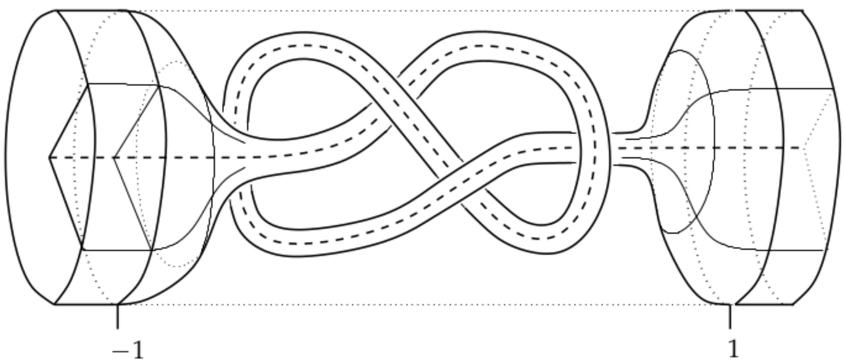

A MODEL IN GROUPOIDS FOR THE SPLICING OPERAD

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Abstract

The splicing operad, introduced by Budney in [Bud12], encodes operations on the space of self-embeddings of manifolds of the form $\mathbb{R}^j \times M$, particularly on the space of 3-dimensional long knots $\mathcal{K}$. Budney establishes that $\mathcal{K}$ forms a free algebra over the splicing operad $\mathcal{SP}_{3,1}$, with an explicit base space $\mathcal{TH}$. This operad extends the little 2-discs operad, which handles connect-sum operations on knots. Remarkably, each connected component of $\mathcal{SP}_{3,1}$ and $\mathcal{K}$ is an Eilenberg-Maclane space $K(G,1)$. This motivates the search for a small groupoid model of $\mathcal{SP}_{3,1}$, meaning an simple operad in groupoids such that its classifying space is equivalent to $\mathcal{SP}_{3,1}$, as for the case of the *parenthesized braids operad* model for the little 2-discs operad [BN98], [Tam03]. This thesis aims to construct a small groupoid model for the splicing operad, offering a combinatorial description of its action on the space of knots.

Contents

Introduction

Knot theory deals with the study of knots in three-dimensional space. More precisely, classical knots are embeddings of S^1 into S^3. A variant of these classical knots is the one of *long* knots, which are embeddings $\mathbb{R} \to \mathbb{R}^3$ that agree with the standard inclusion near infinity. We will call $\mathcal{K}$ this space of embeddings equipped with a suitable topology. A main problem is the one of the *classification* of such knots, up to isotopy. This amounts to understand the structure of $\pi_0(K)$. A useful approach towards this goal is to find interesting operations on these knots. One of the most natural one is the connect-sum: If you take two isotopy classes of long knots, you can glue one after the other and obtain another well defined isotopy class of long knots. This operation is associative and commutative. In 1949, Schubert [Sch49] showed that long knots can be decomposed into a unique sum of prime knots. This results means that $\pi_0(K)$ is a commutative free monoid. However, on the space $\mathcal{K}$ itself, the sum of knots is only homotopy-associative and homotopy-commutative, for similar reasons as concatenation of based loops on a space is only homotopy-associative. To deal with this additional complexity of the operations, the suitable object we need to work with is that of an operad. Roughly, a topological operad is a sequence of spaces $(\mathcal{P}(n))$ that represent a "space of n-ary operations" together with the information of how they compose together. The up-to-homotopy commutativity of the connect-sum of knot could thus be formalized as the fact that if μ is a choice of a connect-sum operation, then the operation $(k_1, k_2) \mapsto \mu(k_1, k_2)$ is connected by a path of operations to the operation $(k_1, k_2) \mapsto \mu(k_2, k_1)$, in a suitable operad encoding operations on the space of knots.
The most famous and central topological operad is the *little discs* operad $\mathcal{D}_n$. In this text we will frequently use the *little cubes* operad $\mathcal{C}_n$, an equivalent operad but better-suited for our context. In [Bud07], Budney shows that $\mathcal{K}$ has the structure of a free algebra over the little 2-cubes operad, with generating space the space of *prime knots* $\mathcal{P}$. This result can be seen as a generalization of Schubert's theorem at the space level: we recover the original theorem remembering only the connected components. This results theoretically allows us to compute the homotopy type of $\mathcal{K}$, knowing the homotopy type of $\mathcal{P}$. However, this latter space is still complicated to understand. In [Bud12], Budney constructs a new operad, the *splicing operad* $\mathcal{SP}_{3,1}$, which encodes additional operations, on the space of knots. These *splicing* operations were defined by Siebenmann [Sie80], inspired by the *satellite operations* defined by Schubert [Sch53]. The article [Bud12] allows then for an operadic description of these operations. The main result is that $\mathcal{K}$ is again a free algebra over $\mathcal{SP}_{3,1}$, with a notably smaller generating space $\mathcal{TH}$. Moreover, $\mathcal{SP}$ is equivalent to a pushout:

$$\begin{array}{ccc} O_2 & \longrightarrow & \overline{\mathcal{C}_1'} \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{TP} & \longrightarrow & \mathcal{SP}_{3,1} \end{array}$$

Where $\mathcal{TP}$ is an "free operad under O_2" (a notion we will define properly) and $\overline{\mathcal{C}_1'}$ is equivalent to a semi-direct product $\mathcal{C}_2 \rtimes O_2$. This time, the problem of determining the homotopy type of $\mathcal{SP}_{3,1}$ and $\mathcal{K}$ reduces to the more specific problem of understanding the isometry group of hyperbolic "knot-generating links", that is, the group of isometries of some links L put in a "maximal symmetry position". Modulo this computation, the homotopy type of $\mathcal{K}$ is entirely computable.

On another level, an interesting fact about the little 2-cubes operad is that it is *aspherical*: all of its connected components are $K(G,1)$, Eilenberg-Maclane spaces of discrete groups. In other words, all the spaces involved are classifying spaces $\mathsf{B}G$ of some groups G. In this case, these groups are the *pure braid groups* PB_n. This fact allowed for a combinatorial description of the little 2-cubes operad, in the following sense: there are discrete and simple-to-describe operads (Br, PaB [Fre17], [Tam03]) in the category of *groupoids*, such that their classifying space is an operad equivalent (in a sense to define) to the little 2-cubes operad $\mathcal{C}_2$. We call such an operad a *groupoid model* for $\mathcal{C}_2$. These groupoid models have proven useful to get informations about the structure of $\mathcal{C}_2$, for example Tamarkin's formality result [Tam03].

It turns out that $\mathcal{S}P$, $\mathcal{TH}$ and $\mathcal{K}$ are also all aspherical: each of their connected components is a $K(G,1)$. This should then allow for a combinatorial description of the splicing operad and $\mathcal{K}$, and find a discrete operad in groupoids $\mathsf{sp}_{3,1}$ such that its classifying spaces is equivalent to $\mathcal{SP}_{3,1}$.

This thesis is divided into three parts: in a first section, we introduce the useful specific algebraic machinery, notably the notion of a $\Sigma^* \wr G$-operad introduced by Budney in the same article. As we will deal with non-topological operads, we define this notion in a broader context and show how this leads to an analogous notion for operads in groupoids. We detail the construction of free $\Sigma^* \wr G$-operads and of the free product of such operads. Secondly, we review and interpret the main results of Budney's article [Bud12] and show that $\mathcal{S}P$ is aspherical and therefore $\mathcal{K}$ too, thus reproving a result of Budney in [Bud06]. In the last part, we develop and motivate the notion of a groupoid model for an operad, illustrating with the case of $\mathcal{D}_2$, the little 2-discs operad. We then proceed to construct an operad in groupoids $\mathsf{sp}_{3,1}$ and show it is a model for the splicing operad. The idea is to construct models of $O_2, \mathcal{TP}, \overline{\mathcal{C}_1'}$ and to then assemble them into a model of $\mathcal{SP}$ by taking their pushout. Finally, we construct a model in groupoids k of the space of knots $\mathcal{K}$ itself. We also include the computation of the rational homology of $\mathcal{SP}_{3,1}$ made by Beatrice Laracca [Lar19] in her master's thesis. We conclude by providing a geometrical interpretation of the generating loops of $\mathcal{K}$ and $\mathcal{SP}_{3,1}$, and suggesting some open questions linked to the splicing operad and the space of knots.

Part I
The theory of operads

1 Symmetric operads

1.1 Definition and some examples

Intuitively, an operad is an algebraic structure which encodes abstract n-ary operations and how they compose together. They were introduced by Peter May to study the structure of iterated loop spaces in [May72].
Formally, let us fix a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$.

Definition 1.1. *A* symmetric sequence *is a sequence $M(n)$ of objects of $\mathcal{C}$, together with a right action of the n-th symmetric group Σ_n on $M(n)$.*

Definition 1.2. *A* symmetric operad *$\mathcal{P}$ in $\mathcal{C}$ is a symmetric sequence equipped with* composition maps

$$\mathcal{P}(r) \otimes \mathcal{P}(n_1) \otimes ...\mathcal{P}(n_r) \to \mathcal{P}(n_1 + ... + n_k)$$

which satisfy equivariance properties with respect to the symmetric group and compatibility with composition, together with a unit morphism $\mathbf{1} \to \mathcal{P}(1)$. *The precise axioms can be found for example in [Fre17], [LV12]. A morphism of operads $\mathcal{P} \to \mathcal{Q}$ is the data of morphisms $\mathcal{P}(n) \to \mathcal{Q}(n)$ in each arity n, which are equivariant with respect to the symmetric group action and commute with the composition maps.*

The object $\mathcal{P}(n)$ must be thought as a family of abstract n-ary operations. The right action of the symmetric group models the permutation of the inputs on these operations, and the composition map $\mathcal{P}(r) \otimes \mathcal{P}(n_1) \otimes ...\mathcal{P}(n_r) \to \mathcal{P}(n_1 + ... + n_k)$ represents formally the fact that if we have a r-ary operation f and a list of operations $g_1, ..., g_r$ with g_i of arity n_i, we can *compose* them and get a $(n_1 + ... + n_r)$-ary operation, concretely :

$$f(g(x_{1,1}, ...x_{1,n_1}), ..., g(x_{r,1}, ..., x_{r,n_r}))$$

Note that such composition maps give rise to *partial composition* operations $\mathcal{P}(m) \otimes \mathcal{P}(n) \xrightarrow{\circ_i} \mathcal{P}(m+n-1)$ by only plugging $\mathcal{P}(n)$ onto the i-th input of $\mathcal{P}(m)$, and plug the unit of the operad on the other inputs. Conversely, from partial composition operations with suitable relations, one can recover the operad structure. The operads in $\mathcal{C}$ and their morphisms form a category which we will call $\mathcal{C}\mathsf{Op}$.

Remark 1.1. *If A is an object of a closed monoidal category $\mathcal{C}$ (meaning it is enriched over itself). The collection $End_n(A) := Hom(A^{\otimes n}, A)$ has a natural structure of a $\mathcal{C}$-operad. We call this operad End(A), the* endomorphism operad *of A.*

As for groups, operads are interesting because there is a notion of a *action* of an operad on an object A. We will say in this case that A is an *algebra* over $\mathcal{P}$. Intuitively, it is a concrete realization of the abstract operations of the operad : an object A equipped with n-ary operations $A \otimes ... \otimes A \to A$ as specified by $\mathcal{P}$, subject to the same composition relations. This can be summarized by :

Definition 1.3. *Let $\mathcal{C}$ be a closed monoidal category and $\mathcal{P}$ a $\mathcal{C}$-operad. An* algebra *over $\mathcal{P}$ is the data of an object A together with a morphism of operads $\mathcal{P} \to End(A)$.*

The algebras over an operad form a category. In this sense, to each operad corresponds a category of *algebras*, and we can think of an operad as an algebraic structure which encodes the data of some specific algebra structure. Let us give some examples in the category of sets and the category of vector spaces.

1. The operad $\mathcal{A}ssoc$ (for *Associative*) is defined by $\mathcal{A}ssoc(n) = \Sigma_n$ the symmetric group, with right action the right multiplication. The composition maps take a n-permutation σ (which we write as an ordered list of integers $\sigma(1), ..., \sigma(n)$) and permutations $\tau_1, ..., \tau_n$ and give the "block permutation" $\tau_{\sigma^{-1}(1)} \square \dots \square \tau_{\sigma^{-1}(n)}$ where by $\square$ we mean concatenating the ordered list representing the permutation and renumbering adequately. For more detail, see [LV12]. Indeed, there is a unique constant $\mathbf{1} \in \mathcal{C}omm(0)$, the *unit*, and one can show this operad is generated by $\mathbf{1}$ and a multiplication $\mu \in \mathcal{A}ssoc(2)$ with the only relation the *associativity*. In operadic language this means

   $$\mu \circ_1 \mu = \mu \circ_2 \mu.$$

   Algebras over this operad are simply *monoids*.

2. The operad $\mathcal{C}omm$ with $\mathcal{C}omm(n) = \{*\}$ for every $n \in \mathbb{N}$ (and obvious structure maps) encodes similarly for *commutative* monoids. Indeed, there is only one 2-ary operation $\mu \in \mathcal{C}omm(2)$ and thus $\mu \circ (12) = \mu$ which means any algebra other $\mathcal{C}omm$ has a commutative (and associative) multiplication. In fact, this is the only structure needed.

3. Similarly, in the category of vector spaces, there exist operads which encode for associative algebras, commutative algebras, and Lie algebras.

1.2 The little disks operad, and May's recognition principle

Peter May introduced the notion of operads in [May72] motivated by the study of *iterated loop spaces*. Consider a pointed topological space (X, x_0) and its loop space $\Omega X = \text{Map}((S^1, 1), (X, x_0))$. There is a natural composition law which takes two loops γ_1, γ_2 and yields a loop $\gamma_1 \cdot \gamma_2$ which travels along γ_1 then γ_2 at double speed for each. This law is compatible with homotopy of loops and makes $\pi_0(\Omega X)$ a group : it is the fundamental group $\pi_1(X, x_0)$. However, at the level of *loops* (not homotopy classes) the composition operation is *not even associative*, only up to homotopy. How to describe in a coherent way the data of these homotopies and all the possible composition operations on loops ? Moreover, if we look more generally at $\Omega^k X$, there is a rich structure of composition laws which, among other things, makes $\pi_0(\Omega^k X) = \pi_k(X, x_0)$ an abelian group for $k \geq 2$. Operads help us to understand this structure on the space of loops.

Definition 1.4. *Let $n \geq 1$ be an integer. The little n-disks operad $\mathcal{D}_n$ is an operad in topological spaces defined by saying that $\mathcal{D}_n(k)$ is the space of embeddings*

$$\underline{k} \times D^n \hookrightarrow D^n$$

where $\underline{k} = \{1, ..., k\}$ and the restriction to each n-ball $\{j\} \times D^n$ is a composition of a translation and a dilatation. The action of the symmetric groups permutes the labels of the balls. The composition maps are given by composition of embeddings.

An algebra over $\mathcal{D}_n$ is then a space A with a continuous family of n-ary operations parametrized by $\mathcal{D}_n(k)$, that satisfy the relations imposed by the structure of $\mathcal{D}_n$. Note that if A is an algebra over $\mathcal{D}_n$, its set of connected component $\pi_0 A$ inherits a structure of a monoid, commutative whenever $n \geq 2$.

Proposition 1.1 ([May72]). *If X is a pointed space, then $\Omega^n X$ has a natural structure of an* algebra *over the little disks operad $\mathcal{D}_n$.*

The remarkable fact is that the converse is true, modulo a small assumption.

Theorem 1 (Recognition Principle [May72]). *Let X be an algebra over $\mathcal{D}_n$. If $\pi_0 X$ is a group (we say X is* grouplike*) then X is homotopy equivalent, as a $\mathcal{D}_n$-algebra, to a n-fold loop space $\Omega^n Y$ for some Y.*

This results extends broadly in some sense the well-known fact that each topological group G has a *classifying space* $\mathsf{B}G$, which by classical results is a *delooping* of G, meaning $\Omega \mathsf{B}G \simeq G$. May's recognition principle says thus that a (grouplike) space that has only a up-to-homotopy associative multiplication can also be delooped.

We see that topological operads are interesting because they encode coherently operations on spaces that are sometimes only homotopy-associative, or homotopy-commutative. The up-to-homotopy associativity (resp. commutativity), in the operadic sense, is then represented by the fact that there is a *path of operations* from $\mu \circ_1 \mu$ to $\mu \circ_2 \mu$ (resp. from μ to $\mu \cdot (12)$) in $\mathcal{D}_n(3)$ (resp. $\mathcal{D}_{n\geq 2}(2)$). Thus, a $\mathcal{D}_1$-algebra is not necessarily commutative, even up to homotopy, while a $\mathcal{D}_2$-algebra is commutative up to homotopy, but the homotopies themselves are not necessarily homotopic. A $\mathcal{D}_\infty (= \operatorname{colim} \mathcal{D}_n)$-algebra is commutative up to any higher homotopy, in other words $\mathcal{D}_\infty(n)$, the space of n-ary operations is (weakly) contractible for each n.

2 The framed disk operad and semi-direct products

We introduce the notion of a semi-direct product of operads following the expositions given in [Wah01], [SW03].
Let $(\mathcal{S}, \otimes, 1)$ be a symmetric monoidal category, and let $(M, \mu, \eta, c, \varepsilon)$ be a bimonoid in $\mathcal{S}$. This means M is an object of $\mathcal{S}$ equipped with an associative, unital multiplication ($\mu : M \otimes M \to M, \eta : 1 \to M$) and a coassociative, counital comultiplication ($c : M \to M \otimes M, \varepsilon : M \to 1$) which is a morphism of algebras. For a bimonoid M, the category of M-modules, denoted M-Mod, is a monoidal category. In this case, we will assume that M is cocommutative (meaning $c \circ \tau = c$ where τ is the twist), thus making the category M-Mod symmetric. Consequently, we can study operads and their algebras within the category M-Mod. We refer to these operads as M-operads.
In other words, M-operads are operads such that the underlying collection is a M-module and the composition maps are M-equivariant.

Definition 2.1. *Let P be an operad in $\mathcal{C}$. The* semi-direct product *of $\mathcal{P}$ by M is defined by*

$$P \rtimes M(n) = P(n) \otimes M^n$$

The symmetric group acts diagonally on the right by permuting the factors of M^n and acting on $P(n)$.The composition maps are "twisted" by the action of M:

$$\begin{array}{c}
(P(k)\otimes M^k)\otimes(P(n_1)\otimes M^{n_1})\otimes\ldots\otimes(P(n_k)\otimes M^{n_k})\to P(\sum n_i)\otimes M^{\sum n_i}\\
\big\downarrow{\scriptstyle shuffle\circ(id\otimes c^k\otimes id\otimes\ldots\otimes id)}\\
P(k)\otimes(M\otimes P(n_1))\otimes\ldots\otimes(M\otimes P(n_k))\otimes M^{\sum n_i}\\
\big\downarrow{\scriptstyle id\otimes left\ actions\otimes id}\\
P(k)\otimes\bigotimes P(n_i)\otimes M^{\sum n_i}\\
\big\downarrow{\scriptstyle structure\ maps\otimes id}\\
P(\sum n_i)\otimes M^{\sum n_i}
\end{array}$$

Proposition 2.1. [*Wah01*] *Let $\mathcal{P}$ and M be as above. An object X of $\mathcal{C}$ is a $\mathcal{P}\rtimes M$-algebra if and only if X is a $\mathcal{P}$-algebra in M-Mod. In other words, X is an M-module and a $\mathcal{P}$-algebra with M-equivariant structure maps.*

The classical example of a semi-direct product of operads is the *framed discs operad* fD_n. The embeddings of discs in D_n are composition of dilatations and translations, while for fD_n we also allow for rotations of the discs. The framed discs operad fD_n can in fact be described as the semi-direct procuct $D_n\rtimes SO(n)$ where $SO(n)$ acts on collections of little discs by rotating the centers. See [SW03] for more details about this semi-direct product. In this text, we will encounter a different semi-direct product of the little discs operad, namely $\mathcal{D}_2\rtimes O_2$ with O_2 acting in a different way.
In [Bud12], the notion of a $\Sigma^*\wr G$ operad is introduced. As we will see, a semi-direct product $\mathcal{P}\rtimes G$ can be seen as a special case of a $\Sigma^*\wr G$-operad.

3 $\Sigma^*\wr G$-operads and free constructions

3.1 Definitions

The splicing operad defined in [Bud12] has a particular structure of "$\Sigma^*\wr G$-operad" introduced in the paper. In this subsection, we define this notion in a general context of a symmetric monoidal category. We also provide a characterization of such a structure over an operad. In all the text, $(\mathcal{C},\otimes,1)$ is a symmetric monoidal category which admits colimits and such that $\otimes$ distributes over colimits. This is the case for all the categories we will encounter in this text: Top and Gpd with the cartesian product, kVect (the category of vector spaces over a field k) for the tensor product. We will call "group object" in a monoidal category a group object for the tensor product, unlike the common definition where a "group object" is defined solely in a cartesian category.

Definition 3.1 (Wreath product)**.** *Let G a group object in $\mathcal{C}$. There is a right action of the symmetric group Σ_n on the product G^n, by permuting the factors. We define formally the wreath product $\Sigma_n\wr G$ as the object $\coprod_{\sigma\in\Sigma_n}G^n_\sigma$ (copies of G^n indexed by permutations) with the group law defined by*

$$G^n_\sigma\otimes G^n_\tau\xrightarrow{(-)\cdot\tau\otimes id}G^n_\sigma\otimes G^n_\tau\xrightarrow{\mu^n}G_{\sigma\tau}$$

Which makes it a group object in $\mathcal{C}$.

Remark 3.1. *For a concrete group G, the wreath product $\Sigma_n \wr G$ is the semi-direct product $\Sigma \ltimes G^n$ with the law :*

$$(\sigma, g_1, ..., g_n) \cdot (\tau, g'_1, ..., g'_n) = (\sigma\sigma', g_{\tau^{-1}(1)}g'_1, ..., g_{\tau^{-1}(n)}g_n).$$

Definition 3.2 ($\Sigma^* \wr G$-operad)**.** *A $\Sigma^* \wr G$-operad is a collection $(P(n))_n$ in $\mathcal{C}$ with a right action of $\Sigma_n \wr G$ and a left action of G, with composition morphisms*

$$P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(k_1 + \ldots k_n)$$

satifying certain equivariance relations, represented by the commutativity of the following diagrams:

$$\begin{array}{ccc}
P(n) \otimes G^n_\sigma \otimes P(k_1) \otimes \cdots \otimes P(k_n) & \xrightarrow{id \otimes L^n \circ \mathrm{shuffle}_\sigma} & P(n) \otimes P(k_{\sigma^{-1}(1)}) \otimes \cdots \otimes P(k_{\sigma^{-1}(n)}) \\
\downarrow{\scriptstyle R \otimes id} & & \downarrow{\scriptstyle \circ} \\
P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) & \xrightarrow{\quad \circ \quad} & P(\sum k_i)
\end{array}$$

$$\begin{array}{ccc}
G \otimes P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) & \xrightarrow{id_G \otimes \circ} & G \otimes P(\sum k_i) \\
{\scriptstyle L \otimes id}\downarrow & & \downarrow \\
P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) & \xrightarrow[\circ]{\quad} & P(\sum k_i)
\end{array}$$

where R and L represent respectively right and left action on the components.The first condition is called the "inner equivariance" while the latter is the "outer equivariance" in the article of Ryan Budney.

Proposition 3.1. *A structure of $\Sigma^* \wr G$-operad over an operad $\mathcal{P}$ is the same thing as a morphism of operads $G \to \mathcal{P}$, when we consider G as an operad concentrated in arity one.*

Proof. Any operad $\mathcal{P}$ with an operad morphism $G \to \mathcal{P}$ inherits a Σ-equivariant G-bimodule structure. This amounts exactly to the data of a left action of G and right action of $\Sigma_n \wr G$ that commute. Moreover, because $G \to \mathcal{P}$ is a morphism of operads, the structure maps of $\mathcal{P}$ are compatible with the composition of operads. These are exactly the coherence conditions for a $\Sigma^* \wr G$-operad.
For the converse, to construct a morphism $G \to \mathcal{P}$, consider the unit map $\mathbf{1} \to \mathcal{P}(1)$. We then construct a map $G \simeq G \otimes \mathbf{1} \to G \otimes \mathcal{P}(1) \to \mathcal{P}(1)$. As G is seen as an operad concentrated in arity one, this is the only necessary data to define a morphism $G \to \mathcal{P}$. This map is indeed compatible with the composition map using the compatibility conditions of Definition 3.2. One can finally check that the $\Sigma^* \wr G$-operad structure is induced by this morphism $G \to \mathcal{P}$. □

Remark 3.2. *If G is a group in $\mathcal{C}$ with a cocommutative comultiplication $\Delta : G \to G^{\otimes 2}$ and $\mathcal{P}$ is an operad with a left action of G compatible with the structure maps, we can construct the semi-direct product operad $\mathcal{P} \rtimes G$. This operad has a natural structure of $\Sigma^* \wr G$-operad. The corresponding morphism $G \to P \rtimes G$ is indeed $G(1) = G = \mathbf{1} \otimes G \to P(1) \otimes G = (P \rtimes G)(1)$. semi-direct products can thus be considered as particular cases of $\Sigma^* \wr G$-operads.*

Remark 3.3. *The fact that G is a group object is not really needed in all the definitions of this section : we really only need G to be a monoid in $\mathcal{C}$. However, in our context G will always be a group. Moreover, the characterization of a $\Sigma^* \wr G$-operad with a morphism $G \to \mathcal{P}$ gives a definition for any symmetric monoidal category (where colimits don't necessarily exist or commute with $\otimes$). In this case a right action of $\Sigma_n \wr G$ must be interpreted as : the combined data of an "external" (e.g defined by a family of automorphisms) Σ_n-action and a "internal" (e.g. defined with $\otimes$) G^n-action that satisfy certain compatibility relations.*

3.2 The construction of free $\Sigma^* \wr G$-operads

We fix a monoidal category $\mathcal{C}$ such that $\otimes$ commutes with colimits.
Let (G, μ, ε) be a group object in $\mathcal{C}$. We denote by $\Sigma^* \wr G$-Seq the category of sequences of objects $(M(n))_{n \in \mathbb{N}}$ with a right action of the group $G^{op} \otimes \Sigma_n \wr G$. In other words, $M(n)$ carries a left action of G and a left action of $\Sigma_n \wr G$. Equivalently, this is the category of G-bimodules, viewing G as an operad. Every such collection can be seen as a symmetric collection by restricting the group action to the symmetric groups (even in a general symmetric monoidal category, there is a morphism $\Sigma_n \to \Sigma_n \wr G$). The goal is to construct explicitly a left adjoint of the forgetful functor $\Sigma^* \wr G$-Op $\to \Sigma^* \wr G$-Seq. The construction is an extension of the classical free operad construction. A similar exposition can be found in [Pet13], where the (internal) group action is replaced by an action of a small (external) category.

When constructing the free operad in the classical case, it is often more convenient to replace the notion of Σ-collection (objects indexed on the integers with an action of the symmetric goup) by the equivalent notion of a functor $\mathbb{S} \to C$, where $\mathbb{S}$ is the core of the category of finite sets. We will use this point of view, as it gives more natural definitions.

Definition 3.3. *A $\mathbb{S}^* \wr G$-collection is a functor $M : \mathbb{S} \to \mathcal{C}$ such that, for every finite set $\underline{r}$ there is a left action of G and a right action of $G^{\underline{r}}$ on $M(\underline{r})$, such that the actions commute and that for every bijection $\underline{r} \xrightarrow{\phi} \underline{s}$, the following diagram commutes:*

$$\begin{array}{ccc} M(\underline{r}) \otimes G^{\underline{r}} & \xrightarrow{\phi \otimes (\phi^{-1})^*} & M(\underline{s}) \otimes G^{\underline{s}} \\ \downarrow & & \downarrow \\ M(\underline{r}) & \xrightarrow{\phi} & M(\underline{s}) \end{array}$$

One can check that restrained to the sets $\{1, ..., n\}$ the conditions amounts to having a right action of the wreath product $\Sigma_n \wr G$ that commutes with a left action of G.

Definition 3.4 (Trees)**.** *Formally, an $\underline{r}$-tree T consists of a set of vertices, denoted by $V(T)$, and a set of edges $e \in E(T)$ oriented from a source $s(e) \in V(T) \sqcup \underline{r}$ towards a target $t(e) \in V(T) \sqcup \{0\}$, such that the following conditions hold:*

1. *There is one and only one edge $e_0 \in E(T)$, the outgoing edge of the tree, such that $t(e_0) = 0$.*
2. *For each $i \in \underline{r}$, there is one and only one edge $e_i \in E(T)$, the ingoing edge of the tree indexed by i, such that $s(e_i) = i$.*

3. *For each vertex $v \in V(T)$, there is one and only one edge $e_v \in E(T)$, the outgoing edge of the vertex v, such that $s(e_v) = v$.*
4. *Each vertex $v \in V(T)$ is connected to the output 0 by a chain of edges $e_v, e_{v_{n-1}}, \ldots, e_{v_1}, e_{v_0}$ such that*

$$v = s(e_v),\ t(e_v) = s(e_{v_{n-1}}),\ t(e_{v_{n-1}}) = s(e_{v_{n-2}}), \ldots, t(e_{v_2}) = s(e_{v_1}), t(e_{v_1}) = s(e_{v_0})$$

and $t(e_{v_0}) = 0$.

Morphisms of $\underline{r}$-trees are compatible bijections of edge and arrows that respect the labeling of the inputs by the set $\underline{r}$. We note the category of $\underline{r}$-trees $\mathsf{Tree}(\underline{r})$.

Definition 3.5 (Grafting)**.** *If $\underline{r}$ and $\underline{s}$ are finite sets, and $i \in \underline{r}$, there is a bifunctor*

$$\mathsf{Tree}(\underline{r}) \times \mathsf{Tree}(\underline{s}) \to \mathsf{Tree}(\underline{r} \circ_i \underline{s})$$

that we will write $\circ_i$, consisting of "grafting" the trees on the input i.

Proof. The construction on objects is obvious. We have to show that it is functorial. Indeed, a pair (f, g) made of an isomorphism of $\underline{r}$-Trees and an isomorphism of $\underline{s}$-Trees induces naturally an isomorphism of $\underline{r} \circ_i \underline{s}$-Trees. □

Once given a $\mathbb{S}^* \wr G$-collection M, we can define a realization of a tree in the following way.

$$M(|) = G$$

$$M(T) := \text{coequalizer}\left(G^{E_{int}(T)} \otimes \bigotimes_{v \in V(T)} M(r_v) \rightrightarrows \bigotimes_{v \in V(T)} M(r_v) \right)$$

induced by both right and left actions of G on the input and output of each internal edge.

Let ϕ be a isomorphism of trees from T to T'. This induces a morphism between the realizations:

$$M(\phi) : M(T) \longrightarrow M(T')$$

the morphism being well defined on the coequalizers because of the compatibility between the right action of $G^{\underline{r}}$ and the symmetric action.

There is a natural grafting operation

$$M(T) \otimes M(T') \xrightarrow{\circ_i} M(T \circ_i T')$$

induced by the tensor product on the realization of the trees. When neither T and T' are the trivial tree, the map is induced by the tensor product:

$$\begin{array}{ccc}
G^{E_{int}}(T) \otimes \bigotimes_{v \in V(T)} M(r_v) \otimes G^{E_{int}}(T') \otimes \bigotimes_{v \in V(T')} M(r_v) & \xrightarrow{\iota} & G^{E_{int}}(T \circ_i T') \otimes \bigotimes_{v \in V(T \circ_i T')} M(r_v) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\bigotimes_{v \in V(T)} M(r_v) \otimes \bigotimes_{v \in V(T')} M(r_v) & \longrightarrow & \bigotimes_{v \in V(T)} M(r_v) \\
\vdots & & \vdots \\
M(T) \otimes M(T') & \xrightarrow{\exists ! \circ_i} & M(T \circ_i T')
\end{array}$$

In the case where T or T' is the trivial tree, this map is simply left or right multiplication by G. Finally, we construct the underlying collection of the free operad ΘM by setting, for a finite set $\underline{r}$:

$$\Theta M(\underline{r}) = \underset{T \in \text{Trees}(\underline{r})}{\text{colim}} M(T)$$

There is a right action of $G^{\underline{r}}$ on $\Theta M(\underline{r})$ by making $G^{\underline{r}_v}$ act on each external edge (those connected to the inputs), and using the canonical isomorphism $\bigotimes G^{\underline{r}_v} \simeq G^{\underline{r}}$. This action is obviously compatible with reindexing: if $\phi : \underline{r} \to \underline{s}$ is a bijection, the respective actions of $G^{\underline{r}}$ and $G^{\underline{s}}$ are compatible in the sense of the diagram of Definition 3.3. There is at the same time a left action of G on the bottom edge, which commutes with the former. These two action make thus $\Theta M(r)$ a $\mathbb{S}^* \wr G$-collection.

Remark 3.4. *The category $Tree(\underline{r})^{iso}$ has the property that every object has no nontrivial automorphism. This implies that the underlying symmetric collection of the free operad $\Theta M(r)$ can be written as*

$$\Theta M(r) = \bigsqcup_{[T] \in \pi_0 Tree(r)^{iso}} M(T)$$

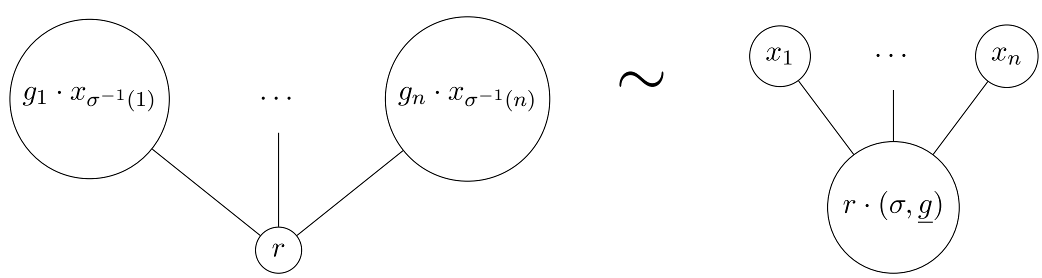
for each finite non-empty set $\underline{r}$, where the coproduct ranges over (a set of representatives of) isomorphism classes of the category of $\underline{r}$-trees $[T] \in \pi_0 Tree(\underline{r})^{iso}$.

The next thing to do is to put an operad structure over this collection. The composition is done by grafting the trees. For $i \in \underline{r}$, there is a partial composition

$$\Theta M(\underline{r}) \otimes \Theta M(\underline{s}) \xrightarrow{\circ_i} \Theta M(\underline{r} \circ_i \underline{s})$$

Induced by the composition morphisms $M(T) \otimes M(T') \to M(T \circ_i T')$.

In the concrete case, when G is a discrete group, we can think of the collection $\Theta M(n)$ as a family of planar trees modded out by the following relation:



Let's show that this construction is indeed a left adjoint to the forgetful functor from G-operads to G-collections. We first construct the counit of the adjunction, that is a natural morphism $\Theta(P) \to P$ for each $\Sigma^* \wr G$-operad P. Let $f : M \to P$ be a morphism in the category of collections with an operad P as target object. For any $\underline{r}$-tree T, where $\underline{r}$ is any (non-empty) finite set, we consider the map

$$M(T) \xrightarrow{f_*} P(T) \xrightarrow{\lambda_T} P(\underline{r})$$

where f_* is the morphism induced by $f : M \to P$ by functoriality of the treewise tensor product construction. λ_T is the treewise composition operation associated to the operad P. The construction of this morphism is almost the same as in [Fre17, A.2.4]. The fact that this constructions extends to $\Sigma^* \wr G$-operads comes from the fact that the composition morphisms $P(r) \otimes P(n_1) \otimes ... \otimes P(n_r) \xrightarrow{\circ_i} P(\sum n_i)$ are equivariant and then induce well-defined morphisms

$$P(m) \otimes_{G^r} P(n_1) \otimes ... \otimes P(n_r) \to P(\sum n_i)$$

that allow to define a morphism $P(T) \to P(\underline{r})$ using the iterative description of the realization of trees (taking inductively coequalizers).

The splicing operad is shown in [Bud12] to be equivalent to a free product (as $\Sigma^* \wr G$-operads) of a free operad $\mathcal{TP}$ with an operad $\overline{\mathcal{C}'_1}$. We are going to describe explicitly the free product of two $\Sigma^* \wr G$-operads, when one of those is free. This construction is widely inspired by [Fre17] in which the corresponding classical case (for symmetric operads) is treated.

3.3 Free products of $\Sigma^* \wr G$-operads

We consider trees T whose set of vertices is equipped with a partition $V(T) = V^\bullet(T) \sqcup V^\circ(T)$ such that $V^\bullet(T)$ defines a subset of vertices marked with a black color, and $V^\circ(T)$ defines a subset of vertices marked with a white color. We can equivalently assume that the set of vertices of our tree T is equipped with a mapping $c : V(T) \to \{\bullet, \circ\}$ to define this coloring:

$$V^\bullet(T) = c^{-1}(\bullet),$$
$$V^\circ(T) = c^{-1}(\circ).$$

We generally specify such a vertex coloring $c : V(T) \to \{\bullet, \circ\}$ by adding a subscript c in the notation of our tree T. We now say that T_c forms a semi-alternate two-colored tree when:
For any inner edge $e \in E_{int}(T)$ with $v = s(e) \in V(T)$ and $u = t(e) \in V(T)$, we have either $(c(u), c(v)) = (\bullet, \circ)$, or $(c(u), c(v)) = (\circ, \bullet)$, or $(c(u), c(v)) = (\circ, \circ)$, but in all cases $(c(u), c(v)) \neq (\bullet, \bullet)$. Thus, the white vertices can form non-trivial subtrees in T_c but the black vertices are all isolated. We adopt the notation $Tree^{\bullet\circ}(r)$ for the class of semi-alternate two-colored $\underline{r}$-trees. We also consider isomorphisms of semi-alternate two-colored $\underline{r}$-trees $f : S_c \to T_d$ which we define as isomorphisms of $\underline{r}$-trees $f : S \to T$ that preserve the color of vertices. We still use the superscript mark iso to denote the category $Tree^{\bullet\circ}(\underline{r})^{iso}$ formed by the semi-alternate two-colored $\underline{r}$-trees and their isomorphisms. Semi-alternate treewise tensor products. Let P be a $\Sigma^* \wr G$-operad. Let M be a $\Sigma^* \wr G$-collection. For any semi-alternate two-colored $\underline{r}$-tree T_c we form the treewise tensor product $M(T_c, P)$ defined by the coequalizer:

$$G^{E_{int}(T)} \otimes \left(\bigotimes_{v \in V(T)} P(\underline{r_v}) \right) \otimes \left(\bigotimes_{v \in V(T)} M(\underline{r}_v) \right) \rightrightarrows \left(\bigotimes_{v \in V(T)} P(\underline{r_v}) \right) \otimes \left(\bigotimes_{v \in V(T)} M(\underline{r}_v) \right)$$

identifying left and right action of G, as in the case of the free $\Sigma^* \wr G$-operad. When T is the trivial tree, we simply put $M(T_c, P) = G$.

The degeneration of semi-alternate treewise tensor products.
We first define degeneration operations on trees which we use to shape our unit insertions. We assume that $e \in E(T)$ is an edge satisfying $s(e), t(e) \notin V_\bullet(T)$ (with possibly $s(e) \in \underline{r}$ or $t(e) = 0$) in a semi-alternate two-colored r-tree $T_c \in Tree_{\bullet\circ}(\underline{r})$. We then consider a tree $s_e(T)_c$ formed by inserting a black vertex v_e on the edge $e \in E(T)$. We formally set $V(s_e(T)) = V(T) \sqcup v_e$, $E(s_e(T)) = E(T) \setminus \{e\} \sqcup \{e^-, e^+\}$, with new edges $e^-, e^+ \in E(s_e(T))$, defined by splitting the edge $e \in E(T)$ and such that we have $s(e^-) = s(e), t(e^-) = s(e^+) = v_e, t(e^+) = t(e)$ in the tree $s_e(T)$. We also assign the color $c(v_e) = \bullet$, as required, to the vertex v_e which we insert on the edge e. We now have

$$M(s_e(T)_c, P) = P(\underline{r}_{v_e}) \otimes \bigotimes_{v \in V_\bullet(T)} P(\underline{r}_v) \bigotimes_{v \in V_\circ(T)} M(\underline{r}_v).$$

implicitly writing $\otimes$ as the tensor product over G for readability (there should be a coequalizer as always). We define the degeneration morphism (at the edge $e \in E(T)$)

$$s_e : M(T_c, P) \to M(s_e(T), P)$$

incuced on the coequalizers by the insertion of the structural morphism $\eta : G \to P(1)$ on the extra factor $P(\underline{r}_{v_e}) = P(1)$ of this tensor product.

The underlying collection of the coproduct with a free operad. We see that the semi-alternate treewise tensor product construction as well as the degeneration operations are functorial with respect to the action of the isomorphisms of the groupoids $\mathrm{Tree}_{\bullet\circ}(\underline{r})^{\mathrm{iso}}$. We moreover have an identity $M(u_* T_c, P) = M(T_c, P)$ when we apply an input reindexing operation $u : \mathrm{Tree}(\underline{r}) \to \mathrm{Tree}(\underline{s})$ to any semi-alternate two-colored $\underline{r}$-tree $T_c \in \mathrm{Tree}_{\bullet\circ}(\underline{r})$. Let $\mathrm{Tree}_{\bullet\circ}(\underline{r})^{\mathrm{iso}}_{\mathrm{T}}$ be the category obtained by adding formal degeneracy operators $s_e : T_c \to s_e(T_c)$ to the isomorphisms of semi-alternate two-colored r-trees.
We finally set:

$$P \vee_G \Theta(M)(\underline{r}) = \operatorname*{colim}_{T_c \in Tree_{\bullet\circ}(\underline{r})^{iso}} M(T_c, P)$$

It is shown in [Fre17] that there is an explicit description of the underlying collection of this free product, analogous to the remark 3.1. The proof adapts naturally to our context because it only involves the structure of the category of trees, which is the same in the classical and the $\Sigma^* \wr G$-case.
We can indeed observe that every semi-alternate two-colored tree T_c admits a maximal degeneration $\widehat{T}c$, obtained by degenerating all edges e satisfying $s(e), t(e) \in V_\bullet(T)$ (and allowable for a degeneration therefore) in the tree T. Such a maximal degeneration can be thought as a "completely alternating tree". We get, as a consequence, that the category of isomorphisms of semi-alternate two-colored r-trees and degeneracies $Tree_{\bullet\circ}(\underline{r})^{iso}$ splits as a coproduct of categories with terminal objects, which are precisely the maximal degenerations of semi-alternate two-colored r-trees $\widehat{T}_c$, for any arity $r > 0$. Let

$$Tree_{\bullet\circ}(\underline{r})^{max} \subset Tree_{\bullet\circ}(\underline{r})^{iso}$$

denote the subcategory of maximal objects. We get an explicit description of the coproduct of free operads:

Proposition 3.2. *The coproduct $P \vee \Theta(M)$ of a non-unitary operad P with the free operad on a non-unitary collection M has a reduced expansion such that:*

$$P \vee \Theta(M)(\underline{r}) = \bigsqcup_{[\widehat{T}_c] \in \pi_0 Tree^{max}_{\bullet\circ}(\underline{r})} M(\widehat{T}_c, P)$$

for each finite (non-empty) set $\underline{r} = 0$, where the coproduct ranges over (a set of representatives of) isomorphism classes of maximal objects $\widehat{T}_c \in Tree_{\bullet\circ}(\underline{r})$ in the category of semi-alternate two-colored r-trees and degeneracies $Tree_{\bullet\circ}(\underline{r})$.

Operad structure. The construction is similar to the classical case, see [Fre17].The proof that the free product is indeed the coproduct in the category of $\Sigma^* \wr G$-operads follows also the same path as in Fresse's book, the only difference being that we need to check everything is $\Sigma^* \wr G$-equivariant.

3.4 An iterated construction of a free product

We collect in this part some technical lemmas that we will use to describe and compute the homotopy type of the splicing operad. We show that the realization of a tree can be constructed inductively using realizations of subtrees. We work in a monoidal category $(\mathcal{C}, \otimes)$ such that $\otimes$ commutes with colimits. This general context will allows us to apply our results both to topological operads and in the category GpdOp of operads in groupoids.

Remark 3.5. *Let $\mathcal{C}$ be monoidal category, G be a group object, X be an object of the form $Y \otimes G$ with G acting by right translation on its copy. Let Z be any left G-object. Then $X \underset{G}{\otimes} Z$, the coequalizer of*

$$X \otimes G \otimes Z \rightrightarrows X \otimes Z$$

is isomorphic to $Y \otimes Z$, the projection map being $X \otimes Z = Y \otimes G \otimes Z \xrightarrow{id_Y \otimes \text{left action}} Y \otimes Z$.

Proof. Let us call p the projection map $X \times Z \to Y \times Z$ We first show that this morphism coequalizes right/left action of G. Indeed, the composition

$$X \otimes G \otimes Z \rightrightarrows X \otimes Z \to Y \otimes Z$$

is equal, for the right action of G on X, simply to the morphism $Y \otimes G^2 \otimes Z \xrightarrow{id_Y \otimes \mu \otimes id_Z} Y \otimes G \otimes Z \xrightarrow{id_X \otimes left} Y \otimes Z$ while the composition with the left action on Z gives

$$Y \otimes G \otimes G \otimes Z \xrightarrow{id_Y \otimes id_G \otimes left} Y \otimes G \otimes Z \xrightarrow{id_Y \otimes left} \otimes Z$$

these two composition are equal by the compatibility of the group action. Now let $X \otimes Z \xrightarrow{\phi} W$ be a cocone. Write s for the natural morphism $Y \otimes Z \xrightarrow{id_Y \otimes \eta \otimes id_Z} X \otimes Z$ with η the unit of G. Note that $ps = id_{Y \otimes Z}$. Define $\psi = \phi s$. The goal is to show that $\psi p = \phi$. Indeed, sp is in fact the top line in this commutative diagram:

$$\begin{array}{ccccc}
Y \otimes G \otimes Z & \xrightarrow{id_Y \otimes left} & Y \otimes Z & \xrightarrow{id_Y \otimes \eta \otimes id_Z} & Y \otimes G \otimes Z \\
\Big\| & & & & \Big\| \\
Y \otimes G \otimes Z & \xrightarrow[id_Y \otimes \eta \otimes id_G \otimes id_Z]{} & Y \otimes G^2 \otimes Z & \xrightarrow[id_Y \otimes id_G \otimes left]{} & Y \otimes G \otimes Z
\end{array}$$

and can then be rewritten as the bottom line. The morphism ϕ coequalizes the action of G, therefore the composition $\phi s p$ is equal to:

$$Y \otimes G \otimes Z \xrightarrow[id_Y \otimes \eta \otimes id_G \otimes id_Z]{} Y \otimes G^2 \otimes Z \xrightarrow[id_Y \otimes \mu]{} Y \otimes G \otimes Z \xrightarrow[\phi]{} W$$

and as the first two arrows compose to the identity, $\phi s p = \phi$. Therefore ψ is a lifting of ϕ and is unique because if ψ' is another such morphism then $\psi' p = \phi$ so $\psi' = \psi' p s = \phi s = \psi$. □

Lemma 3.1. *Let T be a (coloured) $\underline{r}$-tree over $\mathbb{S}^* \wr G$-collections. The left G-action on the root of T induces a left action on the realization $M(T)$.*

Proof. The left and right action on the root commute, so there is a commutative diagram:

$$\begin{array}{ccc}
G \otimes G^{E_{int}(T)} \otimes \bigotimes_v M(\underline{r}_v) & \longrightarrow & G^{E_{int}(T)} \otimes \bigotimes_v M(\underline{r}_v) \\
{\scriptstyle 1\otimes L}\downarrow\ \downarrow{\scriptstyle 1\otimes R} & & {\scriptstyle L}\downarrow\ \downarrow{\scriptstyle R} \\
G \otimes \bigotimes_v M(\underline{r}_v) & \longrightarrow & \bigotimes_v M(\underline{r}_v) \\
\vdots & & \vdots \\
G \otimes M(T) & \overset{\exists ! \phi}{\cdots\cdots\to} & M(T)
\end{array}$$

by universal property of the coequalizer on the left, and the fact that the square made of (red+black) arrows and also the one made from (blue+black) arrows commute (note that this is when the commutativity between left and right action is needed), there exists a well defined morphism $G \otimes M(T) \to M(T)$. One can check that it is indeed a group action. □

Lemma 3.2. *Let T be a (coloured) $\underline{r}$-tree over $\mathbb{S}^* \wr G$-collections. The set of colours is C and so there is a collection M_c for each $c \in C$ and each vertex v of T is labelled by a colour $c(v)$. Let $M(T)$ be its realization (coequalizing the left/right actions of G). We write $T = T(u, T_1, \ldots, T_n)$ when T has a root labeled by an vertex u with colour c of arity n, with T_i the trees grafted on each input of c. Each tree T_i is then a $\underline{r}_i$ tree with $\underline{r} = \bigsqcup \underline{r}_i$. Then:*

$$M(T) \cong \operatorname{coeq}\left(M_c(n) \otimes G^n \otimes \bigotimes M(T_i) \rightrightarrows M_c(n) \otimes \bigotimes M(T_i)\right)$$

where G^n acts on the left on each $M(T_i)$ and on the right on $M_c(n)$.

This simply means that we can take an iterated coequalizer, from the top leaf to the bottom, to construct the realization of a tree.

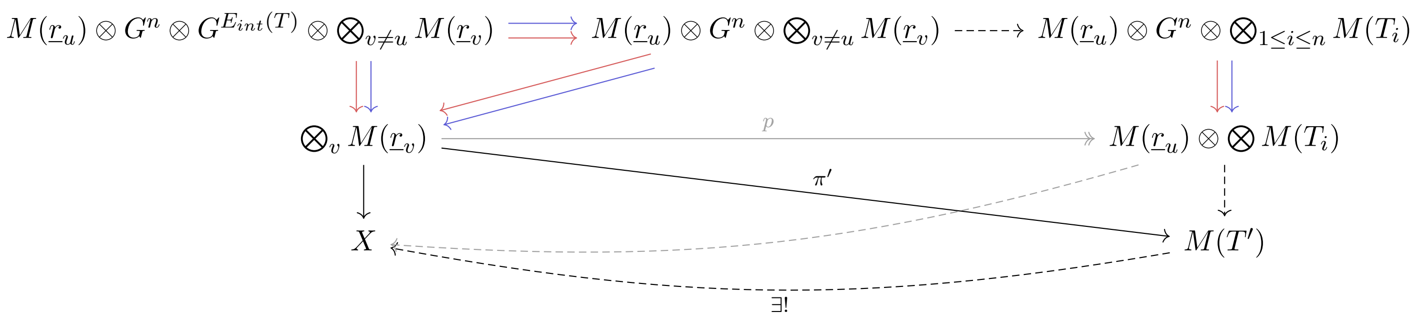
Proof. Define $M'(T)$ as the right-hand side. We are going to show it satisfies the universal property of the coequalizer. Let $\bigotimes M(\underline{r}_v) \xrightarrow{\phi} X$ that coequalizes right and left action. Note that there is a natural projection

$$\bigotimes M(\underline{r}_v) \xrightarrow{p} M(\underline{r}_u) \otimes \bigotimes M(T_i).$$

Composed with the projection of the coequalizer of the action of $G^{\underline{r}_u} = G^n$ between the root and the upper trees, it gives a projection $\bigotimes M(\underline{r}_v) \to M'(T)$.
Now, note that the map ϕ induces a map $M(\underline{r}_u) \otimes G^n \otimes \bigotimes_{v \neq u} M(\underline{r}_v) \to X$ that coequalizes the action associated to non-root edges. Therefore we get a map $M(\underline{r}_u) \otimes G^n \otimes \bigotimes_{1 \leq i \leq n} M(T_i) \to X$ and using the unit morphism of G^n we get a well-defined map

$$M(\underline{r}_u) \otimes \bigotimes_{1 \leq i \leq n} M(T_i) \to X$$

that commutes with the projection p. Finally, this map onto X coequalizes the left/right action of G^n over the root/the upper trees (this is a diagram chase involving the diagram of Lemma 5.1.). We finally get a morphism $M(T') \to X$ and we check that it commutes with the projection $\bigotimes M(\underline{r}_v) \xrightarrow{\pi'} M(T')$. Moreover, $M(T')$ coequalizes the usual left/right action $G^{E_{int}} \otimes \bigotimes M(\underline{r}_v) \to \bigotimes M(\underline{r}_v)$ by another diagram chasing.
We can sum up much of this proof in the following diagram:



□

Part II
The Splicing operad

In 1949 Schubert proved that long knots in $\mathbb{R}^3$ have a unique decomposition into prime knots. Concretely, there is a homotopy-associative pairing $\mathcal{K}_{3,1} \times \mathcal{K}_{3,1} \to \mathcal{K}_{3,1}$ called the connect-sum operation which turns $\pi_0\mathcal{K}_{3,1}$ (the isotopy classes of long knots) into a free commutative monoid. This homotopy-associative pairing has been shown to come from a $\mathcal{C}_2$-algebra structure on $\mathcal{K}_{3,1}$ in [**?**]. Namely, Budney constructs a space of *fat* long knots $\widehat{\mathcal{K}}_{3,1}$, equivalent to $\mathcal{K}_{3,1}$, which is a free algebra over the little 2-cubes operad $\mathcal{C}_2$.

Definition 3.6. *Let M be a manifold. We define $EC(j, M)$ to be the space of self-embeddings $\mathrm{Emb}(\mathbb{R}^j \times M, \mathbb{R}^j \times M)$ that agree with the identity outside $I^j \times M$.*

We are mainly interested in our case to $M = D^2$. The restriction map

$$EC(j, D^2) \to \mathcal{K}_{3,1}$$

given by $f \mapsto f_{|\mathbb{R}^j \times \{0\}}$ is a fibration with fiber $\Omega SO(2) \simeq \mathbb{Z}$. In fact, $EC(1, D^2)$ has the homotopy-type of $\mathcal{K}_{3,1} \times \mathbb{Z}$ since the fibration $EC(1, D^2) \to \mathcal{K}_{3,1}$ splits at the fibre, with splitting given by the linking-number of $f_{|R\times(0,0)}$ and $f_{|R\times(1,0)}$. Thus $\mathcal{K}_{3,1}$ has the homotopy-type of $\widehat{K}_{3,1} \subset EC(1, D^2)$ where $\widehat{K}_{3,1}$ is the subspace of $EC(1, D^2)$ consisting of knots f where the above linking number is zero.

4 The little cubes action on the space of (fat) knots

We introduce first a variant of the little cube operads, the *overlapping intervals* operad defined by Budney in [Bud12].

Definition 4.1. *An increasing affine-linear function $I \to I$ is a little interval. A product of little intervals $I^n \to I^n$ is a little n-cube. A collection of j overlapping n-cubes is an equivalence class of pairs (L, σ) where $L = (L_1, \dots, L_j)$, each L_i is a little n-cube, and $\sigma \in \Sigma_j$. Two collections of j overlapping n-cubes (L, σ) and (L', σ') are said to be equivalent when $L = L'$ and whenever the interiors of L_i and L_k intersect, $\sigma^{-1}(i) < \sigma^{-1}(k) \Leftrightarrow \sigma'^{-1}(i) < \sigma'^{-1}(k)$. Given j overlapping n-cubes, $(L_1, \dots, L_j, \sigma)$, we say the i-th cube L_i is at height $\sigma^{-1}(i)$. $\sigma(1)$ is the index of the bottom cube, and $\sigma(j)$ is the index of the top cube. Let $C'_n(j)$ be the space of all j overlapping n-cubes, with the quotient topology induced by this equivalence relation.*

Informally, a collection of overlapping little cubes is the data of little cubes with an ordering of the cubes, forgetting the relative ordering of two cubes whenever their interiors do not intersects. Note that it is a symmetric collection, any permutation τ acting by relabeling the cubes and sending an ordering σ to $\sigma \circ \tau$. This symmetric collection $C' = \bigsqcup_j C'_n(j)$ can be made an operad by composing the cubes together as in the usual little cubes operad, and patching the orderings togethers [Bud12].

Let f be a little $(n+1)$-cube. The projection $I^{n+1} \to I^n$ along the first component gives the data of a little n-cube. However, a collection of (non-overlapping) little $(n+1)$-cubes may give by projection a collection of potentially overlapping n-cubes. This projection induces actually a map $\mathcal{C}_{n+1}(j) \to \mathcal{C}'_n(j)$ by remembering the ordering in the vertical direction of the cubes whenever they overlap in I^n.

Proposition 4.1. *The map $\mathcal{C}_{n+1} \to \mathcal{C}'_n$ induced by these projections is a morphism of operads. This morphism induces an equivalence between the little $(n+1)$-cubes operad $\mathcal{C}_{n+1}$ and the overlapping little cubes operad $\mathcal{C}'_n$.*

The operad $\mathcal{C}'_n$ is remarkable in the fact that it is the natural operad acting on the space of self embedding $EC(j, M)$. Moreover, it is a *multiplicative* operad equivalent to the little $(n+1)$-discs operad. The action on embedding spaces is described below. This means that there is a well-defined morphism of operad $\mathcal{A}ss \to \mathcal{C}'_1$. In other terms, there exists a *strictly associative* multiplication.

Given a little j-cube L and $f \in EC(j, M)$ the *rescaling* of f by L is $L.f = (L \times Id_M) \circ f \circ (L \times Id_M)^{-1}$. For this to make sense, reinterpret L as its unique affine-linear extension $L : \mathbb{R}^j \to \mathbb{R}^j$. Intuitively, this simply amounts to *compress* the support of the embedding inside the little cube L (image).
Now, define the action of C'_j on $EC(j, M)$ by: $\kappa_n : \mathcal{C}'_j(n) \times EC(j, M)^n \to EC(j, M)$ for $n \in \{1, 2, \cdots\}$ which is given by

$$\kappa_n(L_1, \cdots, L_n, \sigma, f_1, \cdots, f_n) = L_{\sigma(n)}.f_{\sigma(n)} \circ L_{\sigma(n-1)}.f_{\sigma(n-1)} \circ \cdots \circ L_{\sigma(1)}.f_{\sigma(1)}.$$

The action of $\mathcal{C}'_{j+1}$ on $EC(j, M)$ is thus defined as the composite

$$\mathcal{C}_{j+1}(n) \times EC(j, M)^n \xrightarrow{\pi \times id} \mathcal{C}'_j(n) \times EC(j, M)^n \xrightarrow{\kappa} EC(j, M).$$

(include drawing) This action makes $EC(j, M)$ as an algebra over $\mathcal{C}_{j+1}$. In the case $M = D^k$, $j = 1$, this makes the space of *framed* long knots $EC(j, D^k)$ a $\mathbb{E}_{j+1}$-algebra. When $M = D^2$, the subspace $\widehat{\mathcal{K}}_{3,1}$ of fat knots is a subalgebra. This proves that the space of long knots in dimension 3 is indeed a $\mathbb{E}_2$-algebra.
In [Bud07], Budney proves that $\widehat{\mathcal{K}}_{3,1}$ is in fact (homotopy equivalent to) a free algebra over $\mathcal{C}_2$. The base space is the space of prime knots $\mathcal{P}$. However, this subspace is still rather complicated, and the direct computation of its homotopy type is difficult. However, prime knots can still be decomposed into simpler knots using *satellite operations*, which are more general operations that the connect-sum. We could thus hope for a finer operadic description of the space of knots, with a richer class of operations, and a smaller base space. The splicing operad solves this problem.

5 Definition of the splicing operad

The notion of 'splicing' was first described by Siebenmann in his work on the JSJ-decompositions of homology spheres. Splicing has its roots in Schubert's satellite operations, and is closely linked to the JSJ-decomposition of 3-manifolds.

Definition 5.1. *A knot-generating link (KGL) is an $(n+1)$-tuple $(L_0, L_1, \dots, L_n)$ where $L_0 \in \mathcal{K}_{3,1}$ is a thin long knot, $L_i : S^1 \to [-1, 1] \times D^2$ is an embedding for $i \in \{1, 2, \dots, n\}$ such that $(L_0, L_1, \dots, L_n)$ are disjoint and $(L_1, \dots, L_n)$ are disjoint and $\{L_1, \dots, L_n\}$ represents the n-component unlink.*

A splicing diagram is an enhanced or 'fattened' KGL, allowing for a canonical definition of splicing. Intuitively, we replace K_0 by a fat knot, and a link K_i by the data of an embedding of $D^2 \times I$ in the ambient space. The definition work for a large class of embedding spaces:

Definition 5.2. *Let M be a manifold. A splicing diagram for $EC(j,M)$ is an equivalence class of $(n+2)$-tuple $(L_0, L_1, \cdots, L_n, \sigma)$ where $\sigma \in \Sigma_n$ is a permutation, $L_0 \in EC(j,M)$, and $L_i : [-1,1]^j \times M \to [-1,1]^j \times M$ is an embedding for all $i \in \{1,2,\cdots,n\}$. The equivalence relation is given by $(L,\sigma) \sim (L',\sigma') \iff L = L'$ together with the relation that if $L_i(([-1,1]^j)^\circ \times M) \cap L_j(([-1,1]^j)^\circ \times M) \neq \emptyset$ then $\sigma^{-1}(i) < \sigma^{-1}(j) \iff \sigma'^{-1}(i) < \sigma'^{-1}(j)$, where $i,j \in \{1,2,\cdots,n\}$. There is a further continuity constraint on a splicing diagram: considering $\sigma \in \Sigma_n$ as a permutation of $\{0,...,n\}$ with $\sigma(0) = 0$, we require that whenever $0 \leq \sigma^{-1}(i) < \sigma^{-1}(k)$,*

$$\overline{L_i([-1,1]^j \times M)) \setminus L_k([-1,1]^j \times M)} \cap L_k\Big(([-1,1]^j)^\circ \times \partial M\Big) = \emptyset$$

for any $i,j \in \{0,1,\cdots,n\}$.

In what follows, we will always take $M = D^2$. A splicing diagram in this case amounts to a choice of a fat knot L_0, with a list of "hockey pucks" $L_1,...,L_n$ that are embeddings of $D^2 \times I$ in $D^2 \times I$ that intersect in appropriate ways: that is, whenever $\sigma^{-1}(i) < \sigma^{-1}(k)$ the "hockey puck" L_j must not cross the cylindrical part of the boundary of L_k, that is $L_k(]-1,1[\times S^1)$.
We call $\mathcal{SD}_j^M(n)$ the space of splicing diagrams of order n, with the induced topology of embedding spaces.

$(L,\sigma) \in \mathcal{SD}_1^{D^2}(3)$

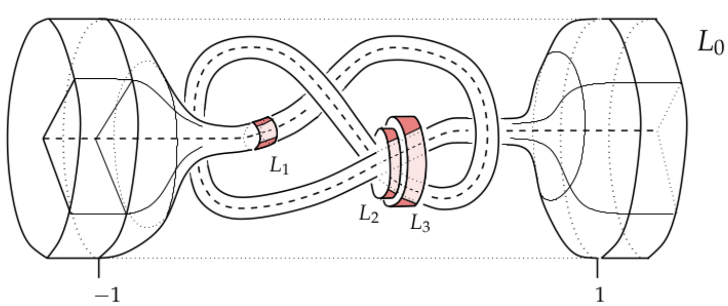


Figure 1: A splicing diagram $(L,\sigma) \in \mathcal{SD}_1^{D^2}$ with a representative σ such that $\sigma^{-1}(2) < \sigma^{-1}(3)$.

5.1 Operad structure and action on embedding spaces

Definition 5.3. *Let $L = (L_0, L_1, \ldots, L_n, \sigma) \in \mathcal{SD}_j^M(n)$ and $F = (f_1, \ldots, f_n) \in \mathcal{EC}(j,M)^n$. The composition $L \cdot F = (L\sigma(n) \circ f_{\sigma(n)}) \circ \ldots \circ (L_{\sigma(2)} \circ f_{\sigma(2)}) \circ (L_{\sigma(1)} \circ f_{\sigma(1)}) \circ L_0 \in EC(j,M)$.*

The collection of splicing diagram is a symmetric collection, the action of the symmetric group simply relabeling the L_i and composing the permutation:

$$(L_0, L_1, ..., L_n, \sigma) \cdot \tau = (L_0, L_{\tau^{-1}(1)}, ..., L_{\tau^{-1}(n)}, \sigma \circ \tau).$$

This action mods out well by the equivalence relation. Now we put an operad structure on this symmetric collection.

Operadic Splicing
Given a collection of composable functions

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$$

their composite will be denoted

$$\bigcirc_{i=1}^n f_i : A_0 \to A_n.$$

Proposition 5.1. *The collection $\mathcal{SD}_j^M = \sqcup_{n=0}^\infty \mathcal{SD}_j^M(n)$ is a multiplicative Σ-operad. With the previous definition, $\mathcal{SD}_j^M$ acts on $EC(j, M)$. The operad's structure map has the form*

$$\mathcal{SD}(k) \times (\mathcal{SD}(j_1) \times \cdots \times \mathcal{SD}(j_k)) \to \mathcal{SD}(j_1 + \cdots + j_k)$$

and is defined below. Let $J = (J_0, J_1, \cdots, J_k, \alpha) \in \mathcal{SD}(k)$ and $(L_i, \sigma_i) \in \mathcal{SD}(j_i)$ for $i = 1, 2, \cdots, k$, then $J.L \in \mathcal{SD}(j_1 + \cdots + j_k)$ has 0-th entry

$$\left(\bigcirc_{i=1}^k (J_{\alpha(i)} L_{\alpha(i)0} J_{\alpha(i)}^{-1})\right) J_0.$$

The (a, b)-th coordinate entry for $a \in \{1, \cdots, k\}$ and $b \in \{1, \cdots, j_a\}$ is given by

$$\left(\bigcirc_{i=\alpha^{-1}(a)+1}^k (J_{\alpha(i)} L_{\alpha(i)0} J_{\alpha(i)}^{-1})\right) J_a L_{a,b}.$$

The operad composition is best understood visually:

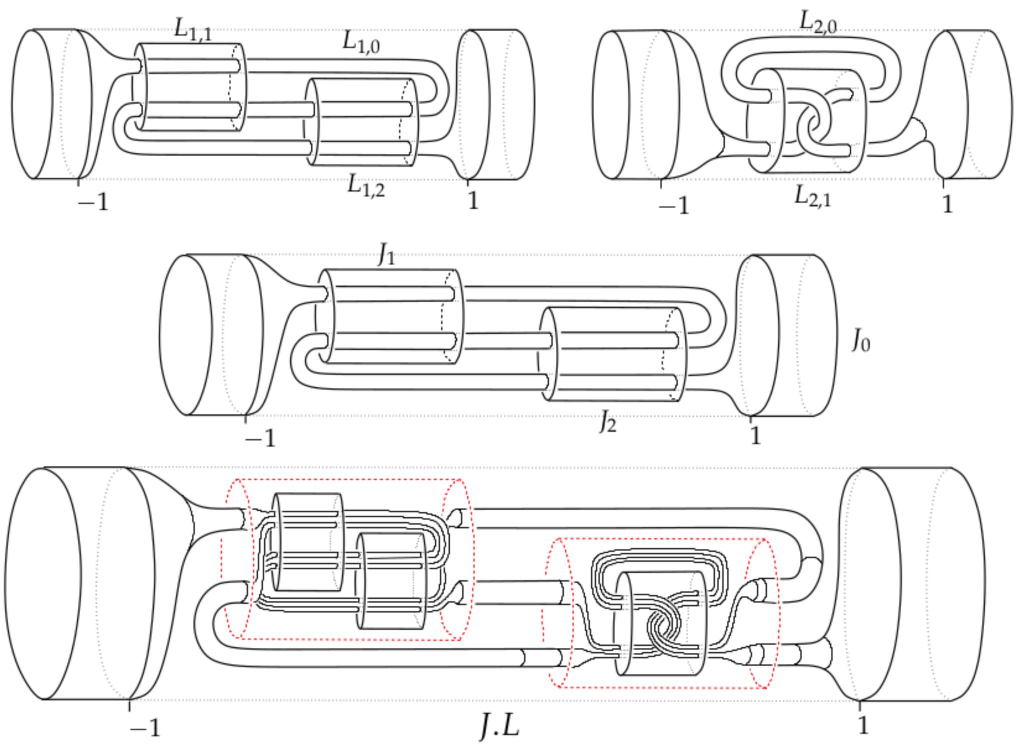


Figure 2: The operadic splicing. This image represents the composition of the arity-two J with the two elements L_1, L_2 respectively of arity 2 and 1. We thus get an arity-3-splicing diagram $J.L$.

5.2 The irreducible splicing operad

The operad $\mathcal{SD}_1^{D^2}$ contains a lot of redundant information ; for example, if (L, σ) is a splicing diagram such that the L_i are disjoint from L_0, the induced operation on knots is a constant one that sends every $(J_1, ..., J_n)$ to the knot L_0. Moreover, $\mathcal{SD}_1^{D^2}$ does not stabilize the subspace $\widehat{K}_{3,1}$ of knots with framing 0. We define then the *irreducible* splicing operad, that solves this redundancy problem and still contains the information of every possible interesting splicing operation on the space of knots.

Definition 5.4 (The irreducible splicing operad $\mathcal{SP}_{3,1}$)**.** *We define the irreducible splicing operad $\mathcal{SP}_{3,1}$ as the suboperad of $\mathcal{SD}_1^{D^2}$ made of elements (L, σ) such that:*

1. *$\mathcal{SP}_{3,1}(0) = \varnothing$ so $\mathcal{SP}_{3,1}$ has no constants (it is a non-unitary operad in the sense of [Fre17])*
2. *$L_0 \in \widehat{K}_{3,1}$ (meaning L_0 has self-linking number zero).*
3. *The embeddings L_i are orientation-preserving.*
4. *The link corresponding to L is irreducible.*
5. *Every incompressible torus in the complement of the link associated to L separates components of L.*

We refer to [Bud12] for the detailed explanation of the last two conditions, which would take us too far.

Proposition 5.2. *$\mathcal{SP}_{3,1}$ (and $\mathcal{SD}_1^{D^2}$) have a structure of $\Sigma^* \wr O_2$ operad.*

Proof. This amounts to construct a morphism of operads $O_2 \to \mathcal{SP}_{3,1}$. Note that O_2 can be seen as a subgroup of $Diff^+(D^1 \times D^2)$ with, for $A \in O_2$,

$$A \cdot (t, x) = (\det(A)t, A \cdot x).$$

Now consider the embedding $Diff^+(D^1 \times D^2) \hookrightarrow \mathcal{SP}_{3,1}(1)$ that sends ϕ to the splicing diagram $(L_0 = id_{D^1 \times D^2}, L_1 = \phi, id)$. We get a morphism $O(2) \to \mathcal{SP}_{3,1}(1)$. It is straightforward that this defines a morphism of operads, and hence gives to $\mathcal{SP}_{3,1}$ the structure of a $\Sigma^* \wr O_2$-operad. □

Remark 5.1. *Note that $\mathcal{SP}_{3,1}$ has in fact a structure of $\Sigma^* \wr Diff(D^1 \times D^2)$-operad if we do not restrict to the subgroup O_2. However, this does not add more information than the O_2-structure: by the work of Hatcher [Hat83], $Diff(D^1 \times D^2)$ has the homotopy-type of its linear subgroup $O_2 \times \mathbb{Z}_2$. When restricting to orientation-preserving diffeomorphisms, we get that $Diff^+(D^1 \times D^2)$ has the homotopy type of O_2.*

Concretely, this $\Sigma^* \wr O_2$ structures amounts to say that we can precompose any hockey puck with an orientationg-preserving isometry, and postcompose any splicing diagram by such an orientation preserving isometry.

5.3 The main results

The main result obtained by Budney is the fact that $\mathcal{SP}_{3,1}$ admits a relatively small set of generators: it is generated by the connect-sum operation and by a family of elementary splicing operations, associated to *Seifert* and *hyperbolic* links. This collection of splicing operations is encoded by an operad $\mathcal{TP}$. The main theorem states that $\mathcal{SP}_{3,1}$ is equivalent to a free product (as $\Sigma^* \wr O_2$-operads) of an operad $\overline{\mathcal{C}_1'}$, which encodes all of the connect-sum operation, and the operad $\mathcal{TP}$ which encodes all of the other splicing operations.

Definition 5.5. *Let the collection of "Generating Links" $\mathcal{GL} = \mathcal{SFL} \sqcup \bigsqcup_{k\in\mathbb{N}^*} \mathcal{HGL}_k$ with:*

- *$\mathcal{SFL}$ the union of components of $\mathcal{SP}_{3,1}(1)$ representing 2-component Seifert links (except the Hopf Link). Once we close these links to be links in S^3, these are the links $\mathcal{S}^{(p,q)}$ from [Bud06] with $(p,q) \in \mathbb{Z}^2$, $GCD(p,q) = 1$ and $p \nmid q$ setting $\mathcal{S}^{(p,q)} = (\{(z_1, z_2) \in \mathbb{C}^2 : z_1^p = z_2^q\} \cap S^3) \cup (S^1 \times \{0\}) \subset S^3$.*
- *$\mathcal{HGL}_k$ is the subspace of $\mathcal{SP}_{3,1}$ corresponding to k-hyperbolic links ($k > 0$), meaning that $L \in \mathcal{SP}_{3,1}(k)$ belongs to $\mathcal{HGL}_k$ if and only if the complement of the corresponding closed link $\hat{L}$ in S^3 has a complete hyperbolic structure of finite-volume.*

Definition 5.6. *We define $\mathcal{TP}$ to be the free $\Sigma^* \wr O_2$-operad generated by the collection $\mathcal{GL}$.*

Note that $\mathcal{C}_1'$, the *overlapping intervals operad*, can be seen as a subset of the (symmetric) operad $\mathcal{SP}_{3,1}$: if $(I_1, ..., I_n, \sigma)$ is a collection of overlapping intervals, take (L, σ) the splicing diagram defined by:

1. L_0 is the identity
2. L_i is the *rescaling* $I_i \cdot id_{D^1 \times D^2}$.

One can visualise such an diagram as a collection of overlapping cylinders:

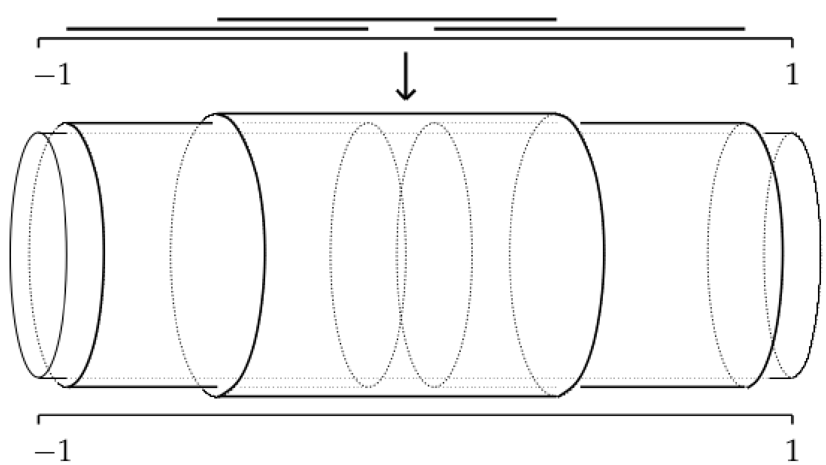


Figure 3: The splicing diagram given by a collection of little intervals forms a collection of overlapping cylinders. We depict these cylinders with slightly different diameters to highlight the ordering induced by σ.

Proposition 5.3. *Let $\overline{\mathcal{C}_1'}$ the $\Sigma^* \wr O_2$-operad generated by the images of $\mathcal{C}_1'$ inside $\mathcal{SP}_{3,1}$. This amounts to allow the pucks to rotate and to be oriented backwards (still being orientation-preserving however). In fact, we have*

$$\overline{\mathcal{C}_1'} \simeq \mathcal{C}_1' \rtimes O_2.$$

Theorem 2 ([Bud12]). *The irreducible splicing operad $\mathcal{SP}_{3,1}$ is equivalent to a free product (in the category of $\Sigma^* \wr O_2$-operads) of $\overline{\mathcal{C}_1'} = \mathcal{C}_1' \rtimes O_2$ with $\mathcal{TP}$. In other words, $\mathcal{SP}_{3,1}$ is equivalent to the pushout*

$$\begin{array}{ccc} O_2 & \longrightarrow & \overline{\mathcal{C}_1'} \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{TP} & \longrightarrow & \mathcal{SP}_{3,1} \end{array}$$

5.4 The homotopy type of $\mathcal{GL}$

In [Bud12], an explicit description of the homotopy type of each component of the generating space $\mathcal{GL}$ is given. For an arity-k link L, we will denote by $\mathcal{SP}_{3,1}(L)$ the connected component of L in $\mathcal{SP}_{3,1}$. We will call $\Sigma^* \wr O_2 \cdot \mathcal{SP}_{3,1}(L)$ the union of the path-components of $\mathcal{SP}_{3,1}$ containing the link L and its image under the action of $\Sigma_k^* \wr O_2$.

Homotopy type of Seifert link components
For a Seifert link $S^{(p,q)}$, there is a $\Sigma^* \wr O_2$-equivariant homotopy equivalence :

$$O_2 \underset{\mathbb{Z}_2}{\times} O_2 \simeq S^1 \times S^1 \times \mathbb{Z}_2 \xrightarrow{\sim} (\Sigma^* \wr O_2) \cdot \mathcal{SP}_{3,1}(S^{(p,q)}$$

where the right (resp. left) action of O_2 is the right (resp.left) multiplication induced on the rightmost (resp. leftmost) O_2 term.

(This is a different presentation than in the original article of Budney; modulo relabeling these are the same.)

Homotopy type of hyperbolic link components
The homotopy type of these components is slightly more complicated.

Definition 5.7. *A hyperbolic link is in* maximal symmetry position *in S^3 whenever*

$$Isom(S^3, L) \to \pi_0 Diff(S^3, L) \to Isom_{\mathcal{H}^3}(S^3 \setminus L)$$

are isomorphisms. The first morphism is the natural projection on the π_0, while the second is induced by Mostow rigidity ($Isom_{\mathcal{H}}$ stands for the isometry group of the hyperbolic manifold $S^3 \setminus L$).

Theorem 3 ([Bud12]). *For an hyperbolic link L seen as a link in S^3, there exists a isotopy of L into a maximal symmetry position. Moreover, if we demand that $(L_1, \cdots, L_k)$ is the trivial link, let B_L denote the subgroup of $\pi_0 Diff(S^3, L)$ that preserves L_0 and the orientation of S^3. Then one can isotope L to ensure B_L acts on (S^3, L) by isometries of S^3, and where L_i are round circles for all $i \in \{1, 2, \cdots, k\}$, that is, the intersection of an affine 2-dimensional subspace of $\mathbb{R}^4$ with S^3.*

The proof uses mainly low-dimensional topology arguments and is an amalgamation of several results. We refer to the original article for the proof.

Remark 5.2. *If we have a $n+1$-stranded link L in S^3 with a distinguished unbased component L_0 (with unordered, unbased other components) we can obtain a knot-generating link $(L_0, L_1, ..., L_n)$ by stereographic projection. This stereographic position is parametrized by :*

- *A choice of basepoint $x_0 \in L_0$ which will be sent to infinity.*
- *A framing of L_0 in x_0 equivalent to a choice of a direction along L_0 and a unit normal vector.*
- *A choice of a numbering of the basepoints $x_i \in L_i$ together with an orientation of L_i.*

The interesting fact is that for a hyperbolic link, we get a continuous family of KGLs by stereographic projections parametrized by a framed basepoint x_0 of L_0, numberings of the other components, basepoints $x_i \in L_i$ and orientations of L_i. More precisely :

Definition 5.8. *Define*

$$F\hat{L} = F\hat{L}_0 \times \Sigma_k \times \prod_{i=1}^{k} UT\hat{L}_i$$

where $UT\hat{L}_i$ is the unit tangent bundle to $\hat{L}_i$, and FL_0 is the frame bundle of L_0, meaning

$F\hat{L}_0 = \{(p, w_1, w_2) : p \in L_0, w_1 \in T_pL_0, w_2 \in \mathbb{R}^4$ *and the triple* (p, w_1, w_2) *is orthonormal*$\}$.

$F\hat{L}$ should be thought of as the minimal data to uniquely describe:

- *a constant-speed diffeomorphism $S^1 \to \hat{L}_0$,*
- *a constant-speed diffeomorphism $\sqcup_k S^1 \to \hat{L}_1 \cup \cdots \cup \hat{L}_k$*
- *a unit-length normal vector field to $\hat{L}_0$ for which its covariant derivative is parallel along $\hat{L}_0$, moreover we demand this normal vector field does not homologically link $\hat{L}_0$. Here 'parallel' means with respect to the connection on the normal bundle induced by orthogonal projection.*

By design there is a left action of B_L on $F\hat{L}$ given by post-composition of these parametrizations with an isometry of S^3. There is also a right action of $Aut(\nu S^1) \times \Sigma_k \wr O_2$ on $F\hat{L}$ given by pre-composition with an isometry of the parametrizing domain $\nu S^1 \sqcup (\sqcup_k S^1)$, moreover these two actions on $F\hat{L}$ commute. We use the convention that νS^1 is the trivial S^1-bundle over S^1, and $Aut(\nu S^1) \equiv (S^1 \times S^1) \rtimes \mathbb{Z}_2$ is automorphisms of the bundle that are orientation-preserving on the total space. Since any two parametrizations differ by precomposition with an element of $Aut(\nu S^1) \times \Sigma_k \wr O_2$, $F\hat{L}$ is an $Aut(\nu S^1) \times \Sigma_k \wr O_2$-torsor. This induces a canonical injection $B_L \to Aut(\nu S^1) \times \Sigma_k \wr O_2$. The composition with the projection $B_L \to Aut(\nu S^1) \times \Sigma_k \wr O_2 \to Aut(\nu S^1)$ is an embedding of groups.

Theorem 4. [*Bud12*] *There is a $\Sigma_k^* \wr O_2$-equivariant homotopy- equivalence*

$$\Pi : F\hat{L}/B_L \to (\Sigma^* \wr O_2) \cdot \mathcal{SP}_{3,1}(L)$$

We will also need :

Theorem 5. *The map $B_L \to O_2$ which remembers only the parametrizations of the knot L_0 is an embedding. Therefore, B_L is cyclic or dihedral.*

Proof. This is a consequence of the "Smith conjecture" [BDG+84], which states that the set of fixed points of a finite-order diffeomorphism of the sphere S^3 is an *unknotted* circle. Consider the kernel K of the projection $B_L \xrightarrow{p} O_2$. Every element of K is an isometry of S^3 which fixes the link L_0.As K is finite, by the Smith conjecture it is trivial what about when L_0 is unknotted ?. Therefore, B_L is isomorphic to a finite subgroup of O_2 and so is dihedral or cyclic. □

6 The splicing operad is aspherical

The goal of this section is to show every connected component of the splicing operad is a $K(G,1)$. We will call such an operad *aspherical*.

Definition 6.1 (Mapping torus)**.** *Let X be a space and ϕ a self-homeomorphism of X. The mapping torus of X along ϕ is the space $I \times X/(0,x) \sim (1,\phi(x))$ which is a bundle over S^1 with fiber X. We write this mapping torus $T_\phi(X)$.*

Note that there is a continuous bijection $X \times [0,1) \to T_\phi(X)$. We will thus describe a point of the mapping torus by a pair (x,t) with $x \in X$ and $0 \leq t < 1$.

Lemma 6.1. *Let X_L a set of components of $\mathcal{GL}$ associated to a hyperbolic link with B_L diedral. Suppose B_L^+ is of order n, and take b a generator of B_L^+, such that its image in O_2 is the rotation by an angle $2\pi/n$ counterclockwise Then $X_L = F\widehat{L}/B_L$ is the mapping torus of $S^1 \times \Sigma_k \wr O_2$ along the diffeomorphism induced by the right action of the generator b on $S^1 \times \Sigma_k \wr O_2$. When B_L is cyclic, X_L is made of two copies of this mapping torus, associated to the generator b of B_L^+.*

Proof. First suppose B_L is diedral. Writing S^1 as $\mathbb{R}/\mathbb{Z}$, first note that each class of $F\widehat{L}$ under B_L has a unique representative $(u,t,\epsilon,\sigma,A \in O_2^k)$ such that $\epsilon = 1$ and $0 \leq t < 1/n$. This is because B_L embeds into O_2, and B_L is diedral.

Now for such a representative, define a diffeomorphism

$$\phi : F\widehat{L}/B_L \to T_b(S^1 \times \Sigma_k \wr O_2)$$

$$\overline{(u,t,1,\sigma,A)} \mapsto (nt,(u,\sigma,A))$$

One can indeed see that this map is well defined and continuous: we see that

$$(0,(u,\sigma,A)) = \phi(\overline{u,0,1,\sigma,A}) = \phi(\overline{b^{-1}\cdot(b\cdot u,\frac{1}{n},b\cdot\sigma,b\cdot A)}) = (1,b\cdot(u,\sigma,A))$$

so the map is well defined in the quotient. There is an inverse map by sending (s,u,σ,A) to the class of $(u,\frac{s}{n},1,\sigma,A)$. The case when B_L is cyclic is similar, except we have to consider the two distinct components of O_2/B_L as base space. □

The next step is to simplify the description of this mapping torus. Recall that the diffeomorphism induced by b is given by *left* multiplication, so this makes X_L a principal *right* $S^1\times\Sigma_k\wr O_2$-bundle. Moreover, the left action of O_2 is also induced by multiplication on the right (see [Bud12]) and so commutes with the action of b. Therefore, the X_L, seen as $\Sigma_k^* \wr O_2$-space, only depends on the connected component of b as an element of $(S^1 \times S^1) \rtimes \mathbb{Z}_2 \times \Sigma_k \wr O_2$. We can thus replace b by its "rectification" b_0 of the form $(1,1,\epsilon,\sigma,\underline{\theta})$ where the θ_i are only the identity or the reflection. We get :

Lemma 6.2. *If X_L is the family of components of $\mathcal{HGL}_k$ associated to a hyperbolic link with symmetry group B_L, then*

$$X_L \simeq \hat{X}_L := T_{b_0}(S^1 \times \Sigma_k \wr O_2)$$

as $\Sigma_k^* \wr O_2$-spaces.

We can obtain now :

Theorem 6. *$\mathcal{SP}_{3,1}$ is aspherical.*

Proof. Let T be a semi-alternate coloured tree on two colours. Such a tree represents a union of components of $\mathcal{SP}_{3,1}$. Each vertex represents either $\overline{\mathcal{C}'_1}(k)$ or the union of the arity-k generating links for $\mathcal{TP}$. Let $h(T)$ be the height of the tree. We do the proof by induction on $h(T)$.
If $h(T) \leq 1$: the realization of T is either O_2, $\overline{\mathcal{C}'_1}(k)$, or $\mathcal{GL}(k)$ for some k, so is aspherical. Now suppose $T = T(u, T_1, ..., T_k)$, with u a vertex and T_1,...,T_k the trees arising from every input edge of u. The realization of T is equal to the previous coequalizer. Moreover, On $M(\underline{r}_u)$ the right action is always free on every possible base space. Let's consider a subset of the root, $R \subset M(\underline{r}_u)$ corresponding to the orbit under $\Sigma^*_{\underline{r}_u} \wr O_2$ of a connected component. R can be either a set of components corresponding to a hyperbolic link, to a Seifert link or to the little overlapping intervals operad. In the latter cases (Seifert or intervals), the underlying space of the root is simply of the form $R = R' \times O_2^k$ with O_2^k acting by right translation, according to the previous description of these spaces. Therefore, the equivariant product $R \underset{O_2^k}{\times} \prod M(T_i)$ is simply isomorphic to

$$R' \times \prod M(T_i)$$

and this component is aspherical by induction.

For the hyperbolic case, the root is not simply of the form $R' \times O_2^k$, but rather a mapping torus with a free action of O_2^k on each fiber. Write $Y = \prod M(T_i)$ for commodity. We have $M(T) = \hat{X}_L \underset{O_2^k}{\times} Y$. To simplify, consider only the case B_L diedral (when B_L is cyclic there is just another isomorphic component). The attachment map of the mapping torus is given by left multiplication by b_0 the "rectification" of b. Let us show that the the coequalizer described above makes $M(T)$ a mapping torus of $S^1 \times \Sigma_k \times \prod M(T_i)$ with a "twisted" attachment map.
Consider the projection $p : M(\underline{r}_u) \twoheadrightarrow S^1$ associated to the principal bundle structure. First note it induces a projection $M(T) \twoheadrightarrow S^1$ because the action of O_2^k stabilizes the fibers. Let U be an open set of S^1 such that we have a trivialization $p^{-1}(U) \simeq U \times S^1 \times \Sigma_k \wr O_2$ with O_2^k acting by right multiplication. Note that because of the wreath product structure, $(\sigma, \underline{\theta}) \cdot (id_{\Sigma_k}, \underline{\theta}') = (\sigma, \underline{\theta\theta}')$, so this action is exactly right translation on the copy of O_2^k. Thus, $p^{-1}(U) \underset{O_2^k}{\times} Y \simeq U \times S^1 \times \Sigma_k \times Y$. This gives local trivializations of the map $M(T) \twoheadrightarrow S^1$. $M(T)$ is then a mapping torus of $S^1 \times \Sigma_k \times Y$. Let's describe the attaching map : write $b_0 = (\sigma_b, \underline{\epsilon}_b) \in \Sigma_k \wr \mathbb{Z}_2$. The point $(0, r, \tau, y) \in I \times S^1 \times \Sigma_k \times Y$ is identified with the class of

$$(1, r, b_0(\tau, 1_{O_2^k})) = (1, r, \sigma_b\tau, (\underline{\epsilon}_b \cdot \tau), y) \in I \times S^1 \times \Sigma_k \wr O_2 \times Y.$$

In the quotient by O_2^k, this gives the point $(1, \sigma_b\tau, (\underline{\epsilon}_b \cdot \tau) \cdot y)$. Therefore the attaching map is the "twisted" homeomorphism that we will call $\hat{b}_0$:

$$S^1 \times \Sigma_k \times Y \mapsto S^1 \times \Sigma_k \times Y$$

$$(r, \tau, y) \mapsto (r, \sigma_b\tau, (\underline{\epsilon}_b \cdot \tau) \cdot y).$$

The long exact sequence of homotopy groups shows then that $M(T)$ is aspherical. Therefore every component of $\mathcal{SP}_{3,1}$ is aspherical. □

We could also have proved this result using the fact that the quotient map $X_L \times Y \to X_L \underset{O_2^k}{\times} Y$ is a fibration, and applying the long exact sequence of homotopy groups. However, the explicit description of the quotient as a mapping torus will be useful when establishing groupoid models for hyperbolic links.

7 The homotopy type of the space of long knots $\mathcal{K}_{3,1}$

In [Bud12], it is shown that the space of long knots is (homotopy equivalent to) a free algebra over the splicing operad $\mathcal{SP}_{3,1}$, generated by the space of torus and hyperbolic knots. We say that a long knot f is *invertible* whenever $r \cdot f \in K_f$, where $r \in O_2$ is the reflexion. In other words, it means the "backwards" image of f is isotopic to f.

Theorem 7. *Let $\mathcal{TH} \subset \widehat{\mathcal{K}}_{3,1}$ be the subspace consisting of knots which are either non-trivial torus knots, or hyperbolic knots. Then the action of $\mathcal{SP}_{3,1}$ on $\widehat{\mathcal{K}}_{3,1}$ induces an O_2-equivariant homotopy-equivalence*

$$\mathcal{SP}_{3,1}(\mathcal{TH}) = \coprod_{j=0}^{\infty} \mathcal{SP}_{3,1} \underset{\Sigma_j \wr O_2}{\times} \mathcal{TH}^j \to \widehat{\mathcal{K}}_{3,1}$$

The base space $\mathcal{TH}$ is explicitly described by Budney [Bud12], together with the O_2-action:

1. If f is a torus knot, then $O_2 \cdot K_f$ is homotopy equivalent to S^1, and the O_2-action is the standard left action on it.
2. If f is an *invertible* hyperbolic knot, then $O_2 \cdot K_f$ is equivalent to $S^1 \times S^1$, with O_2 acting diagonally.
3. I ff is a non-invertible hyperbolic knot, then $O_2 \cdot K_f$ is equivalent to $S^1 \times S^1 \times S^0$, with O_2 acting diagonally (the action of O_2 on S^0 is simply the determinant : a reflexion permutes the component, a rotation does not).

Corollary 7.1. *For every long knot $f \in K$, its connected component K_f is a $K(\pi, 1)$.*

Proof. Every component of the base space $\mathcal{TH}$ component is a $K(G, 1)$. Moreover the right action of $\Sigma_j \wr O_2$ on every component of the generators $\mathcal{SP}_{3,1}$ is free and is consequently free on the full operad. the equivariant quotient is thus also a $K(G, 1)$, using the long exact sequence of a fibration and the fact that $\pi_1(O_2) \to \pi_1(\mathcal{SP}_{3,1}(k) \times \mathcal{TH})$ is injective, by the explicit description of the action of O_2 on $\mathcal{TH}$. This is needed to prove that the π_2 of $\mathcal{K}$ is zero.· □

Part III
Construction of a model for $\mathcal{SP}_{3,1}$

8 Nerves, fundamental groupoids and models for operads

This section aims to develop the notion of a *groupoid model* of a topological operad.

Definition 8.1. *A* groupoid *is a category where every morphism is invertible.*

In our case all of our groupoids will be small. Groupoids form a category Gpd with functors as morphisms. This category has a symmetric monoidal structure for the cartesian product of groupoids. Moreover, it is cartesian closed as $\mathsf{Func}(\mathcal{G}, \mathcal{H})$, the set of functors from $\mathcal{G}$ to $\mathcal{H}$, has a structure of a groupoid, arrows being natural transformations. Indeed any such natural transformation is invertible because it is given pointwise by morphisms in $\mathcal{H}$.
Therefore, there is a well defined notion of an operad in groupoids (or even in general categories), and of an algebra over such an operad. We will call this category of operads GpdOp.

Definition 8.2 (Nerve of a category)**.** *Let $\mathcal{C}$ be a small category. Its nerve $N_\bullet\mathcal{C}$ is a simplicial set given in degree n by the set of all sequences of n composable morphisms*

$$\bullet \xrightarrow{f_1} \bullet \cdots \bullet \xrightarrow{f_n} \bullet$$

The face maps d_i are defined by

$$d_0((f_1, ..., f_n)) = (f_2, ..., f_n)$$
$$d_i((f_1, ..., f_n)) = (f_1, ..., f_{i+1} \circ f_i, ..., f_n) \text{ for } 1 \leq i < n$$
$$d_n((f_1, ..., f_n)) = (f_1, ..., f_{n-1})$$

Degeneracies are defined by $s_i((f_k)) =$ insert the identity in i-th position.

Definition 8.3. *The* classifying space $\mathsf{B}\mathcal{C}$ *of a category is the geometric realization of its nerve.*

We recall that the geometric realization of a simplicial set $\mathfrak{X}$ is the topological space defined by

$$|\mathfrak{X}| = \bigsqcup_{n \in \mathbb{N}} \Delta^n \times \mathfrak{X}_n /$$

Where Δ^n is the standard geometric n-simplex, and $\sim$ is the equivalence relation generated by $(\delta^i(x), u) \sim (x, d_i u)$ and $(\sigma^i(x), u) \sim (x, s_i u)$ where δ^i and σ^i are the standard coface and codegeneracies of the cosimplicial space $\Delta^\bullet$. An interesting feature of the classifying space functor B is that it is *strictly monoidal* : therefore, if P is an operad in groupoids, its classifying space BP is a topological operad.

Definition 8.4. *We say two topological operads $\mathcal{P}$ and $\mathcal{Q}$ are* equivalent *(and we write $\mathcal{P} \simeq \mathcal{Q}$) when they are connected by a zigzag*

$$\mathcal{P} \xleftarrow{\sim} \bullet \cdots \bullet \xrightarrow{\sim} \mathcal{Q}$$

each arrow being a morphism of operads and a weak homotopy equivalence aritywise.

There is an analogous notion for operads in groupoids : ‘

Definition 8.5. *A categorical equivalence is a morphism of operads in groupoids which is an equivalence of categories in each arity. Two operads* P *and* Q *are said to be categorically equivalent (and we write* $\mathsf{P} \simeq \mathsf{Q}$*) if they can be connected by a zigzag of categorical equivalences:*

$$\mathsf{P} \xleftarrow{\sim} \bullet \xrightarrow{\sim} \dots \xleftarrow{\sim} \bullet \xrightarrow{\sim} \mathsf{Q}.$$

Finally, we can define what is a groupoid model of an operad :

Definition 8.6. *Let* $\mathcal{P}$ *be a topological operad. An operad in groupoids* p *is said to be a* model *of* $\mathcal{P}$ *whenever* $\mathsf{Bp} \simeq \mathcal{P}$.

The interesting point is that an operad in groupoids is sometimes simpler (finite in each arity, with an explicit set of morphisms) than the topological operad it models. We now prove an equivalent characterization of a model for an operad, which will be more comfortable for us to work with.

Let $\mathcal{P}$ be any topological operad. Its fundamental groupoid πP is an operad in groupoids, as π commutes with products. Moreover, there is a chain of adjunctions

$$\mathsf{Top} \underset{|-|}{\overset{\mathrm{Sing}_\bullet}{\rightleftarrows}} \mathsf{sSet} \underset{\pi}{\overset{\mathrm{N}_\bullet}{\leftrightarrows}} \mathsf{Gpd}$$

which links the category of topological spaces and the category of groupoids. The fundamental groupoid π is indeed isomorphic to the composition

$$\mathsf{Top} \xrightarrow{Sing_\bullet} \mathsf{sSet} \xrightarrow{\pi} \mathsf{Gpd}.$$

Note that it does not induce an adjunction between these two categories (the two adjunctions above do not go in the same directions). The unit (respectively, the counit) of the adjunction between simplicial sets and topological spaces is a weak-equivalence. Moreover, the counit of the adjunction between simplicial sets and groupoids defines an isomorphism $G \xrightarrow{\sim} \pi N_\bullet G$, for each groupoid G , while the unit of this adjunction defines a weak-equivalence $X \to N_\bullet \pi X$ in simplicial sets as soon as X is a Kan complex with a trivial homotopy in degree > 1. All of these facts lift to the corresponding adjunctions between the respective categories of operads [Fre17]. From these observations we deduce

Proposition 8.1. *Let* $\mathcal{P}$ *be a topological operad. Then, an operad in groupoids* p *is a model of* P *if and only if* p *is equivalent to* πP *in* GpdOp *and* $\mathcal{P}$ *is aspherical.*

Proof. First let's note that $B\pi P$ is equivalent to P under the assumption that P is aspherical, as we get the following zigzag of equivalences:

$$B\pi P = |N_\bullet \pi \mathrm{Sing}_\bullet P| \xleftarrow{\sim} |\mathrm{Sing}_\bullet P| \xrightarrow{\sim} P$$

If p is equivalent to πP, then their realization also are and we get that p is a model of P. Conversely, if $B\mathsf{p}$ is equivalent to $\mathcal{P}$, then $\mathcal{P}$ is aspherical and taking the fundamental groupoid gives a chain

$$\pi P \simeq \pi B\mathsf{p} = \pi \mathrm{Sing}_\bullet |N_\bullet \mathsf{p}| \xleftarrow{\sim} \pi N_\bullet \mathsf{p} \xrightarrow{\sim} \mathsf{p}.$$

□

We will sometimes write BG for the one-object groupoid with automorphism group G. This notation conflicts with B being the classifying space functor, but the meaning will be clear from the context.

9 Motivation: the colored braid operads

We now construct a model of the little discs operad $\mathcal{D}_2$, following the exposition by Fresse in [Fre17]. He calls this model the *colored braid operad* CoB, also denoted $\mathcal{C}^\beta$ by Wahl [Wah01]. In our case we will use the notation Br to avoid confusion with other objects. The object set $\mathrm{Ob}(\mathsf{Br}(r))$ of the groupoid of colored braids on r strands $\mathsf{Br}(r)$ is the set of permutations $w \in \Sigma_r$ which we regard as orderings $(w(1), \ldots, w(r))$ of the values $(1, \ldots, r)$. The morphism set $\mathrm{Hom}(w, w')$ consists of formal braids α together with a bijection $i \mapsto \alpha_i$ between the index set $i \in 1, \ldots, r$ and the strands $\alpha_i \in \{\alpha_1, \ldots, \alpha_r\}$ of the braid α such that w (resp w') corresponds to the permutation induced by the coloring on the bottom (resp. the top) of the strands. Intuitively, this bijection assigns a color $i \in \{1, \ldots, r\}$ to each strand α_i.

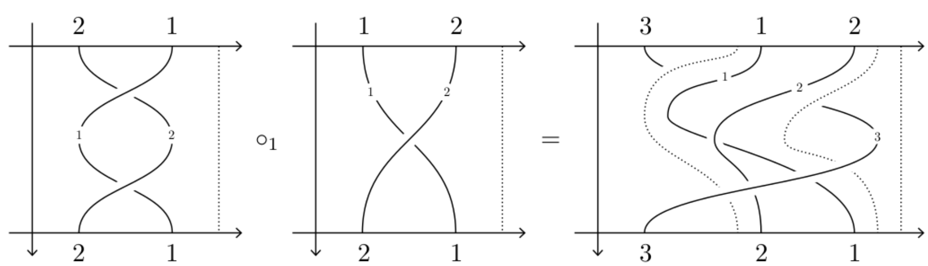


Figure 4: The operadic composition of colored braids ([Fre17])

The symmetric structure of the colored braid groupoids.
Each groupoid of colored braids $\mathsf{Br}(r)$ inherits a natural right action of permutations. Therefore the collection $\mathsf{Br} = \{\mathsf{Br}(r), r > 0\}$ forms a symmetric sequence of groupoids. Explicitly, to each permutation $s \in \Sigma_r$, and $\{1, ..., r\} \to$ Strands(α) a coloring of the braid α, we can relabel the strands by precomposing with s. This induces an action of the symmetric group both on the object set and on the morphism set of the colored braid groupoid, that preserves the groupoid structure.

The operadic composition operations on colored braids.
We have an identity $\mathsf{Br}(1) = \mathrm{pt}$. The identity map of the one-point set pt endows the collection of colored braid groupoids with an operadic unit $\eta : \mathrm{pt} \to \mathsf{Br}(1)$. We define operadic composition operations $\circ_k : \mathsf{Br}(m) \times \mathsf{Br}(n) \to \mathsf{Br}(m + n - 1)$ by the operadic composition of permutations at the object set level. Indeed, the collection of object-sets of our operad $\mathrm{Ob}(\mathsf{Br})$ is identified with the associative operad A$ssoc$ in the category of sets). We use the operadic composition operation for braids in order to define the value of this operadic composition operation on the morphism sets of our groupoids. Note that there is an inclusion of operads $\mathcal{C}_1 \hookrightarrow \mathcal{C}_2$ by sending a little interval collection to the corresponding collection of little squares located on the horizontal central line

of the unit square. This map commutes indeed with the operad structure and $\mathcal{C}_1$ can thus be seen as a suboperad of $\mathcal{C}_2$.

Definition 9.1 (Fundamental groupoid with choice of basepoints)**.** *Let X be a space and $B \subset X$ a subspace. We write $\pi(X)_{|B}$ for the restriction of the fundamental groupoid πX to points of B. For example, if $x \in X$, we can identify $\pi(X)_{|\{x\}} \simeq B\pi_1(X, x)$. Note that $\pi(X)_{|B}$ is equivalent to the full groupoid πX as long as B intersects every connected component of X.*

Note that if $\mathcal{Q}$ is a suboperad of $\mathcal{P}$, the operad $\pi(\mathcal{P})_{|\mathcal{Q}}$ defined by restricting the fundamental groupoid aritywise is indeed an operad in groupoids.

Lemma 9.1 ([Fre17] p.189)**.** *1. The inclusion*

$$\pi(\mathcal{C}_2)_{|\mathcal{C}_1} \to \pi\mathcal{C}_2$$

is an equivalence.

2. There is an categorical equivalence of operads

$$\pi(\mathcal{C}_2)_{|\mathcal{C}_1} \to \mathsf{Br}$$

given by the following map: we send a little interval collections to the corresponding permutation, and a path from two little intervals to the corresponding element of the braid group, induced by the path made by the center of each interval.

Proof. For the point 1., it suffices to see that $\mathcal{C}_2$ is connected, so $\mathcal{C}_1$ seen as a suboperad of $\mathcal{C}_2$ intersects every component of $\mathcal{C}_2$. The inclusion is then automatically an equivalence and an operad morphism. For the point 2., see [Fre17]. □

Theorem 8 ([Fre17])**.** Br *is a model of $\mathcal{C}_2$.*

Proof. We have constructed a zig-zag of operad equivalences

$$\mathsf{Br} \xleftarrow{\sim} (\pi\mathcal{C}_2)_{|\mathcal{C}_1} \xrightarrow{\sim} \pi\mathcal{C}_2$$

and we can conclude by the fact that the little 2-cubes operad is aspherical ($\mathcal{C}_2(n) \sim \mathrm{Conf}_n(\mathbb{R}^2) \sim K(P_n, 1)$ where P_n is the pure braid group on n strands. □

Why care about groupoid models ?
Generally, groupoid models allow for a combinatorial description of aspherical operads, which can help computations. Another interesting point is that an operad p in groupoids *encodes additional structure on categories*. In particular, it provides a link between the theory of symmetric/braided monoidal categories and topological operad. For example, we have that:

- Algebras over the operad PaB of parenthesized braids [Tam03] are exactly braided monoidal categories with strict unit.
- Algebras over the operad Br are braided monoidal categories with strictly associative tensor product (i.e. the associator is the identity). In particular, this gives that the geometric realization of a braided monoidal category is an $\mathbb{E}_2$-algebra (meaning an algebra over an operad equivalent to $\mathcal{D}_2$).
- Algebras over the operad $\mathsf{Br} \rtimes \mathsf{B}\mathbb{Z}$ (which is a model of $f\mathcal{D}_2$) are ribbon braided strict monoidal categories [Wah01].

- More simply, (more or less strictly) symmetric monoidal categories can be seen as algebras over some models of D_∞, (like the operad Com or the parenthesized operad [Fre17]) and non-symmetric monoidal categories are algebras over models of D_1 (for example the associative operad Ass)

Moreover, these models have been used successfully in understanding the homotopy type of operads: Tamarkin used PaB to prove the formality of $\mathcal{D}_2$ [Tam03]. The proof uses the fact that because of the simplicial structure of BPaB, we can replace the big singular chain complex $C_*^{Sing}(\mathcal{D}_2)$ with the simplicial chain complex $C_*(\mathsf{PaB})$ which has richer structure. More recently Horel used this same operad PaB to compute the automorphism group of the profinite completion of $\mathcal{D}_2$ [Hor17] and link it with the Grothendieck-Teichmuller group, and Idrissi [Idr17] constructed a groupoid model for the Swiss-Cheese operad and studied algebras over it.

10 Group objects in groupoids and a model for O_2

As every connected component of the splicing operad is a $K(G,1)$, our goal is to find a simple operad in groupoids $\mathsf{sp}_{3,1}$ such that its classifying space is equivalent to $\mathcal{SP}_{3,1}$. The model we construct is a $\Sigma^* \wr \mathsf{o}_2$-operad, where o_2 will be a group object in groupoids that models the action of O_2. Indeed, we will show that there is multiplicative equivalence $\mathsf{o}_2 \overset{\sim}{\hookrightarrow} \pi O_2$. As we are going to model equivariant operads in groupoids, we need to talk about group objects in groupoids. In particular, we will be looking for a small model of O_2 in groupoids. Note that as for operads, if G is a group in groupoids, the classigying space BG is a topological group.

Definition 10.1. *A group in groupoids* G *is a model of a topological group G if there is a zigzag that connects* BG *to G, each arrow being a group morphism and a homotopy equivalence. We call such zigzags between topological groups* multiplicative equivalences*. Such a notion can be defined for group in groupoids, replacing homotopy equivalence by categorical equivalence.*

Note that a multiplicatively equivalent group in groupoids yield multiplicatively equivalent classifying spaces.

Remark 10.1. *This also shows that if G is a connected, aspherical topological group, then it is multiplicatively equivalent to an abelian topological group.*

Proof. Indeed, by the Eckmann-Hilton argument, $B\pi_1(G,1)$ is an abelian topological group by construction, which is multiplicatively equivalent to G. □

A simple way to obtain models of topological groups is to restrict the fundamental groupoid :

Lemma 10.1. *If H is a subgroup of a topological group G that intersects every connected component of G, $\pi(G)_{|H}$ is a group object multiplicatively equivalent to πG, by the inclusion $\pi(G)_{|H} \overset{\sim}{\hookrightarrow} \pi G$. In particular, let G be a topological group of the form $G_0 \rtimes D$ where G_0 is connected and a $K(\pi,1)$, and D is a discrete group. Then $\pi(G)_{|D}$ is a model of G.*

Note that in the latter case it is the smallest model we could construct of G, as this subgroupoid of πG intersects every component exactly once.

Corollary 10.1. *O_2 has a model $\mathsf{o}_2 := \pi(O_2)_{|\mathbb{Z}_2}$ with two objects $+, -$, $Aut(+) = Aut(-) = \mathbb{Z}$ and with the laws*

$$\begin{aligned} m^+ \cdot n^+ &= (m+n)^+ \\ m^- \cdot n^+ &= (m-n)^- \\ m^+ \cdot n^- &= (m+n)^- \\ m^- \cdot n^- &= (m-n)^+ \end{aligned}$$

with $m^\pm$ denoting the morphism $m \in \mathbb{Z} = Aut(\pm)$.

11 A model of $\overline{\mathcal{C}_1'}$

As shown in [Bud12], the operad $\overline{\mathcal{C}_1'}$ is isomorphic to the semi-direct product $\mathcal{C}_1' \rtimes O_2$ as a $\Sigma^* \wr O_2$-operad. We recall that $\mathcal{C}_1'$ is the "overlapping intervals operad" which is equivalent to $\mathcal{C}_2$ of little 2-cubes. The action of O_2 factors via $\mathbb{Z}_2$, which acts by mirror reflection on the intervals. In this section we construct a model of this operad, as a semi-direct product of the colored braid operad with the group object o_2. The construction is quite similar as the construction of the model for the framed disc operad in [Wah01]. One helpful fact with the overlapping intervals operad $\mathcal{C}_1'$ is that it is a *multiplicative* operad. For the classical little discs operad, there is no direct morphism between Br (which is multiplicative) and $\pi\mathcal{C}_2$, and this is why a zigzag was needed in the previous section. However the multiplicativity of $\mathcal{C}_1$ allows to define a direct morphism $\mathsf{Br} \to \pi\mathcal{C}_1'$ as we will show.

Definition 11.1. *Let Br be the colored braids operad in groupoids [Wah01]. There is an action of the group object o_2 on Br that factors through the action of $\mathbb{Z}_2$ given by mirror reflection of the strands as on the figure:*

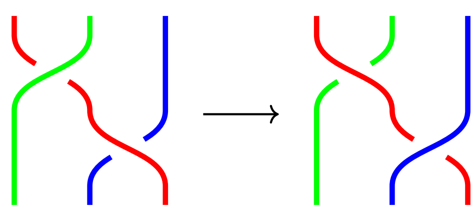


Figure 5: Action of the non-trivial element $(-1) \in \mathbb{Z}_2$ on the groupoid of colored braids (here in arity 3).

On the objects, the action sends a permutation σ to itself. On the morphisms, it sends the generator σ to σ^{-1}. Concretely, this action must be though as the "mirror" image of the braid along the normal axis to the sheet. We can then construct the operad $\mathsf{MBr} = \mathsf{Br} \rtimes \mathbb{Z}_2$. The discussion above also allows us to define $\mathsf{MBr}' = \mathsf{Br} \rtimes \mathsf{o}_2$ where the action of o_2 factors through the projection $\mathsf{o}_2 \to \mathbb{Z}_2$ (forgetting every non-identity morphism)
MBr' could be also described as a pullback in operads $\mathsf{MBr} \underset{\mathbb{Z}_2}{\times} \mathsf{o}_2$ (where $\mathbb{Z}_2$ and o_2 are here considered as the "translation operads" with an arity k term equal to $\mathbb{Z}_2^k$ (resp. o_2^k)).

Let us describe with more detail the operad structure maps of MBr'. The objects of $\mathsf{MBr}'(k)$ are "signed permutations" $\sigma \in \Sigma_k \wr \mathbb{Z}_2$ that we will write as ordered lists of the integers $1, ..., n$ with a given sign for each, written as an index. For example:

$$2_+ \; 1_- \; 3_+$$

A morphism between two signed permutations is the data of a *colored* braid, that is a braid which connects two equal numbers with the same sign. Each strand s_i is moreover labeled with an integer $m_i \in \mathbb{Z}$.

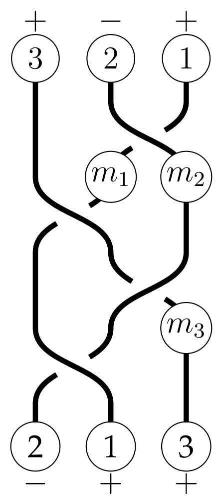


Figure 6: A morphism between $(2_-, 1_+, 3_+)$ and $(3_+, 2_-, 1_+)$ in $\mathsf{MBr}'(3)$. The braids are labeled by integers m_1, m_2, m_3 that represent morphisms in o_2.

The groupoid composition law simply concatenates the braids and sums the labels m_i. Let us describe the operadic composition. The partial i-th composition: $\mathsf{MBr}'(n) \times \mathsf{MBr}'(m) \xrightarrow{\circ_i} \mathsf{MBr}'(n+m-1)$ "plugs" the m-stranded braid onto the i-th braid in $\mathsf{MBr}'(n)$, renumbers the ends of the braids, multiplies the signs and composes the labels accordingly:

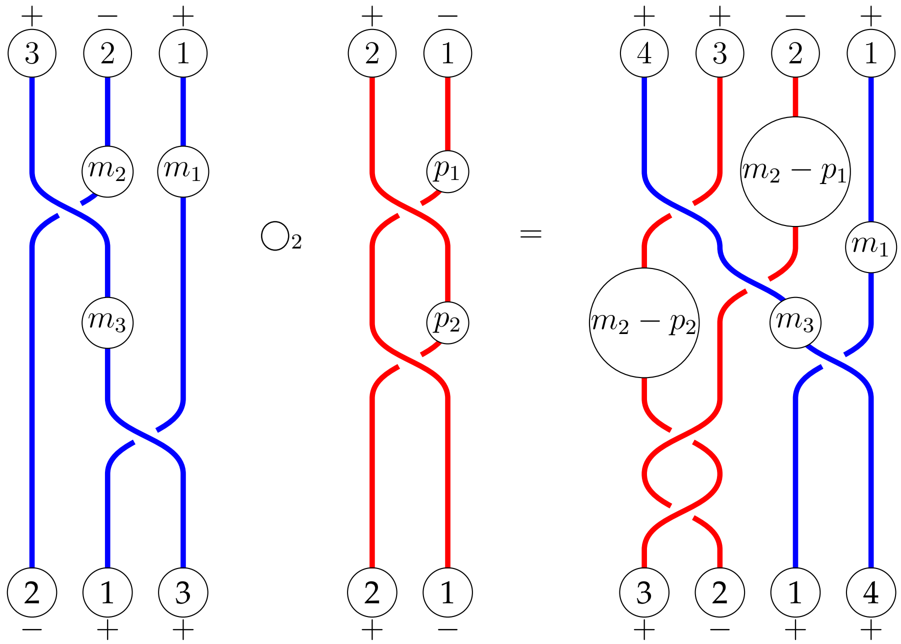


Figure 7: The operadic composition in MBr'. We plug the red braid onto the blue strand numbered by 2. As 2 is labeled by a minus sign, we flip the red braid before plugging in, multiply the signs, and compose the labeling of the braids according to the group law of o_2.

The labels of the braids compose according to the composition law of o_2. If we want instead to describe $\mathsf{MBr} = \mathsf{Br} \rtimes \mathbb{Z}_2$, we simply need to remove the labelings of the braids.

Proposition 11.1. *There is a direct equivalence of operads*

$$\mathsf{MBr} \overset{\sim}{\hookrightarrow} \pi(\mathcal{C}_1')$$

which is an embedding. On the objects, we send $\sigma \in \Sigma_n$ to the collection of little intervals $L = (I, I, ..., I, \sigma)\cdot$ On morphisms, we send a braid β to the unique class of paths between overlapping intervals that induce this braid on their centers (recall that $\mathcal{C}_1'(n)$ is equivalent to $\mathcal{C}_2$ which is a $K(PB_n, 1)$).

Corollary 11.1. MBr' *is a model of $\overline{\mathcal{C}_1'}$, and we even have an embedding $\mathsf{MBr}' \to \pi\overline{\mathcal{C}_1'}$.*

Proof. In fact, the image of MBr can simply be seen as $\pi\mathcal{C}'_{1|\mathcal{A}ssoc}$, the restriction of $\pi(\mathcal{C}_1')$ to the suboperad generated by the multiplication $\mu \in \mathcal{C}_1'(2)$ i.e. the suboperad composed of little intervals which are all the full interval I. Therefore, we see that at the object level, this map is a morphism of operads (both are the associative operad $\mathcal{A}ssoc$) and an equivalence of groupoids. At the morphism level, the composition of paths of intervals is exactly the operadic composition for braids. This is more obvious if we represent the overlapping intervals as a collection of stacked intervals in I^2, which project onto I, remembering the ordering.
Secondly, we note that if $\mathcal{Q} = \mathcal{P} \rtimes G$ is a semi-direct product of topological operad with a topological group G then we have an identification $\pi\mathcal{Q} \simeq \pi\mathcal{P} \rtimes \pi G$. This is simply

because π commutes with products, and the group action of G induces a group action of πG. We then get a map

$$\mathsf{Br} \rtimes \mathsf{o}_2 \xrightarrow{\sim} \pi(\mathcal{C}'_1) \rtimes \pi O_2 = \pi(\overline{\mathcal{C}'_1})$$

which is an equivalence aritywise. Moreover, this map is a morphism of operads, as the action of O_2 on the suboperad $\mathcal{A}ssoc \subset \mathcal{C}'_1$ restricts exactly to the expected action of $\mathsf{o}_2 \subset \pi O_2$ on $\pi\mathcal{C}'_{1|\mathcal{A}ssoc}$: as (-1) acts by horizontal mirror reflection (left-right), a braid is sent to its mirror braid along the vertical axis, without changing the position of any interval : we recover exactly the action of o_2 on $\mathsf{Br}\cdot$.

□

12 A model of $\mathcal{TP}$

In this section, we construct groupoid models for hyperbolic links.

12.1 Fundamental groupoid of a mapping torus, and semi-direct product of groupoids

As the $\Sigma_k^* \wr O_2$-spaces associated to hyperbolic links can be described by mapping tori, as seen previously, we begin this section by describing the fundamental groupoid of such a mapping torus.

Proposition 12.1. *Let X be a space (non necessarily connected) and $b \in Homeo(X)$. Then $\pi(T_bX)$ has the following description:*

1. *The underlying set is T_bX.*
2. *The set of morphism is the set of triplets*

$$(\gamma, t_0, s) \in \pi X \times [0,1) \times \mathbb{R}$$

Intuitively, such a triplet is the path that starts from the point $(\gamma(0), t_0)$ and moves along γ in $X \times \{t_0\}$ then follows the horizontal line $(\gamma(1), t)$ in the quotient, for a duration $s \in \mathbb{R}$. Note that the endpoint of (γ, t_0, s) is

$$(b^{\lfloor t_0+s \rfloor}(\gamma(1)), t_0 + s \mod 1).$$

3. *Composition of morphisms is the following: if (γ, t_0, s) and (γ', t'_0, s') are two composable morphisms, their composition is*

$$(b^{-\lfloor t_0+s \rfloor}(\gamma') \circ \gamma, t_0 + s + s' \mod 1, s + s')$$

We use the Van Kampen theorem for groupoids (see [Far04]). Let us call $\mathcal{G}$ the groupoid we designed earlier. We show that this groupoid is indeed the pushout of the diagram

$$\begin{array}{ccc} \pi X \times \{0, \frac{1}{2}\} & \xrightarrow{(b\times 1)\sqcup\iota} & \pi(X \times [\frac{1}{2}, 1]) \\ \iota\big\downarrow & & \\ \pi(X \times [0, \frac{1}{2}]) & & \end{array}$$

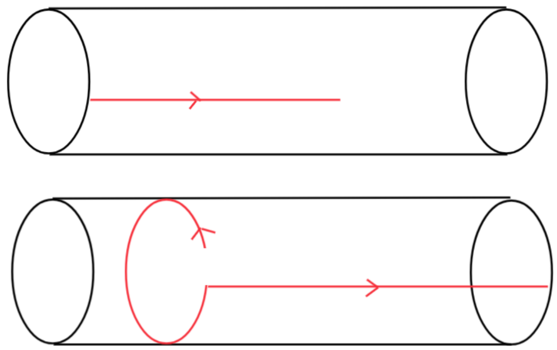


Figure 8: A path (γ, t_0, s) in the mapping torus of two circles, with gluing map the map that exchanges the two circles. γ is a path in the bottom circle, $t_0 = \frac{1}{3}$, $s = \frac{4}{3}$.

with ι the natural inclusion. There is an obvious cocone to $\mathcal{G}$ sending for example a path $(\gamma, \lambda) \in \pi(X \times [0, \frac{1}{2}])$ to the triplet $(\gamma, \lambda(0), \lambda(1) - \lambda(0))$. Note that a path (γ, t_0, s) can split as a composition of paths living in either $X \times [0, \frac{1}{2}]$ or $X \times [\frac{1}{2}, 1]$. For example, we can decompose (γ, t_0, s) in the following way: the first part is $\gamma \times t_0$ then multiple "horizontal" paths of lenght $1/2$ then a horizontal path of length $< 1/2$.). Now, if X is a cocone of this diagram, we can construct a lift $\mathcal{G} \to X$ by using this decomposition. This implies that the lift is unique.

12.2 Models for knot-generating links

The goal is now to establish a model of the free $\Sigma^* \wr O_2$-operad over the base space $\mathcal{BL} = \bigsqcup_{k \in \mathbb{N}} \mathcal{HGL}_k \sqcup \mathcal{SFL}$.

Groupoid models for Seifert links.
We define sfl the groupoid corresponding to Seifert links as a disjoint union of groupoids of the form

$$B\mathbb{Z}^2 \times \mathbb{Z}_2 = \mathsf{o}_2 \times_{\mathbb{Z}_2} \mathsf{o}_2,$$

with one copy for every seifert link in $\mathcal{SFL}$. The action of $\Sigma_1^* \wr \mathsf{o}_2 = \mathsf{o}_2^{op} \times \mathsf{o}_2$ is simply by left and right multiplication on the left and right o_2 term. More concretely the action is given by

$$\mathsf{o}_2 \times (B\mathbb{Z}^2 \times \mathbb{Z}_2) \times \mathsf{o}_2 \to B\mathbb{Z}^2 \times \mathbb{Z}_2$$
$$(\epsilon_1, \epsilon, \epsilon_2) \mapsto \epsilon_1 \epsilon \epsilon_2$$

on objects, and

$$(k_{\epsilon_1}, (m, n)_\epsilon, l_{\epsilon_2}) \mapsto (k + \epsilon_1 m, n + \epsilon l)_{\epsilon_1 \epsilon \epsilon_2}$$

Note that this groupoid can be seen as $\pi(S^1 \times S^1 \times \mathbb{Z}_2, \{*\} \times \{*\} \times \mathbb{Z}_2)$, which amounts to choose exactly one point in each connected component. The resulting groupoid is then the smallest possible we could get to represent these components of the space of links. Note then that for Seifert links, the right action of o_2 is free in the sense that we can write each component as $B\mathbb{Z} \times \mathsf{o}_2$ with o_2 acting by right translation on its copy.

Groupoid models for hyperbolic links. The situation is a bit more complicated than before because to have a good model in groupoids we need to restrict the fundamental groupoid to a small subset of basepoints that is stable under the action of $\Sigma_k^* \wr \mathsf{o}_2$. Our construction uses the point of view that X_L fibers over O_2/B_L. The idea is to restrict the fundamental groupoid to a specific subset of the fiber in $\pm 1 \in O_2/B_L$. We recall that we have seen that X_L is equivalent to a mapping torus with attaching map the "rectification" $b_0 \in \Sigma_k^* \wr \mathbb{Z}_2$ of $b \in B_L^+$.

We now define hgl_k the groupoid corresponding to hyperbolic links in arity k.

1. We start with the case B_L diedral. then $O_2/B_L \simeq S^1$. The space $F\widehat{L}/B_L$ is then fibered over a circle. The fiber over $\overline{1} \in O_2/B_L$ is canonically identified with $S^1 \times \Sigma_k \wr O_2$, via the map

$$S^1 \times \Sigma_k \wr O_2 \hookrightarrow (S^1 \times S^1) \rtimes \mathbb{Z}_2 \times \Sigma_k \wr O_2 = F\widehat{L} \to X_L$$

$$(z, \sigma, \underline{\theta}) \mapsto \overline{(z, 1, 1, \sigma, \underline{\theta})}.$$

   We define then

$$\mathsf{x}_L := \pi(\hat{X}_L)_{|\{1\} \times \Sigma_k \wr \mathbb{Z}_2}$$

   where $\{1\} \times \Sigma_k \wr \mathbb{Z}_2$ is seen as a subset of $\hat{X}_L$ via the inclusion

$$\{1\} \times \Sigma_k \wr \mathbb{Z}_2 \to S^1 \times \Sigma_k \wr O_2.$$

2. If B_L is cyclic, then $O_2/B_L \simeq O_2$. In this case we need to consider the fiber in 1 but also in -1 to keep track of the other connected component of $\hat{X}_L$.
   The union of these two fibers are identified with
   $S^1 \times \mathbb{Z}_2 \times \Sigma_k \wr O_2$, via the map

$$S^1 \times \mathbb{Z}_2 \times \Sigma_k \wr O_2 \hookrightarrow (S^1 \times S^1) \times \mathbb{Z}_2 \times \Sigma_k \wr O_2 = F\widehat{L} \to X_L$$

$$(z, \epsilon, \sigma, \underline{\theta}) \mapsto \overline{(z, 1, \epsilon, \sigma, \underline{\theta})}$$

   We set therefore:

$$\mathsf{x}_L := \pi(X_L)_{|\{1\} \times \mathbb{Z}_2 \times \Sigma_k \wr \mathbb{Z}_2}$$

   via the inclusion

$$\{1\} \times \mathbb{Z}_2 \times \Sigma_k \wr \mathbb{Z}_2 \hookrightarrow S^1 \times \mathbb{Z}_2 \times \Sigma_k \wr O_2$$

In both cases, these two subsets:

1. intersect every connected component of X_L
2. are stable under the right action of $\Sigma_k \wr \mathbb{Z}_2 \subset \Sigma_k \wr O_2$, and the action is free.
3. are stable under the left action of $\mathbb{Z}_2 \subset O_2$.

At the level of groupoids, we then get:

1. x_L inherits an action of $\Sigma_k^* \wr \mathsf{o}_2$, which is free when restricted to $\Sigma_k \wr \mathsf{o}_2$.

2. $\mathsf{x}_L \hookrightarrow \pi X_L$ is a $\Sigma^* \wr \mathsf{o}_2$-equivalence.

Proposition 12.2 (Explicit description of x_L)**.** *Let L be an k-hyperbolic generating link, B_L its symmetry group.*

1. *If B_L is diedral, call m the order of the cyclic group B_L^+, equivalently the order of b. Then x_L is a groupoid with underlying set:*

$$\Sigma_k \times \mathbb{Z}_2^k.$$

If $u = (\sigma, \underline{\epsilon}), v = (\sigma', \underline{\epsilon}')$ are in the same orbit under the representation, write $j(u, v)$ for the class modulo m such that $b^j \cdot u = v$. The morphisms between u and v can be described by the set :

$$Hom(u, v) = \mathbb{Z} \times \mathbb{Z}^k \times (\mathbb{Z} + \frac{j}{m})$$

The composition law of morphisms is given by addition : if $(p_0, p_1, ..., p_k, q) = f \in Hom(u, v)$ and $(p'_0, p'_1, ..., p'_k, q') = f' \in Hom(v, w)$, then $j(u, w) = j(u, v) + j(v, w)$ and $f' \circ f = (p_0 + p'_0, ..., p_k + p'_k, q + q')$ which is indeed a morphism from u to w. If u and v are not in the same orbit under b, there are simply no morphisms between them. It is equipped with a right action of $\Sigma \wr \mathsf{o}_2$: at the object level and for the morphisms in the $\mathbb{Z}^k$ part, the action is induced by the right action of the group object $\Sigma_k \wr \mathsf{o}_2$ on itself, leaving invariant the other parts of the hom-sets. The action of o_2 on the left is induced by the left O_2 action : a morphism $m^+ \in (+)$ acts by translation on the leftmost copy of $\mathbb{Z}$ in the hom-sets, while $0^- = id_- \in (-)$ acts by the following action : it acts on objects as r, the symmetry of B_L which maps to $-id$ in O_2, and acts like r on morphisms except on the $\mathsf{B}\mathbb{Z}$ part where we additionally have to multiply by (-1). Indeed, $(-1) \in O_2$ sends $(s, t, 1, \sigma, \underline{\theta}) \in F\hat{L}$ to $(-s, t, -1, \sigma, \underline{\theta})$ which is in the class of $r \cdot (-s, t, -1, \sigma, \underline{\theta})$ and thus we can deduce the action.

2. *If B_L is cyclic, then x_L is a groupoid with object set:*

$$\mathbb{Z}_2 \times \Sigma_k \times \mathbb{Z}_2^k.$$

The structure is simply two copies of the previous description. The left action of o_2 is the following : the negative object $(-)$ permutes the two connected components, while the morphism $m \in (+)$ acts by translation of m on the left-hand side copy of $\mathbb{Z}$ in the hom-sets.

Proof. The idea is simply to restrict the fundamental groupoid of the mapping torus $T_{b_0}(S^1 \times \Sigma_k \wr O_2)$ to the specific subset of the fiber in $1 \in S^1$, using the Proposition 12.1, and compute the induced action of $\Sigma_k^* \wr O_2$, which is easy to describe. □

12.3 Semi-direct products of groupoids and models of hyperbolic links

This subsection proposes a refined and more compact description of the groupoid model for hyperbolic links. This point of view will help us to compute $\mathsf{sp}_{3,1}$. We need some definitions :

Definition 12.1. *Let $\mathcal{G}$ be a groupoid and G a group acting on $\mathcal{G}$. The* semi-direct product *$\mathcal{G} \rtimes G$ is the groupoid defined by*

$$\mathsf{Ob}(\mathcal{G} \rtimes G) = \mathcal{G}$$

$$\mathsf{Hom}(\mathcal{G} \rtimes G) = \mathsf{Hom}(\mathcal{G}) \times G$$

The source of (γ, g) is the one of γ, while the target is $g \cdot t(\gamma)$ with $t(\gamma)$ the target of γ. If (γ_1, g_1) and (γ_2, g_2) are two composable morphisms, their composition is given by

$$(\gamma_2, g_2) \circ (\gamma_1, g_1) = ((g_1^{-1} \cdot \gamma_2) \circ \gamma_1, g_2 g_1).$$

which is indeed a morphism from $s(\gamma_1)$ to $t(\gamma_2, g_2) = g_2 t(\gamma_2)$.

This definition coincides with the usual semi-direct product if $\mathcal{G}$ is of the form $\mathsf{B}H$.

Definition 12.2. *Let $\mathcal{G}$ be a groupoid and $\phi \in Aut(\mathcal{G})$. The* groupoid mapping torus $\hat{T}_\phi \mathcal{G}$ *is defined as the semi-direct product $\mathcal{G} \rtimes \mathbb{Z}$ given by the action of ϕ.*

This definition is a sort of smaller equivalent version of the mapping torus for groupoids. In fact we have :

Proposition 12.3. *Let X be a space and ϕ a self-homeomorphism of X. Then the groupoid mapping torus $\hat{T}_{\pi\phi}(\pi X)$ is isomorphic to $\pi T_\phi X_{|\{1\} \times X}$ and so is equivalent to $\pi T_\phi X$. Conversely, if $\mathcal{G}$ is a groupoid and $\phi \in Aut(\mathcal{G})$, the groupoid mapping torus $\hat{T}_\phi \mathcal{G}$ is equivalent to $\pi T_{\phi_*}(\mathsf{B}\mathcal{G})$, where ϕ_* is the induced self-homeomorphism of $\mathsf{B}\mathcal{G}$.*

Proof. The explicit description of the fundamental groupoid of the mapping torus made earlier gives this result by restricting to morphisms of the form $(\gamma, t = 0, s \in \mathbb{Z})$. The second point uses additionally the fact that the unit $\mathcal{G} \to \pi\mathsf{B}\mathcal{G}$ is an equivalence of groupoids. □

This point of view gives a nice interpretation of our groupoid models for generating links in terms of groupoid mapping tori :

Proposition 12.4. *The groupoid model for a k-ary hyperbolic link with symmetry group $B_L \simeq D_n$ is a groupoid mapping torus :*

$$\mathsf{x_L} = (\mathsf{B}\mathbb{Z} \times \Sigma_k \times \mathsf{o}_2^k) \rtimes \mathbb{Z}$$

where the action of $\mathbb{Z}$ is given by left multiplication by b_0 (the "rectification" of the generator b of B_L). When $B_L = C_n$ the cyclic group of order n, $\mathsf{x_L}$ is made of two such copies.

We define then:

$$\mathsf{hgl}_k = \bigsqcup_{L \text{ hyperbolic link}} \mathsf{x}_L$$

and finally construct the operad tp being the free $\Sigma^* \wr \mathsf{o}_2$-operad over the base $\Sigma^* \wr \mathsf{o}_2$-collection

$$\mathsf{bl} := \bigsqcup_k \mathsf{hgl}_k \sqcup \mathsf{sfl}.$$

Theorem 9. tp *is a model of $\mathcal{TP}$.*

Proof. As the fact that tp is a model of $\mathcal{TP}$ is not necessary to construct $\mathsf{sp}_{3,1}$, we refer the reader to the full proof for the model $\mathsf{sp}_{3,1}$ we will construct later, because the arguments are the same.

□

12.4 Some examples

We treat completely two examples mentioned in the original article to see what is going on. Our first example is this 3-component hyperbolic link:

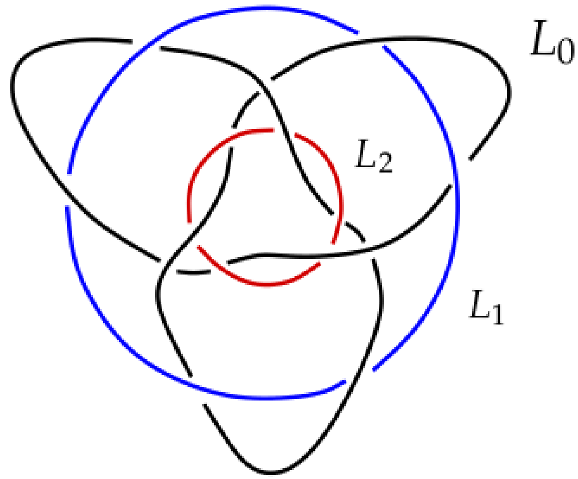


Figure 9: A hyperbolic link in maximal symmetry position.

According to [Bud12], the symmetry group of this link L is $B_L = D_3$ the dihedral group of order 3. It is generated by a rotation r of $2\pi/3$ counterclockwise and by a rotation s of π along the vertical axis laying inside the plane. Once chosen basepoints and orientations of the link components, we can compute the induced group morphism $B_L \to (S^1 \times S^1) \rtimes \mathbb{Z}_2 \times \Sigma_2 \ltimes O_2^2$ explicitly: it sends the rotation r to

$$(1, e^{-2i\pi/3}, 1, \mathrm{id}_{\Sigma_2}, (e^{2i\pi/3}, 1), (e^{2i\pi/3}, 1))$$

and the symmetry s to

$$(1, 1, -1, \mathrm{id}_{\Sigma_2}, (1, -1), (1, -1))$$

with multiplicative notation for each group. The quotient $X_L = F\widehat{L}/B_L$ has therefore $\pi_0(F\widehat{L})/\pi_0(Im(B_L)) = 16/2 = 8$ connected components. As r, the generator of B_L^+ is isotopic to the identity, the principal bundle $X_L \to O_2/B_L \simeq S^1$ is indeed a trivial principal fiber bundle of the form

$$S^1 \times S^1 \times \Sigma_2 \ltimes (O_2)^2 \to S^1$$

$$(u, v, \sigma, A_1, A_2) \mapsto v.$$

Let's compute the left action of O_2. A rotation $(e^{i\theta}, 1) \in O_2$ simply rotates the left circle in the quotient, but the symmetry $(1, -1) \in O_2$ permutes path components : indeed, its action on the identity:

$$(1, -1) \cdot \overline{(1, 1, +1_{\mathbb{Z}_2}, id_{\Sigma_2}, (1, 1), (1, 1))} = \overline{(1, 1, (-1)_{\mathbb{Z}_2}, id_{\Sigma_2}), (1, 1), (1, 1))}$$

is in the class of $(1, 1, +1_{\mathbb{Z}_2}, id_{\mathbb{Z}_2}, (1, -1), (1, -1))$ (applying the symmetry $s \in B_L^-$). The groupoid model is then discrete (there are no non-trivial morphisms between two different points)), its set of points is

$$\Sigma_2 \times \mathbb{Z}_2^2 = \mathbb{Z}_2^3$$

and each point has automorphisms $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}^2$. The right action of $\Sigma_2 \rtimes \mathsf{o}_2^2$ is free and the left action of o_2 follows from the description of the O_2-action. Specifically, $\mathsf{Ob}(\mathsf{o}_2) = \mathbb{Z}_2$ acts diagonally on the two copies of $\mathbb{Z}_2$.
We could in fact write $\mathsf{x}_L = B\mathbb{Z}^2 \times \Sigma_2 \wr \mathsf{o}_2$ with right translation action of $\Sigma_2 \wr \mathsf{o}_2$.

Sakuma's example In Sakuma's example, $B_L = D_{10}$. The generator b of B_L^+ is a rotation by $\frac{2\pi}{5}$ counterclockwise followed by a rotation of π along a circular axis on the drawing: the set of fixed points of this rotation is a circle contained in the plane. The symmetry s is simply a rotation by π along the vertical axis. Here, the representation $B_L^+ \to \Sigma_5 \wr \mathbb{Z}_2$ is a cycle $(1,2,3,4,5)-$. The orbit of each pair $(\sigma, \underline{\epsilon})$ has then 10 elements.

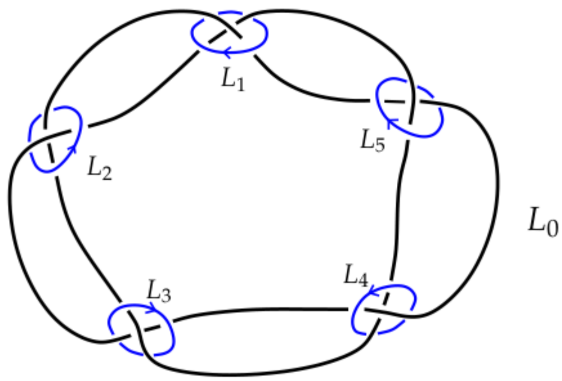


Figure 10: Sakuma's example. An hyperbolic link with $B_L = D_{10}$.

Let us describe the corresponding groupoid. Its set of points is $\Sigma_5 \times \mathbb{Z}_2^5$. Points are connected when they are in the same orbit under B_L. The automorphism group of each point is $\mathbb{Z} \oplus \mathbb{Z}^5 \oplus \mathbb{Z}$. More precisely, the morphisms between $(\sigma, \underline{\epsilon})$ and $b^j \cdot (\sigma, \underline{\epsilon})$ with $j \in \mathbb{Z}_{10}$ are described by

$$\mathbb{Z} \oplus \mathbb{Z}^5 \oplus (\mathbb{Z} + \frac{j}{10})$$

with the natural addition law for composable morphisms. The right action of $\Sigma_5 \wr \mathsf{o}_2$ is the natural action on the objects and translation on the $\mathbb{Z}^5$ part of the morphisms (induced by the action of $\Sigma_5 \wr O_2$ on O_2^5). The left action of o_2 permutes the path components : for example, (-1) sends a permutation σ to $\tau\sigma$ with $\tau = (25)(34)$.

13 The reduced splicing operad $\mathsf{sp}_{3,1}$

We define finally our model for the splicing operad:

Definition 13.1. *The **reduced splicing operad** $\mathsf{sp}_{3,1}$ is defined as*

$$\mathsf{sp}_{3,1} = \mathsf{MBr}' \underset{\mathsf{o}_2}{\vee} \mathsf{tp}$$

Remark 13.1. *Another potentially interesting model for the splicing operad would be the same free product but replacing* Br *with the equivalent* parenthesized braids operad PaB ([*Fre17*], [*Tam03*]) *which can be interpreted as a cofibrant replacement of* Br*. It is still a discrete but combinatorially more complex model.*

13.1 Plan of the proof and discussion of some arguments

The goal is now to show that the reduced splicing operad is indeed a model in groupoids for the splicing operad. As seen in Part I, we want to find a zigzag of equivalences

$$\mathsf{sp}_{3,1} \leftarrow \bullet \rightarrow ... \leftarrow \bullet \rightarrow \pi\mathcal{SP}_{3,1}.$$

1. For this, the first step is to show that $\pi\mathcal{SP}_{3,1}$ is also a free product of $\Sigma^* \wr \pi O_2$-operads in groupoids, namely, we want

$$\pi\mathcal{SP}_{3,1} = \pi\mathcal{TP} \underset{\pi O_2}{\vee} \pi\overline{\mathcal{C}_1'}.$$

   As π does not commute with colimits, we need a bit of work. We know by [Far04] that π commutes with homotopy colimits, but even there we would need to 1) lift this result to the category of operads in Top, and hope that the pushout describing the splicing operad is an homotopy pushout, for the usual model structure on topological operads. This is not obvious. However, the components of the operad are realizations of trees $M(T)$ which are quotients of products of components by free actions of powers of O_2. The fundamental groupoid does not commutes with quotients, but we should expect it to commute with quotients by a free action.

2. Once we get this, we notice that both the reduced splicing operad and $\pi SP_{3,1}$ can be written as pushouts:

$$\begin{array}{ccc} \mathsf{o}_2 & \longrightarrow & \mathsf{MBr}' \\ \downarrow & \ulcorner & \downarrow \\ \mathsf{tp} & \longrightarrow & \mathsf{sp}_{3,1} \end{array} \qquad \begin{array}{ccc} \pi O_2 & \longrightarrow & \pi\overline{\mathcal{C}_1'} \\ \downarrow & \ulcorner & \downarrow \\ \pi\mathcal{TP} & \longrightarrow & \pi\mathcal{SP}_{3,1} \end{array}$$

   We already know that:

$$\mathsf{o}_2 \overset{\sim}{\hookrightarrow} \pi O_2$$
$$\mathsf{MBr}' \overset{\sim}{\hookrightarrow} \pi\overline{\mathcal{C}_1'}$$

   And we have an equivalence :

$$\mathsf{gl} \overset{\sim}{\hookrightarrow} \pi\mathcal{GL}$$

   We could hope that these equivalences give freely $\mathsf{sp}_{3,1} \simeq \pi\mathcal{SP}_{3,1}$ but this is not automatic, because equivalence of categories are not necessarily preserved by

pushouts.
Indeed, it would be the case if we could show these two pushouts are *homotopy pushouts* in a suitable model structure in GrpOp. Such a model structure exists and can be constructed using the axiomatic machinery of [BM03]. See [Hor17] for more on this model structure.

We know that a pushout is a homotopy pushout whenever one of the two arrows is a cofibration [Qui67] . The good candidate is the map $\mathsf{o}_2 \to \mathsf{tp}$ because the latter operad is "almost free". However, the left action of o_2 on tp has fixed points for hyperbolic components with B_L diedral, so this may not be the case. Actually, the way we prove tp is a model of $\mathcal{TP}$ adapts in a large part to show $\mathsf{sp}_{3,1}$ is a model of $\mathcal{SP}_{3,1}$. We will then proceed more explicitely.

13.2 The equivalence $\mathsf{sp}_{3,1} \simeq \pi\mathcal{SP}_{3,1}$

We know show that $\pi\mathcal{SP}_{3,1}$ is a free product of the expected operads. A useful tool is the description of the fundamental groupoid of a mapping torus.

Lemma 13.1. *Let X be a principal bundle over S^1 with fiber $H \ltimes G$, with H acting on the right on G. It can be identified with a mapping torus of $H \rtimes G$ with homeomorphism given by* left *multiplication with an element $(h_0, g_0) \in H \ltimes G$. Let Y be a left G-space. Then*

$$\pi(X \times_G Y) = \pi X \times_{\pi G} \pi Y$$

This simply comes from the explicit description of the fundamental groupoid of each space, and computing the quotient under the action of πG. More formally: We have already seen that $X \times_G Y$ is a mapping torus with fiber $H \times Y$ and "twisted" attaching map, following the proof of the asphericity of $\mathcal{SP}_{3,1}$ for hyperbolic components. The morphisms are the quadruplets $(\gamma_H, \gamma_Y, t_0, s)$ with γ_H (resp γ_G) a path in H (resp G), $t_0 \in [0,1)$, $s \in \mathbb{R}$, with a "twisted" composition law. Now let's compute $\pi X \times_{\pi G} \pi Y$. and the level of morphisms this amounts to identify $(\gamma_H, \gamma_G \cdot \gamma'_G, t_0, s, \gamma_Y)$ with $(\gamma_H, \gamma_G, t_0, s, \gamma'_G \cdot \gamma_Y)$. The induced composition law on this quotient yields:

$$(\gamma_H, t_0, s, \gamma_Y)\cdot(\gamma'_H, t'_0, s', \gamma'_Y) = (\gamma_H \cdot \gamma'_H h^{-\lfloor t_0+s \rfloor}, t_0+s+s' \mod 1, s+s', g^{-\lfloor t_0+s \rfloor}\cdot \gamma_Y \gamma'_Y)$$

which is the multiplication law for the fundamental groupoid of the mapping torus $X \times_G Y$. Therefore $\pi X \times_{\pi G} \pi Y$ is indeed the fundamental groupoid of the expected mapping torus and so is equal to $\pi(X \times_G Y)$.

Proposition 13.1 ($\pi\mathcal{SP}_{3,1}$ is a free product)**.** *The operad $\pi\mathcal{SP}_{3,1}$ has a natural structure of $\Sigma^* \wr \pi O_2$-operad as well as $\pi\mathcal{TP}$ and $\pi\overline{\mathcal{C}'_1}$. Moreover, $\pi\mathcal{TP}$ is free as a $\Sigma^* \wr \pi O_2$-operad with base $\pi\mathcal{GL}$ and there is in fact an identification $\pi\mathcal{SP}_{3,1} \simeq \pi\overline{\mathcal{C}'_1} \vee_{\pi O_2} \pi\mathcal{TP}$.*

Proof. The $\Sigma^* \wr \pi O_2$-operad structure comes from the induced operad morphism $\pi O_2 \to \pi\mathcal{SP}_{3,1}$ by functoriality. To show that $\mathcal{SP}_{3,1}$ is a free product, we proceed by induction on the height of the trees. The Lemma 11.1 makes it work in the case when the root component is hyperbolic. In the other cases, the root component of the tree is of the form $R = R' \times O_2^k$ with right action of O_2^k by translation, so $\pi(R \times_{O_2^k} \prod M(T_i))$ is indeed

$$\pi(R' \times \prod M(T_i)) = \pi(R') \times \pi(\prod M(T_i)) = \pi(R) \times_{\pi O_2^k} \pi \prod M(T_i))$$

The same argument shows that $\pi\mathcal{TP}$ is a free $\Sigma^* \wr \pi O_2$-operad with base $\pi\mathcal{GL}$. □

The idea is now to show that $\mathsf{sp}_{3,1}$ is a model of $\mathcal{SP}_{3,1}$ by constructing an explicit *fully faithful embedding*

$$\mathsf{sp}_{3,1} \overset{\sim}{\hookrightarrow} \pi\mathcal{SP}_{3,1}.$$

Let us construct the map by induction on the height of the trees. The main point we have to be careful to is that we are comparing a $\Sigma^* \wr \mathsf{o}_2$-operad to a $\Sigma^* \wr \pi O_2$-operad. The equivalence therefore has only a meaning as an equivalence between *symmetric* operads. Take a two-colored semi-alternate tree T. This tree represents a union of some connected components of $\mathsf{sp}_{3,1}$ or $\mathcal{SP}_{3,1}$. From now on, when we say "component" we mean : *union of orbits of a connected component under the action of* $\Sigma^* \wr O_2$.
Write X for a component of $\mathcal{SP}_{3,1}$ corresponding to the root of T, and x the corresponding model in groupoids we constructed. Write Y for the product of the realization of the higher trees so that $M(T) = X \underset{O_2^k}{\times} Y$. By induction hypothesis, we have a model $\mathsf{y} \overset{\sim}{\hookrightarrow} Y$.
When X is not associated to an hyperbolic link, we have seen $X = X' \times O_2^k$ for some k, and $\mathsf{x} = \mathsf{x}' \times \mathsf{o}_2^k$ with $\mathsf{x}' \overset{\sim}{\hookrightarrow} X'$), the map is easy to write down: it is simply

$$\mathsf{x} \times_{\mathsf{o}_2^k} \pi Y = x' \times Y \overset{\sim}{\hookrightarrow} \pi X' \times \pi Y = \pi X \times_{\pi O_2^k} \pi Y$$

Now we treat the case when the base is a k-hyperbolic component. We will suppose B_L dihedral as always. If B_L is cyclic the situation is the same but with two copies of the groupoid.

We first show that $\mathsf{Ob}(\mathsf{x}_L \underset{\mathsf{o}_2^k}{\times} \mathsf{y})$ identifies as a subset of $\mathsf{Ob}(\pi X_L \underset{\pi O_2^k}{\times} \pi Y)$: note that colimits at the object level are computed as colimits of sets (because the forgetful functor $\mathsf{Gpd} \to \mathsf{Set}$ has a left adjoint and thus commutes with colimits). Now note that $\mathsf{Ob}(\mathsf{x}_L) = \Sigma_k \times \mathbb{Z}_2^k$ and then we have

$$\mathsf{Ob}(\mathsf{x}_L \underset{\mathsf{o}_2^k}{\times} \mathsf{y}) = \Sigma_k \times \mathbb{Z}_2^k \underset{\mathbb{Z}_2^k}{\times} \mathsf{Ob}(\mathsf{y}) = \Sigma_k \times \mathsf{Ob}(\mathsf{y}) \hookrightarrow \Sigma_k \times Y$$

by induction hypothesis and the right-hand side can be identified with the fiber in $1 \in S^1$ of $X_L \underset{O_2^k}{\times} Y$. We then use the following fact :

Lemma 13.2. *1.* $\mathsf{x}_L \underset{\mathsf{o}_2^k}{\times} \mathsf{y}$ *is a groupoid mapping torus* $\hat{T}_\phi(\mathsf{B}\mathbb{Z} \times \Sigma_k \times \mathsf{y})$ *with* ϕ *acting by the induced "twisted" automorphism of* $\mathsf{B}\mathbb{Z} \times \Sigma_k \times \mathsf{y}$ *:*

$$\mathsf{B}\mathbb{Z} \times \Sigma_k \times \mathsf{y} \to \mathsf{B}\mathbb{Z} \times \Sigma_k \times \mathsf{y}$$

$$(\tau, y) \mapsto (\sigma_b \tau, (\underline{\epsilon}_b \cdot \tau) \cdot y) \text{ on objects}$$

$$(m, id_\tau, \gamma_y) \mapsto (m, id_{\sigma_b \tau}, (\underline{\epsilon}_b \cdot \tau) \cdot \gamma_y) \text{ on morphisms.}$$

2. Moreover, $(\pi X_L \underset{\pi O_2^k}{\times} \pi Y)_{|\text{fiber over } 1}$ *is a groupoid mapping torus* $\hat{T}_{\pi \hat{b}_0}(\pi S^1 \times \pi \Sigma_k \times Y)$ *with* $\pi \hat{b}_0$ *the induced "twisted" automorphism of* $\pi S^1 \times \Sigma_k \times \pi Y$.

Proof. For the first point, we write explicitly the morphism set of $\mathsf{x}_L \times \mathsf{y}$ using the fact that x_L is a groupoid mapping torus (proposition 12.4). The quotient by the action of o_2^k yields a groupoid with object set $\Sigma_k \times \mathsf{y}$ as expected. The morphisms of $\mathsf{x}_L \times \mathsf{y}$ are described by multiplets :

$$(r, id_\sigma, \underline{m}^{\underline{\epsilon}}, t, \gamma_y)$$

With r an integer as a morphism of $\mathsf{B}\mathbb{Z}$, writing $\underline{m}^{\underline{\epsilon}}$ for the morphism given by $(m_1, ..., m_n) \in (\epsilon_1, ..., \epsilon_n) \in \mathsf{o_2}^k$, t for an integer that represents the morphism coming for the semi-direct product structure, and γ_y for a morphism in y. We claim the projection map to the quotient by o_2^k is equal to the following :

$$\Pi : x_L \times \mathsf{y} \to (\Sigma_k \times \mathsf{y}) \rtimes \mathbb{Z}$$

$$(r, id_\sigma, \underline{m}^{\underline{\epsilon}}, t, \gamma_y) \mapsto (r, id_\sigma, \underline{m}_{\underline{\epsilon}} \cdot \gamma_y, t)$$

Realizing $\mathsf{x}_L \underset{\mathsf{o}_2^k}{\times} \mathsf{y}$ as the semi-direct product on the right, given by the "twisted" automorphism written in Lemma 13.2.1. This map obviously coequalizes the left/right action of o_2^k. We check this is a groupoid morphism by a tedious calculation. To begin with, we check that the morphism Π is compatible with source and target: Let $f = (r, id_\sigma, \underline{m}_{\underline{\epsilon}}, t,_y \gamma_{y'})$ be a morphism in $\mathsf{x}_L \times \mathsf{y}$. We can take $t = 1$ because all morphisms are generated by morphisms with $t = 1$, so it suffices to check compatibility with source and target for $t = 1$.
The source of f is $s(t) = (\sigma, \epsilon, y)$ while its target is $(\sigma_b\sigma, (\epsilon_b \cdot \sigma)\epsilon, y')$. The source of $\Pi(f)$ is $(\sigma, \epsilon \cdot y)$ while its target is

$$\Pi(t(f)) = (\sigma_b\sigma, (\epsilon_b \cdot \sigma)\sigma') \cdot \gamma') = (\sigma_b\sigma, (\epsilon_b \cdot \sigma) \cdot \sigma' \cdot y)$$

which yields the expected target in the semi-direct product structure of $\mathsf{x_L} \underset{\mathsf{o}_2^k}{\times} \mathsf{y}$. The next point is to show that it commutes with composition to write properly.

The second point is obtained combining Proposition 12.3 and Lemma 13.1. □

Comparing these two, as the "twisted" action on the small groupoid is the restriction of the twisted action on the big one, we get :

Corollary 13.1. *The inclusion*

$$\mathsf{x}_L \underset{\mathsf{o}_2}{\times} \mathsf{y} \hookrightarrow (\pi X_L \underset{\pi O_2^k}{\times} \pi Y)_{|\textit{fiber over } 1}$$

is fully faithful.

This concludes the proof in the hyperbolic case, and we finally get :

Theorem 10. $\mathsf{sp}_{3,1}$ *is a model of* $\mathcal{SP}_{3,1}$.

13.3 Model for space of knots in groupoids

We have now constructed a small model $\mathsf{sp}_{3,1}$ of the splicing operads in groupoids. We now show that the space of knots itselfs admits a models k, which will be a free $\mathsf{sp}_{3,1}$-algebra over a base space th.

Definition 13.2. *Let* th *be the groupoid defined as a disjoint union of:*

- $\mathsf{B}\mathbb{Z}$ *for each torus knot component* $f \in \mathcal{TH}$. *There is a left action of* o_2 *by* $m^{\pm\epsilon} \cdot n = \epsilon n + m$. *This groupoid is indeed simply the restriction* $\pi S^1_{|\{1\}}$ *with the induced action of* o_2, *which stabilizes* $1 \in S^1$.
- $\mathsf{B}\mathbb{Z}^2 = \mathsf{B}\mathbb{Z} \times \mathsf{B}\mathbb{Z}$ *for each invertible hyperbolic knot component.* o_2 *acts diagonally on the left. This is the restriction* $\pi(S^1 \times S^1)_{|\{(1,1)\}}$ *with the induced action of* o_2.
- $\mathsf{B}\mathbb{Z}^2 \times \mathbb{Z}_2$ *for each non-invertible hyperbolic knot component.* o_2 *acts diagonally (with the action* $o_2 \circlearrowright \mathbb{Z}_2$ *that permutes the components). Once again this is the restriction* $\pi(S^1 \times S^1 \times \mathbb{Z}_2)_{|\{(1,1,+1),(1,1,-1)\}}$ *with the induced action of* o_2.

Definition 13.3. *We define* k *the model for long knots, as the free* $\Sigma^* \wr \mathsf{o}_2$*-algebra over* th.

Theorem 11. k *is a model of* $\mathcal{K}$, *and we have an* o_2*-equivariant, fully faithful embedding* $\mathsf{k} \overset{\sim}{\hookrightarrow} \pi\mathcal{K}$.

Proof. The proof follows the same steps as the proof that $\mathsf{sp}_{3,1}$ is a model of $\mathcal{SP}_{3,1}$: There is a natural morphism

$$\mathsf{k} = \mathsf{sp}_{3,1}(\mathsf{th}) \overset{\sim}{\hookrightarrow} \pi K = \pi\mathcal{SP}_{3,1}(\pi K)$$

and we show it is a fully faithful inclusion, using the equivalence $\mathsf{sp}_{3,1}$ with $\pi\mathcal{SP}_{3,1}$ and th with $\pi\mathcal{TH}$. □

14 Computing the rational homology of $\mathcal{SP}_{3,1}$

We expose here the computation of the homology of $\mathcal{SP}_{3,1}$ made by Beatrice Laracca [Lar19].

The rational homology of the splicing operad has a natural structure of a module over the wreath algebra $\Sigma^* \wr H_\bullet(O_2)$, which is simply the homology of the corresponding wreath product. Let us make this structure more explicit. As O_2 is a (Lie) group, its homology $H_*(O_2)$ is a cocommutative Hopf algebra. It is generated by the elements $r \in H_0(O_2)$ and $\Delta \in H_1(O_2)$ which correspond respectively to the connected component of $-Id$ and to the generator of the H_1 of the connected component of Id. These generator satisfy the relations

$$r\Delta = \Delta r$$
$$r^2 = 1$$
$$\Delta^2 = 0$$

The first and second ones by direct computations, the third one by dimensionality reasons. Therefore an action of $H_*(O_2)$ is entirely determined by the action of r and Δ satisfying these relations.

14.1 Computing the homology

The homology of $\mathcal{C}_1' \rtimes O_2$. Our first task is to determine the homology of $\overline{\mathcal{C}_1'} = \mathcal{C}_1' \rtimes O_2$. We know well the homology of the little discs operad . It is the *Gerstenhaber operad* which encodes the structure of Gerstenhaber algebras [LV12]. It is generated by:

1. An arity 2, degree 0 element μ (the multiplication) and an arity 2, degree 1 element $\{\cdot,\cdot\}$ (the *Poisson bracket*)
2. The following relations hold:

   (Commutativity)
   $$\mu \circ_1 \mu = \mu \circ_2 \mu$$

   (Jacobi)
   $$\lambda \circ_1 \lambda = \lambda \circ_2 \lambda + \lambda \cdot (23) \circ_1 \lambda$$

   (Poisson)
   $$\lambda \circ_1 \mu = \mu \circ_2 \lambda + \mu \cdot (23) \circ_1 \lambda$$

The semi-direct product adds other operators, namely $s \in H_0(O_2)$ and $\Delta \in H_1(O_2)$. We have in fact :

Theorem 12 ([Lar19])**.** *An algebra X over $H_*(\overline{\mathcal{C}_1'})$ is a Gerstenhaber algebra with a linear involution $r : X \to X$ of degree 0 and a linear endomorphism $\Delta : X \to X$ of degree 1 such that $\Delta^2 = 0$ and, for each $x, y \in X$, the following relations hold:*

$$\begin{aligned} r(\mu(x,y)) &= \mu(r(x), r(y)) \\ r([x,y]) &= -[r(x), r(y)] \\ \Delta(\mu(x,y)) &= \mu(\Delta(x), y) + (-1)^{|x|}\mu(x, \Delta(y)) \\ \Delta[x,y] &= [\Delta(x), y] + (-1)^{|x|+1}[x, \Delta(y)] \end{aligned}$$

Together with the computation of the homology of $\mathcal{GL}$, we can describe $H_*(\mathcal{SP})$:

Theorem 13 ([Lar19])**.** *The homology of the splicing operad $H_\bullet(\mathcal{SP}_{3,1})$ is a free product of the operad $H_*(\overline{\mathcal{C}_1'}) = \mathsf{Gerst} \rtimes H_*(O_2)$ and of $H_*(\mathcal{TP})$ which is a free $\Sigma * \wr H_*(O_2)$-operad on the generators $H_*(\mathcal{GL})$. The following generators are:*

1. *Operations of arity 1, $P_{p,q} \in H_*(\mathcal{SL})$ of degree 0, one for each $(p,q) \in \mathbb{Z}^2$ with $p \not| q$ and $p\hat{q} = 1$, with the relation $sP_{p,q}s = P_{p,q}$ for $s \in H_0(O_2)$.*
2. *Operations of arity k, for $k \in \mathbb{N}$, $P_L \in H_*(\mathcal{HGL})$ of degree 0, one for each hyperbolic link $L \in \mathcal{HGL}_k$. The relations are $P_L\rho(b) = P_L$, for $b \in B_L$. '*
3. *Operations of arity k, for $k \in \mathbb{N}$, $\iota_L \in H_*(\mathcal{HGL})$ of degree 1, one for each hyperbolic link $L \in \mathcal{HL}_k$. They satisfy $\iota_L\rho(b) = \pm\iota_L$, for $b \in B_L$.*

Together with :

Theorem 14 ([Lar19])**.** *The rational homology of the space of long knot is the free $H_*(\mathcal{SP})$-algebra on the $H_*(O_2)$-module with one 0-dimensional generator $T_{p,q}$ for each torus knot, two generators P_K, ι_K of degree 0 and 1 respectively for each hyperbolic knot up to inversion, and relations $sT_{p,q} = T_{p,q}$ and $rP_K = P_K$ whenever the knot K is invertible.* $\cdot$

15 Geometric interpretation

We have seen that every component of the splicing operad is a $K(G,1)$, as for the space of long knots $\mathcal{K}_{3,1}$. Therefore, to understand the homotopy type of these space it suffices to identify the generators of the fundamental group. This part aims to provide a geometric interpretation of some generators of $\pi_1(\mathcal{SP}_{3,1})$, and the induced action on the space of long knots.

15.1 Generators of $\overline{\mathcal{C}_1'}$

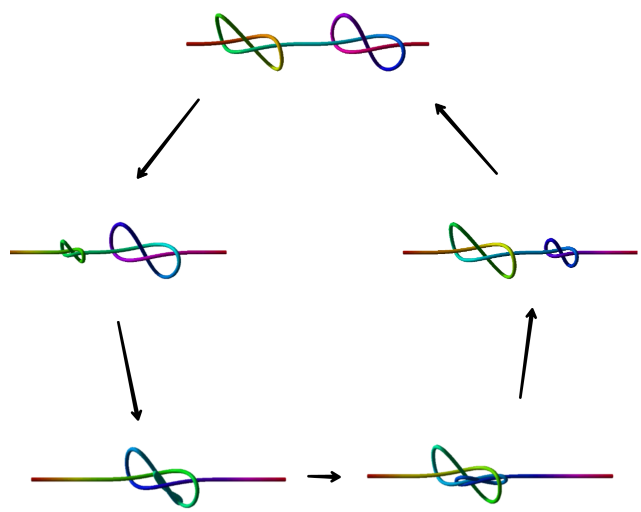


Figure 11: The loop induced by a path in $\pi_1(\overline{\mathcal{C}}_1'(2)))$ corresponding to a generator of the braid groups (which is simply $\mathbb{Z}$ for two strands). The concrete operation on the spaces of knots is described geometrically in the figure.

Note that $\overline{\mathcal{C}_1'(2)} \simeq S^1 \times O_2^2$. The image above describes geometrically the path corresponding to the left circle. Generators associated to a component of O_2^2 can be described as the ones that rotates one of the knots of the input, but not the other like in the following example:

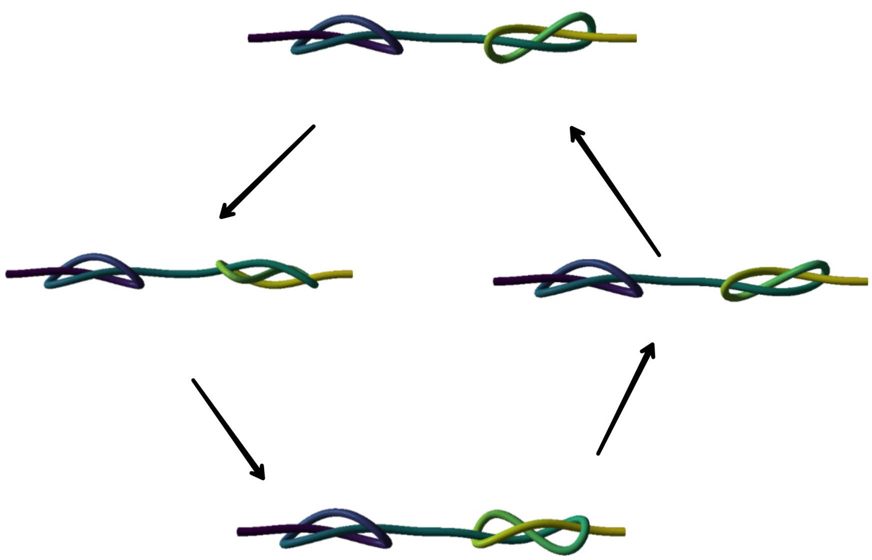
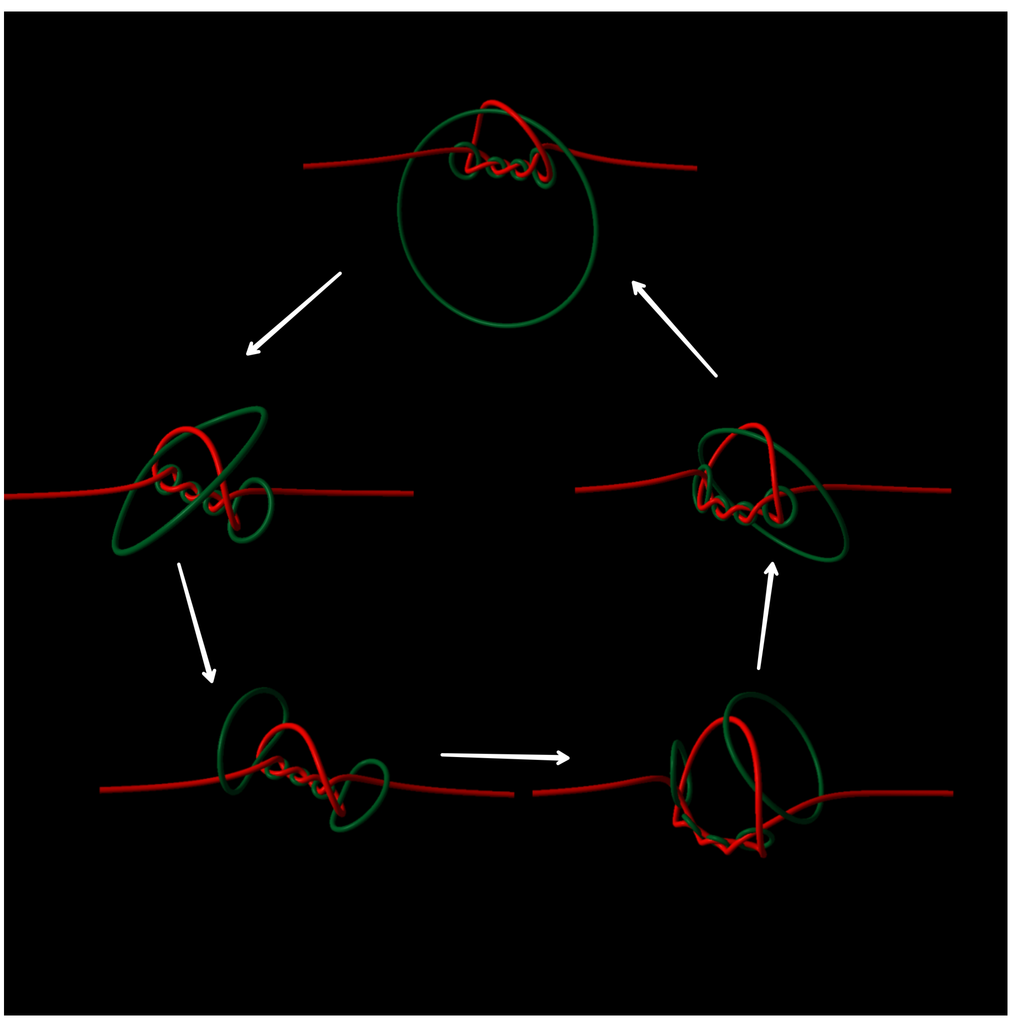


Figure 12: The loop induced on the image in $\mathcal{K}_{3,1}$ by one of the generators of $\pi_1(O_2^2, 1) \subset \pi_1(\overline{\mathcal{C}}_1'(2))$. The inputs are a trefoil knot and an eight knot.

15.2 Generators for the hyperbolic components



The path represents a possible monodromy along the base circle of the fibration $F\hat{L} \to S^1$. It corresponds to reparametrization of Sakuma's link along L_0. The image is the result of stereographic projection of the maximal symmetry position of L with base point moving along L_0 with an arbitrary choice of framing. The choice was made to make L_0 almost fixed in the animation.
Note that this path permutes the hockey pucks cyclically and reverses their direction (coherently with the fact that $B_L^+ = \mathbb{Z}_{10}$). This illustrates that $F\hat{L}$ is a nontrivial bundle, and that this path represents in fact $1/10$th of a loop in $\mathcal{SP}_{3,1}$.

15.3 Some further questions

We suggest here some related questions.

Question 1 (Formality of $\mathcal{SP}_{3,1}$)**.** *The proof of the formality of $\mathcal{C}_2$ by Tamarkin [Tam03] uses the equivalent operad* BPaB*. Could we adapt his argument to show the splicing operad is formal, considering the model* $(\mathsf{PaB} \rtimes \mathsf{o}_2) \vee_{\mathsf{o}_2} \mathsf{tp}$*, or even simply* $\mathsf{sp}_{3,1}$ *?*

Question 2. *Using the model description, could we compute explicitely the fundamental groups of the components of $\mathcal{SP}_{3,1}$ and $\mathcal{K}$?*

Question 3. *For a group, the free $\Sigma^* \wr G$-operad construction shows that there is a free-forgetful adjunction*

$$\Sigma^* \wr G\mathsf{Op} \leftrightarrows \Sigma^* \wr G\mathsf{Coll}$$

Or, equivalently, seing G as an operad concentrated in arity 1, the adjunction is

$$G \downarrow \mathsf{Op} \leftrightarrows G\mathsf{Mod}G$$

Where $G \downarrow \mathsf{Op}$ is the category of operads under G, and $G\mathsf{Mod}G$ the category of (G, G)-bimodules. If we consider a general operad $\mathcal{P}$, non-necessarily concentrated in arity one, there is still a forgetful functor

$$\mathcal{P} \downarrow \mathsf{Op} \to P\mathsf{Mod}P$$

What are the conditions on $\mathcal{P}$ under which this forgetful functor has a left adjoint?

Question 4. *Does the splicing operad in higher dimensions gives interesting operations on (codimension 2, framed) knots ?*

Question 5. *In [Sin06], Dev Sinha constructs a little 2-cubes action on the space of (framed) knots in codimension ≥ 3 using embedding calculus. Budney's geometric action extends to higher dimensional framed knots and yields another little 2-cubes action. It is still conjectured whether the two actions are equivalent.*

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