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Behavior near the extinction time in self-similar fragmentations I: The stable case

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Abstract. The stable fragmentation with index of self-similarity $\alpha \in [-1/2, 0)$ is derived by looking at the masses of the subtrees formed by discarding the parts of a $(1+\alpha)^{-1}$–stable continuum random tree below height t, for $t \geq 0$. We give a detailed limiting description of the distribution of such a fragmentation, $(F(t), t \geq 0)$, as it approaches its time of extinction, ζ. In particular, we show that $t^{1/\alpha}F((\zeta - t)^+)$ converges in distribution as $t \to 0$ to a non-trivial limit. In order to prove this, we go further and describe the limiting behavior of (a) an excursion of the stable height process (conditioned to have length 1) as it approaches its maximum; (b) the collection of open intervals where the excursion is above a certain level; and (c) the ranked sequence of lengths of these intervals. Our principal tool is excursion theory. We also consider the last fragment to disappear and show that, with the same time and space scalings, it has a limiting distribution given in terms of a certain size-biased version of the law of ζ.

In addition, we prove that the logarithms of the sizes of the largest fragment and last fragment to disappear, at time $(\zeta - t)^+$, rescaled by $\log(t)$, converge almost surely to the constant $-1/\alpha$ as $t \to 0$.

Résumé. La fragmentation stable d'incice $\alpha \in [-1/2, 0)$ est construite à partir des masses des sous-arbres de l'arbre continu aléatoire stable d'indice $(1+\alpha)^{-1}$ obtenus en ne gardant que les feuilles situées à une hauteur supérieure à t, pour $t \geq 0$. Nous donnons une description détaillée du comportement asymptotique d'une telle fragmentation, $(F(t), t \geq 0)$, au voisinage de son point d'extinction, ζ. En particulier, nous montrons que $t^{1/\alpha}F((\zeta - t)^+)$ converge en loi lorsque $t \to 0$ vers une limite non triviale. Pour obtenir ce résultat, nous allons plus loin et décrivons le comportement asymptotique en loi, après normalisation, (a) d'une excursion du processus de hauteur stable (conditionnée à avoir une longueur 1) au voisinage de son maximum; (b) des intervalles ouverts où l'excursion est au-dessus d'un certain niveau; et (c) de la suite décroissante des longueurs de ces intervalles. Notre outil principal est la théorie des excursions. Nous nous intéressons également au dernier fragment à disparaître et montrons, qu'avec les mêmes normalisations en temps et espace, la masse de ce fragment a une distribution limite construite à partir d'une certaine version biaisée de ζ.

Enfin, nous montrons que les logarithmes des masses du plus gros fragment et du dernier fragment à disparaître, au temps $(\zeta - t)^+$, divisés par $\log(t)$, convergent presque sûrement vers la constante $-1/\alpha$ lorsque $t \to 0$.

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1. Introduction

The subject of this paper is a class of random fragmentation processes which were introduced by Bertoin [5], called the self-similar fragmentations. In fact, we will find it convenient to have two slightly different notions of a fragmentation process. By an *interval fragmentation*, we mean a process $(O(t), t \geq 0)$ taking values in the space of open subsets of

$(0, 1)$ such that $O(t) \subseteq O(s)$ whenever $0 \le s \le t$. We refer to a connected interval component of $O(t)$ as a *block*. Let $F(t) = (F_1(t), F_2(t), \dots)$ be an ordered list of the lengths of the blocks of $O(t)$. Then $F(t)$ takes values in the space

$$\mathcal{S}_1^{\downarrow} = \left\{ \mathbf{s} = (s_1, s_2, \dots): \ s_1 \ge s_2 \ge \cdots \ge 0, \sum_{i=1}^{\infty} s_i \le 1 \right\}.$$

We call the process $(F(t), t \ge 0)$ a *ranked fragmentation*. A ranked fragmentation is called *self-similar with index* $\alpha \in \mathbb{R}$ if it is a time-homogeneous Markov process which satisfies certain *branching* and *self-similarity* properties. Roughly speaking, these mean that every block should split into a collection of sub-blocks whose relative lengths always have the same distribution, but at a rate which is proportional to the length of the original block raised to the power α. (Rigorous definitions will be given in Section 2.) Clearly the sign of α has a significant effect on the behavior of the process. If $\alpha > 0$ then larger blocks split faster than smaller ones, which tends to act to balance out block sizes. On the other hand, if $\alpha < 0$ then it is the smaller blocks which split faster. Indeed, small blocks split faster and faster until they are reduced to *dust*, that is blocks of size 0.

The asymptotic behavior of self-similar fragmentations has been studied quite extensively. In one sense, it is trivial, in that $F(t) \to 0$ a.s. as $t \to \infty$, whatever the value of the index α (provided the process is not trivially constant, i.e. equal to its initial value for all times t). For $\alpha \ge 0$, rescaled versions of the empirical measures of the lengths of the blocks have law of large numbers-type behavior (see Bertoin [6] and Bertoin and Rouault [8]). For $\alpha < 0$, however, the situation is completely different. Here, there exists an almost surely finite random time ζ, called the *extinction time*, when the state is entirely reduced to dust. The manner in which mass is lost has been studied in detail in [13] and [14].

The purpose of this article is to investigate the following more detailed question when α is negative: *how does the process* $F((\zeta - t)^+)$ *behave as* $t \to 0$*?* We provide a detailed answer for a particularly nice one-parameter family of self-similar fragmentations with negative index, called the *stable fragmentations*.

The simplest of the stable fragmentations is the *Brownian fragmentation*, which was first introduced and studied by Bertoin [5]. Suppose that $(\mathbf{e}(x), 0 \le x \le 1)$ is a standard Brownian excursion. Consider, for $t \ge 0$, the sets

$$O(t) = \{ x \in [0, 1]: \ \mathbf{e}(x) > t \}$$

and let $F(t) \in \mathcal{S}_1^{\downarrow}$ be the lengths of the interval components of $O(t)$ in decreasing order. Then it can be shown that $(F(t), t \ge 0)$ is a self-similar fragmentation with index $-1/2$. Miermont [20] generalized this construction by replacing the Brownian excursion with an excursion of the height process associated with the stable tree of index $\beta \in (1, 2)$, introduced and studied by Duquesne, Le Gall and Le Jan [11,19]. The corresponding process is a self-similar fragmentation of index $\alpha = 1/\beta - 1$.

The behavior near the extinction time in the Brownian fragmentation can be obtained via a decomposition of the excursion at its maximum. We discuss this case in Section 3. Abraham and Delmas [1] have recently proved a generalized Williams' decomposition for the excursions which code stable trees. This provides us with the necessary tools to give a complete description of the behavior of the stable fragmentations near their extinction time, which is detailed in Section 4. In every case, we obtain that

$$t^{1/\alpha} F\big((\zeta - t)^+\big) \xrightarrow{d} F_\infty \quad \text{as } t \to 0,$$

where F_∞ is a random limit which takes values in the set of non-increasing non-negative sequences with finite sum. The limit F_∞ is constructed from a self-similar function H_∞ on $\mathbb{R}$, which itself arises when looking at the scaling behavior of the excursion in the neighborhood of its maximum. See Theorems 4.1 and 4.2 and Corollary 4.3 for precise statements.

In Corollary 4.4, we also consider the process of the *last fragment*, that is the size of the (as it turns out unique) block which is the last to disappear. We call this size $F_*(t)$ and prove that, scaled as before, $F_*((\zeta - t)^+)$ also has a limit in distribution as $t \to 0$ which, remarkably, can be expressed in terms of a certain size-biased version of the distribution of ζ.

Sections 5–8 are devoted to the proofs of these results.

Finally, in Section 9, we consider the logarithms of the largest and last fragments and show that

$$\frac{\log F_1((\zeta - t)^+)}{\log(t)} \to -1/\alpha \quad \text{and} \quad \frac{\log F_*((\zeta - t)^+)}{\log(t)} \to -1/\alpha$$

almost surely as $t \to 0$. In fact, these results hold for a more general class of self-similar fragmentations with negative index.

We will investigate the limiting behavior of $F((\zeta - t)^+)$ as $t \to 0$ for general self-similar fragmentations with negative index α in future work, starting in [12]. In general, as indicated by the results for the logarithms of F_1 and F_* above, the natural conjecture is that $t^{1/\alpha}$ is the correct re-scaling for non-trivial limiting behavior. However, since the excursion theory tools we use here are not available in general, we are led to develop other methods of analysis.

2. Self-similar fragmentations

Define $\mathcal{O}_{(0,1)}$ to be the set of open subsets of $(0, 1)$. We begin with a rather intuitive notion of a fragmentation process.

Definition 2.1 (Interval fragmentation). *An interval fragmentation is a process* $(O(t), t \geq 0)$ *taking values in* $\mathcal{O}_{(0,1)}$ *such that* $O(t) \subseteq O(s)$ *whenever* $0 \leq s \leq t$.

In this paper we will be dealing with interval fragmentations which derive from excursions. Here, an *excursion* is a continuous function $f : [0, 1] \to \mathbb{R}^+$ such that $f(0) = f(1) = 0$ and $f(x) > 0$ for all $x \in (0, 1)$. The associated interval fragmentation, $(O(t), t \geq 0)$ is defined as follows: for each $t \geq 0$,

$$O(t) = \left\{x \in [0, 1]:\ f(x) > t\right\}.$$

An example is given in Fig. 1.

We need to introduce a second notion of a fragmentation process. We endow the space $\mathcal{S}_1^{\downarrow}$ with the topology of pointwise convergence.

Definition 2.2 (Ranked self-similar fragmentation). *A ranked self-similar fragmentation* $(F(t), t \geq 0)$ *with index* $\alpha \in \mathbb{R}$ *is a càdlàg Markov process taking values in* $\mathcal{S}_1^{\downarrow}$ *such that*

- $(F(t), t \geq 0)$ *is continuous in probability*;
- $F(0) = (1, 0, 0, \ldots)$;
- *Conditionally on* $F(t) = (x_1, x_2, \ldots)$, $F(t + s)$ *has the law of the decreasing rearrangement of the sequences* $x_i F^{(i)}(x_i^{\alpha} s)$, $i \geq 1$, *where* $F^{(1)}, F^{(2)}, \ldots$ *are independent copies of the original process* F.

Let $r : \mathcal{O}_{(0,1)} \to \mathcal{S}_1^{\downarrow}$ be the function which to an open set $O \subseteq (0, 1)$ associates the ranked sequence of the lengths of its interval components. We say that an interval fragmentation is *self-similar* if it possesses branching and self-similarity properties which entail that $(r(O(t)), t \geq 0)$ is a ranked self-similar fragmentation. See [2,5,7] for this and further background material.

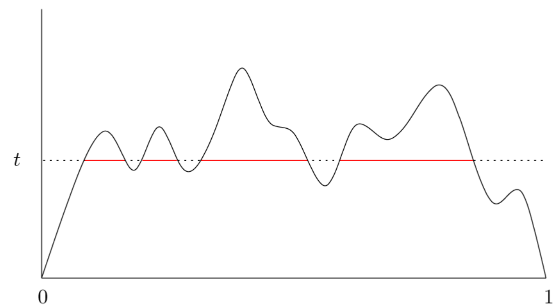

Fig. 1. An interval fragmentation derived from a continuous excursion: the set $O(t)$ is represented by the solid lines at level t.

Bertoin [5] has proved that a ranked self-similar fragmentation can be characterized by three parameters, (α, ν, c). Here, $\alpha \in \mathbb{R}$; ν is a measure on $\mathcal{S}_1^{\downarrow}$ such that $\nu((1, 0, \ldots)) = 0$ and $\int_{\mathcal{S}_1^{\downarrow}} (1 - s_1)\nu(\mathrm{d}\mathbf{s}) < \infty$; and $c \in \mathbb{R}^+$. The parameter α is the *index of self-similarity*. The measure ν is the *dislocation measure*, which describes the way in which fragments suddenly dislocate; heuristically, a block of mass m splits at rate $m^\alpha \nu(\mathrm{d}\mathbf{s})$ into blocks of masses $(ms_1, ms_2, \ldots)$. The real number c is the *erosion coefficient*, which describes the rate at which blocks continuously melt. Note that ν may be an infinite measure, in which case the times at which dislocations occur form a dense subset of $\mathbb{R}^+$. When $c = 0$, the fragmentation is a pure jump process.

In the context of an interval fragmentation derived from an excursion, it is easy to see that the extinction time of the fragmentation is just the maximum height of the excursion:

$$\zeta = \max_{0 \le x \le 1} f(x).$$

In the examples we treat, this maximum will be attained at a unique point, x_*. In this case, let $O_*(t)$ be the interval component of $O(t)$ containing x_* at time t, and let $F_*(t)$ be its length, i.e. $F_*(t) = |O_*(t)|$. We call both O_* and F_* the *last fragment process*.

3. The Brownian fragmentation

We begin by discussing the special case of the Brownian fragmentation. The sketch proofs in this section are not rigorous, but can be made so, as we will demonstrate later in the paper. Our intention is to introduce the principal ideas in a framework which is familiar to the reader.

Let $(\mathbf{e}(x), 0 \le x \le 1)$ be a normalized Brownian excursion with length 1 and for each $t \ge 0$ define the associated interval fragmentation by

$$O(t) := \big\{x \in [0, 1]\colon \mathbf{e}(x) > t\big\}.$$

See Fig. 2 for a picture. The associated ranked fragmentation process, $(F(t), t \ge 0)$, has index of self-similarity $\alpha = -1/2$, binary dislocation measure specified by $\nu(s_1 + s_2 < 1) = 0$ and

$$\nu(s_1 \in \mathrm{d}x) = 2\big(2\pi x^3(1 - x)^3\big)^{-1/2} \mathbb{1}_{[1/2,1)}(x)\,\mathrm{d}x,$$

Fig. 2. The Brownian interval fragmentation with the open intervals which constitute the state at times $t = 0, 0.15, 0.53$ and 0.92 indicated.

and erosion coefficient 0. See Bertoin [5] for a proof. The extinction time of this fragmentation process is the maximum of the Brownian excursion. In particular, from Kennedy [18] we have

$$\mathbb{P}(\zeta > t) = 2\sum_{n=1}^{\infty}\big(4t^2n^2 - 1\big)\exp\big(-2t^2n^2\big), \quad t \geq 0.$$

To the best of our knowledge, this is the only fragmentation for which the law of ζ is known. It is well known that the maximum of $\mathbf{e}$ is reached at a unique point $x_* \in [0,1]$ a.s., and so the mass, $F_*(t)$, of the last fragment to survive is well defined.

There is a complete characterization of the limit in law of the rescaled fragmentation near to its extinction time.

Theorem 3.1 (Brownian fragmentation). *If O is the Brownian interval fragmentation then*

$$\frac{O((\zeta - t)^+) - x_*}{t^2} \xrightarrow{d} O_\infty \quad \text{as } t \to 0,$$

where $O_\infty = \{x \in \mathbb{R}\colon H_\infty(x) < 1\}$,

$$H_\infty(x) = R_+(x)\mathbb{1}_{\{x\geq 0\}} + R_-(-x)\mathbb{1}_{\{x<0\}}$$

and R_+ and R_- are two independent Bes(3) *processes.*

A full discussion of the topology in which the above convergence in distribution occurs is deferred until Section 5.1. A proof of this theorem is given in Uribe Bravo [23]. We give here a sketch, since a rigorous proof will be given in the wider setting of general stable fragmentations.

Sketch of proof of Theorem 3.1. We decompose the excursion $\mathbf{e}$ at its maximum ζ. Define

$$\tilde{\mathbf{e}}(x) = \begin{cases} \zeta - \mathbf{e}(x_* + x), & 0 \leq x \leq 1 - x_*, \\ \zeta - \mathbf{e}(x - 1 + x_*), & 1 - x_* < x \leq 1. \end{cases}$$

Then by Williams' decomposition for the Brownian excursion [22], Section VI.55, we have that $\tilde{\mathbf{e}}$ is again a standard Brownian excursion. Moreover, if $t < \zeta$ then

$$\begin{aligned} &t^{-2}\big(O(\zeta - t) - x_*\big) \\ &\quad = \big\{y \in \big[0, (1 - x_*)t^{-2}\big]\colon \tilde{\mathbf{e}}\big(yt^2\big) < t\big\} \cup \big\{y \in \big[-x_*t^{-2}, 0\big]\colon \tilde{\mathbf{e}}\big(1 + yt^2\big) < t\big\}. \end{aligned}$$

Now by the scaling property of Brownian motion, $(t^{-1}\tilde{\mathbf{e}}(xt^2), 0 \leq x \leq t^{-2})$ has the distribution of a Brownian excursion of length t^{-2}, which we will denote $(b^t(x), 0 \leq x \leq t^{-2})$. So the above set has the same distribution as

$$\big\{x \in \big[0, (1 - x_*)t^{-2}\big]\colon b^t(x) < 1\big\} \cup \big\{x \in \big[-x_*t^{-2}, 0\big]\colon b^t\big(t^{-2} + x\big) < 1\big\}.$$

Fix $n \in \mathbb{R}^+$. As $t \to 0$, the length of the excursion goes to ∞ and $(b^t(x), 0 \leq x \leq n) \xrightarrow{d} (R_+(x), 0 \leq x \leq n)$, where $(R_+(x), x \geq 0)$ is a 3-dimensional Bessel process started at 0 (see, for example, Theorem 0.1 of Pitman [21]). Moreover, by symmetry, $(b^t(t^{-2} - x), 0 \leq x \leq n) \xrightarrow{d} (R_-(x), 0 \leq x \leq n)$, where R_- is (in fact) an independent copy of R_+. Thus we obtain

$$t^{-2}\big(O\big((\zeta - t)^+\big) - x_*\big) \xrightarrow{d} O_\infty,$$

as claimed. □

Theorem 3.1 enables us to deduce an explicit limit law for the associated ranked fragmentation. See Corollary 4.3 for a precise statement and note that, as detailed at the end of Section 4, the passage from the convergence of open sets to that of these ranked sequences is not immediate. We also have the following limit law for the last fragment, F_*.

Corollary 3.2. *If F is the Brownian fragmentation then*

$$\frac{F_*((\zeta - t)^+)}{t^2} \xrightarrow{d} \left(\frac{2\zeta}{\pi}\right)^2 \quad \text{as } t \to 0,$$

or, equivalently,

$$t\bigl(F_*\bigl((\zeta - t)^+\bigr)\bigr)^{-1/2} \xrightarrow{d} \zeta_*,$$

where ζ_* is a size-biased version of ζ, i.e. $\mathbb{E}[f(\zeta_*)] = \mathbb{E}[\zeta f(\zeta)]/\mathbb{E}[\zeta]$ for every test function f.

Proof. Let $T_+ = \inf\{t \geq 0\colon\ R_+(t) = 1\}$ and $T_- = \inf\{t \geq 0\colon\ R_-(t) = 1\}$, where R_+ and R_- are the independent Bes(3) processes from the statement of Theorem 3.1. Then by Theorem 3.1,

$$\frac{F_*((\zeta - t)^+)}{t^2} \xrightarrow{d} T_+ + T_-.$$

By Proposition 2.1 of Biane et al. [9],

$$T_+ + T_- \stackrel{d}{=} \left(\frac{2\zeta}{\pi}\right)^2$$

and, moreover, if we define $Y = \sqrt{\frac{2}{\pi}}\zeta$ then Y satisfies

$$\mathbb{E}\bigl[f(1/Y)\bigr] = \mathbb{E}\bigl[Yf(Y)\bigr]$$

for any test function f (in particular, Y has mean 1). Hence,

$$\mathbb{E}\left[f\left(\frac{\pi}{2\zeta}\right)\right] = \sqrt{\frac{2}{\pi}}\mathbb{E}\bigl[\zeta f(\zeta)\bigr] = \mathbb{E}\bigl[\zeta f(\zeta)\bigr]/\mathbb{E}[\zeta],$$

which completes the proof. □

Remark 1. *As noted by Uribe Bravo* [23], *the random variable* $(2\zeta/\pi)^2$ *has Laplace transform* $2\lambda(\sinh\sqrt{2\lambda})^{-2}$. *He also uses another result in Biane et al.* [9] *to show that the Lebesgue measure of the set O_∞ has Laplace transform* $(\cosh\sqrt{2\lambda})^{-2}$. *Let $M(t)$ be the total mass of the fragmentation at time t, that is the Lebesgue measure of $O(t)$. Then this entails that*

$$\frac{M((\zeta - t)^+)}{t^2} \xrightarrow{d} M_\infty,$$

where M_∞ has Laplace transform $(\cosh\sqrt{2\lambda})^{-2}$.

Remark 2. *The Bes*(3) *process encodes a fragmentation process with immigration which arises naturally when studying rescaled versions of the Brownian fragmentation near $t = 0$* (*see Haas* [15]). *This is closely related to our approach*: *using Williams' decomposition of the Brownian excursion*, *we obtain results on the behavior of the fragmentation near its extinction time by studying the sets of* $\{x \in [0, 1]\colon\ \mathbf{e}(x) < t\}$ *for small t. This duality between the behavior of the fragmentation near* 0 *and near its extinction time seems to be specific to the Brownian case.*

4. General stable fragmentations

There is a natural family which generalizes the Brownian fragmentation: the *stable fragmentations*, constructed and studied by Miermont in [20]. The starting point is the stable height processes H with index $1 < \beta \leq 2$ which were introduced by Duquesne, Le Gall and Le Jan [11,19] in order to code the genealogy of continuous state branching processes with branching mechanism λ^{β} via stable trees. We do not give a definition of these processes here, since it is rather involved; full definitions will be given in the course of the next section. Here, we simply recall that it is possible to consider a normalized excursion of H, say $\mathbf{e}$, which is almost surely continuous on $[0,1]$. When $\beta = 2$, this is the normalized Brownian excursion (up to a scaling factor of $\sqrt{2}$).

Once again, let

$$O(t) := \{x \in [0,1]:\ \mathbf{e}(x) > t\}.$$

For $1 < \beta < 2$, Miermont [20] proved that the corresponding ranked fragmentation is a self-similar fragmentation of index $\alpha = 1/\beta - 1$ and erosion coefficient 0. The dislocation measure is somewhat harder to express than that of the Brownian fragmentation. Let T be a stable subordinator of Laplace exponent $\lambda^{1/\beta}$ and write $\Delta T_{[0,1]}$ for the sequence of its jumps before time 1, ranked in decreasing order. Then for any non-negative measurable function f,

$$\int_{\mathcal{S}_1^{\downarrow}} f(\mathbf{s})\nu(d\mathbf{s}) = \frac{\beta(\beta-1)\Gamma(1-\frac{1}{\beta})}{\Gamma(2-\beta)}\mathbb{E}\big[T_1 f\big(T_1^{-1}\Delta T_{[0,1]}\big)\big].$$

As we will discuss in Section 5.2, there is a unique point $x_* \in [0,1]$ at which $\mathbf{e}$ attains its maximum (this maximum is denoted ζ to be consistent with earlier notation, so that $\zeta = \mathbf{e}(x_*)$). So the size of the last fragment, F_*, is well defined for the stable fragmentations. We first state a result on the behavior of the stable height processes near their maximum.

Theorem 4.1. *Let* $\mathbf{e}$ *be a normalized excursion of the stable height process with parameter* β *and extend its definition to* $\mathbb{R}$ *by setting* $\mathbf{e}(x) = 0$ *when* $x \notin [0,1]$. *Then there exists an almost surely positive continuous self-similar function* H_∞ *on* $\mathbb{R}$, *which is symmetric in distribution* (*in the sense that* $(H_\infty(-x), x \geq 0) \stackrel{d}{=} (H_\infty(x), x \geq 0)$) *and converges to* $+\infty$ *as* $x \to +\infty$ *or* $x \to -\infty$, *and which is such that*

$$t^{-1}\big(\zeta - \mathbf{e}\big(x_* + t^{-1/\alpha}\cdot\big)\big) \xrightarrow{d} H_\infty \quad \text{as } t \to 0,$$

where $\alpha = 1/\beta - 1$. *The convergence holds with respect to the topology of uniform convergence on compacts.*

A precise definition of H_∞ is given in Section 5.3. Intuitively, we can think of it as an excursion of the height process of infinite length, centered at its "maximum" and flipped upside down.

Theorem 4.1 leads to the following generalization of Theorem 3.1.

Theorem 4.2 (Stable interval fragmentation). *Let* O *be a stable interval fragmentation with parameter* β *and consider the corresponding self-similar function* H_∞ *introduced in Theorem* 4.1. *Then*

$$t^{1/\alpha}\big(O\big((\zeta - t)^+\big) - x_*\big) \xrightarrow{d} \{x \in \mathbb{R}:\ H_\infty(x) < 1\} \quad \text{as } t \to 0.$$

The topology on the bounded open sets of $\mathbb{R}$ in which this convergence occurs will be discussed in the next section. Define

$$\mathcal{S}^{\downarrow} := \left\{\mathbf{s} = (s_1, s_2, \ldots):\ s_1 \geq s_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} s_i < \infty\right\}.$$

We endow this space with the distance

$$d_{\mathcal{S}^{\downarrow}}(\mathbf{s}, \mathbf{s}') = \sum_{i=1}^{\infty} |s_i - s_i'|.$$

We also have the following ranked counterpart of Theorem 4.2: let $F(t)$ be the decreasing sequence of lengths of the interval components of $(O(t), t \geq 0)$ and, similarly, let F_∞ be the decreasing sequence of lengths of the interval components of $\{x \in \mathbb{R}:\ H_\infty(x) < 1\}$. Then $F_\infty \in \mathcal{S}^{\downarrow}$.

Corollary 4.3 (Ranked stable fragmentation). *As* $t \to 0$,

$$t^{1/\alpha} F\big((\zeta - t)^+\big) \xrightarrow{d} F_\infty.$$

In particular, this gives the behavior of the total mass $M(t) := \sum_{i=1}^{\infty} F_i(t)$ of the fragmentation near its extinction time.

Finally, as in the Brownian case, the distribution of the limit of the size of the last fragment can be expressed in terms of a size-biased version of the height ζ.

Corollary 4.4 (Behavior of the last fragment). *As* $t \to 0$,

$$t\big(F_*\big((\zeta - t)^+\big)\big)^\alpha \xrightarrow{d} \zeta_{*^\alpha}, \tag{4.1}$$

where ζ_{*^α} *is a "*$(-1/\alpha - 1)$*-size-biased" version of* ζ, *which means that for every test-function* f,

$$\mathbb{E}\big[f(\zeta_{*^\alpha})\big] = \frac{\mathbb{E}[\zeta^{-1/\alpha-1} f(\zeta)]}{\mathbb{E}[\zeta^{-1/\alpha-1}]}. \tag{4.2}$$

Moreover,

(i) *there exist positive constants* $0 < A < B$ *such that*

$$\exp\big(-Bt^{1/(1+\alpha)}\big) \le \mathbb{P}(\zeta_{*^\alpha} \ge t) \le \exp\big(-At^{1/(1+\alpha)}\big)$$

for all t *sufficiently large*;

(ii) *for any* $q < 1 - 1/\alpha$,

$$\mathbb{P}(\zeta_{*^\alpha} \le t) \le t^q$$

for all $t \ge 0$ *sufficiently small.*

The proof of Theorem 4.1 is based on the "Williams' decomposition" of the height function H given by Abraham and Delmas [1], Theorem 3.2, and can be found in Section 6. We emphasise the fact that uniform convergence on compacts of a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ to $f : \mathbb{R} \to \mathbb{R}$ does not imply in general that the sets $\{x \in \mathbb{R}: f_n(x) < 1\}$ converge to $\{x \in \mathbb{R}: f(x) < 1\}$. Take, for example, f constant and equal to 1 and f_n constant and equal to $1 - 1/n$ (see the next section for the topology we consider on open sets of $\mathbb{R}$). Less trivial examples show that there may exist another kind of problem when passing from the convergence of functions to that of ranked sequences of lengths of interval components: take, for example, a continuous function $f : \mathbb{R} \to \mathbb{R}^+$ which is strictly larger than 1 on $\mathbb{R} \backslash [-1, 1]$ and then consider continuous functions $f_n : \mathbb{R} \to \mathbb{R}^+$ such that $f_n = f$ on $[-n, n]$, $f_n = 0$ on $[n + 1, 2n]$. Clearly, f_n converges uniformly on compacts to f, but the lengths of the longest interval components of $\{x \in \mathbb{R}: f_n(x) < 1\}$ converge to ∞.

However, we will see that the random functions we work with do not belong to the set of "problematic" counter-examples that can arise and that it will be possible to use Theorem 4.1 to get Theorem 4.2 and Corollary 4.3. Preliminary work will be done in Section 5, where an explicit construction of the limit function H_∞ via Poisson point measures is also given. Theorem 4.2 and Corollary 4.3 are proved in Section 7. Then we will see in Section 8 that the limit ζ_{*^α} arising in (4.1) is actually distributed as the length to the power α of an excursion of H, conditioned to have its maximum equal to 1. It will then be easy to check that this is a size-biased version of ζ as defined in (4.2). The bounds for the tails $\mathbb{P}(\zeta_{*^\alpha} \ge t)$ will also be proved in Section 8, as well as the following remark.

Remark. *In the Brownian case* ($\alpha = -1/2$), *the distribution of* ζ (*and consequently that of* ζ_{*^α}), *is known*; *see Section 3. We do not know the distribution of* ζ_{*^α} *explicitly when* $\alpha \in (-1/2, 0)$. *However, in this case, if we set, for* $\lambda \ge 0$,

$$\Phi(\lambda) := \mathbb{E}\big[\exp\big(-\lambda \zeta_{*^\alpha}^{1/\alpha}\big)\big] = \frac{\mathbb{E}[\exp(-\lambda \zeta^{1/\alpha}) \zeta^{-1/\alpha-1}]}{\mathbb{E}[\zeta^{-1/\alpha-1}]},$$

then it can be shown that Φ satisfies the following equation:

$$\Phi(\lambda) = \exp\biggl(-\int_{\mathbb{R}^+\times[0,\lambda^{-\alpha}]}\bigl(1-\mathrm{e}^{-(-\alpha/(\alpha+1))^{-1/\alpha}r\int_0^t(1-\Phi(v^{-1/\alpha}))v^{1/\alpha}\,\mathrm{d}v}\bigr)$$
$$\times \frac{-\alpha}{(1+\alpha)^2\Gamma(\frac{1+2\alpha}{1+\alpha})}\mathrm{e}^{-r(-\alpha t/(\alpha+1))^{1+1/\alpha}}r^{-1/(\alpha+1)}\,\mathrm{d}r\,\mathrm{d}t\biggr). \tag{4.3}$$

Finally, we recall that the almost sure logarithmic results for F_1 and F_* will be proved in Section 9.

5. Technical background

We start by detailing the topology on open sets which will give a proper meaning to the statement of Theorem 4.2. We then recall some facts about height processes and prove various useful lemmas. Finally, we introduce the decomposition result of Abraham and Delmas [1], in a form suitable for our purposes.

5.1. *Topological details*

When dealing with interval fragmentations, we will work in the set $\mathcal{O}$ of bounded open subsets of $\mathbb{R}$. This is endowed with the following distance:

$$d_{\mathcal{O}}(A,B) = \sum_{k\in\mathbb{N}} 2^{-k} d_{\mathcal{H}}\bigl(\bigl(A\cap(-k,k)\bigr)^{\mathrm{c}}\cap[-k,k],\bigl(B\cap(-k,k)\bigr)^{\mathrm{c}}\cap[-k,k]\bigr),$$

where S^{c} denotes the closed complement of $S\in\mathcal{O}$ and $d_{\mathcal{H}}$ is the Hausdorff distance on the set of compact sets of $\mathbb{R}$. For $A\neq\mathbb{R}$, let $\chi_A(x)=\inf_{y\in A^{\mathrm{c}}}|x-y|$. If we define, for $x\in[-k,k]$, $\chi_A^k(x)=\chi_{A\cap(-k,k)}(x)=\inf_{y\in(A\cap(-k,k))^{\mathrm{c}}}|x-y|$ then we also have

$$d_{\mathcal{O}}(A,B) = \sum_{k\in\mathbb{N}} 2^{-k}\sup_{x\in[-k,k]}\bigl|\chi_A^k(x)-\chi_B^k(x)\bigr|$$

(see p. 69 of Bertoin [7]). The open sets we will deal with in this paper arise as excursion intervals of continuous functions. In particular, we will need to know that if we have a sequence of continuous functions converging (in an appropriate sense) to a limit, then the corresponding open sets converge.

Consider the space $C(\mathbb{R},\mathbb{R}^+)$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}^+$. By *uniform convergence on compacts*, we mean convergence in the metric

$$d(f,g) = \sum_{k\in\mathbb{N}} 2^{-k}\Bigl(\sup_{t\in[-k,k]}\bigl|f(t)-g(t)\bigr|\wedge 1\Bigr).$$

The name is justified by the fact that convergence in d is equivalent to uniform convergence on all compact sets.

Suppose $f\in C(\mathbb{R},\mathbb{R}^+)$. We say that $a\in\mathbb{R}^+$ is a *local maximum* of f if there exist $s\in\mathbb{R}$ and $\varepsilon>0$ such that $f(s)=a$ and $\max_{s-\varepsilon\le t\le s+\varepsilon} f(t)=a$. Note that this includes the case where f is constant and equal to a on some interval, even if f never takes values smaller than a. We define a *local minimum* analogously.

Proposition 5.1. *Let $f_n:\mathbb{R}\to\mathbb{R}^+$ be a sequence of continuous functions and let $f:\mathbb{R}\to\mathbb{R}^+$ be a continuous function such that $f(0)=0$, $f(x)\to\infty$ as $x\to+\infty$ or $x\to-\infty$. Suppose also that* 1 *is not a local maximum of f and that* $\mathrm{Leb}\{x\in\mathbb{R}\colon f(x)=1\}=0$. *Define $A=\{x\in\mathbb{R}\colon f(x)<1\}$ and $A_n=\{x\in\mathbb{R}\colon f_n(x)<1\}$. Suppose now that f_n converges to f uniformly on compact subsets of $\mathbb{R}$. Then $d_{\mathcal{O}}(A_n,A)\to 0$ as $n\to\infty$.*

Define

$$g(x)=\sup\bigl\{y\le x\colon f(y)\ge 1\bigr\}$$

and

$$d(x) = \inf\{y \ge x:\ f(y) \ge 1\}.$$

Then

$$\chi_A(x) = \bigl(x - g(x)\bigr) \wedge \bigl(d(x) - x\bigr). \tag{5.1}$$

Define g_n and d_n to be the analogous quantities for f_n. The proof of the following lemma is straightforward.

Lemma 5.2. *For* $x, y \in \mathbb{R}$,

$$\bigl|\chi_A(x) - \chi_A(y)\bigr| \le |x - y|.$$

Moreover,

$$\bigl|\chi_{A_n}(x) - \chi_A(x)\bigr| \le \max\bigl\{\bigl|g_n(x) - g(x)\bigr|, \bigl|d_n(x) - d(x)\bigr|\bigr\}.$$

Proof of Proposition 5.1. It suffices to prove that $d_{\mathcal{H}}((A_n \cap (-1,1))^{\mathrm{c}} \cap [-1,1], (A \cap (-1,1))^{\mathrm{c}} \cap [-1,1]) \to 0$ as $n \to \infty$ for $f, f_n : [-1,1] \to \mathbb{R}^+$ such that $f(0) = 0$, 1 is not a local maximum of f, $\mathrm{Leb}\{x \in [-1,1]:\ f(x) = 1\} = 0$ and $f_n \to f$ uniformly. In other words, we need to show that $\sup_{x \in [-1,1]} |\chi^1_{A_n}(x) - \chi^1_A(x)| \to 0$ as $n \to \infty$. Note that the appropriate definitions of $g(x)$ and $d(x)$ in order to make (5.1) true for χ^1_A are

$$g(x) = \sup\{y \le x:\ f(y) \ge 1\} \vee -1$$

and

$$d(x) = \inf\{y \ge x:\ f(y) \ge 1\} \wedge 1,$$

and we adopt these definitions for the rest of the proof.

Let $\varepsilon > 0$. For $r > 0$ let

$$E^{\uparrow}_r = \bigl\{x \in (-1,1):\ x \in (a,b) \text{ such that } f(y) > 1, \forall y \in (a,b), |b-a| > r\bigr\},$$
$$E^{\downarrow}_r = \bigl\{x \in (-1,1):\ x \in (a,b) \text{ such that } f(y) < 1, \forall y \in (a,b), |b-a| > r\bigr\}.$$

These are the collections of excursion intervals of length exceeding r above and below 1. Take $0 < \delta < \varepsilon/2$ small enough that $\mathrm{Leb}(E^{\uparrow}_\delta \cup E^{\downarrow}_\delta) > 2 - \varepsilon/2$ (we can do this since $\mathrm{Leb}\{x \in [-1,1]:\ f(x) = 1\} = 0$). Set $R = [-1,1] \setminus (E^{\uparrow}_\delta \cup E^{\downarrow}_\delta)$. Each of $E^{\uparrow}_\delta$ and $E^{\downarrow}_\delta$ is composed of a finite number of open intervals.

- We will first deal with $E^{\uparrow}_\delta$. On this set, $\chi^1_A(x) = 0$. Take hereafter $0 < \eta < \delta/6$ and let

$$E^{\uparrow}_{\delta,\eta} = \bigl\{x \in E^{\uparrow}_\delta:\ (x - \eta, x + \eta) \subseteq E^{\uparrow}_\delta\bigr\}.$$

Then $\inf_{x \in E^{\uparrow}_{\delta,\eta}} f(x) > 1$. Since $f_n \to f$ uniformly, it follows that there exists n_1 such that $f_n(x) > 1$ for all $x \in E^{\uparrow}_{\delta,\eta}$ whenever $n \ge n_1$. Then for $n \ge n_1$, we have $\chi^1_{A_n}(x) = 0$ for all $x \in E^{\uparrow}_{\delta,\eta}$. Since $|\chi^1_{A_n}(x) - \chi^1_{A_n}(y)| \le |x - y|$, it follows that $\chi^1_{A_n}(x) < \eta$ for all $x \in E^{\uparrow}_\delta$. So $\sup_{x \in E^{\uparrow}_\delta} |\chi^1_A(x) - \chi^1_{A_n}(x)| < \eta$ whenever $n \ge n_1$.

- Now turn to $E^{\downarrow}_\delta$. As before, define

$$E^{\downarrow}_{\delta,\eta} = \bigl\{x \in E^{\downarrow}_\delta:\ (x - \eta, x + \eta) \subseteq E^{\downarrow}_\delta\bigr\}.$$

Then $\sup_{x \in E^{\downarrow}_{\delta,\eta}} f(x) < 1$. Since $f_n \to f$ uniformly, it follows that there exists n_2 such that $f_n(x) < 1$ for all $x \in E^{\downarrow}_{\delta,\eta}$ whenever $n \ge n_2$. Now, for each excursion below 1, there exists a left end-point g and a right end-point d. For all x

in the same excursion, $g(x)=g$ and $d(x)=d$. Suppose first that we have $g\neq -1$, $d\neq 1$ (in this case we say that the excursion *does not touch the boundary*). Since 1 is not a local maximum of f, there must exist $z_g<g$ and $z_d>d$ such that $|z_g-g|<\eta$, $|z_d-d|<\eta$, $f(z_g)>1$ and $f(z_d)>1$.

Suppose there are N_δ excursions below 1 of length greater than δ which do not touch the boundary. To excursion i there corresponds a left end-point g_i, a right end-point d_i and points z_{g_i}, z_{d_i}, $1\le i\le N_\delta$. Write

$$\tilde{E}_\delta^{\downarrow}=\bigcup_{i=1}^{N_\delta}(g_i,d_i)\quad\text{and}\quad \tilde{E}_{\delta,\eta}^{\downarrow}=\tilde{E}_\delta^{\downarrow}\cap E_{\delta,\eta}^{\downarrow}=\bigcup_{i=1}^{N_\delta}(g_i+\eta,d_i-\eta).$$

Then $\min_{1\le i\le N_\delta}(f(z_{g_i})\wedge f(z_{d_i}))>1$. Since $f_n\to f$ uniformly, there exists n_3 such that $\min_{1\le i\le N_\delta}(f_n(z_{g_i})\wedge f_n(z_{d_i}))>1$ for all $n\ge n_3$. So for $n\ge n_2\vee n_3$ and any $x\in\tilde{E}_{\delta,\eta}^{\downarrow}$, by the intermediate value theorem, there exists at least one point $a_n(x)\in(g(x)-\eta,g(x)+\eta)$ such that $f_n(a_n(x))=1$ and at least one point $b_n(x)\in(d(x)-\eta,d(x)+\eta)$ such that $f_n(b_n(x))=1$. Since $g(x)$ and $d(x)$ are constant on excursion intervals, it follows that $\sup_{x\in\tilde{E}_{\delta,\eta}^{\downarrow}}|g_n(x)-g(x)|<\eta$ and $\sup_{x\in\tilde{E}_{\delta,\eta}^{\downarrow}}|d_n(x)-d(x)|<\eta$ for $n\ge n_2\vee n_3$. Hence, by Lemma 5.2,

$$\sup_{x\in\tilde{E}_{\delta,\eta}^{\downarrow}}\left|\chi_A^1(x)-\chi_{A_n}^1(x)\right|<\eta$$

whenever $n\ge n_2\vee n_3$. Since $|\chi_A^1(x)-\chi_A^1(y)|\le|x-y|$ and $|\chi_{A_n}^1(x)-\chi_{A_n}^1(y)|\le|x-y|$, by using the triangle inequality we obtain that

$$\sup_{x\in\tilde{E}_{\delta}^{\downarrow}}\left|\chi_A^1(x)-\chi_{A_n}^1(x)\right|<3\eta.$$

It remains to deal with any excursions in $E_\delta^{\downarrow}$ which touch the boundary. Clearly, there is at most one excursion in $E_\delta^{\downarrow}$ touching the left boundary and at most one excursion touching the right boundary. For these excursions, we can argue as before at the non-boundary end-points. At the boundary end-points, the argument is, in fact, easier since we have (by construction) $\chi_A^1(-1)=\chi_{A_n}^1(-1)=0$ and $\chi_A^1(1)=\chi_{A_n}^1(1)=0$. So, there exists n_4 such that for all $n\ge n_2\vee n_3\vee n_4$,

$$\sup_{x\in E_{\delta}^{\downarrow}}\left|\chi_A^1(x)-\chi_{A_n}^1(x)\right|<3\eta.$$

• For any $x\in R$, we have $\chi_A^1(x)\le\delta/2$. Moreover, since $\mathrm{Leb}(E_\delta^{\uparrow}\cup E_\delta^{\downarrow})>2-\varepsilon/2$, there must exist a point $z(x)\in R$ such that $|z(x)-x|<\varepsilon/2$ which is the end-point of an excursion interval (above or below 1) of length exceeding δ. So for all $x\in R$ and all n we have

$$\begin{aligned}\left|\chi_A^1(x)-\chi_{A_n}^1(x)\right|&\le\left|\chi_A^1(x)\right|+\left|\chi_{A_n}^1(x)-\chi_{A_n}^1\big(z(x)\big)\right|+\left|\chi_{A_n}^1\big(z(x)\big)-\chi_A^1\big(z(x)\big)\right|\\&\le\delta/2+\left|x-z(x)\right|+\sup_{y\in E_\delta^{\uparrow}\cup E_\delta^{\downarrow}}\left|\chi_{A_n}^1(y)-\chi_A^1(y)\right|\\&<\delta/2+\varepsilon/2+\sup_{y\in E_\delta^{\uparrow}\cup E_\delta^{\downarrow}}\left|\chi_{A_n}^1(y)-\chi_A^1(y)\right|\end{aligned}$$

(note that at the second inequality we use the continuity of χ_A^1 and $\chi_{A_n}^1$ and the fact that $\chi_A^1(z(x))=0$).

• Finally, let $n_0=n_1\vee n_2\vee n_3\vee n_4$. Then since $\eta<\delta/6$ and $\delta<\varepsilon/2$, for $n\ge n_0$ we have

$$\sup_{x\in[-1,1]}\left|\chi_A^1(x)-\chi_{A_n}^1(x)\right|<\varepsilon.$$

The result follows. □

The following lemma will be used implicitly in Section 6.

Lemma 5.3. *Suppose that $a > 0$, $\alpha \in \mathbb{R}$ and that $f : \mathbb{R} \to \mathbb{R}^+$ is a continuous function. Let $g(t) = a^\alpha f(at)$. Then the function $(a, f) \mapsto g$ is continuous for the topology of uniform convergence on compacts.*

Proof. Let $f_n : \mathbb{R} \to \mathbb{R}^+$ be a sequence of continuous functions with $f_n \to f$ uniformly on compacts. Suppose that a_n is a sequence of reals with $a_n \to a > 0$. Suppose $K \subseteq \mathbb{R}$ is a compact set. Then we have

$$\begin{aligned}
&\sup_{t\in K}\left|a^\alpha f(at) - a_n^\alpha f_n(a_n t)\right| \\
&\quad\le \sup_{t\in K}\left|a^\alpha f(at) - a_n^\alpha f(at)\right| + \sup_{t\in K}\left|a_n^\alpha f(at) - a_n^\alpha f(a_n t)\right| + \sup_{t\in K}\left|a_n^\alpha f(a_n t) - a_n^\alpha f_n(a_n t)\right| \\
&\quad\le \sup_{t\in aK}\left|f(t)\right|\left|a^\alpha - a_n^\alpha\right| + a_n^\alpha \sup_{t\in K}\left|f(at) - f(a_n t)\right| + a_n^\alpha \sup_{t\in K}\left|f(a_n t) - f_n(a_n t)\right|.
\end{aligned}$$

The set aK is compact and so f is bounded on it; it follows that the first term converges to 0. Take $0 < \varepsilon < a$. Since $a_n \to a$, there exists n sufficiently large that $|a_n - a| < \varepsilon$. Define $\tilde{K} = \{bt\colon t \in K, b \in [a - \varepsilon, a + \varepsilon]\}$. Then $\tilde{K}$ is also a compact set. The second term converges to 0 because f is uniformly continuous on $\tilde{K}$. The third term is bounded above by $((a-\varepsilon)^\alpha \vee (a+\varepsilon)^\alpha)\sup_{t\in\tilde{K}}|f(t) - f_n(t)|$ and so, since $f_n \to f$ uniformly on compacts, this converges to 0. □

Finally, we will need the following lemma in Section 7.

Lemma 5.4. *Let $f : \mathbb{R} \to \mathbb{R}^+$ be a continuous function such that*

$$\mathrm{Leb}\bigl\{x \in \mathbb{R}\colon f(x) = 1\bigr\} = 0.$$

Suppose $f_n : \mathbb{R} \to \mathbb{R}^+$ is a sequence of continuous functions that converges to f uniformly on compacts. Then, for all $K > 0$, as $n \to \infty$,

$$\mathrm{Leb}\bigl\{x \in [-K, K]\colon f_n(x) < 1\bigr\} \to \mathrm{Leb}\bigl\{x \in [-K, K]\colon f(x) < 1\bigr\}.$$

Proof. Let $K > 0$ and fix $\varepsilon > 0$. For all n sufficiently large

$$f(x) - \varepsilon \le f_n(x) \le f(x) + \varepsilon \quad \text{for all } x \in [-K, K],$$

hence

$$\begin{aligned}
\mathrm{Leb}\bigl\{x \in [-K, K]\colon f(x) < 1 - \varepsilon\bigr\} &\le \mathrm{Leb}\bigl\{x \in [-K, K]\colon f_n(x) < 1\bigr\} \\
&\le \mathrm{Leb}\bigl\{x \in [-K, K]\colon f(x) < 1 + \varepsilon\bigr\}.
\end{aligned}$$

When $\varepsilon \to 0$, the left-hand side of this inequality converges to $\mathrm{Leb}\{x \in [-K, K]\colon f(x) < 1\}$ and the right-hand side to $\mathrm{Leb}\{x \in [-K, K]\colon f(x) \le 1\}$, which are equal by assumption. □

5.2. *Height processes*

Here, we define the stable height process and recall some of its properties. We refer to Le Gall and Le Jan [19] and Duquesne and Le Gall [11] for background. (All of the definitions and results stated without proof below may be found in these references.)

Suppose that X is a spectrally positive stable Lévy process with Laplace exponent λ^β, $\beta \in (1, 2]$. That is, $\mathbb{E}[\exp(-\lambda X_t)] = \exp(t\lambda^\beta)$ for all $\lambda, t \ge 0$ and, therefore, if $\beta \in (1, 2)$, the Lévy measure of X is $\beta(\beta-1)(\Gamma(2-\beta))^{-1}x^{-\beta-1}\,\mathrm{d}x$, $x > 0$. Let $I(t) := \inf_{0\le s\le t} X(s)$ be the infimum process of X. For each $t > 0$, consider the time-reversed process defined for $0 \le s < t$ by

$$\hat{X}^{(t)}(s) := X(t) - X\bigl((t-s)-\bigr),$$

and let $(\hat{S}^{(t)}(s), 0 \le s \le t)$ be its supremum, that is $\hat{S}^{(t)}(s) = \sup_{u \le s} \hat{X}^{(t)}(u)$. Then the height process $H(t)$ is defined to be the local time at 0 of $\hat{S}^{(t)} - \hat{X}^{(t)}$.

It can be shown that the process H possesses a continuous version (Theorem 1.4.3 of [11]), which we will implicitly consider in the following. The scaling property of X is inherited by H (see Section 3.3 of [11]) as follows: for all $a > 0$,

$$\left(a^{1/\beta-1} H(ax), x \ge 0\right) \stackrel{d}{=} \left(H(x), x \ge 0\right).$$

When $\beta = 2$, the height process is, up to a scaling factor, a reflected Brownian motion.

The excursion measure of $X - I$ away from 0 is denoted by $\mathbb{N}$. *In the following, we work under this excursion measure*. Let $\mathcal{E}$ be the space of excursions; that is, continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(0) = 0$, $\inf\{t > 0\colon f(t) = 0\} < \infty$ and if $f(s) = 0$ for some $s > 0$ then $f(t) = 0$ for all $t > s$. The lifetime of $H \in \mathcal{E}$ is then denoted by σ, that is

$$\sigma := \inf\left\{x > 0\colon H(x) = 0\right\}.$$

We define its maximum to be

$$H_{\max} := \max_{x \in [0,\sigma]} H(x).$$

We will also deal with (regular versions of) the probability measures $\mathbb{N}(\cdot|\sigma = v)$, $v > 0$ and $\mathbb{N}(\cdot|H_{\max} = m)$, $m > 0$. The following proposition is implicit in Section 3 of Abraham and Delmas [1] and is also a consequence of Theorem 9.1(ii) below.

Proposition 5.5. *For any $v > 0$, under $\mathbb{N}(\cdot|\sigma = v)$ there exists an almost surely unique point $x_{\max}$ at which H attains its maximum, that is*

$$x_{\max} := \inf\left\{x \in [0, \sigma]\colon H(x) = H_{\max}\right\}.$$

Note that $\mathbf{e}$, ζ and x_* (see Section 4) have the distributions of H, $H_{\max}$ and $x_{\max}$ respectively under $\mathbb{N}(\cdot|\sigma = 1)$.
First we note the tails of certain measures.

Proposition 5.6. *For all $m > 0$,*

$$\mathbb{N}(H_{\max} > m) = (\beta - 1)^{1/(1-\beta)} m^{1/(1-\beta)}, \tag{5.2}$$

$$\mathbb{N}(\sigma > m) = \left(\Gamma(1 - 1/\beta)\right)^{-1} m^{-1/\beta}. \tag{5.3}$$

Proof. For (5.2) see, for example, Corollary 1.4.2 and Section 2.7 of Duquesne and Le Gall [11]. It is well known (Theorem 1, Section VII.1 of [3]) that the right inverse process $J = (J(t), t \ge 0)$ of I defined by $J(t) := \inf\{u \ge 0\colon -I(u) > t\}$ is a stable subordinator with a Lévy measure $(\beta\Gamma(1 - 1/\beta))^{-1} x^{-1-1/\beta}\,\mathrm{d}x$, $x > 0$. Since $\mathbb{N}(\sigma \in \mathrm{d}m)$ is equal to this Lévy measure, (5.3) follows. □

Recall that $\alpha = 1/\beta - 1$. We will, henceforth, primarily work in terms of α rather than β. We will make extensive use of the scaling properties of the height function under the excursion measure. For $m > 0$, let $H^{[m]}(x) = m^{-\alpha} H(x/m)$. Note that if H has lifetime σ then $H^{[m]}$ has lifetime $m\sigma$ and maximum height $m^{-\alpha} H_{\max}$. Note also that $(H^{[m]})^{[a]} = H^{[ma]}$, for all $m, a > 0$. The following proposition, which summarizes results from Section 3.3 of Duquesne and Le Gall [11], gives a version of the scaling property for the height process conditioned on its lifetime.

Proposition 5.7. *For any test function $f : \mathcal{E} \to \mathbb{R}$ and any $x, m > 0$, we have*

$$\mathbb{N}\left[f\left(H^{[m]}\right)|\sigma = x/m\right] = \mathbb{N}\left[f(H)|\sigma = x\right].$$

Moreover, for any $\eta > 0$,

$$\mathbb{N}\big[f(H)|\sigma = x\big] = \mathbb{N}\big[f\big(H^{[x/\sigma]}\big)|\sigma > \eta\big].$$

We now state two lemmas that will play an essential role in the proof of Theorem 4.1. The first lemma gives the scaling property for H conditioned on its maximum.

Lemma 5.8. *Let* $f : \mathcal{E} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *be any test function. For all* $m > 0$,

$$\mathbb{N}\big[f\big(H^{[m]}, m\sigma, m^{-\alpha} H_{\max}\big)\big] = m^{1+\alpha}\mathbb{N}\big[f(H, \sigma, H_{\max})\big].$$

Moreover, for all $x, a > 0$,

$$\mathbb{N}\big[f\big(H^{[x/\sigma]}, \sigma, H_{\max}\big)\big] = a^{-1-\alpha}\mathbb{N}\big[f\big(H^{[x/\sigma]}, a\sigma, a^{-\alpha} H_{\max}\big)\big].$$

In particular, for any test function $g : \mathcal{E} \times \mathbb{R}^+ \to \mathbb{R}$,

$$\mathbb{N}\big[g\big(H^{[x]}, \sigma\big)|H_{\max} = u\big] = \mathbb{N}\big[g\big(H, x^{-1}\sigma\big)|H_{\max} = x^{-\alpha} u\big]$$

and

$$\mathbb{N}\big[g\big(H^{[x/\sigma]}, \sigma\big)|H_{\max} = u\big] = \mathbb{N}\big[g\big(H^{[x/\sigma]}, x^{-1}\sigma\big)|H_{\max} = x^{-\alpha} u\big].$$

Proof. By conditioning on the value of σ and the tails in Proposition 5.6, we have

$$\mathbb{N}\big[f\big(H^{[m]}, m\sigma, m^{-\alpha} H_{\max}\big)\big] = c\int_{\mathbb{R}^+} \mathbb{N}\big[f\big(H^{[m]}, mb, m^{-\alpha} H_{\max}\big)|\sigma = b\big]b^{-\alpha-2}\,\mathrm{d}b$$

for some constant c. By Proposition 5.7,

$$\int_{\mathbb{R}^+} \mathbb{N}\big[f\big(H^{[m]}, mb, m^{-\alpha} H_{\max}\big)|\sigma = b\big]b^{-\alpha-2}\,\mathrm{d}b = \int_{\mathbb{R}^+} \mathbb{N}\big[f(H, mb, H_{\max})|\sigma = mb\big]b^{-\alpha-2}\,\mathrm{d}b.$$

Changing variable with $a = mb$ gives

$$cm^{1+\alpha}\int_{\mathbb{R}^+} \mathbb{N}\big[f(H, a, H_{\max})|\sigma = a\big]a^{-\alpha-2}\,\mathrm{d}a = m^{1+\alpha}\mathbb{N}\big[f(H, \sigma, H_{\max})\big],$$

which yields the first statement. The second statement is a consequence of the first and the conditioned statements follow easily. □

Finally, we relate the law of H conditioned on its lifetime and the law of H conditioned on its maximum. For the rest of the paper, c denotes a positive finite constant that may vary from line to line.

Lemma 5.9. *For all test functions* $f : \mathcal{E} \to \mathbb{R}$ *and all* $x > 0$,

$$\mathbb{N}\big[f(H)|\sigma = x\big] = \Gamma(-\alpha)\left(\frac{-\alpha}{\alpha+1}\right)^{1/\alpha}\int_{\mathbb{R}^+} \mathbb{N}\big[f\big(H^{[x/\sigma]}\big)\mathbb{1}_{\{\sigma > x\}}|H_{\max} = x^{-\alpha} u\big]u^{1/\alpha}\,\mathrm{d}u.$$

Moreover, for all non-negative test functions $g : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathbb{N}\big[g\big(\sigma^{\alpha}\big)|H_{\max} = 1\big] = \frac{\mathbb{N}[g(H_{\max})H_{\max}^{-1/\alpha-1}|\sigma = 1]}{\mathbb{N}[H_{\max}^{-1/\alpha-1}|\sigma = 1]}.$$

Proof. Taking $\eta = 1$ in the second statement of Proposition 5.7, we see that $\mathbb{N}[f(H)|\sigma = x]$ is equal to $\mathbb{N}[f(H^{[x/\sigma]})\mathbb{1}_{\{\sigma>1\}}]/\mathbb{N}(\sigma > 1)$. Then, conditioning on the value of $H_{\max}$ and using (5.2), we have

$$\mathbb{N}\big[f\big(H^{[x/\sigma]}\big)\mathbb{1}_{\{\sigma>1\}}\big] = \left(\frac{-\alpha}{\alpha+1}\right)^{1/\alpha} \int_{\mathbb{R}^+} \mathbb{N}\big[f\big(H^{[x/\sigma]}\big)\mathbb{1}_{\{\sigma>1\}}|H_{\max} = u\big]u^{1/\alpha}\,\mathrm{d}u.$$

By the final statement of Lemma 5.8, we see that this is equal to

$$\left(\frac{-\alpha}{\alpha+1}\right)^{1/\alpha} \int_{\mathbb{R}^+} \mathbb{N}\big[f\big(H^{[x/\sigma]}\big)\mathbb{1}_{\{\sigma>x\}}|H_{\max} = x^{-\alpha}u\big]u^{1/\alpha}\,\mathrm{d}u.$$

The first statement follows by noting from (5.3) that $\mathbb{N}(\sigma > 1) = \Gamma(-\alpha)^{-1}$.

In order to prove the second statement in the lemma, note that by the first statement we have

$$\mathbb{N}\big[g(H_{\max})H_{\max}^{-1/\alpha-1}|\sigma = 1\big] = c\int_{\mathbb{R}^+} \mathbb{N}\big[g\big(\sigma^\alpha H_{\max}\big)\sigma^{-1-\alpha}H_{\max}^{-1/\alpha-1}\mathbb{1}_{\{\sigma>1\}}|H_{\max} = u\big]u^{1/\alpha}\,\mathrm{d}u.$$

By Lemma 5.8, we have that

$$\mathbb{N}\big[g\big(\sigma^\alpha H_{\max}\big)\mathbb{1}_{\{\sigma>1\}}\sigma^{-1-\alpha}H_{\max}^{-1/\alpha-1}|H_{\max} = u\big] = \mathbb{N}\big[g\big(\sigma^\alpha\big)\sigma^{-1-\alpha}\mathbb{1}_{\{\sigma>u^{1/\alpha}\}}|H_{\max} = 1\big]$$

for all $u > 0$. Hence, by Fubini's theorem,

$$\begin{aligned}\mathbb{N}\big[g(H_{\max})H_{\max}^{-1/\alpha-1}|\sigma = 1\big] &= c\mathbb{N}\left[g\big(\sigma^\alpha\big)\sigma^{-1-\alpha}\int_{\sigma^\alpha}^\infty u^{1/\alpha}\,\mathrm{d}u\,\middle|\,H_{\max} = 1\right]\\ &= c\mathbb{N}\big[g\big(\sigma^\alpha\big)|H_{\max} = 1\big].\end{aligned}$$

The result follows. □

5.3. *Williams' decomposition*

Except in the Brownian case, the height process is not Markov. However, a version of it can be reconstructed from a measure-valued Markov process ρ, called the *exploration process* (see Le Gall and Le Jan [19] or Section 0.3 of Duquesne and Le Gall [11] for a definition), by taking $H(t)$ to be the supremum of the topological support of $\rho(t)$. Abraham and Delmas [1] give a decomposition of the height process H (more precisely, of the continuum random tree coded by this height process) in terms of this exploration process. This decomposition is the analogue of Williams' decomposition for the Brownian excursion discussed earlier. We recall their result below but we state it in terms of the height process. This is somewhat less precise, but is sufficient for our purposes and easier to state.

Notation

Take an arbitrary point measure $\mu = \sum_{i\geq 1} a_i\delta_{t_i}$ on $(0,\infty)$. Now consider, for each $i \geq 1$, an independent Poisson point process on $\mathbb{R}^+ \times \mathcal{E}$ of intensity $\mathrm{d}u\,\mathbb{N}(\cdot, H_{\max} \leq t_i)$, having points $\{(u_j^{(i)}, f_j^{(i)}), j \geq 1\}$. For each $i \geq 1$, define a subordinator $\tau^{(i)}$ by

$$\tau^{(i)}(u) = \sum_{j:u_j^{(i)}\leq u} \sigma\big(f_j^{(i)}\big), \quad u \geq 0,$$

where for any excursion f, $\sigma(f)$ denotes its length. Note that the Lévy measure of this subordinator integrates the function $1 \wedge x$ on $\mathbb{R}^+$ since $\mathbb{N}[1 \wedge \sigma, H_{\max} \leq t_i] \leq \mathbb{N}[1 \wedge \sigma]$, which is finite by Proposition 5.6. Hence $\tau^{(i)}(u) < \infty$ for all $u \geq 0$ a.s.

We will need the function $H^{(i)}$, defined on $[0, \tau^{(i)}(a_i)]$ by

$$H^{(i)}(x) := \sum_{j:u_j^{(i)} \le a_i} f_j^{(i)}\big(x - \tau^{(i)}\big(u_j^{(i)}-\big)\big)\mathbb{1}_{\{\tau^{(i)}(u_j^{(i)}-) < x \le \tau^{(i)}(u_j^{(i)})\}}. \tag{5.4}$$

The process $H^{(i)}$ can be viewed as a collection of excursions of H conditioned to have heights lower than t_i and such that the local time of $H^{(i)}$ at 0 is a_i.

We will now use this set-up in the situation of interest. Let $\rho := \sum_{i\ge1} \delta_{(v_i, r_i, t_i)}$ be a Poisson point measure on $[0, 1] \times \mathbb{R}^+\backslash\{0\} \times \mathbb{R}^+\backslash\{0\}$ with intensity

$$\frac{\beta(\beta-1)}{\Gamma(2-\beta)} \exp\big(-rc_\beta t^{1/(1-\beta)}\big)r^{-\beta}\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}t,$$

where $c_\beta = (\beta-1)^{1/(1-\beta)}$. Conditionally on the point measures $\sum_{i\ge1}(1-v_i)r_i\delta_{t_i}$ and $\sum_{i\ge1} v_i r_i \delta_{t_i}$, use them to define two independent collections of independent processes $\{H_+^{(i)}, i \ge 1\}$ and $\{H_-^{(i)}, i \ge 1\}$ respectively, as in (5.4). We now glue these together in order to define a function H_∞ on $\mathbb{R}$. For $u \ge 0$, set

$$\eta^+(u) := \sum_{i:t_i\le u} \tau_+^{(i)}\big((1-v_i)r_i\big), \qquad \eta^-(u) := \sum_{i:t_i\le u} \tau_-^{(i)}(v_i r_i).$$

It is not obvious that $\eta^+(u)$ and $\eta^-(u)$ are almost surely finite for all $u \ge 0$. This is a consequence of the forthcoming Theorem 5.10. It can also be proved via Campbell's theorem for Poisson point processes. Now set

$$\begin{aligned} H_\infty(x) := {}& \left(\sum_{i\ge1}\big[t_i - H_+^{(i)}\big(x - \eta^+(t_i-)\big)\big]\mathbb{1}_{\{\eta^+(t_i-)<x\le\eta^+(t_i)\}}\right)\mathbb{1}_{\{x\ge0\}} \\ & + \left(\sum_{i\ge1}\big[t_i - H_-^{(i)}\big(-x - \eta^-(t_i-)\big)\big]\mathbb{1}_{\{\eta^-(t_i-)<-x\le\eta^-(t_i)\}}\right)\mathbb{1}_{\{x<0\}}. \end{aligned}$$

Note that almost surely for all $u > 0$,

$$\eta^+(u) = \inf\big\{x > 0\colon\ H_\infty(x) > u\big\}$$

or, equivalently, the right inverse of η^+ is equal to the supremum process

$$\Big(\sup_{0\le y\le x} H_\infty(y), x \ge 0\Big).$$

Symmetrically,

$$\eta^-(u) = \sup\big\{x < 0\colon\ H_\infty(x) > u\big\}$$

and the right inverse of η^- is the supremum process

$$\Big(\sup_{-x\le y\le 0} H_\infty(y), x \ge 0\Big).$$

Roughly, the construction of H_∞ works as follows: conditional on the supremum (and for each value of the supremum), we take a collection of independent excursions below that supremum which are conditioned not to go below the x-axis, and which have total local time r_i for an appropriate value r_i. This local time is split with a uniform random variable into a proportion v_i which goes to the left of the origin and a proportion $(1-v_i)$ which goes to the right of the origin. See Fig. 3 for an idea of what H_∞ looks like (note that the times t_i should be dense on the vertical axis). Note that we may always choose a continuous version of H_∞. Roughly, this is because the processes η^+ and η^- almost surely have no intervals where they are constant. This implies that the one-sided suprema of H_∞ are continuous. Finally, the excursions that we glue beneath these suprema can be assumed to be continuous.

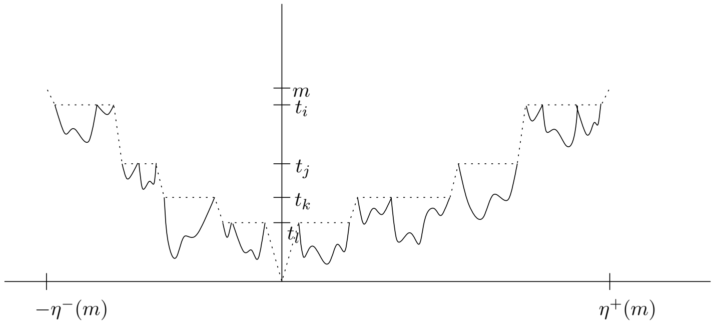

Fig. 3. Schematic drawing of H_∞, with the one-sided running suprema indicated by dotted lines.

The following theorem says that if we flip this picture over we obtain an excursion of the height process which is conditioned on its maximum height. The proof follows easily from Lemma 3.1 and Theorem 3.2 of Abraham and Delmas [1].

Theorem 5.10 (Abraham and Delmas [1] (Stable case, $1 < \beta < 2$)). *For all $m > 0$,*

$$\big(m - H_\infty\big(x - \eta^-(m)\big), 0 \le x \le \eta^-(m) + \eta^+(m)\big) \stackrel{d}{=} \big(H(x), 0 \le x \le \sigma\big) \quad \textit{under } \mathbb{N}(\cdot | H_{\max} = m).$$

We note that, in particular,

$$\eta(m) := \eta^-(m) + \eta^+(m)$$

has the distribution of σ under $\mathbb{N}(\cdot|H_{\max} = m)$ and that $\eta^-(m)$ has the distribution of $x_{\max}$ under the same measure.

Theorem 3.2 of Abraham and Delmas [1] also entails a Brownian counterpart of this result. Much of the complication in the description of H_∞ for $1 < \beta < 2$ came from the fact that the stable height process has repeated local minima. Here the construction of H_∞ is simpler since local minima are unique in the Brownian excursion. Let $\sum_{i\ge1} \delta_{(t_i,h_i)}$ and $\sum_{i\ge1} \delta_{(u_i,f_i)}$ be two independent Poisson point measures on $\mathbb{R}_+ \times \mathcal{E}$, both with intensity $\mathrm{d}t\, \mathbb{N}(\cdot, H_{\max} \le t)$. For $s \ge 0$, set

$$\eta^+(s) := \sum_{i:t_i \le s} \sigma(h_i), \qquad \eta^-(s) := \sum_{i:u_i \le s} \sigma(f_i).$$

Finally, set

$$\begin{aligned} H_\infty(x) := {} & \Big(\sum_{i\ge1} \big[t_i - h_i\big(x - \eta^+(t_i-)\big)\big] \mathbb{1}_{\{\eta^+(t_i-) < x \le \eta^+(t_i)\}}\Big) \mathbb{1}_{\{x\ge0\}} \\ & + \Big(\sum_{i\ge1} \big[u_i - f_i\big(-x - \eta^-(u_i-)\big)\big] \mathbb{1}_{\{\eta^-(u_i-) < -x \le \eta^-(u_i)\}}\Big) \mathbb{1}_{\{x<0\}}. \end{aligned}$$

Theorem 5.11 (Abraham and Delmas [1], Williams [22] (Brownian case, $\beta = 2$)). *For all $m > 0$,*

$$\big(m - H_\infty\big(x - \eta^-(m)\big), 0 \le x \le \eta^-(m) + \eta^+(m)\big) \stackrel{d}{=} \big(H(x), 0 \le x \le \sigma\big) \quad \textit{under } \mathbb{N}(\cdot | H_{\max} = m).$$

In the sequel, we will need various properties of the processes $(H_\infty(x), x \in \mathbb{R})$ and $(\eta(m), m \ge 0)$. We start with some properties of H_∞.

Lemma 5.12. *For all $\beta \in (1, 2]$, the process H_∞ is self-similar: for all $m \in \mathbb{R}$,*

$$\big(m^\alpha H_\infty(mx), x \in \mathbb{R}\big) \stackrel{d}{=} \big(H_\infty(x), x \in \mathbb{R}\big).$$

Moreover, with probability 1, $H_\infty(x) \to +\infty$ *as* $x \to +\infty$ *or* $x \to -\infty$.

Proof. The self-similarity property is an easy consequence of the identity in law stated in Theorems 5.10 and 5.11 and of the scaling property of H conditioned on its maximum (Lemma 5.8).

Now, we will show that for each $A > 0$, a.s. $H_\infty(t) > A$ for t sufficiently large. This will imply that $\liminf_{x\to+\infty} H_\infty(x)$ is almost surely larger than A for all $A > 0$, and hence is infinite. So consider $A > 0$ and recall the construction of H_∞ from Poisson point measures in the stable cases $1 < \beta < 2$ (the proof can be done in a similar way in the Brownian case). By construction of H_∞, we will have that $H_\infty(t) > A$ for t sufficiently large if and only if, conditional on the Poisson point measure $\sum_{i\geq 1}(1-v_i)r_i\delta_{t_i}$, the number of i such that $t_i > A+1$ and

$$A_i := \max_{x\in[0,\tau_+^{(i)}((1-v_i)r_i)]} H_+^{(i)}(x) \geq t_i - A$$

is almost surely finite. By the Borel–Cantelli lemma, it is therefore sufficient to check that the sum $\sum_{t_i>A+1}\mathbb{P}(A_i \geq t_i - A | t_i, (1-v_i)r_i)$ is almost surely finite. By standard rules of Poisson measures,

$$\begin{aligned}\mathbb{P}\bigl(A_i \geq t_i - A | t_i, (1-v_i)r_i\bigr) &= 1 - \exp\bigl(-(1-v_i)r_i\mathbb{N}[t_i - A \leq H_{\max} \leq t_i]\bigr)\\ &= 1-\exp\bigl(-(1-v_i)r_i c_\beta\bigl((t_i-A)^{1/(1-\beta)} - t_i^{1/(1-\beta)}\bigr)\bigr)\\ &\leq (1-v_i)r_i c_\beta\bigl((t_i-A)^{1/(1-\beta)} - t_i^{1/(1-\beta)}\bigr).\end{aligned}$$

Finally,

$$\begin{aligned}&\mathbb{E}\Bigl[\sum_{t_i>A+1}(1-v_i)r_i\bigl((t_i-A)^{1/(1-\beta)} - t_i^{1/(1-\beta)}\bigr)\Bigr]\\ &\quad= \int_0^1\int_0^\infty\int_{A+1}^\infty (1-v)r\bigl((t-A)^{1/(1-\beta)} - t^{1/(1-\beta)}\bigr)\frac{\beta(\beta-1)}{\Gamma(2-\beta)}\exp\bigl(-c_\beta r t^{1/(1-\beta)}\bigr)r^{-\beta}\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}v,\end{aligned}$$

which is clearly finite. This gives the desired result. The proof is identical for the behavior as $x \to -\infty$. □

From their construction from Poisson point measures, it is immediate that the processes η^+ and η^- both have independent (but not stationary) increments. The following lemma is an obvious corollary of the self-similarity of H_∞.

Lemma 5.13. *Let* $1 < \beta \leq 2$. *Then for all* $x \geq 0$ *and* $m > 0$,

$$\bigl(m^{1/\alpha}\bigl(\eta^+(u+x) - \eta^+(u)\bigr), u \geq 0\bigr) \stackrel{d}{=} \left(\eta^+\left(\frac{u+x}{m}\right) - \eta^+\left(\frac{u}{m}\right), u \geq 0\right).$$

In particular, for any $a > 0$,

$$m^{1/\alpha}\bigl(\eta^+(ma+u) - \eta^+(u)\bigr) \stackrel{d}{\to} \eta^+(a)$$

as $m \to \infty$. *The same holds by replacing the process* η^+ *by* η^- *and then by* η.

6. Convergence of height processes

Let $\mathcal{E}^*$ be the set of excursions $f : \mathbb{R} \to \mathbb{R}^+$ such that f has a unique maximum. We write $f_{\max} = \max_{x\in\mathbb{R}} f(x)$ and $x_{\max} = \operatorname{argmax}_{x\in\mathbb{R}} f(x)$. Let $\phi : \mathcal{E}^* \to C(\mathbb{R}, \mathbb{R}^+)$ be defined by

$$\phi(f)(t) = f_{\max} - f(x_{\max} + t).$$

Let $(H(x), 0 \le x \le \sigma)$ be an excursion with a unique maximum. Extend this to a function in $\mathcal{E}^*$ by setting H to be zero outside the interval $[0, \sigma]$. Now put

$$\bar{H} = \phi(H).$$

The aim of this section is to prove Theorem 4.1, which, using the scaling property of the stable height process, can be re-stated as

Theorem 6.1. *Let H_x have the distribution of $\bar{H}$ under $\mathbb{N}(\cdot|\sigma = x)$. Then*

$$H_x \xrightarrow{d} H_\infty$$

as $x \to \infty$, in the sense of uniform convergence on compact intervals.

Write $H_\infty^{(m)}$ for the function which is H_∞ capped at level m:

$$H_\infty^{(m)}(x) = \begin{cases} H_\infty(x) & \text{if } -\eta^-(m) \le x \le \eta^+(m), \\ m & \text{otherwise.} \end{cases}$$

Then we can re-state Theorem 5.10 as

Theorem 6.2. *For all $m > 0$,*

$$H_\infty^{(m)} \overset{d}{=} \bar{H} \quad \text{under } \mathbb{N}(\cdot | H_{\max} = m).$$

We will need the following technical lemma.

Lemma 6.3. *For all $a > 0$,*

$$\left(m^{1/\alpha}\eta(ma), H_\infty^{(ma)}\right) \xrightarrow{d} \left(\tilde{\eta}(a), H_\infty\right),$$

as $m \to \infty$, where $\tilde{\eta}(a)$ has the same distribution as $\eta(a)$ and is independent of H_∞. Here, the convergence is for the topology associated with the metric

$$d\big((a_1, f_1), (a_2, f_2)\big) = |a_1 - a_2| + \sum_{k \in \mathbb{N}} 2^{-k} \Big(\sup_{t \in [-k,k]} \big|f_1(t) - f_2(t)\big| \wedge 1 \Big)$$

on $\mathbb{R} \times C(\mathbb{R}, \mathbb{R}^+)$.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and let $g : C([-n, n], \mathbb{R}^+) \to \mathbb{R}$ be a bounded continuous function for some $n \in \mathbb{N}$. To ease notation, whenever $h : \mathbb{R} \to \mathbb{R}^+$ we will write $g(h)$ for $g(h|_{[-n,n]})$.

It follows from Lemma 5.13 that $m^{1/\alpha}\eta(ma) \overset{d}{=} \eta(a)$ for all $m, a > 0$, but this is insufficient to give the required asymptotic independence. Since the processes η^+, η^- have independent increments, by construction we have that $m^{1/\alpha}(\eta(ma) - \eta(u))$ is independent of $(H_\infty(y), y \in [-\eta^-(u), \eta^+(u)])$ and $\eta^+(u), \eta^-(u)$, whenever $ma > u$. By Lemma 5.13,

$$m^{1/\alpha}\big(\eta(ma) - \eta(u)\big) \xrightarrow{d} \eta(a)$$

as $m \to \infty$. So when $ma > u$ we certainly have

$$\begin{aligned}
&\mathbb{E}\big[f\big(m^{1/\alpha}\big(\eta(ma) - \eta(u)\big)\big)g\big(H_\infty^{(ma)}\big)\mathbb{1}_{\{\eta^-(u) > n, \eta^+(u) > n\}}\big] \\
&\quad = \mathbb{E}\big[f\big(m^{1/\alpha}\big(\eta(ma) - \eta(u)\big)\big)\big]\mathbb{E}\big[g\big(H_\infty^{(ma)}\big)\mathbb{1}_{\{\eta^-(u) > n, \eta^+(u) > n\}}\big] \\
&\quad \to \mathbb{E}\big[f\big(\eta(a)\big)\big]\mathbb{E}\big[g(H_\infty)\mathbb{1}_{\{\eta^-(u) > n, \eta^+(u) > n\}}\big].
\end{aligned}$$

Without loss of generality, the functions $|f|$ and $|g|$ are bounded by 1. To ease notation, put $G_m = g(H_\infty^{(ma)}) \times \mathbb{1}_{\{\eta^-(u)>n,\eta^+(u)>n\}}$. We will first show that

$$\bigl|\mathbb{E}\bigl[f\bigl(m^{1/\alpha}\bigl(\eta(ma)-\eta(u)\bigr)\bigr)G_m\bigr]-\mathbb{E}\bigl[f\bigl(m^{1/\alpha}\eta(ma)\bigr)G_m\bigr]\bigr|\to 0$$

as $m\to\infty$. Clearly, this absolute value is smaller than

$$\mathbb{E}\bigl[\bigl|f\bigl(m^{1/\alpha}\bigl(\eta(ma)-\eta(u)\bigr)\bigr)-f\bigl(m^{1/\alpha}\eta(ma)\bigr)\bigr|\bigr]=\mathbb{E}\bigl[\bigl|f\bigl(\bigl(\eta(a)-\eta(u/m)\bigr)\bigr)-f\bigl(\eta(a)\bigr)\bigr|\bigr],$$

by the self-similarity property of η (Lemma 5.13). The function f is bounded and continuous and so, by dominated convergence, this last quantity converges to 0. So we obtain that

$$\mathbb{E}\bigl[f\bigl(m^{1/\alpha}\eta(ma)\bigr)g\bigl(H_\infty^{(ma)}\bigr)\mathbb{1}_{\{\eta^-(u)>n,\eta^+(u)>n\}}\bigr]\to\mathbb{E}\bigl[f\bigl(\eta(a)\bigr)\bigr]\mathbb{E}\bigl[g(H_\infty)\mathbb{1}_{\{\eta^-(u)>n,\eta^+(u)>n\}}\bigr].$$

We must now remove the $\mathbb{1}_{\{\eta^-(u)>n,\eta^+(u)>n\}}$. Again using self-similarity, we have that $\eta^-(u)\to\infty$ and $\eta^+(u)\to\infty$ almost surely as $u\to\infty$. Take $\varepsilon>0$. Then there exists a u such that

$$\mathbb{P}\bigl(\eta^-(u)\le n\bigr)<\frac{\varepsilon}{2},\qquad \mathbb{P}\bigl(\eta^+(u)\le n\bigr)<\frac{\varepsilon}{2}.$$

So for all $m>0$,

$$\begin{aligned}&\bigl|\mathbb{E}\bigl[f\bigl(m^{1/\alpha}\eta(ma)\bigr)g\bigl(H_\infty^{(ma)}\bigr)\mathbb{1}_{\{\eta^-(u)>n,\eta^+(u)>n\}}\bigr]-\mathbb{E}\bigl[f\bigl(m^{1/\alpha}\eta(ma)\bigr)g\bigl(H_\infty^{(ma)}\bigr)\bigr]\bigr|\\&\quad\le\mathbb{P}\bigl(\eta^-(u)\le n\bigr)+\mathbb{P}\bigl(\eta^+(u)\le n\bigr)<\varepsilon.\end{aligned}$$

It is now straightforward to conclude that

$$\mathbb{E}\bigl[f\bigl(m^{1/\alpha}\eta(ma)\bigr)g\bigl(H_\infty^{(ma)}\bigr)\bigr]\to\mathbb{E}\bigl[f\bigl(\eta(a)\bigr)\bigr]\mathbb{E}\bigl[g(H_\infty)\bigr]$$

as $m\to\infty$. □

Proof of Theorem 6.1. Let $g:C([-n,n],\mathbb{R}^+)\to\mathbb{R}^+$ be a bounded continuous function for some $n\in\mathbb{N}$. As before, if $h:\mathbb{R}\to\mathbb{R}^+$, we will write $g(h)$ for $g(h|_{[-n,n]})$. Then it will be sufficient for us to show that

$$\mathbb{E}\bigl[g(H_x)\bigr]\to\mathbb{E}\bigl[g(H_\infty)\bigr]$$

as $x\to\infty$.

Without loss of generality, take $|g|\le 1$. Then

$$\begin{aligned}\mathbb{E}\bigl[g(H_x)\bigr]&=\mathbb{N}\bigl[g(\bar{H})|\sigma=x\bigr]\\&=\mathbb{N}\bigl[g\bigl(\phi(H)\bigr)|\sigma=x\bigr]\\&=c\int_{\mathbb{R}^+}\mathbb{N}\bigl[g\bigl(\phi\bigl(H^{[x/\sigma]}\bigr)\bigr)\mathbb{1}_{\{\sigma\ge x\}}|H_{\max}=x^{-\alpha}u\bigr]u^{1/\alpha}\,\mathrm{d}u,\end{aligned}$$

by Lemma 5.9. Suppose that we could show that

$$\mathbb{N}\bigl[g\bigl(\phi\bigl(H^{[x/\sigma]}\bigr)\bigr)\mathbb{1}_{\{\sigma\ge x\}}|H_{\max}=x^{-\alpha}u\bigr]\to\mathbb{E}\bigl[g(H_\infty)\bigr]\mathbb{N}[\mathbb{1}_{\{\sigma\ge 1\}}|H_{\max}=u]\tag{6.1}$$

as $x\to\infty$ for all $u>0$. Since $|g|\le 1$,

$$\mathbb{N}\bigl[g\bigl(\phi\bigl(H^{[x/\sigma]}\bigr)\bigr)\mathbb{1}_{\{\sigma\ge x\}}|H_{\max}=x^{-\alpha}u\bigr]\le\mathbb{N}\bigl[\mathbb{1}_{\{\sigma\ge x\}}|H_{\max}=x^{-\alpha}u\bigr]=\mathbb{N}[\mathbb{1}_{\{\sigma\ge 1\}}|H_{\max}=u],$$

by Lemma 5.8 and we also have that

$$\int_{\mathbb{R}^+}\mathbb{N}[\mathbb{1}_{\{\sigma\ge 1\}}|H_{\max}=u]u^{1/\alpha}\,\mathrm{d}u=c\mathbb{N}[\sigma\ge 1]<\infty.$$

So then by the dominated convergence theorem, we would be able to conclude that

$$\mathbb{N}\big[g\big(\phi(H)\big)|\sigma = x\big] \to \mathbb{E}\big[g(H_\infty)\big].$$

It remains to prove (6.1). By Theorem 6.2,

$$\begin{aligned}&\mathbb{N}\big[g\big(\phi\big(H^{[x/\sigma]}\big)\big)\mathbb{1}_{\{\sigma\geq x\}}|H_{\max} = x^{-\alpha}u\big]\\ &\quad= \mathbb{E}\big[g\big(\big(x^{-1}\eta\big(x^{-\alpha}u\big)\big)^{\alpha}H_\infty^{(x^{-\alpha}u)}\big(x^{-1}\eta\big(x^{-\alpha}u\big)\cdot\big)\big)\mathbb{1}_{\{\eta(x^{-\alpha}u)\geq x\}}\big].\end{aligned}$$

Now, by Lemma 6.3 we have that

$$\big(x^{-1}\eta\big(x^{-\alpha}u\big), H_\infty^{(x^{-\alpha}u)}\big) \xrightarrow{d} \big(\tilde{\eta}(u), H_\infty\big)$$

as $x \to \infty$, where $\tilde{\eta}(u)$ and H_∞ are independent. By the Skorokhod representation theorem, we may suppose that this convergence is almost sure. But then, by the bounded convergence theorem, since $\mathbb{P}(\tilde{\eta}(u) = 1) = \mathbb{N}(\sigma = 1|H_{\max} = u) = 0$, we have

$$\begin{aligned}&\mathbb{E}\big[g\big(\big(x^{-1}\eta\big(x^{-\alpha}u\big)\big)^{\alpha}H_\infty^{(x^{-\alpha}u)}\big(x^{-1}\eta\big(x^{-\alpha}u\big)\cdot\big)\big)\mathbb{1}_{\{\eta(x^{-\alpha}u)\geq x\}}\big]\\ &\quad\to \mathbb{E}\big[g\big(\tilde{\eta}(u)^{\alpha}H_\infty\big(\tilde{\eta}(u)\cdot\big)\big)\mathbb{1}_{\{\tilde{\eta}(u)\geq 1\}}\big]\end{aligned}$$

as $x \to \infty$. By the scaling property of H_∞ (Lemma 5.12) and the independence of $\tilde{\eta}(u)$ and H_∞, we see that this last is equal to

$$\mathbb{E}\big[g(H_\infty)\big]\mathbb{P}\big(\tilde{\eta}(u) \geq 1\big).$$

Since $\mathbb{P}(\tilde{\eta}(u) \geq 1) = \mathbb{N}[\mathbb{1}_{\{\sigma\geq 1\}}|H_{\max} = u]$, the result follows. □

7. Convergence of open sets and their sequences of ranked lengths

In this section, we prove Theorem 4.2 and Corollary 4.3.

We will need the concept of a *tagged fragment*, that is, the size $(\lambda(t))_{t\geq 0}$ of a block of the fragmentation which is tagged uniformly at random. Then, according to [5], $(-\log(\lambda(t)), t \geq 0)$ is a time-change of a subordinator ξ with Laplace exponent

$$\phi(t) = \int_{\mathcal{S}_1^{\downarrow}}\left(1 - \sum_{i=1}^{\infty} s_i^{1+t}\right)\nu(\mathrm{d}\mathbf{s}), \quad t \geq 0. \tag{7.1}$$

More specifically, $\lambda(t) = \exp(-\xi(\rho(t)))$, where

$$\rho(t) := \inf\left\{u \geq 0\colon \int_0^u \exp\big(\alpha\xi(r)\big)\,\mathrm{d}r \geq t\right\}.$$

Lemma 7.1.

(i) *For all* $a > 0$,

$$\mathrm{Leb}\big\{x \in \mathbb{R}\colon H_\infty(x) = a\big\} = 0 \quad a.s.$$

(ii) *For all* $a > 0$, *almost surely*, *a is not a local maximum of* H_∞.

Proof. (i) By Proposition 1.3.3 of Duquesne and Le Gall [11], the height process $(H(x), x \geq 0)$ possesses a collection of local times $(L_s^x, s \geq 0, x \geq 0)$, which we can assume is jointly measurable, and continuous and non-decreasing in s. Moreover, for any non-negative measurable function $g : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\int_0^s g(H_r)\,\mathrm{d}r = \int_{\mathbb{R}^+} g(x) L_s^x \,\mathrm{d}x \quad \text{a.s.}$$

Taking $g(x) = \mathbb{1}_{\{x=a\}}$, we see that for any $s \geq 0$,

$$\mathrm{Leb}\bigl\{t \in [0, s]:\ H(t) = a\bigr\} = 0 \quad \text{a.s.}$$

Since the height process is built out of excursions, the same is true for H under the excursion measure $\mathbb{N}$. In other words, for all $a > 0$,

$$\mathbb{N}\bigl(\bigl\{\mathrm{Leb}\bigl\{x \in [0, \sigma]:\ H(x) = a\bigr\} \neq 0\bigr\}\bigr) = 0. \tag{7.2}$$

Moreover, from the construction of the process H_∞ in Section 5.3, and using the same notation, we see that

$$\mathrm{Leb}\bigl\{x \in \mathbb{R}:\ H_\infty(x) = a\bigr\} = \sum_{i \geq 1} \sum_{j : u_j^{(i)} \leq r_i} \mathrm{Leb}\bigl\{x \in \bigl[0, \sigma\bigl(f_j^{(i)}\bigr)\bigr]:\ f_j^{(i)}(x) = t_i - a\bigr\},$$

where $\sum_{i \geq 1} \delta_{(r_i, t_i)}$ is a Poisson point measure of intensity

$$\beta(\beta - 1)\bigl(\Gamma(2 - \beta)\bigr)^{-1} \exp\bigl(-r c_\beta t^{1/(1-\beta)}\bigr) r^{-\beta}\,\mathrm{d}r\,\mathrm{d}t$$

on $\mathbb{R}^+ \setminus \{0\} \times \mathbb{R}^+ \setminus \{0\}$ and, for each $i \geq 1$, $\{(u_j^{(i)}, f_j^{(i)}), j \geq 1\}$ are the points of a Poisson point measure of intensity $\mathrm{d}u\,\mathbb{N}(\cdot, H_{\max} \leq t_i)$ on $\mathbb{R}^+ \times \mathcal{E}$. From this representation and (7.2), it is clear that the Lebesgue measure of $\{x \in \mathbb{R}:\ H_\infty(x) = a\}$ is equal to 0 almost surely.

(ii) By Lemma 2.5.3 of Duquesne and Le Gall [11], for all $a > 0$,

$$\mathbb{N}\bigl(\{a \text{ is a local minimum of } H\}\bigr) = 0.$$

With notation as in part (i), we have

$$\{a \text{ is a local maximum of } H_\infty\} = \bigcup_{i \geq 1} \Biggl(\{t_i = a\} \bigcup_{j : u_j^{(i)} \leq r_i} \bigl\{t_i - a \text{ is a local minimum of } f_j^{(i)}\bigr\}\Biggr)$$

(recall that in the construction of H_∞, the excursions $f_j^{(i)}$ are glued upside down at levels t_i, so that local minima of these excursions correspond to local maxima of H_∞). The conclusion is now obvious.

□

The proof of Theorem 4.2 follows immediately from Theorem 6.1, Proposition 5.1 and Lemma 7.1.

It remains to prove Corollary 4.3. We will need the following generalization of Theorem 6.1 and the forthcoming Lemma 7.3.

Theorem 7.2. *Let H_x be as in Theorem 6.1, and denote by $H_x|_K$ its restriction to the compact $[-K, K]$ (and similarly for H_∞). Then, for all $K > 0$,*

$$\bigl(H_x|_K, \mathrm{Leb}\bigl\{y \in [-K, K]:\ H_x(y) < 1\bigr\}\bigr) \xrightarrow{d} \bigl(H_\infty|_K, \mathrm{Leb}\bigl\{y \in [-K, K]:\ H_\infty(y) < 1\bigr\}\bigr),$$

as $x \to \infty$. Here, the convergence is in the topology associated with the metric

$$d\bigl((f_1, a_1), (f_2, a_2)\bigr) = \sup_{x \in [-K, K]} \bigl|f_1(x) - f_2(x)\bigr| + |a_1 - a_2|$$

on $C([-K,K],\mathbb{R}^+)\times\mathbb{R}^+$.

Proof. By the Skorokhod representation theorem, there exists a probability space such that the convergence in Theorem 6.1 is almost sure. Then the result follows from Lemma 5.4 and Lemma 7.1(i). □

Lemma 7.3. *Given* $\varepsilon>0$, *there exists* $K>0$ *such that*

$$\mathbb{P}\Big(\inf_{y\in(-\infty,K]\cup[K,+\infty)} H_x(y)<1\Big)\le\varepsilon$$

for all x *sufficiently large.*

Proof. By symmetry, it is sufficient to show that there exists $K>0$ such that $\mathbb{P}(\inf_{y\in[K,+\infty)} H_x(y)<1)\le\varepsilon$ for all x sufficiently large. As in the proof of Theorem 6.1, we combine Lemma 5.9 and Theorem 5.10 to get

$$\begin{aligned}&\mathbb{P}\Big(\inf_{y\in[K,+\infty)} H_x(y)<1\Big)\\&\quad=c\int_{\mathbb{R}^+}\mathbb{P}\Big(\inf_{y\in[K,+\infty)} H_\infty^{(x^{-\alpha}u)}\big(x^{-1}\eta\big(x^{-\alpha}u\big)y\big)<x^\alpha\eta^{-\alpha}\big(x^{-\alpha}u\big),\eta\big(x^{-\alpha}u\big)\ge x\Big)u^{1/\alpha}\,\mathrm{d}u.\end{aligned}\tag{7.3}$$

For all $L\ge1$, the probability in the integral can be bounded above by

$$\mathbb{P}\Big(\inf_{y\in[x^{-1}\eta(x^{-\alpha}u)K,+\infty)} H_\infty^{(x^{-\alpha}u)}(y)<x^\alpha\eta^{-\alpha}\big(x^{-\alpha}u\big),Lx>\eta\big(x^{-\alpha}u\big)\ge x\Big)+\mathbb{P}\big(\eta\big(x^{-\alpha}u\big)\ge Lx\big).$$

The first probability in this sum is in turn smaller than

$$\mathbb{P}\Big(\inf_{y\in[K,+\infty)} H_\infty^{(x^{-\alpha}u)}(y)<L^{-\alpha},Lx>\eta\big(x^{-\alpha}u\big)\ge x\Big),$$

and so is also smaller than

$$\mathbb{P}\Big(\inf_{y\in[K,+\infty)} H_\infty^{(x^{-\alpha}u)}(y)<L^{-\alpha}\Big)\wedge\mathbb{P}\big(Lx>\eta\big(x^{-\alpha}u\big)\ge x\big).\tag{7.4}$$

Recall that by self-similarity,

$$\mathbb{P}\big(Lx>\eta\big(x^{-\alpha}u\big)\ge x\big)=\mathbb{P}\big(L>\eta(u)\ge1\big)\le\mathbb{P}\big(\eta(u)\ge1\big).$$

Hence, (7.4) can be bounded from above by $\mathbb{P}(\eta(u)\ge1)$ when $x^{-\alpha}u\le L^{-\alpha}$, and by

$$\mathbb{P}\Big(\inf_{y\in[K,+\infty)} H_\infty(y)<L^{-\alpha}\Big)\wedge\mathbb{P}\big(\eta(u)\ge1\big)$$

when $x^{-\alpha}u>L^{-\alpha}$. From the identity (7.3) and these observations, we have

$$\begin{aligned}\mathbb{P}\Big(\inf_{y\in[K,+\infty)} H_x(y)<1\Big)&\le c\int_0^{L^{-\alpha}x^\alpha}\mathbb{P}\big(\eta(u)\ge1\big)u^{1/\alpha}\,\mathrm{d}u\\&\quad+c\int_0^\infty\mathbb{P}\Big(\inf_{y\in[K,+\infty)} H_\infty(y)<L^{-\alpha}\Big)\wedge\mathbb{P}\big(\eta(u)\ge1\big)u^{1/\alpha}\,\mathrm{d}u\\&\quad+c\int_0^\infty\mathbb{P}\big(\eta(u)\ge L\big)u^{1/\alpha}\,\mathrm{d}u.\end{aligned}\tag{7.5}$$

Recall that $c\int_0^\infty\mathbb{P}(\eta(u)\ge L)u^{1/\alpha}\,\mathrm{d}u=\mathbb{N}[\sigma\ge L]<\infty$. Then fix L large enough that the third integral in the right-hand side of (7.5) is smaller than $\varepsilon/3$. This L being fixed, note that the first integral on the right-hand side of (7.5) is smaller than $\varepsilon/3$ for all x sufficiently large. Since $H_\infty(y) \to \infty$ a.s. as $y \to +\infty, -\infty$, by dominated convergence we have that the second integral (which does not depend on x) is smaller than $\varepsilon/3$ for K sufficiently large. Hence, there exists some $K > 0$ such that $\mathbb{P}(\inf_{y\in[K,+\infty)} H_x(y) < 1) \le \varepsilon$ for all x sufficiently large. □

Proof of Corollary 4.3. Let $\varepsilon > 0$ and consider some real number K such that, for x large enough, the events $E_x = \{\inf_{y\in(-\infty,K]\cup[K,+\infty)} H_x(y) < 1\}$ all have probability smaller than $\varepsilon/5$ (such a K exists by Lemma 7.3). Taking K larger if necessary, we also have that E_∞ (defined in a similar way using H_∞) has probability smaller than $\varepsilon/5$.

Re-stated in terms of the functions H_x, our goal is to check that the decreasing sequence of lengths of the interval components of $O_x := \{y \in \mathbb{R}\colon H_x(y) < 1\}$, say $|O_x|^\downarrow$, converges in distribution as $x \to \infty$ to the decreasing sequence of lengths, $|O_\infty|^\downarrow$, of the interval components of $O_\infty := \{y \in \mathbb{R}\colon H_\infty(y) < 1\}$. We recall that the topology considered on $\mathcal{S}^\downarrow$ is given by the distance $d_{\mathcal{S}^\downarrow}(\mathbf{s}, \mathbf{s}') = \sum_{i=1}^\infty |s_i - s_i'|$. Now, let O_x^K be the restriction of O_x to the open set $(-K, K)$, for $x \in (0, \infty]$. Let $|O_x^K|^\downarrow$ be the corresponding ranked sequence of lengths of interval components. By the Skorokhod representation theorem, we may suppose that the convergence stated in Theorem 7.2 holds almost surely. From this, Lemma 7.1 and Proposition 5.1, we have that O_x^K converges to O_∞^K, in the sense that their complements in $[-K, K]$ converge in the Hausdorff distance. Moreover, the total length of O_x^K converges to that of O_∞^K. We deduce that $|O_x^K|^\downarrow$ converges to $|O_\infty^K|^\downarrow$ in the *pointwise* distance on $\mathcal{S}^\downarrow$ (see Proposition 2.2 of Bertoin [7]). But since we also have convergence of the total length of the open sets, the convergence actually holds in the $d_{\mathcal{S}^\downarrow}$ distance.

Now, let $f : \mathcal{S}^\downarrow \to \mathbb{R}$ be any continuous function such that $\sup_{\mathbf{s}\in\mathcal{S}^\downarrow} |f(\mathbf{s})| \le 1$. Since $O_x = O_x^K$ on E_x^c, we have

$$\begin{aligned}\bigl|\mathbb{E}\bigl[f\bigl(|O_x|^\downarrow\bigr)\bigr] - \mathbb{E}\bigl[f\bigl(|O_\infty|^\downarrow\bigr)\bigr]\bigr| &\le \bigl|\mathbb{E}\bigl[f\bigl(\bigl|O_x^K\bigr|^\downarrow\bigr)\mathbb{1}_{E_x^c}\bigr] - \mathbb{E}\bigl[f\bigl(\bigl|O_\infty^K\bigr|^\downarrow\bigr)\mathbb{1}_{E_\infty^c}\bigr]\bigr| + \mathbb{P}(E_x) + \mathbb{P}(E_\infty)\\ &\le \bigl|\mathbb{E}\bigl[f\bigl(\bigl|O_x^K\bigr|^\downarrow\bigr)\bigr] - \mathbb{E}\bigl[f\bigl(\bigl|O_\infty^K\bigr|^\downarrow\bigr)\bigr]\bigr| + 2\mathbb{P}(E_x) + 2\mathbb{P}(E_\infty).\end{aligned}$$

Using the fact that $|O_x^K|^\downarrow$ converges in distribution to $|O_\infty^K|^\downarrow$, we get that for all x sufficiently large, $|\mathbb{E}[f(|O_x|^\downarrow)] - \mathbb{E}[f(|O_\infty|^\downarrow)]| \le \varepsilon$. The result follows. □

8. Behavior of the last fragment

In this section, we prove the results stated in Corollary 4.4 and the remark which follows it.

First, it is implicit in the proofs in the previous sections that the distribution of the length of the interval component of $\{y \in \mathbb{R}\colon H_x(y) < 1\}$ that contains 0 (i.e. the distribution of $xF_*((\zeta - x^\alpha)^+)$) converges as $x \to \infty$ to the distribution of the length of the interval component of $\{y \in \mathbb{R}\colon H_\infty(y) < 1\}$ that contains 0. By construction of the function H_∞ (see Section 5.3), this interval component is distributed as $\eta(1)$. Indeed, almost surely 1 is not one of the t_i's and therefore $H_\infty(y) < 1$ for every $y \in (-\eta^-(1), \eta^+(1))$. Moreover, it is easy to see that $H_\infty(\eta^+(1)) = H_\infty(\eta^-(1)) = 1$. In other words, we have that

$$t\bigl(F_*\bigl((\zeta - t)^+\bigr)\bigr)^\alpha \xrightarrow{d} \bigl(\eta(1)\bigr)^\alpha \quad \text{as } t \to 0.$$

Recall then that $(\eta(1))^\alpha$ is distributed as σ^α under $\mathbb{N}(\cdot|H_{\max} = 1)$ which, by Lemma 5.9, is easily seen to be distributed as the $(-1/\alpha - 1)$-size-biased version of ζ defined in (4.2).

We now turn to the bounds given in (i) and (ii), Corollary 4.4, for the tails of this size-biased version of ζ.

(i) When $t \to \infty$, the given bounds are easy consequences of Proposition 14 of [13] on the asymptotic behavior of $\mathbb{P}(\zeta > t)$ as $t \to \infty$. Indeed, according to that result, there exist $A, B > 0$ such that, for all t large enough,

$$\exp\bigl(-B\Psi(t)\bigr) \le \mathbb{P}(\zeta \ge t) \le \exp\bigl(-A\Psi(t)\bigr),$$

where Ψ is the inverse of the bijection $t \in [1, \infty) \to t/\phi(t) \in [1/\phi(1), \infty)$ and ϕ is defined at (7.1). Miermont [20], Section 3.2, shows that in the case of the stable fragmentation,

$$\phi(t) = (1 + \alpha)^{-1}\frac{\Gamma(t - \alpha)}{\Gamma(t)}.$$

Now let c be a positive constant that may vary from line to line. Using the fact that $\Gamma(z)$ is proportional to $z^{z-1/2}\exp(-z)$ for large z, we get that $\phi(t) \sim ct^{-\alpha}$ as $t \to \infty$. So $\Psi(t) \sim ct^{1/(1+\alpha)}$ as $t \to \infty$. We just need to convert the statements about the tails of ζ into statements about the tails of ζ_{*^α}. On the one hand, note that

$$\mathbb{P}(\zeta_{*^\alpha} \geq t) = \frac{\mathbb{E}[\zeta^{-1/\alpha-1}\mathbb{1}_{\{\zeta \geq t\}}]}{\mathbb{E}[\zeta^{-1/\alpha-1}]} \geq \frac{t^{-1/\alpha-1}\mathbb{P}(\zeta \geq t)}{\mathbb{E}[\zeta^{-1/\alpha-1}]} \geq \frac{\exp(-ct^{1/(1+\alpha)})}{\mathbb{E}[\zeta^{-1/\alpha-1}]}$$

for t sufficiently large. On the other hand, using the Cauchy–Schwarz inequality and the fact that ζ has positive moments of all orders, we easily get that

$$\mathbb{P}(\zeta_{*^\alpha} \geq t) = \frac{\mathbb{E}[\zeta^{-1/\alpha-1}\mathbb{1}_{\{\zeta \geq t\}}]}{\mathbb{E}[\zeta^{-1/\alpha-1}]} \leq \frac{c\mathbb{P}(\zeta \geq t)^{1/2}}{\mathbb{E}[\zeta^{-1/\alpha-1}]}$$

for all $t \geq 0$. The result follows immediately.

(ii) The second assertion is as straightforward to prove. First,

$$\mathbb{P}(\zeta_{*^\alpha} \leq t) = \frac{\mathbb{E}[\zeta^{-1/\alpha-1}\mathbb{1}_{\{\zeta \leq t\}}]}{\mathbb{E}[\zeta^{-1/\alpha-1}]} \leq \frac{t^{-1/\alpha-1}\mathbb{P}(\zeta \leq t)}{\mathbb{E}[\zeta^{-1/\alpha-1}]}. \tag{8.1}$$

In Section 4.2.1 of [13] it is proved that for all $q < 1 + \underline{p}/(-\alpha)$, $\mathbb{P}(\zeta \leq t) \leq t^q$ for small t, where

$$\underline{p} = \sup\left\{q \geq 0: \int_{(1,\infty)} \exp(qx)L(\mathrm{d}x) < \infty\right\}.$$

Here, L is the Lévy measure of the subordinator ξ with Laplace exponent ϕ (see the beginning of Section 7 for the definition). Using Miermont's results [20], Section 3.2, again, the measure L is proportional to $\exp(x)(\exp(x) - 1)^{\alpha-1}\mathbb{1}_{\{x>0\}}\,\mathrm{d}x$. It follows that $\underline{p} = -\alpha$. Combining this with (8.1) gives the desired result.

We finish this section with the proof of Eq. (4.3), which is based on the fact that $\Phi(\lambda) := \mathbb{E}[\exp(-\lambda\eta(1))] = \mathbb{E}[\exp(-\eta(\lambda^{-\alpha}))]$, $\lambda \geq 0$. We recall that it is assumed that $\alpha \in (-1/2, 0)$, i.e. $\beta \in (1, 2)$. Using the Poisson construction of $\eta(1)$, the expression for the Laplace transform of a subordinator, the self-similarity of the process η and Campbell's formula for Poisson point processes, we obtain

$$\begin{aligned}
\Phi(\lambda) &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\sum_{i:t_i \leq \lambda^{-\alpha}} \left(\tau_+^{(i)}\left((1-v_i)r_i\right) + \tau_-^{(i)}(v_ir_i)\right)\right)\middle| v_i, r_i, t_i, i \geq 1\right]\right] \\
&= \mathbb{E}\left[\exp\left(-\sum_{i:t_i \leq \lambda^{-\alpha}} r_i \int_0^\infty \left(1 - \mathrm{e}^{-u}\right)\mathbb{N}[\sigma \in \mathrm{d}u, H_{\max} \leq t_i]\right)\right] \\
&= \mathbb{E}\left[\exp\left(-(\beta-1)^{\beta/(\beta-1)} \sum_{i:t_i \leq \lambda^{-\alpha}} r_i \int_0^{t_i}\int_0^\infty \left(1 - \mathrm{e}^{-u}\right)\mathbb{N}[\sigma \in \mathrm{d}u | H_{\max} = v]v^{\beta/(1-\beta)}\,\mathrm{d}v\right)\right] \\
&= \mathbb{E}\left[\exp\left(-(\beta-1)^{\beta/(\beta-1)} \sum_{i:t_i \leq \lambda^{-\alpha}} r_i \int_0^{t_i} \mathbb{E}\left[1 - \mathrm{e}^{-\eta(v)}\right]v^{\beta/(1-\beta)}\,\mathrm{d}v\right)\right] \\
&= \mathbb{E}\left[\exp\left(-(\beta-1)^{\beta/(\beta-1)} \sum_{i:t_i \leq \lambda^{-\alpha}} r_i \int_0^{t_i} \left(1 - \Phi\left(v^{-1/\alpha}\right)\right)v^{\beta/(1-\beta)}\,\mathrm{d}v\right)\right] \\
&= \exp\left(-\int_{\mathbb{R}^+ \times [0,\lambda^{-\alpha}]} \left(1 - \mathrm{e}^{-(\beta-1)^{\beta/(\beta-1)} r \int_0^t (1-\Phi(v^{-1/\alpha}))v^{\beta/(1-\beta)}\,\mathrm{d}v}\right)\frac{\beta(\beta-1)}{\Gamma(2-\beta)}\mathrm{e}^{-c_\beta r t^{1/(1-\beta)}} r^{-\beta}\,\mathrm{d}r\,\mathrm{d}t\right),
\end{aligned}$$

where $c_\beta = (\beta-1)^{1/(1-\beta)}$. Substituting $\beta = (1+\alpha)^{-1}$, we obtain the desired expression.

9. Almost sure logarithmic asymptotics

The following result describes the almost sure logarithmic behavior near the extinction time of the largest fragment and the last fragment processes. It is valid for general self-similar fragmentations with parameters $\alpha < 0$, $c = 0$ and $\nu(\sum_{i=1}^{\infty} s_i < 1) = 0$. We recall that for such fragmentations, the extinction time ζ possesses exponential moments (see [13], Proposition 14).

Theorem 9.1.

(i) *Suppose there exists* $\eta > 0$ *such that* $\int_{\mathcal{S}_1^{\downarrow}} s_1^{-\eta} \mathbb{1}_{\{s_1<1/2\}} \nu(\mathrm{d}\mathbf{s}) < \infty$. *Then,*

$$\frac{\log(F_1((\zeta - t)^+))}{\log(t)} \overset{a.s.}{\to} -1/\alpha \quad \text{as } t \to 0.$$

(ii) *If, moreover,* $\alpha \geq -\gamma_\nu := -\inf\{\gamma > 0\colon \lim_{\varepsilon\to 0} \varepsilon^{\gamma} \nu(s_1 < 1 - \varepsilon) = 0\}$, *then the last fragment process* F_* *is well defined* (*i.e. the fragmentation* F *can be encoded into an interval fragmentation for which there exists a unique point* $x \in (0, 1)$ *which is reduced to dust at time* ζ) *and*

$$\frac{\log(F_*((\zeta - t)^+))}{\log(t)} \overset{a.s.}{\to} -1/\alpha \quad \text{as } t \to 0.$$

Corollary 9.2. *For the stable fragmentation with index* $-1/2 \leq \alpha < 0$ (*or, equivalently,* $1 < \beta \leq 2$), *with probability* 1,

$$\lim_{t\to 0} \frac{\log(F_1((\zeta - t)^+))}{\log(t)} = \lim_{t\to 0} \frac{\log(F_*((\zeta - t)^+))}{\log(t)} = -\frac{1}{\alpha} = \frac{\beta}{\beta - 1}.$$

Proof of Theorem 9.1. (i) We first prove this result in the case where there exists some real number $a > 0$ such that $\nu(s_1 < a) = 0$. We will then explain how to adapt it to the more general assumption $\int_{\mathcal{S}_1^{\downarrow}} s_1^{-\eta} \mathbb{1}_{\{s_1<1/2\}} \nu(\mathrm{d}\mathbf{s}) < \infty$. By the first Borel–Cantelli lemma, it is sufficient to show that, for any $\varepsilon > 0$,

$$\sum_{i=1}^{\infty} \mathbb{P}\bigl(F_1\bigl(\bigl(\zeta - \mathrm{e}^{-i}\bigr)^+\bigr) > \exp\bigl(\bigl(\alpha^{-1} + \varepsilon\bigr)i\bigr)\bigr) < \infty, \tag{9.1}$$

$$\sum_{i=1}^{\infty} \mathbb{P}\bigl(F_1\bigl(\bigl(\zeta - \mathrm{e}^{-i}\bigr)^+\bigr) \leq \exp\bigl(\bigl(\alpha^{-1} - \varepsilon\bigr)i\bigr)\bigr) < \infty. \tag{9.2}$$

(Note that if $i^{-1}\log(F_1((\zeta - \mathrm{e}^{-i})^+))$ converges almost surely to $1/\alpha$ as $i \to \infty$, then almost surely for all sequences $(t_n, n \geq 0)$ converging to 0, $\log(F_1((\zeta - t_n)^+))/\log(t_n)$ converges to $-1/\alpha$, since

$$\frac{F_1((\zeta - \mathrm{e}^{-(i_n+1)})^+)}{-i_n} \leq \frac{F_1((\zeta - t_n)^+)}{\log(t_n)} \leq \frac{F_1((\zeta - \mathrm{e}^{-i_n})^+)}{-(i_n + 1)},$$

where $i_n = \lfloor -\log(t_n)\rfloor$).

Let $\mathcal{F}_t = \sigma(F(s)\colon s \leq t)$ and suppose that T is a $(\mathcal{F}_t)_{t\geq 0}$-stopping time such that $T < \zeta$ a.s. According to [5], the branching and self-similarity properties of F hold for $(\mathcal{F}_t)_{t\geq 0}$-stopping times, hence

$$\zeta - T = \sup_{j\geq 1}\bigl\{F_j(T)^{-\alpha}\zeta_j\bigr\},$$

where $(\zeta_j, j \geq 1)$ are independent copies of ζ, independent of $F(T)$. Let

$$T_1 = \inf\bigl\{t \geq 0\colon\ F_1(t) \leq \exp\bigl(\bigl(\alpha^{-1} + \varepsilon\bigr)i\bigr)\bigr\},$$

$$T_2 = \inf\bigl\{t \geq 0\colon\ F_1(t) \leq \exp\bigl(\bigl(\alpha^{-1} - \varepsilon\bigr)i\bigr)\bigr\}.$$

We start by proving (9.1). We have

$$\begin{aligned}\mathbb{P}\big(F_1\big(\big(\zeta-\mathrm{e}^{-i}\big)^+\big)>\exp\big(\big(\alpha^{-1}+\varepsilon\big)i\big)\big)&=\mathbb{P}\big(T_1>\zeta-\mathrm{e}^{-i}\big)\\&=\mathbb{P}\Big(\sup_{j\ge1}F_j(T_1)^{-\alpha}\zeta_j<\mathrm{e}^{-i}\Big)\\&\le\mathbb{P}\big(F_1(T_1)^{-\alpha}\zeta_1<\mathrm{e}^{-i}\big).\end{aligned}$$

Since we have assumed that there exists $a>0$ such that $\nu(s_1<a)=0$, we have $F_1(T_1)\ge aF_1(T_1-)\ge a\exp((\alpha^{-1}+\varepsilon)i)$ a.s., and so

$$\begin{aligned}\mathbb{P}\big(F_1(T_1)^{-\alpha}\zeta_1<\mathrm{e}^{-i}\big)&\le\mathbb{P}\big(a^{-\alpha}\exp\big(-\alpha\big(\alpha^{-1}+\varepsilon\big)i\big)\zeta<\mathrm{e}^{-i}\big)\\&=\mathbb{P}\big(\zeta<a^{\alpha}\mathrm{e}^{\alpha\varepsilon i}\big).\end{aligned}\tag{9.3}$$

Let $(\lambda(t))_{t\ge0}$ be the size of a tagged fragment as defined at the beginning of Section 7, and let ξ be the related subordinator. Then consider the first time at which λ reaches 0, i.e.

$$\sigma=\inf\big\{t\ge0\colon\ \lambda(t)=0\big\}=\int_0^\infty\exp\big(\alpha\xi(r)\big)\,\mathrm{d}r.$$

Of course, $\sigma\le\zeta$. Moreover, by Proposition 3.1 of Carmona, Petit and Yor [10], σ has moments of all orders strictly greater than -1; this implies, in particular, that $\mathbb{E}[\zeta^{-\gamma}]<\infty$ for $0<\gamma<1$. So, by Markov's inequality, we have that

$$\mathbb{P}\big(F_1\big(\big(\zeta-\mathrm{e}^{-i}\big)^+\big)>\exp\big(\big(\alpha^{-1}+\varepsilon\big)i\big)\big)\le a^{\alpha/2}\mathrm{e}^{\alpha\varepsilon i/2}\mathbb{E}\big[\zeta^{-1/2}\big],$$

which is summable in i.

Now turn to (9.2). Arguing as before, we have

$$\begin{aligned}\mathbb{P}\big(F_1\big(\big(\zeta-\mathrm{e}^{-i}\big)^+\big)\le\exp\big(\big(\alpha^{-1}-\varepsilon\big)i\big)\big)&=\mathbb{P}\big(T_2\le\zeta-\mathrm{e}^{-i}\big)\\&=\mathbb{P}\Big(\sup_{j\ge1}F_j(T_2)^{-\alpha}\zeta_j\ge\mathrm{e}^{-i}\Big).\end{aligned}$$

Take $q>-1/\alpha$. Then, since $\zeta_1,\zeta_2,\dots$ are independent and identically distributed and independent of $F(T_2)$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{j\ge1}F_j(T_2)^{-\alpha}\zeta_j\ge\mathrm{e}^{-i}\Big)&\le\sum_{j\ge1}\mathbb{P}\big(F_j(T_2)^{-\alpha}\zeta_j\ge\mathrm{e}^{-i}/2\big)\\&\le2^q\mathbb{E}\big[\zeta^q\big]\mathrm{e}^{iq}\mathbb{E}\Big[\sum_{j\ge1}F_j(T_2)^{-\alpha q}\Big].\end{aligned}$$

The expectation $\mathbb{E}[\zeta^q]$ is finite and, since $-\alpha q-1>0$, we have

$$\sum_{j\ge1}F_j(T_2)^{-\alpha q}\le F_1(T_2)^{-\alpha q-1}\sum_{j\ge1}F_j(T_2)\le F_1(T_2)^{-\alpha q-1}.$$

But then

$$\mathbb{P}\big(F_1\big(\big(\zeta-\mathrm{e}^{-i}\big)^+\big)\le\exp\big(\big(\alpha^{-1}-\varepsilon\big)i\big)\big)\le2^q\mathbb{E}\big[\zeta^q\big]\exp\big(i\big(\varepsilon\alpha q-\alpha^{-1}+\varepsilon\big)\big),$$

which is summable in i for large enough q.

It remains to adapt this proof to the more general case where $\int_{\mathcal{S}_1^{\downarrow}}s_1^{-\eta}\mathbb{1}_{\{s_1<1/2\}}\nu(\mathrm{d}\mathbf{s})<\infty$ for some $\eta>0$. The key inequality (9.3) is then no longer valid and we have to check that $\mathbb{P}(F_1(T_1)^{-\alpha}\zeta_1<\mathrm{e}^{-i})$ is still summable in i. The rest of the proof remains unchanged. So, denote by $\Delta(T_1)$ the size of the "multiplicative" jump of F_1 at time T_1 (i.e.

$\Delta(T_1) := F_1(T_1)/F_1(T_1-))$ and recall that ζ_1 denotes a random variable with the same distribution as ζ and independent of $F_1(T_1)$. Note that we may, and will, suppose that ζ_1 is independent of the whole fragmentation F. Then,

$$\begin{aligned}
&\mathbb{P}\big(F_1^{-\alpha}(T_1)\zeta_1 < \mathrm{e}^{-i}\big)\\
&\quad= \mathbb{P}\big(F_1^{-\alpha}(T_1-)\big(\Delta(T_1)\big)^{-\alpha}\zeta_1 < \mathrm{e}^{-i}\big)\\
&\quad\le \mathbb{P}\big(\mathrm{e}^{-\alpha(\alpha^{-1}+\varepsilon)i}\big(\Delta(T_1)\big)^{-\alpha}\zeta_1 < \mathrm{e}^{-i}\big)\\
&\quad= \mathbb{P}\big(\big(\Delta(T_1)\big)^{-\alpha}\zeta_1 < \mathrm{e}^{\alpha\varepsilon i}, \Delta(T_1) < 1/2\big) + \mathbb{P}\big(\big(\Delta(T_1)\big)^{-\alpha}\zeta_1 < \mathrm{e}^{\alpha\varepsilon i}, \Delta(T_1) \ge 1/2\big).
\end{aligned} \tag{9.4}$$

The second term in this last line is bounded from above, for all $\gamma > 0$, by

$$\mathrm{e}^{\gamma\alpha\varepsilon i}\mathbb{E}\big[\zeta^{-\gamma}\big]2^{-\alpha\gamma},$$

which is finite and summable in i if we take $0 < \gamma < 1$. To bound the first term in (9.4), introduce $\mathcal{D}_1$, the set of jump times of F_1. For $t \in \mathcal{D}_1$, let $\mathbf{s}(t)$ be the relative mass frequencies obtained by the dislocation of $F_1(t-)$, and let $\Delta(t) := F_1(t)/F_1(t-)$. Since the largest fragment coming from F_1 at the time of a split may not be the largest block overall after the split, $s_1(t) \le \Delta(t)$. Then,

$$\begin{aligned}
&\mathbb{P}\big(\big(\Delta(T_1)\big)^{-\alpha}\zeta_1 < \mathrm{e}^{\alpha\varepsilon i}, \Delta(T_1) < 1/2\big)\\
&\quad= \mathbb{E}\bigg[\sum_{t\in\mathcal{D}_1} \mathbb{1}_{\{(\Delta(t))^{-\alpha}\zeta_1<\mathrm{e}^{\alpha\varepsilon i}, \Delta(t)<1/2\}}\mathbb{1}_{\{F_1(t-)\ge \mathrm{e}^{(\alpha^{-1}+\varepsilon)i}, F_1(t)\le \mathrm{e}^{(\alpha^{-1}+\varepsilon)i}\}}\bigg]\\
&\quad\le \mathrm{e}^{\gamma\alpha\varepsilon i}\mathbb{E}\bigg[\sum_{t\in\mathcal{D}_1} (s_1(t))^{\alpha\gamma}\mathbb{1}_{\{s_1(t)<1/2\}\}}\mathbb{1}_{\{F_1(t-)\ge \mathrm{e}^{(\alpha^{-1}+\varepsilon)i}\}}\bigg]\mathbb{E}\big[\zeta_1^{-\gamma}\big]
\end{aligned}$$

for all $\gamma > 0$. The expectation $\mathbb{E}[\zeta_1^{-\gamma}]$ is finite when $\gamma < 1$, which we assume for the rest of the proof. Now, the process $(\varSigma(u), u \ge 0)$ defined by

$$\varSigma(u) = \sum_{t\in\mathcal{D}_1, t\le u} \big(s_1(t)\big)^{\alpha\gamma}\mathbb{1}_{\{s_1(t)<1/2\}}\mathbb{1}_{\{F_1(t-)\ge \mathrm{e}^{(\alpha^{-1}+\varepsilon)i}\}}$$

is increasing, càdlàg, and adapted to the filtration $(\mathcal{F}_t, t \ge 0)$ (and hence optional). So, according to the Doob–Meyer decomposition [22], Section VI, [17], Theorem 1.8, Chapter 2, it possesses an increasing predictable compensator $(A(u), u \ge 0)$ such that $\mathbb{E}[\varSigma(\infty)] = \mathbb{E}[A(\infty)]$. Moreover, since the fragmentation process is a pure jump process where each block of size m splits at rate $m^\alpha\nu(\mathrm{d}\mathbf{s})$ into blocks of lengths $(ms_1, ms_2, \ldots)$, this compensator is given by

$$A(u) = \int_0^u F_1(t)^\alpha \mathbb{1}_{\{F_1(t)\ge \mathrm{e}^{(\alpha^{-1}+\varepsilon)i}\}}\,\mathrm{d}t \int_{\mathcal{S}_1^\downarrow} s_1^{\alpha\gamma}\mathbb{1}_{\{s_1<1/2\}}\nu(\mathrm{d}\mathbf{s}), \quad u \ge 0.$$

The second integral in this product is finite for small enough $\gamma > 0$, by assumption. The first is clearly smaller than $\mathrm{e}^{-\delta(\alpha^{-1}+\varepsilon)i}\int_0^s F_1(t)^{\alpha+\delta}\mathbf{1}_{\{F_1(t)>0\}}\,\mathrm{d}t$, for all $\delta \ge 0$. This leads us to

$$\mathbb{P}\big(\big(\Delta(T_1)\big)^{-\alpha}\zeta_1 < \mathrm{e}^{\alpha\varepsilon i}, \Delta(T_1) < 1/2\big) \le C_\gamma \mathrm{e}^{(\gamma\alpha\varepsilon-\delta(\alpha^{-1}+\varepsilon))i}\mathbb{E}\bigg[\int_0^\zeta F_1(r)^{\alpha+\delta}\,\mathrm{d}r\bigg], \tag{9.5}$$

where C_γ is a constant depending only on $\gamma > 0$ and which is finite provided γ is sufficiently small. To finish, we claim that for all $0 < \delta < -\alpha$, $\mathbb{E}[\int_0^\zeta F_1(r)^{\alpha+\delta}\,\mathrm{d}r] < \infty$ (note this finiteness is obvious for $\delta \ge -\alpha$, since $\mathbb{E}[\zeta] < \infty$). Indeed, from Bertoin [5], we know that the α-self-similar fragmentation process F (and its interval counterpart) can be transformed through (somewhat complicated) time-changes into a $(-\delta)$-self-similar fragmentation process with same dislocation measure. We refer to Bertoin's paper for details. In particular, if $|O_x(t)|$ denotes the length of the fragment containing $x \in (0,1)$ at time t in the interval α-self-similar fragmentation, and if ζ_x denotes the first time at which this length reaches 0, we have

$$\int_0^{\zeta_x} \left|O_x(r)\right|^{\alpha+\delta} \mathrm{d}r = \zeta_x^{(\delta)} \le \zeta^{(\delta)},$$

where $\zeta_x^{(\delta)}$ is the time at which the point x is reduced to dust in the fragmentation with parameter $-\delta$ and $\zeta^{(\delta)} := \sup_x \zeta_x^{(\delta)}$ is the time at which the whole fragmentation with parameter $-\delta$ is reduced to dust. Now, since $\alpha+\delta<0$, we have $(F_1(r))^{\alpha+\delta} \le |O_x(r)|^{\alpha+\delta}$ for all $x \in (0,1)$, $r \ge 0$ and, therefore,

$$\mathbb{E}\left[\int_0^{\zeta} F_1(r)^{\alpha+\delta}\,\mathrm{d}r\right] = \mathbb{E}\left[\sup_x \int_0^{\zeta_x} F_1(r)^{\alpha+\delta}\,\mathrm{d}r\right] \le \mathbb{E}\left[\sup_x \int_0^{\zeta_x} \left|O_x(r)\right|^{\alpha+\delta}\,\mathrm{d}r\right] \le \mathbb{E}\left[\zeta^{(\delta)}\right] < \infty.$$

Hence, if we choose δ small enough that $\gamma\alpha\varepsilon - \delta(\alpha^{-1}+\varepsilon) < 0$, we get from (9.5) that $\mathbb{P}((\Delta(T_1))^{-\alpha}\zeta_1 < \mathrm{e}^{\alpha\varepsilon i}, \Delta^{(1)}(T_1) < 1/2)$ is summable in i.

(ii) The additional assumption on ν implies that it is infinite, hence we know (see Haas and Miermont [16]) that the fragmentation can be encoded into a continuous function G which is, moreover, γ-Hölder, for all $\gamma < (-\alpha) \wedge \gamma_\nu$ ($= -\alpha$ here). In particular, the maximum, ζ, of G is attained for some $x \in (0,1)$. More precisely, we claim it is attained at a *unique* point, which is denoted by x_*. See the end of the proof for an explanation of this uniqueness. It implies, in particular, that the last fragment process is well defined: for each $t < \zeta$, we denote by $O_*(t)$ the interval component of $\{x \in (0,1)\colon G(x) > t\}$ which contains x_*, and by $F_*(t)$ the length of this interval. For $t<\zeta$, let

$$x_t^- = \sup\bigl\{x \le x_*\colon G(x) \le \zeta - t\bigr\},$$
$$x_t^+ = \inf\bigl\{x \ge x_*\colon G(x) \le \zeta - t\bigr\},$$

so that $O_*(\zeta - t) = (x_t^-, x_t^+)$. Then, for all $0 \le \gamma < -\alpha$, there exists some constant C such that

$$t = G(x_*) - G\bigl(x_t^-\bigr) \le C\bigl(x_* - x_t^-\bigr)^\gamma,$$
$$t = G(x_*) - G\bigl(x_t^+\bigr) \le C\bigl(x_t^+ - x_*\bigr)^\gamma,$$

and, consequently, $F_*(\zeta - t) = x_t^+ - x_t^- \ge 2(t/C)^{1/\gamma}$. This implies that

$$\limsup_{t\to 0}\left(\frac{\log(F_*((\zeta-t)^+))}{\log(t)}\right) \le -1/\alpha.$$

For the liminf, use part (i) and the fact that $F_*(t) \le F_1(t)$ for all $t \ge 0$.

Finally, we have to prove that there is a unique $x \in (0,1)$ such that $G(x) = \zeta$. Note that

$$\mathbb{P}(\exists t < \zeta\colon \text{ at least two fragments present at time } t \text{ die at } \zeta)$$
$$= \mathbb{P}(\exists t \in \mathbb{Q}, t < \zeta\colon \text{ at least two fragments present at time } t \text{ die at } \zeta)$$

and this latter probability is equal to 0 if, for all $t \in \mathbb{Q}$,

$$\mathbb{P}(\text{at least two fragments present at time } t \text{ die at } \zeta, t < \zeta)$$
$$= \mathbb{P}\bigl(\exists i \ne j\colon F_i^{-\alpha}(t)\zeta_i = F_j^{-\alpha}(t)\zeta_j \text{ and } F_i(t) \ne 0\bigr) = 0,$$

where (ζ_i, ζ_j) are independent and distributed as ζ, independently of $F(t)$. Clearly, this is satisfied if the distribution of ζ has no atoms. Now recall that we are in the case where ν is infinite and suppose that there exists $t > 0$ such that $\mathbb{P}(\zeta = t) > 0$. Recall also that, conditional on $u < \zeta$, $\zeta = u + \sup_{i \ge 1} F_i(u)^{-\alpha}\zeta_i$ where $(\zeta_i, i \ge 1)$ are independent copies of ζ, independent of $F(u)$. Moreover, the supremum is actually a maximum, since we know there exists $x \in (0,1)$ such that $G(x) = \zeta$. Then for all $0 < u < t$,

$$\begin{aligned}
&\mathbb{P}\bigl(\exists i\colon F_i^{-\alpha}(u)\zeta_i = t-u\bigr) > 0 \\
&\quad \Leftrightarrow \quad \exists i\colon \mathbb{P}\bigl(F_i^{-\alpha}(u)\zeta = t-u\bigr) > 0 \quad \bigl(\text{with } \zeta \text{ independent of } F(u)\bigr) \\
&\quad \Leftrightarrow \quad \mathbb{P}\bigl(\lambda^{-\alpha}(u)\zeta = t-u\bigr) > 0,
\end{aligned}$$

where λ denotes the tagged fragment process. Recall that $\lambda(u) = \exp(-\xi(\rho(u)))$, where ξ is a subordinator with Laplace exponent given by (7.1). Now, for any $b > 0$,

$$\mathbb{P}\bigl(\xi\bigl(\rho(u)\bigr) = b\bigr) \le \mathbb{P}\bigl(\exists v \ge 0\colon \xi(v) = b\bigr).$$

But we know from Kesten's theorem (Proposition 1.9 in Bertoin [4]) that the right-hand side is 0 because the Lévy measure of the subordinator ξ is infinite and it has no drift. Hence, $\mathbb{P}(\lambda^{-\alpha}(u)\zeta = t-u) = 0$ for all $0 < u < t$, and we can deduce the claimed uniqueness. $\square$

Proof of Corollary 9.2. It has been proved in Haas and Miermont [16], Section 3.5, that the dislocation measure ν of any stable fragmentation satisfies

$$\int_{\mathcal{S}_1^{\downarrow}} s_1^{-1} \mathbb{1}_{\{s_1 < 1/2\}} \nu(\mathrm{d}\mathbf{s}) < \infty.$$

(Note that this is obvious in the Brownian case since the fragmentation is binary and so $\nu(s_1 < 1/2) = 0$.) From [16], Section 4.4, we know that the parameter γ_ν (defined in Theorem 9.1(ii)) associated with the dislocation measure ν of the stable fragmentation with index α is given by $\gamma_\nu = -\alpha$. Hence, both assumptions of Theorem 9.1(i) and (ii) are satisfied. $\square$

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References

[1] R. Abraham and J.-F. Delmas. Williams' decomposition of the Lévy continuous random tree and simultaneous extinction probability for populations with neutral mutations. *Stochastic Process. Appl.* **119** (2009) 1124–1143.

[2] A.-L. Basdevant. Fragmentation of ordered partitions and intervals. *Electron. J. Probab.* **11** (2006) 394–417 (electronic). MR2223041

[3] J. Bertoin. *Lévy Processes*. *Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge, 1996. MR1406564

[4] J. Bertoin. Subordinators: Examples and applications. In *Lectures on Probability Theory and Statistics (Saint-Flour, 1997)* 1–91. *Lecture Notes in Math.* **1717**. Springer, Berlin, 1999. MR1746300

[5] J. Bertoin. Self-similar fragmentations. *Ann. Inst. H. Poincaré Probab. Statist.* **38** (2002) 319–340. MR1899456

[6] J. Bertoin. The asymptotic behavior of fragmentation processes. *J. Eur. Math. Soc. (JEMS)* **5** (2003) 395–416. MR2017852

[7] J. Bertoin. *Random Fragmentation and Coagulation Processes*. *Cambridge Studies in Advanced Mathematics* **102**. Cambridge Univ. Press, Cambridge, 2006. MR2253162

[8] J. Bertoin and A. Rouault. Discretization methods for homogeneous fragmentations. *J. London Math. Soc. (2)* **72** (2005) 91–109. MR2145730

[9] P. Biane, J. Pitman and M. Yor. Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. *Bull. Amer. Math. Soc. (N.S.)* **38** (2001) 435–465 (electronic). MR1848256

[10] P. Carmona, F. Petit and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential Functionals and Principal Values Related to Brownian Motion* 73–130. *Bibl. Rev. Mat. Iberoamericana*. Rev. Mat. Iberoamericana, Madrid, 1997. MR1648657

[11] T. Duquesne and J.-F. Le Gall. Random trees, Lévy processes and spatial branching processes. *Astérisque* **281** (2002) 1–147. MR1954248

[12] C. Goldschmidt and B. Haas. Behavior near the extinction time in self-similar fragmentations II: Finite dislocation measures. In preparation, 2010.

[13] B. Haas. Loss of mass in deterministic and random fragmentations. *Stochastic Process. Appl.* **106** (2003) 245–277. MR1989629
[14] B. Haas. Regularity of formation of dust in self-similar fragmentations. *Ann. Inst. H. Poincaré Probab. Statist.* **40** (2004) 411–438. MR2070333
[15] B. Haas. Fragmentation processes with an initial mass converging to infinity. *J. Theoret. Probab.* **20** (2007) 721–758. MR2359053
[16] B. Haas and G. Miermont. The genealogy of self-similar fragmentations with negative index as a continuum random tree. *Electron. J. Probab.* **9** (2004) 57–97 (electronic). MR2041829
[17] J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes*, 2nd edition. *Grundlehren der Mathematischen Wissenschaften* **288**. Springer, Berlin, 2003. MR1943877
[18] D. P. Kennedy. The distribution of the maximum Brownian excursion. *J. Appl. Probab.* **13** (1976) 371–376. MR0402955
[19] J.-F. Le Gall and Y. Le Jan. Branching processes in Lévy processes: The exploration process. *Ann. Probab.* **26** (1998) 213–252. MR1617047
[20] G. Miermont. Self-similar fragmentations derived from the stable tree. I. Splitting at heights. *Probab. Theory Related Fields* **127** (2003) 423–454. MR2018924
[21] J. Pitman. *Combinatorial Stochastic Processes*. *Lecture Notes in Math.* **1875**. Springer, Berlin, 2006. Lectures from the 32nd Saint-Flour Summer School on Probability Theory. MR2245368
[22] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales, Vol. 2: Itô Calculus*. Cambridge Univ. Press, Cambridge, 2000. MR1780932
[23] G. Uribe Bravo. The falling apart of the tagged fragment and the asymptotic disintegration of the Brownian height fragmentation. *Ann. Inst. H. Poincaré Probab. Statist.* To appear, 2010. Available at arXiv:0811.4754.