard n-simplex, $\partial\Delta^n$ its interior, S^n the quotient of Δ^n by its interior and Λ^n_k the k^{th}-horn.

(1) We write $\mathcal{I}$ for the set of maps $\partial\Delta^n \to \Delta^n$.
   (a) Show that a map of simplicial sets $f : X \to S$ is injective in all degrees if and only if it belongs to $LLP(RLP(\mathcal{I}))$, i.e. it is a cofibration.

(2) Suppose X is a Kan complex, show that for any simplicial set S, $\mathrm{Map}(S, X)$ is also a Kan complex.
   Let X be a simplicial set and $x \in X$. Recall that if X is a Kan complex, we have denoted $\pi_n(X, x)$ the quotient of $\mathrm{Map}(S^n, X)$ by the equivalence relation $\sim$ generated by $f \sim g$ if there is a map $\varphi : \Delta^1 \to \mathrm{Map}(S^n, X)$ whose source and target are f and g.

(3) Let $f : K \to L$ be a map of Kan complexes. Show that f is a weak equivalence if and only if f induces an isomorphism on every homotopy group as defined above.

(4) Let X be a topological space and denote $\mathrm{Sing}_\bullet X$ the simplicial set $\mathrm{Hom}(\Delta^\bullet, X)$. Show that $\mathrm{Sing}_\bullet X$ is a Kan complex and $\pi_n(X, x) \simeq \pi_n(\mathrm{Sing}_\bullet X, x)$.

(5) Let X be a Kan complex. Show that $\pi_0(X) \simeq \pi_0(\mathrm{Sing}_\bullet |X|)$. Deduce inductively that $\pi_n(X, x) \simeq \pi_n(\mathrm{Sing}_\bullet |X|, x)$.

(6) Conclude to show that the Kan model structure on **sSet** is Quillen-equivalent to the classical model structure on **Top**.

Exercice 2 (*Modules over cdgas*)**.** Let A be a cdga (commutative differential graded algebra) over $\mathbb{Q}$. Let $\mathbf{Mod}(A)$ denote the category of dg modules over A. An object in $\mathbf{Mod}(A)$ is thus a cochain complex M together with a morphism of complexes $A \otimes_{\mathbb{Q}} M \to M$ satisfying the module axioms (i.e. $(a \cdot b) \cdot m = a \cdot (b \cdot m)$, $1 \cdot m = m$).

(1) Show that the forgetful functor $U : \mathbf{Mod}(A) \to \mathrm{Ch}(\mathbb{Q})$ is a right adjoint and describe its left adjoint F.

(2) Show that there is a model structure on $\mathbf{Mod}(A)$ where
   - weak equivalences are the morphisms f such that $U(f)$ is a quasi-isomorphism,
   - fibrations are the morphisms f such that $U(f)$ is surjective.

(3) Show that the functor $- \underset{A}{\otimes} - : \mathbf{Mod}(A) \times \mathbf{Mod}(A) \longrightarrow \mathbf{Mod}(A)$ admits a total left derived functor $- \overset{\mathbb{L}}{\underset{A}{\otimes}} - : \mathbf{Ho}(\mathbf{Mod}(A) \times \mathbf{Mod}(A)) \cong \mathbf{Ho}(\mathbf{Mod}(A)) \times \mathbf{Ho}(\mathbf{Mod}(A)) \longrightarrow \mathbf{Ho}(\mathbf{Mod}(A))$.

(4) Let $f : A \to B$ be a morphism of cdgas. Show that the functor $f_* : \mathbf{Mod}(B) \to \mathbf{Mod}(A)$, given by $A \otimes_{\mathbb{Q}} M \overset{f\otimes id}{\to} B \otimes_{\mathbb{Q}} M \to M$, is a right Quillen functor.

(5) Assume $f : A \to B$ is a quasi-isomorphism of cdgas. Show that f_* is a Quillen equivalence.

Solution 1. One possible reference is the book *Modules over operads and functors*, by Benoît Fresse (sections 11.2.5 – 11.2.10).

Exercice 3 (Loops and Suspensions *(by Victor Saunier)*)**.** Let $\mathcal{C}$ be a model category with a zero object. For $X \in \mathcal{C}$, we denote ΣX the homotopy colimit of the following diagram $0 \longleftarrow X \longrightarrow 0$ and ΩX the homotopy limit of $0 \longrightarrow X \longleftarrow 0$.

(1) Compute ΩX in $\mathbf{sSet}_*$, $\mathbf{Top}_*$, $\mathbf{Ch}(\mathbb{Z})$.

(2) Compute ΣX in $\mathbf{sSet}_*$, $\mathbf{Top}_*$, $\mathbf{Ch}(\mathbb{Z})$.

(3) Show that $\Sigma : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$ is adjoint to $\Omega : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C})$. In which of the previous cases is this adjunction an equivalence?

Exercice 4 (Detailled construction of the Nerve of a category)**.** In this exercise, we establish a link between the theory of categories and the theory of simplicial sets. More precisely, we check that we can translate the information provided by a category $\mathcal{C}$ into a simplicial set, called the nerve of $\mathcal{C}$ and denoted by $N(\mathcal{C})$. We will see that this translation does not lose any information and that in fact the theory of categories can be seen as a sub-theory of that of simplicial sets.

(1) The category of simplexes Δ can be canonically identified with a full subcategory of Cat, spanned by the categories of the form $[n] := [0 \to 1 \to \cdots \to n]$. Use this inclusion and the previous exercise to produce an adjunction sending $\tau(\Delta[n]) = [n]$.

(2) Let $\mathcal{C}$ be a small category. Check that the functor N is characterized as follows: $N(\mathcal{C})_n$ consists of composable strings of morphims in $\mathcal{C}$ of lenght n: $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} X_n$. In particular, the 0-simplexes of $N(\mathcal{C})$ are the objects of $\mathcal{C}$ and the 1-cells are morphisms in $\mathcal{C}$. Describe the face and degeneracy maps in terms of compositions and identity morphims.

(3) Show that the canonical morphism induced by the inclusion $\tau(\partial\Delta[n]) \to \tau(\Delta[n]) = [n]$ is an isomorphism of categories for $n \geq 3$. Describe both $\tau(\partial\Delta[1])$ and $\tau(\partial\Delta[2])$.
(*Hint:* use the construction of $\partial\Delta[n]$ as a cokernel).

(4) Deduce that the canonical map $\tau(\mathrm{sk}_2(X)) \to \tau(X)$ is an isomorphism of categories for every simplicial set X. In other words, the category $\tau(X)$ only depends on the 2-skeleton of X.

(5) Let X be a simplicial set. Check that the category $\tau(\mathrm{sk}_2(X))$ is isomorphic to the quotient of the free category with X_0 as objects and X_1 as morphisms under the following relation on morphisms:
- for every 2-simplex $\sigma : \Delta[2] \to X$, we identify $\partial_1(\sigma)$ with the composition $\partial_0(\sigma) \circ \partial_2(\sigma)$.
- for every $x \in X_0$, identify $\epsilon_0(x)$ with Id_x

(6) Let $\mathcal{C}$ be a category and describe the category $\tau(\mathrm{sk}_2(N(\mathcal{C}))$. Conclude that the adjunction map $\tau(N(\mathcal{C})) \to \mathcal{C}$ is an isomorphism of categories and that N is fully faithful.

(7) Let I_n denote the sub-simplicial set (subfunctor) of $\Delta[n]$ given by $\bigcup_i^n \mathrm{im}\ \alpha_i \subseteq \Delta[n]$ where $\alpha_i : \Delta[1] \to \Delta[n]$ is the map sending $0 \to i$ and $1 \mapsto i+1$. Show that I_n is the colimit of the diagram

$$\begin{tikzcd}
\Delta[1] & & \Delta[1] & & \cdots & & \Delta[1] \\
& \Delta[0] \arrow[ul, "\partial_1"'] \arrow[ur, "\partial_0"] & & \Delta[0] \arrow[ul, "\partial_1"'] \arrow[ur, "\partial_0"] & \cdots & \Delta[0] \arrow[ul, "\partial_1"'] \arrow[ur, "\partial_0"] &
\end{tikzcd}$$

where $\Delta[1]$ appears n times.

(8) Let $\mathcal{C}$ be a category and let $N(\mathcal{C})$ denote its nerve. Show that the composition with the inclusion $I_n \subseteq \Delta[n]$ produces a bijection

$$\mathrm{Hom}_{\mathbf{sEns}}(\Delta[n], N(\mathcal{C})) \cong \mathrm{Hom}_{\mathbf{sEns}}(I_n, N(\mathcal{C}))$$

for all $n \geq 2$. Conclude that the canonical map $\tau(I_n) \to \tau(\Delta[n]) = [n]$ is an isomorphism of categories for $n \geq 2$.

Note that the category $[n] := [0 \to 1 \to \ldots \to n]$ is just the category associated to the poset $0 < 1 \cdots < n$. In particular an order preserving map $[n] \to [m]$ is a functor from the category $[n]$ to the category $[m]$ (as one can simply check by hand).

1. Let us define $N : \mathrm{Cat} \to \mathrm{sSet}$ as the functor defined as follows. To a small category $\mathcal{C}$ we associate the family of sets $N(\mathcal{C})_n := \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$, in other words the set of functors from the category $[n]$ to $\mathcal{C}$. Since the morphisms of Δ are precisely the non-decreasing application which as we have seen are functors between categories of the form $[n]$, it is immediate that any non-decreasing map $f[n] \to [m]$ induces a map $f^* : N(\mathcal{C})_m = \mathrm{Hom}_{\mathrm{Cat}}([m], \mathcal{C}) \xrightarrow{-\circ f} \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}) = N(\mathcal{C})_m$ by composition of functors. By functoriality of composition of functors, we obtain a well defined functor $N(\mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}}(-, \mathcal{C}) : \Delta^{op} \to \mathrm{sSet}$ given by the collection of the $(N(\mathcal{C})_n)_{n \in \mathbb{N}}$. The construction shall really look like the construction of $\mathrm{Sing}_\bullet(X)$ for a space. We use the collection of the categories $[n]$ as an natural cosimplicial category where in the latter we were using the natural cosimplicial space $(\Delta^n)_{n \geq 0}$. That being seen, it is natural to find the left adjoint of the functor N by mimicking the definition of the geometric realization. More concretely, we set $\tau : \mathrm{sSet} \to \mathrm{Cat}$ by setting $\tau(X_\bullet) :=$

$(\coprod_{n\in\mathbb{N}} X_n \times [n]) \coprod_{(\coprod_{f:[n]\to[m]\in\Delta} X_m\times[n])} (\coprod_{m\in\mathbb{N}} X_m \times [m])$ to be the pushout[1] in Cat given by the diagram

$$\tau(X_\bullet) := \coprod_{n\in\mathbb{N}} X_n \times [n] \xleftarrow{f^*} \coprod_{f:[n]\to[m]\in\Delta} X_m \times [n] \xrightarrow{f_*} \coprod_{m\in\mathbb{N}} X_m \times [m]$$

where f^* is just induced by the map $f^* : X_m \to X_n$ given by the simplicial structure of $X_\bullet$ and f_* is just induced by $f : [n] \to [m]$. Note that since $\Delta[n]$ has a unique non-degenerate n-simplex ($id : [n] \to [n]$) and all others non-degenerate simplices are faces of it, then, it is immediate that the pushout defining $\tau(\Delta[n])$ is nothing more than the category $[n]$ itself. The proof that the two constructions are indeed an adjunction can be done in a similar way to the proof of the geometric realisation case seen in class; one only needs to replace continuous maps $\Delta^n \to Y$ by functors, and elements $\bar{t} \in \Delta^n$ by objects of $[n]$.

A companion proof is to use that every simplicial set is the colimit $X_\bullet \cong \mathrm{colim}_{\Delta[n]\to X_\bullet} \Delta[n]$. Since τ is defined by a colimit, it shows that to prove the adjunction it is enough to check it on all $\Delta[n]$. But then we have

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta[n], N(\mathcal{C})) \cong N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Cat}}(\tau(\Delta[n]), \mathcal{C})$$

where the first identity is given by the Yoneda Lemma for $\Delta[n]$ as seen in class.
Remark: the pushout formula defining τ shows that for every degenerate simplex $\sigma \in X_n$, the category $\{\sigma\} \times [n]$ is collapsed into the category $\{y\} \times [j]$ corresponding to the unique non-degenerate simplex $y \in X_j$ that σ is an iterate degeneracy of. Hence, as for the geometric realisation of spaces, the category $\tau(X_\bullet)$ is uniquely defined by the non-degenerate simplices. Further, the set of objects of $\tau(X_\bullet)$ is exactly the set X_0 of vertices and every non-degenerate 1-simplex σ yields a morphism $d_1(\sigma) \to d_0(\sigma)$ in $\tau(X_\bullet)$.

2. We have seen that $N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$. Since $[n]$ has only $n+1$ objects (the integers $0, \dots, n$) and exactly one non-identity morphisms $i \to j$ between two objects i, j such that $i < j$ (and no such morphism for $j \geq i$, a functor is given by $n+1$-objects $X_0, \dots X_n \in \mathcal{C}$ and one morphism $f_i : X_i \to X_{i+1}$ for any $i < n$.

3. Pour $n = 1$, $\partial\Delta[1]$ est un ensemble discret à deux éléments, donc $\tau(\partial\Delta[1]) = * \sqcup *$ est une catégorie discrète à deux objets, tandis que $\tau(\Delta[1]) = * \to *$. Pour $n = 2$, $\tau\partial\Delta[2]$ est la catégorie libre engendrée par trois objets $0, 1, 2$ et trois morphismes $0 \to 1, 1 \to 2, 0 \to 2$. En particulier, rien n'impose que $0 \to 1 \to 2$ soit égal au morphisme générateur $0 \to 2$. Ainsi $\tau(\partial\Delta[2])$ est différent de $\tau\Delta[2]$ (qui est égal à $[2]$ d'après les questions précédentes).
Passons au cas $n \geq 3$. On va utiliser le fait qu'on peut écrire $\partial\Delta[n]$ comme un coégalisateur :

$$\bigsqcup_{0\leq i<j\leq n} \Delta[n-2] \rightrightarrows \bigsqcup_{0\leq k\leq n} \Delta[n-1]$$

(On recolle les faces de Δ_n le long de leurs faces communes). Comme τ est un adjoint à gauche, il commute avec les colimites et donc on peut écrire :

$$\tau(\partial\Delta[n]) = \mathrm{coeq}\left(\bigsqcup_{0\leq i<j\leq n} [n-2] \rightrightarrows \bigsqcup_{0\leq k\leq n} [n-1]\right)$$

Où les deux flèches sont les inclusions de $[n-2]_{(i,j)}$ dans $[n-1]_i$ et $[n-1]_j$ respectivement en évitant la position j et la position i. Ici $[n-2]_{(i,j)}$ désigne la copie de $[n-2]$ indexée par (i,j). Pour y voir plus clair, on va réindexer les objets de ces catégories. On va remplacer $[n-2]_{(i,j)}$ par $0 \to \cdots \to \hat{i} \to \cdots \to \hat{j} \to \cdots \to n$ où $\hat{i}$ désigne qu'on omet l'objet i. De même on remplacera $[n-1]_i$ par la catégorie $(0 \to \cdots \to \hat{i} \to \cdots \to n)$. Avec ces notations, les flèches dans le coégalisateur sont simplement les inclusions naturelles de ces catégories, vues comme sous-catégorie de $[n]$. Comme les objets d'une colimite de catégorie se calculent par la colimite ensembliste, on voit déjà que la colimite est une catégorie à $n+1$ objets $0, ..., n$, munie de morphismes $i \to i+1$ induits par les morphisems

[1] this pushout is precisely the coequalizer of the maps f_*, f^*

dans $(0 \to \cdots \to \hat{k} \to \cdots \to n)$ pour un k différent de i et $i+1$. Tous les choix de k donnent le même morphisme dans la colimite car on a dans le coégalisateur un diagramme

$$(0 \to \cdots \to \hat{k}_2 \to \cdots \to n) \leftarrow (0 \to \cdots \hat{k}_1 \to \cdots \to \hat{k}_2 \to \cdots \to n) \to (0 \to \cdots \hat{k}_1 \to \cdots \to n)$$

qui montre que les deux flèches $i \to i+1$ dans les catégories à gauche et à droite s'envoient sur le même morphisme dans la colimite. Il reste à montrer que le morphisme $i \to i+2$ provenant de

$$(0 \to \cdots \to \widehat{i+1} \to \cdots \to n)$$

est bien égal à la composition $i \to i+1 \to i+2$ bien définie dans la colimite. Ce n'était pas le cas pour $n = 2$, mais c'est vrai pour $n = 3$: il suffit de prendre $j \in [n] - \{i, i+1, i+2\}$ (supposons par exemple $j < i$, l'autre cas se fait par symétrie) et de considérer le diagramme

$$(0 \to \cdots \to \hat{j} \to \cdots \to n) \leftarrow (0 \to \cdots \hat{j} \to \cdots \to \widehat{i+1} \to \cdots \to n) \to (0 \to \cdots \widehat{i+1} \to \cdots \to n)$$

ce qui montre que la flèche $i \to i+2$ à droite s'identifie avec la même flèche dans la catégorie de gauche : or dans cette catégorie elle est égale à la composée $i \to i+1 \to i+2$ ce qui implique que c'est le cas dans la colimite aussi.

4. We know that X is the colimit of its skeletons and that each skeleton is built by induction via the pushouts along the inclusions $\partial\Delta[n] \to \Delta[n]$. As τ commutes with colimits the previous exercise solves the question.

5. Since $Sk_2(X)$ has no non-degenerates simplices of degree ≥ 3, we only have to understand the contributions of non-degenerate simplices of degree 1 and 2. We have seen in question 1 that the objects of $\tau(Sk_2(X)$ are X_0 and that X_1 generates morphisms. Note that if $\sigma \in X_1$ is degenerate, that is $\sigma = \epsilon_0(x)$, then, in the colimit defining $\tau(Sk_2(X)$, we have that $\tau(\sigma) = \epsilon_0(\tau(x)) = Id_x$. Now it remains to understand the two simplices. But in the two simplex [2] we have that the unique morphism $0 \to 2$ is the composition $0 \to 1 \to 2$. But $0 \to 2$ is just the image $d_1([1])$ by the functor associated to d_1 while the subcategory $0 \to 1 \subset [2]$ is the image of d_2 and $1 \to 2$ the one of d_0. Hence the explicit formula of the colimit defining τ shows that every two simplex σ imposes a relation $\partial_1(\sigma) = \partial_0(\sigma) \circ \partial_2(\sigma)$. We have no other relations since we only need to consider non-degenerates simplices of degree less than 2.

6. By question 2., $N(\mathcal{C})_0$ is the set of objects of $\mathcal{C}$ and $N(\mathcal{C})_1$ is the set of morphisms in $\mathcal{C}$ and $N(\mathcal{C})_2$ is the set of all composable two arrows. Its faces are given by $\partial_0(X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2) = f_1$, $\partial_1(X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2) = f_1 \circ f_0$ and $\partial_2(X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2) = f_0$. Hence by question 5., we have that $\tau(Sk_2(N(\mathcal{C})))$ is the free category generated by the objects and arrows of $\mathcal{C}$ quotiented by the relation of composition in $\mathcal{C}$. It is thus isomorphic to $\mathcal{C}$ itself. By direct inspection, the adjunction map $\tau(N(\mathcal{C})) \to \mathcal{C}$ is the map taking the category $\tau(N(\mathcal{C}))$ which is the identity on objects and maps string of arrows to their class in $\mathcal{C}$. By the previous computations it is thus an isomorphism. This being proved we thus have the isomorphisms

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{D}, \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Cat}}(\tau(N(\mathcal{D})), \mathcal{C}) \cong_{\mathrm{sSet}} (N(\mathcal{D}), N(\mathcal{C}))$$

which proves the fully faithfulness of N.

7. This essentially reduces to a computation of colimits of sets.

8. The simplicial set I_n has exactly $n+1$ non-degenerate 1-simplices, denoted $\alpha_i \cong \{i, i+1\}$, and no higher non-degenerates ones. The only relation between these 1-simplices are that $d_1(\alpha_{i+1}) = d_0(\alpha_i)$ hence, a simplicial set map from I_n to $X_\bullet$ is given by a n-tuple $(x_1, \ldots, x_n)$ satisfying that $d_0(x_1) = d_1(x_2)$ and so on. In other words, $\mathrm{Hom}_{\mathrm{sSet}}(I_n, X) \cong X_1 \times_{X_0} X_1 \times \cdots \times_{X_0} X_1$. Applying this to $X_\bullet = N(\mathcal{C})$, we obtain that a map from I_n to $N(\mathcal{C})$ is exactly a string of n-composable arrows, hence the claimed isomorphism. We take $\mathcal{C} = [n] = \tau(\Delta[n])$. The canonical map $\tau(I_n) \to \tau(\Delta[n]) = [n]$ is by definition the image of the identity of $[n]$ under the map $\mathrm{Hom}_{\mathrm{Cat}}([n], [n]) \xrightarrow{-\circ\tau(i)} \mathrm{Hom}_{\mathrm{Cat}}(\tau(I_n), [n])$ where $i : I_n \hookrightarrow \Delta[n]$ is the inclusion. But by adjunction, and since $\tau(\Delta[n]) = [n]$, we have a commutative diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Cat}}([n],[n]) \arrow[r, "\cong"] \arrow[d, "-\circ\tau(i)"'] & \mathrm{Hom}_{\mathbf{sEns}}(\Delta[n], N([n])) \arrow[d, "-\circ i"] \\
\mathrm{Hom}_{\mathrm{Cat}}(\tau(I_n),[n]) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbf{sEns}}(I_n, N([n]))
\end{tikzcd}
$$

where the right vertical map is a bijection by above. Hence the left vertical one is bijective too.