Bernoulli **17**(4), 2011, 1217–1247
DOI: 10.3150/10-BEJ312

Self-similar scaling limits of non-increasing Markov chains

BÉNÉDICTE HAAS and GRÉGORY MIERMONT

Université Paris-Dauphine, Paris, France. E-mail: haas@ceremade.dauphine.fr
Université Paris-Sud, Paris, France. E-mail: gregory.miermont@math.u-psud.fr

We study scaling limits of non-increasing Markov chains with values in the set of non-negative integers, under the assumption that the large jump events are rare and happen at rates that behave like a negative power of the current state. We show that the chain starting from n and appropriately rescaled, converges in distribution, as $n \to \infty$, to a non-increasing self-similar Markov process. This convergence holds jointly with that of the rescaled absorption time to the time at which the self-similar Markov process reaches first 0.

We discuss various applications to the study of random walks with a barrier, of the number of collisions in Λ-coalescents that do not descend from infinity and of non-consistent regenerative compositions. Further applications to the scaling limits of Markov branching trees are developed in our paper, Scaling limits of Markov branching trees, with applications to Galton–Watson and random unordered trees (2010).

Keywords: absorption time; Λ-coalescents; random walks with a barrier; regenerative compositions; regular variation; self-similar Markov processes

1. Introduction and main results

Consider a Markov chain taking values in the set of non-negative integers $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, and with non-increasing paths. We are interested in the asymptotic behavior in distribution of the chain started from n, as n tends to ∞. Our main assumption is (roughly speaking) that the chain, when in state n, has a "small" probability, of order $c_\varepsilon n^{-\gamma}$ for some $\gamma > 0$ and some $c_\varepsilon > 0$, of accomplishing a negative jump with size in $[n\varepsilon, n]$, where $0 < \varepsilon < 1$. A typical example is constructed from a random walk $(S_k, k \geq 0)$ with non-negative steps with tail distribution proportional to $n^{-\gamma}$ as n tends to ∞, for some $\gamma \in (0, 1)$, by considering the Markov chain starting from n: $(\max(n - S_k, 0), k \geq 0)$. An explicit example is provided by the step distribution $q_n = (-1)^{n-1}\binom{\gamma}{n}$, $n \geq 1$.

Under this main assumption, we show in Theorem 1 that the chain started from n, and properly rescaled in space and time, converges in distribution in the Skorokhod space to a non-increasing self-similar Markov process. These processes were introduced and studied by Lamperti [15, 16], under the name of *semi-stable* processes, and by many authors since then. Note that Stone [20] discusses limit theorems for birth-and-death chains and diffusions that involve self-similar Markov processes, but in a context that is very different from ours.

A quantity of particular interest is the absorption time of the chain, that is, the first time after which the chain remains constant. We show in Theorem 2 that jointly with the convergence of Theorem 1, the properly rescaled absorption time converges to the first time the limiting self-similar Markov process hits 0. In fact, we even show that all positive moments of the rescaled absorption time converge.

1350-7265

These results have applications to a number of problems considered in the literature, such as the random walk with a barrier [14] when the step distribution is in the domain of attraction of a stable random variable with index in (0, 1), or the number of coalescing events in a Λ-coalescent that does not come down from infinity [7,14]. It also allows us to recover some results by Gnedin, Pitman and Yor [9] for the number of blocks in regenerative composition structures, and to extend this result to the case of "non-consistent compositions". One of the main motivations for the present study was to provide a unified framework to treat such problems, which can all be translated in terms of absorption times of non-increasing Markov chains. Moreover, the convergence of the rescaled Markov chain as a process, besides the convergence of the absorption time, provides new insights on these results. Finally, our main results are also a starting point for obtaining the scaling limits of a large class of random trees satisfying a simple discrete branching property. This is the object of the paper [11].

Let us now present our main results and applications in a more formal way. Implicitly, all the random variables in this paper are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Notation. *For two positive sequences $x_n, y_n, n \geq 0$, the notation $x_n \sim y_n$ means that x_n/y_n converges to* 1 *as $n \to \infty$.*

1.1. Scaling limits of non-increasing Markov chains

For every $n \geq 0$, consider a non-negative sequence $(p_{n,k}, 0 \leq k \leq n)$ that sums to 1,

$$\sum_{k=0}^{n} p_{n,k} = 1.$$

We view the latter as a probability distribution on $\{0, 1, \dots, n\}$, and view the family $(p_{n,k}, 0 \leq k \leq n)$ as the transition probabilities for a discrete-time Markov chain, which takes integer values and has non-increasing paths. We will denote by $(X_n(k), k \geq 0)$ such a Markov chain, starting at the state $X_n(0) = n$. For every $n \geq 1$, we let p_n^* be the law on [0, 1] of $X_n(1)/n$, so that

$$p_n^*(\mathrm{d}x) = \sum_{k=0}^{n} p_{n,k} \delta_{k/n}(\mathrm{d}x).$$

Our main assumption all throughout the paper will be the following hypothesis.

(H). *There exist*:

- *a sequence $(a_n, n \geq 1)$ of the form $a_n = n^\gamma \ell(n)$, where $\gamma > 0$ and $\ell : \mathbb{R}_+ \to (0, \infty)$ is a function that is slowly varying at ∞*,
- *a non-zero*, *finite*, *non-negative measure μ on* [0, 1],

such that the following weak convergence of finite measures on [0, 1] *holds*:

$$a_n (1 - x) p_n^*(\mathrm{d}x) \xrightarrow[n \to \infty]{(w)} \mu(\mathrm{d}x). \tag{1}$$

This means that a jump of the process X_n/n from 1 to $x \in (0,1)$ occurs with a small intensity $a_n^{-1}\mu(\mathrm{d}x)/(1-x)$, and indicates that an interesting scaling limit for the Markov chain X_n should arise when rescaling space by n and time by a_n. Also, note that $\mu([0,1])/a_n$ is equivalent as $n \to \infty$ to the expectation of the first jump of the chain X_n/n, and this converges to 0 as $n \to \infty$. The role of the factor $(1-x)$ in (1) is to temper the contribution of very small jumps in order to evaluate the contribution of larger jumps.

Of course, in (H), the sequence $a = (a_n, n \geq 1)$, the function ℓ and the measure μ are not uniquely determined. One can simultaneously replace a by ca and μ by $c\mu$ for any given $c > 0$. Also, one can replace ℓ by any function that is equivalent to it at infinity. However, it is clear that μ is determined up to a positive multiplicative constant (with a simultaneous change of the sequence a as depicted above), and that γ is uniquely determined.

We will soon see that hypothesis (H) appears very naturally in various situations. It is also very general, in the sense that there are no restrictions on the sequences $(a_n, n \geq 1)$ or measures μ that can arise. Here is a formal statement, which is proved at the end of Section 4.

Proposition 1. *For any finite measure μ on $[0,1]$ and any sequence of the form $a_n = n^\gamma \ell(n)$ where $\gamma > 0$ and $\ell : \mathbb{R}_+ \to (0,\infty)$ is slowly varying at ∞, one can find a sequence of probability vectors $((p_{n,k}, 0 \leq k \leq n), n \geq 0)$ such that (1) holds.*

We now describe the objects that will arise as scaling limits of X_n. For $\lambda > 0$ and $x \in [0,1)$, let

$$[\lambda]_x = \frac{1-x^\lambda}{1-x}, \qquad 0 \leq x < 1, \tag{2}$$

and set $[\lambda]_1 = \lambda$. For each $\lambda > 0$, this defines a continuous function $x \mapsto [\lambda]_x$ on $[0,1]$. If μ is a finite measure on $[0,1]$, then the function ψ defined for $\lambda > 0$ by

$$\psi(\lambda) := \int_{[0,1]} [\lambda]_x \mu(\mathrm{d}x) \tag{3}$$

and extended at 0 by $\psi(0) := \lim_{\lambda \downarrow 0} \psi(\lambda) = \mu(\{0\})$ is the Laplace exponent of a subordinator. To see this, let $\mathtt{k} = \mu(\{0\})$, $\mathtt{d} = \mu(\{1\})$, so that ψ can be written in the usual Lévy–Khintchine form:

$$\psi(\lambda) = \mathtt{k} + \mathtt{d}\lambda + \int_{(0,1)} (1-x^\lambda)\frac{\mu(\mathrm{d}x)}{1-x} = \mathtt{k} + \mathtt{d}\lambda + \int_0^\infty (1-\mathrm{e}^{-\lambda y})\omega(\mathrm{d}y),$$

where ω is the push-forward of the measure $(1-x)^{-1}\mu(\mathrm{d}x)\mathbf{1}_{\{0<x<1\}}$ by the mapping $x \mapsto -\log x$. Note that ω is a σ-finite measure on $(0,\infty)$ that integrates $y \mapsto y \wedge 1$, as it ought. Conversely, any Laplace exponent of a (possibly killed) subordinator can be put in the form (3) for some finite measure μ.

Now, let ξ be a subordinator with Laplace exponent ψ. This means that the process $(\xi_t, t \geq 0)$ is a non-decreasing Lévy process with

$$\mathbb{E}[\exp(-\lambda\xi_t)] = \exp(-t\psi(\lambda)), \qquad t, \lambda \geq 0.$$

Note in particular that the subordinator is killed at rate $\mathtt{k} \geq 0$. The function $t \in [0, \infty) \to \int_0^t \exp(-\gamma\xi_r)\,\mathrm{d}r$ is continuous, non-decreasing and its limit at infinity, denoted by

$$I := \int_0^\infty \exp(-\gamma\xi_r)\,\mathrm{d}r,$$

is a.s. finite. Standard properties of this random variable are studied in [4]. We let $\tau : [0, I) \to \mathbb{R}_+$ be its inverse function, and set $\tau(t) = \infty$ for $t \geq I$. The process

$$Y(t) := \exp\bigl(-\xi_{\tau(t)}\bigr), \qquad t \geq 0, \tag{4}$$

is a non-increasing self-similar Markov process starting from 1. Recall from [16] that if $\mathbb{P}_x$ is the law of an $\mathbb{R}_+$-valued Markov process $(M_t, t \geq 0)$ started from $M_0 = x \geq 0$, then the process is called self-similar with exponent $\alpha > 0$ if the law of $(r^{-\alpha} M_{rt}, t \geq 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{r^{-\alpha}x}$, for every $r > 0$ and $x \geq 0$.

In this paper, all processes that we consider belong to the space $\mathcal{D}$ of càdlàg, non-negative functions from $[0, \infty)$ to $\mathbb{R}$. This space is endowed with the Skorokhod metric, which makes it a Polish space. We refer to [6], Chapter 3.5, for background on the topic. We recall that $\lfloor r \rfloor$ denotes the integer part of the real number r.

Theorem 1. *For all $t \geq 0$ and all $n \in \mathbb{N}$, we let*

$$Y_n(t) := \frac{X_n(\lfloor a_n t \rfloor)}{n}.$$

Then, under the assumption (H), *we have the following convergence in distribution*

$$Y_n \xrightarrow[n\to\infty]{(d)} Y$$

for the Skorokhod topology on $\mathcal{D}$, where Y is defined at (4).

A theorem by Lamperti [16] shows that any càdlàg, non-increasing, non-negative, self-similar Markov process (started from 1) can be written in the form (4) for some subordinator ξ and some $\gamma > 0$. In view of this, Theorem 1, combined with Proposition 1, implies that every non-increasing, càdlàg, self-similar Markov process is the weak scaling limit of a non-decreasing Markov chain with rare large jumps.

In fact, as the proof of Theorem 1 will show, a more precise result holds. With the above notations, for every $t \geq 0$, we let $Z(t) = \exp(-\xi_t)$, so that $Y(t) = Z(\tau(t))$. Let also

$$\tau_n^{-1}(t) = \inf\left\{u \geq 0 : \int_0^u Y_n^{-\gamma}(r)\,\mathrm{d}r > t\right\}, \qquad t \geq 0,$$

and $Z_n(t) := Y_n(\tau_n^{-1}(t))$.

Proposition 2. *Under the same hypotheses and notations as Theorem 1, one has the joint convergence in distribution*

$$(Y_n, Z_n) \xrightarrow[n\to\infty]{(d)} (Y, Z)$$

for the product topology on $\mathcal{D}^2$.

1.2. Absorption times

Let $\mathcal{A}$ be the set of absorbing states of the chain, that is,

$$\mathcal{A} := \{k \in \mathbb{Z}_+ : p_{k,k} = 1\}.$$

Under assumption (H) it is clear that $\mathcal{A}$ is finite, and not empty since it contains at least 0. It is also clear that the absorbing time

$$A_n := \inf\{k \in \mathbb{Z}_+ : X_n(k) \in \mathcal{A}\}$$

is a.s. finite. For $(Y, Z) = (\exp(-\xi_\tau), \exp(-\xi))$ defined as in the previous subsection, we let $\sigma = \inf\{t \geq 0 : Y(t) = 0\}$. Then it holds that

$$\sigma = \int_0^\infty \exp(-\gamma \xi_r)\,\mathrm{d}r, \tag{5}$$

which is a general fact that we recall (21) in Section 3.2 below.

Theorem 2. *Assume* (H). *Then, as $n \to \infty$,*

$$\frac{A_n}{a_n} \xrightarrow{(d)} \sigma,$$

and this holds jointly with the convergence in law of (Y_n, Z_n) to (Y, Z) as stated in Proposition 2. Moreover, for all $p \geq 0$,

$$\mathbb{E}\left[\left(\frac{A_n}{a_n}\right)^p\right] \to \mathbb{E}[\sigma^p].$$

When $p \in \mathbb{Z}_+$, the limiting moment $\mathbb{E}[\sigma^p]$ is equal to $p!/\prod_{i=1}^p \psi(\gamma i)$.

Note that even the first part of this result is not a direct consequence of Theorem 1 since convergence of functions in $\mathcal{D}$ does not lead, in general, to the convergence of their absorption times (when they exist).

1.3. Organization of the paper

We start in Section 2 with a series of applications of Theorems 1 and 2 to random walks with a barrier, Λ-coalescents and non-consistent regenerative compositions. Most of the proofs of these results, as well as further developments, are postponed to Sections 5 (for the random walks with a barrier) and 6 (for Λ-coalescents).

In the preliminary Section 3, we gather some basic facts needed for the proofs of Theorems 1 and 2 and Proposition 2, which are undertaken in Section 4. The proof of Proposition 2 and Theorem 1 will be obtained by a classical two-step approach: first, we show that the laws of $(Y_n, Z_n), n \geq 1$ form a tight family of probability distributions on $\mathcal{D}^2$. Then, we will show that the only possible limiting distribution is that of (Y, Z). This identification of the limit will be obtained via a simple martingale problem. Tightness is studied in Section 4.1 and the characterization of the limits in Section 4.2. In both cases, we will work with some sequences of martingales related to the chains X_n, which are introduced in Section 3.3. The convergence of (Y_n, Z_n) to (Y, Z) is a priori not sufficient to get the convergence of the absorption times, as stated in Theorem 2. This will be obtained in Section 4.3, by first showing that $t^\beta \mathbb{E}[Z_n(t)^\lambda]$ is uniformly bounded for every $\beta > 0$.

Last, a proof of Proposition 1 is given at the end of Section 4.

2. Applications

2.1. Random walk with a barrier

Let $q = (q_k, k \geq 0)$ be a non-negative sequence with total sum $\sum_k q_k = 1$, which is interpreted as a probability distribution on $\mathbb{Z}_+$. We assume that $q_0 < 1$ in order to avoid trivialities. For $n \geq 0$, we let

$$\overline{q}_n = \sum_{k>n} q_k, \qquad n \geq 0.$$

The random walk with a barrier is a variant of the usual random walk with step distribution q. Informally, every step of the walk is distributed as q, but conditioned on the event that it does not bring the walk to a level higher than a given value n. More formally, for every n, we define the random walk with barrier n as the Markov chain $(S_k^{(n)}, k \geq 0)$ starting at 0, with values in $\{0, 1, 2, \dots, n\}$ and with transition probabilities

$$q_{i,j}^{(n)} = \begin{cases} \dfrac{q_{j-i}}{1-\overline{q}_{n-i}}, & \text{if } \overline{q}_{n-i} < 1, \\ \mathbf{1}_{\{j=i\}}, & \text{if } \overline{q}_{n-i} = 1, \end{cases} \qquad 0 \leq i \leq j \leq n.$$

(This definition is not exactly the same as in [14], but the absorption time A_n is exactly the random variable M_n, which is the main object of study in this paper. We will comment further on this point in Section 5.)

To explain the definition, note that when $\overline{q}_r < 1$, $(q_k/(1-\overline{q}_r), 0 \leq k \leq r)$ is the law of a random variable with distribution q, conditioned to be in $\{0, \dots, r\}$. When $\overline{q}_r = 1$, the quotient is not well defined, and we choose the convention that the conditioned law is the Dirac measure at $\{0\}$. In other words, when the process arrives at a state i such that $\overline{q}_{n-i} = 1$, so that every jump with distribution q would be larger than $n-i$, we choose to let the chain remain forever at state i. Of course, the above discussion is not needed when $q_0 > 0$.

As a consequence of the definition, the process

$$X_n(k) = n - S_k^{(n)}, \qquad k \geq 0,$$

is a Markov process with non-increasing paths, starting at n, and with transition probabilities

$$p_{i,j} = \frac{q_{i-j}}{1-\overline{q}_i}, \qquad 0 \leq j \leq i, \tag{6}$$

with the convention that $p_{i,j} = \mathbf{1}_{\{j=i\}}$ when $\overline{q}_i = 1$. The probabilities (6) do not depend on n, so this falls under our basic framework. As before, we let A_n be the absorbing time for X_n.

Theorem 3. (i) *Let $\gamma \in (0,1)$, and assume that $\overline{q}_n = n^{-\gamma}\ell(n)$, where $\gamma \in (0,1)$ and ℓ is slowly varying at ∞. Let ξ be a subordinator with Laplace exponent*

$$\psi(\lambda) = \int_0^\infty (1 - \mathrm{e}^{-\lambda y}) \frac{\gamma \mathrm{e}^{-y}\,\mathrm{d}y}{(1-\mathrm{e}^{-y})^{\gamma+1}}, \qquad \lambda \geq 0,$$

and let

$$\tau(t) = \inf\left\{u \geq 0 : \int_0^u \exp(-\gamma\xi_r)\,\mathrm{d}r > t\right\}, \qquad t \geq 0.$$

Then,

$$\left(\frac{X_n(\lfloor t/\overline{q}_n \rfloor)}{n}\right) \xrightarrow[n\to\infty]{(d)} \left(\exp(-\xi_{\tau(t)}), t \geq 0\right),$$

jointly with the convergence

$$\overline{q}_n A_n \xrightarrow[n\to\infty]{(d)} \int_0^\infty \exp(-\gamma\xi_t)\,\mathrm{d}t.$$

For the latter, the convergence of all positive moments also holds.

(ii) *Assume that $m := \sum_{k=0}^\infty k q_k$ is finite. Then*

$$\left(\left(\frac{X_n(\lfloor tn \rfloor)}{n}, t \geq 0\right), \frac{A_n}{n}\right) \xrightarrow[n\to\infty]{(P)} \left(\left(\left((1-mt) \vee 0\right), t \geq 0\right), 1/m\right),$$

in probability in $\mathcal{D} \times \mathbb{R}_+$. Convergence of all positive moments also holds for the second components.

Of course, this will be proved by checking that (H) holds for transition probabilities of the particular form (6), under the assumption of 3. This result encompasses Theorems 1.1 and 1.4 in [14]. Note that Theorems 1.2 and 1.5 in the latter reference give information about the deviation for A_n around n/m in case (ii) of Theorem 3 above, under some assumptions on q (saying essentially that a random variable with law q is in the domain of attraction of a stable law with index in $[1,2]$, as opposed to $(0,1)$ in Theorem 3). See also [5] for related results in a different context.

The Lévy measure of the subordinator $\gamma\xi$ involved in Theorem 3 is clearly given by

$$\exp(-x/\gamma)\big(1-\exp(-x/\gamma)\big)^{-\gamma-1}\,\mathrm{d}x\,\mathbf{1}_{\{x\geq 0\}}.$$

Bertoin and Yor [1] show that the variable $\int_0^\infty \exp(-\gamma\xi_r)\,\mathrm{d}r$ is then distributed as $\Gamma(1-\gamma)^{-1}\tau_\gamma^{-\gamma}$, where τ_γ is a stable random variable with Laplace transform $\mathbb{E}[\exp(-\lambda\tau_\gamma)] = \exp(-\lambda^\gamma)$.

2.2. On collisions in Λ-coalescents that do not come down from infinity

We first briefly recall the definition and basic properties of a Λ-coalescent, referring the interested reader to [17,18] for more details.

Let Λ be a finite measure on $[0,1]$. For $r\in\mathbb{N}$, a (Λ,r)-coalescent is a Markov process $(\Pi_r(t), t\geq 0)$ taking values in the set of partitions of $\{1,2,\dots,r\}$, which is monotone in the sense that $\Pi_r(t')$ is coarser than $\Pi_r(t)$ for every $t'>t$. More precisely, Π_r only evolves by steps that consist of merging a certain number (at least 2) of blocks of the partition into one, the other blocks being left unchanged. Assuming that $\Pi_r(0)$ has n blocks, the rate of a collision event involving $n-k+1$ blocks, bringing the process to a state with k blocks, for some $1\leq k\leq n-1$, is given by

$$g_{n,k} = \binom{n}{k-1}\int_{[0,1]} x^{n-k-1}(1-x)^{k-1}\Lambda(\mathrm{d}x),$$

and the blocks that intervene in the merging event are uniformly selected among the $\binom{n}{k-1}$ possible choices of $n-k+1$ blocks out of n. Note that these transition rates depend only on the number of blocks present at the current stage. In particular, they do not depend on the particular value of r.

A Λ-coalescent is a Markov process $(\Pi(t), t\geq 0)$ with values in the set of partitions of $\mathbb{N}$, such that for every $r\geq 1$, the restriction $(\Pi|_{[r]}(t), t\geq 0)$ of the process to $\{1,2,\dots,r\}$ is a (Λ,r)-coalescent. The existence (and uniqueness in law) of such a process is discussed in [17]. The most celebrated example is the Kingman coalescent obtained for $\Lambda=\delta_0$.

The Λ-coalescent $(\Pi(t), t\geq 0)$ is said to *come down from infinity* if, given that $\Pi(0) = \{\{i\}, i\geq 1\}$ is the partition of $\mathbb{N}$ that contains only singletons, $\Pi(t)$ a.s. has a finite number of blocks for every $t>0$. When the coalescent does not come down from infinity, it turns out that $\Pi(t)$ has a.s. infinitely many blocks for every $t\geq 0$, and we say that the coalescent *stays infinite*. See [19] for more details and a nice criterion for the property of coming down from infinity. By Lemma 25 in [17], the Λ-coalescent stays infinite if $\int_{[0,1]} x^{-1}\Lambda(\mathrm{d}x)<\infty$.

Starting with n blocks in a (Λ,r)-coalescent (or in a Λ-coalescent), let $X_n(k)$ be the number of blocks after k coalescing events have taken place. Due to the above description, the process

$(X_n(k), k \geq 0)$ is a Markov chain with transition probabilities given by

$$\begin{aligned} p_{n,k} &= \mathbb{P}\big(X_n(1) = k\big) = \frac{g_{n,k}}{g_n} \\ &= \frac{1}{g_n}\binom{n}{k-1}\int_{[0,1]} x^{n-k-1}(1-x)^{k-1}\Lambda(\mathrm{d}x), \qquad 1 \leq k \leq n-1, \end{aligned} \tag{7}$$

where g_n is the total transition rate $g_n = \sum_{k=1}^{n-1} g_{n,k}$. This chain always gets absorbed at 1.

The total number of collisions in the coalescent coincides with the absorption time $A_n := \inf\{k : X_n(k) = 1\}$. There have been many studies on the asymptotic behavior of A_n as $n \to \infty$ [7,10,13,14], in contexts that mostly differ from ours (see the comments below). For $u \in (0, 1]$ we let

$$h(u) = \int_{[u,1]} x^{-2}\Lambda(\mathrm{d}x). \tag{8}$$

We are interested in cases where $\lim_{u\downarrow 0} h(u) = \infty$ but $\int_0^1 x^{-1}\Lambda(\mathrm{d}x) < \infty$, so the coalescent stays infinite by the above discussion.

Theorem 4. *Let $\gamma \in (0, 1)$. We assume that the function h is regularly varying at 0 with index $-\gamma$. Let ξ be a subordinator with Laplace exponent*

$$\psi(\lambda) = \frac{1}{\Gamma(2-\gamma)}\int_0^1 \big(1-(1-x)^{\lambda}\big)x^{-2}\Lambda(\mathrm{d}x), \qquad \lambda \geq 0, \tag{9}$$

and let

$$\tau(t) := \inf\left\{u \geq 0 : \int_0^u \exp(-\gamma\xi_r)\,\mathrm{d}r > t\right\}, \qquad t \geq 0. \tag{10}$$

Then,

$$\left(\frac{X_n(\lfloor h(1/n)t\rfloor)}{n}, t \geq 0\right) \xrightarrow[n\to\infty]{(d)} \exp(-\xi_\tau). \tag{11}$$

Moreover, jointly with (11), *it holds that*

$$\frac{A_n}{h(1/n)} \xrightarrow[n\to\infty]{(d)} \int_0^\infty \exp(-\gamma\xi_r)\,\mathrm{d}r, \tag{12}$$

and there is also a convergence of moments of orders $p \geq 0$.

Note that a result related to (12) is announced in [7] (remark following Theorem 3.1 therein).

Of course, the statement of Theorem 4 remains true if we simultaneously replace h and ψ in (8) and (9) with ch and $c\psi$ for any $c > 0$. Also, the statement remains true if we change h with any of its equivalents at 0 in (11) or (12). Theorem 4 specialises to yield the following results on beta coalescents. Recall that the beta-coalescent with parameters $a, b > 0$, also denoted by $\beta(a, b)$-coalescent, is the Λ-coalescent associated with the measure

$$\Lambda(\mathrm{d}x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x\,\mathbf{1}_{[0,1]}(x).$$

Corollary 1. *For the beta-coalescent $\beta(a, b)$ with parameters $1 < a < 2$ and $b > 0$, the process of numbers of collisions satisfies*

$$\left(\frac{X_n(\lfloor n^{2-a}t\rfloor)}{n}, t \geq 0\right) \xrightarrow[n\to\infty]{(d)} \exp(-\xi_\tau),$$

where ξ is a subordinator with Laplace exponent

$$\psi(\lambda) = \frac{2-a}{\Gamma(a)} \int_0^\infty (1-\mathrm{e}^{-\lambda y}) \frac{\mathrm{e}^{-by}}{(1-\mathrm{e}^{-y})^{3-a}}\,\mathrm{d}y$$

and τ the time change defined from ξ by (10), *replacing there γ with $2-a$. Moreover, the total number A_n of collisions in such a beta-coalescent satisfies, jointly with the previous convergence,*

$$\frac{A_n}{n^{2-a}} \xrightarrow[n\to\infty]{(d)} \int_0^\infty \exp\big(-(2-a)\xi_r\big)\,\mathrm{d}r.$$

The convergence of all positive moments also holds.

When $b = 2 - a$, we know from the particular form of the Laplace exponent of ξ that the range of $\exp(-\xi)$ is identical in law with the zero set of a Bessel bridge of dimension $2 - 2b$ (see [8]). When, moreover, $b \in (0, 1/2]$, the time changed process $\exp(-\xi_\tau)$ is distributed as the tagged fragment in a $1/(1-b)$-stable fragmentation (with a dislocation measure suitably normalized). More generally, when $b \in (0, 1)$ and $a > 1 + b$, the time changed process $\exp(-\xi_\tau)$ is distributed as the tagged fragment in a Poisson–Dirichlet fragmentation with a dislocation measure proportional to $PD^*(1-b, a+b-3)$ as defined in [12], Section 3. In such cases, the Laplace exponent of ξ can be explicitly computed. See Corollary 8 of [12].

When $b = 1$ (and still $1 < a < 2$), the asymptotic behavior of A_n is proved by Iksanov and Möhle in [14], using there the connection with this model and random walks with a barrier. As mentioned at the end of the previous section, the limit random variable $\int_0^\infty \exp(-(2-a)\xi_t)\,\mathrm{d}t$ is then distributed as $(a-1)\tau_{2-a}^{a-2}$, where τ_{2-a} is a $(2-a)$-stable variable, with Laplace transform $\mathbb{E}[\exp(-\lambda\tau_{2-a})] = \exp(-\lambda^{2-a})$.

Besides, Iksanov, Möhle and co-authors obtain various results on the asymptotic behavior of A_n for beta coalescents when $a \notin (1, 2)$. See [13] for a summary of these results.

2.3. Regenerative compositions

A *composition* of $n \in \mathbb{N}$ is a sequence $(c_1, c_2, \ldots, c_k)$, $c_i \in \mathbb{N}$, with sum $\sum_{i=1}^k c_i = n$. The integer k is called the *length* of the composition. If X_n is a Markov chain taking values in $\mathbb{Z}_+$, *strictly* *decreasing* on $\mathbb{N}$ and such that $X_n(0) = n$, the random sequence

$$C_i^{(n)} := X_n(i-1) - X_n(i), \qquad 1 \le i \le K^{(n)} := \inf\{k : X_n(k) = 0\},$$

clearly defines a random composition of n, of length $K^{(n)}$. Thanks to the Markov property of X, the random sequence $(C^{(n)}, n \ge 1)$ has the following *regenerative* property:

$$\left(C_2^{(n)}, C_3^{(n)}, \ldots, C_{K^{(n)}}^{(n)}\right) \quad \text{conditional on} \quad \left\{C_1^{(n)} = c_1\right\} \overset{\text{law}}{=} C^{(n-c_1)} \qquad \forall 1 \le c_1 < n.$$

This is called a *regenerative composition*. Conversely, starting from a regenerative composition $(C^{(n)}, n \ge 1)$, we build, for each $n \ge 1$, a strictly decreasing Markov chain X_n starting at n by setting

$$X_n(k) = n - \sum_{i=1}^{k} C_i^{(n)}, \qquad 1 \le k \le K^{(n)}, \quad \text{and} \quad X_n(k) = 0 \qquad \text{for } k \ge K^{(n)}.$$

The transition probabilities of the chain are $p_{n,k} = \mathbb{P}(C_1^{(n)} = n-k)$ for $0 \le k < n$, $p_{n,n} = 0$ for $n \ge 1$ and $p_{0,0} = 1$.

Regenerative compositions have been studied in great detail by Gnedin and Pitman [8] under the additional following *consistency* property: For all $n \ge 2$, if n balls are thrown at random into an ordered series of boxes according to $C^{(n)}$, then the composition of $n-1$ obtained by deleting one ball uniformly at random is distributed according to $C^{(n-1)}$. Gnedin and Pitman [8] show in particular that regenerative consistent compositions can be constructed via (unkilled) subordinators through the following procedure. Let ξ be such a subordinator and $(U_i, i \ge 1)$ be an independent sequence of i.i.d. random variables uniformly distributed on $(0, 1)$. Construct from this an ordered partition of $[n]$, say, $(B_1^{(n)}, \ldots, B_{K^{(n)}}^{(n)})$, by declaring that i and j are in the same block if and only if U_i and U_j are in the same open interval component of $[0, 1] \backslash \{1 - \exp(-\xi_t), t \ge 0\}^{\text{cl}}$. The order of blocks is naturally induced by the left-to-right order of open interval components. Then $((\#B_1^{(n)}, \ldots, \#B_{K^{(n)}}^{(n)}), n \ge 1)$ defines a regenerative consistent composition. Conversely, each regenerative consistent composition can be constructed in that way from a subordinator.

In cases where the subordinator has no drift and its Lévy measure ω has a tail that varies regularly at 0, that is, $\overline{\omega}(x) := \int_x^\infty \omega(\mathrm{d}y) = x^{-\gamma}\ell(x)$, where $\gamma \in (0, 1)$ and ℓ is slowly varying at 0, Gnedin, Pitman and Yor [9] show that

$$\frac{K^{(n)}}{\Gamma(1-\gamma) n^\gamma \ell(1/n)} \overset{\text{a.s.}}{\to} \int_0^\infty \exp(-\gamma \xi_r) \, \mathrm{d}r.$$

The duality between regenerative compositions and strictly decreasing Markov chains, coupled with Theorem 2, allows us to extend this result by Gnedin, Pitman and Yor to the largest setting of regenerative compositions that do not necessarily follow the consistency property, provided hypothesis (H) holds. Note, however, that in this more general context we can only obtain a convergence in distribution.

Let us check here that in the consistent cases, the assumption of regular variation on the tail of the Lévy measure associated with the composition entails (H). Following [8], the transition probabilities of the associated chain X are then given by

$$p_{n,k} = \mathbb{P}\big(C_1^{(n)} = n-k\big) = \frac{1}{Z_n}\binom{n}{k}\int_0^1 x^k(1-x)^{n-k}\tilde{\omega}(\mathrm{d}x), \qquad 0 \le k \le n-1,$$

where $\tilde{\omega}$ is the push-forward of ω by the mapping $x \mapsto \exp(-x)$ and Z_n is the normalizing constant $Z_n = \int_0^1 (1-x^n)\tilde{\omega}(\mathrm{d}x)$. It is easy to see that (1) is satisfied with $a_n = Z_n$ and $\mu(\mathrm{d}x) = (1-x)\tilde{\omega}(\mathrm{d}x)$ since the distributions $(p_{n,k}, 0 \le k \le n-1), n \ge 1$, are mixtures of binomial-type distributions (we refer to the proof of Proposition 1 or to that of the forthcoming Lemma 9 for detailed arguments in a similar context). The Laplace transform defined via μ by (3) is then that of a subordinator with Lévy measure ω, no drift and killing rate $\mathtt{k} = 0$. Besides, by Karamata's Tauberian theorem [3], Theorem 1.7.1′, the assumption $\overline{\omega}(x) = x^{-\gamma}\ell(x)$, $\gamma \in (0,1)$, where ℓ is slowly varying at 0, implies that

$$Z_n = \int_0^\infty (1-\mathrm{e}^{-nx})\omega(\mathrm{d}x) = n\int_0^\infty \mathrm{e}^{-nu}u^{-\gamma}\ell(u)\,\mathrm{d}u \sim \Gamma(1-\gamma)n^\gamma\ell(1/n) \qquad \text{as } n \to \infty,$$

and we have indeed (H) with the correct parameters $(a_n, n \ge 1)$ and μ.

Last, we rephrase Theorem 1 in terms of regenerative compositions.

Theorem 5. *Let $(C^{(n)}, n \ge 1)$ be a regenerative composition.*

(i) *Assume that it is consistent, constructed via a subordinator ξ with no drift and a Lévy measure with a tail $\overline{\omega}$ that varies regularly at* 0 *with index $-\gamma$, $\gamma \in (0,1)$. Then,*

$$\left(\sum_{k \le t\overline{\omega}(1/n)\Gamma(1-\gamma)} \frac{C_k^{(n)}}{n}, t \ge 0\right) \overset{(\mathrm{d})}{\to} \big(1-\exp\big(-\xi_{\tau(t)}\big), t \ge 0\big),$$

where τ is the usual time change defined as the inverse of $t \mapsto \int_0^t \exp(-\gamma\xi_r)\,\mathrm{d}r$.

(ii) *When the regenerative composition is non-consistent, assume that $\mathbb{E}[C_1^{(n)}]/n$ varies regularly as $n \to \infty$ with index $-\gamma$, $\gamma \in (0,1]$ and that*

$$\frac{\mathbb{E}[C_1^{(n)} f(1-C_1^{(n)}/n)]}{\mathbb{E}[C_1^{(n)}]} \to \int_{[0,1]} f(x)\mu(\mathrm{d}x)$$

for a probability measure μ on $[0,1]$ *and all continuous functions $f : [0,1] \to \mathbb{R}_+$. Then,*

$$\left(\sum_{k \le tn/\mathbb{E}[C_1^{(n)}]} \frac{C_k^{(n)}}{n}, t \ge 0\right) \overset{(\mathrm{d})}{\to} \big(1-\exp\big(-\xi_{\tau(t)}\big), t \ge 0\big),$$

where ξ is the subordinator with Laplace exponent defined via μ by (3) *and τ the usual time change.*

As was pointed out to us by a referee, the assertion (i) in this statement actually holds in the almost-sure sense. This is an easy consequence of [9], Theorem 4.1.

3. Preliminaries

Our goal now is to prove Theorems 1 and 2 and Proposition 2. We start in this section with some preliminaries. From now on and until the end of Section 4, we suppose that assumption (H) is in force. Consider the generating function defined for all $\lambda \geq 0$ by

$$G_n(\lambda) = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^{\lambda} p_{n,k} = \mathbb{E}\left[\left(\frac{X_n(1)}{n}\right)^{\lambda}\right] \tag{13}$$

with the convention $G_0(\lambda) = 0$. Then

$$1 - G_n(\lambda) = \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n}\right)^{\lambda}\right) p_{n,k} = \int_{[0,1]} [\lambda]_x (1-x) p_n^*(\mathrm{d}x), \tag{14}$$

where $[\lambda]_x$ was defined around (2). Thanks to (H), we immediately get

$$a_n\big(1 - G_n(\lambda)\big) \underset{n\to\infty}{\longrightarrow} \psi(\lambda), \qquad \lambda > 0, \tag{15}$$

the limit being the Laplace exponent defined at (3). In fact, if this convergence holds for every $\lambda > 0$, then (1) holds.

Proposition 3. *Assume that there exists a sequence of the form $a_n = n^{\gamma}\ell(n), n \geq 1$ for some slowly varying function $\ell : \mathbb{R}^+ \to (0,\infty)$, such that (15) holds for some function ψ and every $\lambda > 0$, or only for an infinite set of values of $\lambda \in (0,\infty)$ having at least one accumulation point. Then there exists a unique finite measure μ on $[0,1]$ such that $\psi(\lambda) = \int_0^1 [\lambda]_x \mu(\mathrm{d}x)$ for every $\lambda > 0$, and (H) holds for the sequence $(a_n, n \geq 1)$ and the measure μ.*

Proof. For any given $\lambda > 0$, the function $x \mapsto [\lambda]_x$ is bounded from below on $[0,1]$ by a positive constant $c_\lambda > 0$. Therefore, if (15) holds, then, using (14), we obtain that

$$\sup_{n\geq 1} \int_{[0,1]} a_n (1-x) p_n^*(\mathrm{d}x) \leq \frac{1}{c_\lambda} \sup_{n\geq 1} a_n\big(1 - G_n(\lambda)\big) < \infty.$$

Together with the fact that the measures $a_n(1-x)p_n^*(\mathrm{d}x), n \geq 1$ are all supported on $[0,1]$, this implies that all subsequences of $(a_n(1-x)p_n^*(\mathrm{d}x), n \geq 1)$ have a weakly convergent subsequence. Using (15) again, we see that any possible weak limit μ satisfies $\psi(\lambda) = \int_0^1 [\lambda]_x \mu(\mathrm{d}x)$. This function is analytic in $\lambda > 0$, and uniquely characterizes μ. The same holds if we only know this function on an infinite subset of $(0,\infty)$ having an accumulation point, by analytic continuation. $\square$

For some technical reasons, we need for the proofs to work with sequences $(a_n, n \geq 0)$ rather than sequences indexed by $\mathbb{N}$. We therefore complete all the sequences $(a_n, n \geq 1)$ involved in (H) or (15) with an initial term $a_0 = 1$. This is implicit in the whole Sections 3 and 4.

3.1. Basic inequalities

Let $\lambda > 0$ be fixed. By (15), there exists a finite constant $c_1(\lambda) > 0$ such that for every $n \geq 0$,

$$1 - G_n(\lambda) \leq \frac{c_1(\lambda)}{a_n}. \tag{16}$$

In particular, $G_n(\lambda) > 1/2$ for n large enough. Together with the fact that $a_n > 0$ for every $n \geq 0$, this entails the existence of an integer $n_0(\lambda) \geq 0$ and finite constants $c_2(\lambda), c_3(\lambda) > 0$ such that, for every $n \geq n_0(\lambda)$,

$$-\ln(G_n(\lambda)) \leq \frac{c_2(\lambda)}{a_n} \leq c_3(\lambda). \tag{17}$$

When, moreover, $p_{n,n} < 1$ for all $n \geq 1$ (or, equivalently, $G_n(\lambda) < 1$ for all $n \geq 1$), we obtain the existence of a finite constant $c_4(\lambda) > 0$ such that, for every $n \geq 1$

$$\ln(G_n(\lambda)) \leq -\frac{c_4(\lambda)}{a_n}. \tag{18}$$

Last, since $(a_n, n \geq 0)$ is regularly varying with index γ and since $a_n > 0$ for all $n \geq 0$, we get from Potter's bounds [3], Theorem 1.5.6, that for all $\varepsilon > 0$, there exist finite positive constants $c'_1(\varepsilon)$ and $c'_2(\varepsilon)$ such that, for all $1 \leq k \leq n$

$$c'_1(\varepsilon)\left(\frac{n}{k}\right)^{\gamma-\varepsilon} \leq \frac{a_n}{a_k} \leq c'_2(\varepsilon)\left(\frac{n}{k}\right)^{\gamma+\varepsilon}. \tag{19}$$

3.2. Time changes

Let $f : [0, \infty) \to [0, 1]$ be a càdlàg non-increasing function. We let $\sigma_f := \inf\{t \geq 0 : f(t) = 0\}$, with the convention $\inf\{\varnothing\} = \infty$. Now fix $\gamma > 0$. For $0 \leq t < \sigma_f$, we let

$$\tau_f(t) := \int_0^t f(r)^{-\gamma}\, \mathrm{d}r,$$

and $\tau_f(t) = \infty$ for $t \geq \sigma_f$. Then $(\tau_f(t), t \geq 0)$ is a right-continuous, non-decreasing process with values in $[0, \infty]$, and which is continuous and strictly increasing on $[0, \sigma_f)$. Note that $\tau_f(\sigma_f-) = \int_0^{\sigma_f} f(r)^{-\gamma}\, \mathrm{d}r$ might be finite or infinite. We set

$$\tau_f^{-1}(t) = \inf\{u \geq 0 : \tau_f(u) > t\}, \qquad t \geq 0,$$

which defines a continuous, non-decreasing function on $\mathbb{R}_+$, that is strictly increasing on $[0, \tau_f(\sigma_f-))$, constant equal to σ_f on $[\tau_f(\sigma_f-), \infty)$, with limit $\tau_f^{-1}(\infty) = \sigma_f$. The functions τ_f and τ_f^{-1}, respectively restricted to $[0, \sigma_f)$ and $[0, \tau_f(\sigma_f-))$, are inverses of each other. The function τ_f is recovered from τ_f^{-1} by the analogous formula $\tau_f(t) = \inf\{u \geq 0 : \tau_f^{-1}(u) > t\}$, for any $t \geq 0$.

We now consider the function

$$g(t) := f(\tau_f^{-1}(t)), \qquad t \geq 0,$$

which is also càdlàg, non-increasing, with values in [0, 1], and satisfies $\sigma_g = \tau_f(\sigma_f -)$. Note also that $f(t) = g(\tau_f(t))$, $t \geq 0$, where, by convention, $g(\infty) = 0$. Finally, we have

$$\mathrm{d}\tau_f(t) = f(t)^{-\gamma}\,\mathrm{d}t \qquad \text{on } [0, \sigma_f)$$

and

$$\tau_f^{-1}(t) = f(\tau_f^{-1}(t))^{\gamma}\,\mathrm{d}t = g(t)^{\gamma}\,\mathrm{d}t \qquad \text{on } [0, \sigma_g).$$

Now, for $c > 0$, we will often use the change of variables $u = \tau_f(r/c)$ to get that when $g(t) > 0$ (i.e., $t < \sigma_g$), for any measurable, non-negative function h,

$$\int_0^{c\tau_f^{-1}(t)} h(f(r/c))\,\mathrm{d}r = c\int_0^t h(g(u))g(u)^{\gamma}\,\mathrm{d}u. \tag{20}$$

In particular $\tau_f^{-1}(t) = \int_0^t g(r)^{\gamma}\,\mathrm{d}r$ for $t < \tau_f(\sigma_f -)$. This remains true for $t \geq \tau_f(\sigma_f -)$ since $g(t) = 0$ for $t \geq \tau_f(\sigma_f -)$. Consequently, $\tau_f^{-1}(t) = \int_0^t g^{\gamma}(r)\,\mathrm{d}r$ for all $t \geq 0$ and

$$\sigma_f = \int_0^{\infty} g(r)^{\gamma}\,\mathrm{d}r. \tag{21}$$

This also implies that $\tau_f(t) = \inf\{u \geq 0 : \int_0^u g^{\gamma}(r)\,\mathrm{d}r > t\}$ for every $t \geq 0$.

3.3. Martingales associated with X_n

We finally recall the very classical fact that if P is the transition function of a Markov chain X with countable state space M, then for any non-negative function f, the process defined by

$$f(X(k)) + \sum_{i=0}^{k-1}(\mathrm{Id} - P)f(X(i)), \qquad k \geq 0,$$

is a martingale, provided all the terms of this process are integrable. When, moreover, $f^{-1}(\{0\})$ is an absorbing set (i.e., $f(X(k)) = 0$ implies $f(X(k+1)) = 0$), the process defined by

$$f(X(k))\prod_{i=0}^{k-1}\frac{f(X(i))}{Pf(X(i))}, \qquad k \geq 0,$$

with the convention $0 \cdot \infty = 0$ is also a martingale (absorbed at 0), provided all the terms are integrable. From this, we immediately obtain the following.

Proposition 4. *For every $\lambda > 0$ and every integer $n \geq 1$, the processes defined by*

$$\left(\frac{X_n(k)}{n}\right)^{\lambda} + \sum_{i=0}^{k-1}\left(\frac{X_n(i)}{n}\right)^{\lambda}\left(1 - G_{X_n(i)}(\lambda)\right), \qquad k \geq 0, \tag{22}$$

and

$$\Upsilon_n^{(\lambda)}(k) = \left(\frac{X_n(k)}{n}\right)^{\lambda}\left(\prod_{i=0}^{k-1} G_{X_n(i)}(\lambda)\right)^{-1}, \qquad k \geq 0, \tag{23}$$

are martingales with respect to the filtration generated by X_n, with the convention that $\Upsilon_n^{(\lambda)}(k) = 0$ whenever $X_n(k) = 0$.

4. Scaling limits of non-increasing Markov chains

We now start the proof of Theorems 1 and 2 and Proposition 2. As mentioned before, this is done by first establishing tightness for the processes $Y_n(t) = X_n(\lfloor a_n t\rfloor)/n$, $t \geq 0$. We recall that (H) is assumed throughout the section, except in the last subsection, which is devoted to the proof of Proposition 1.

4.1. Tightness

Lemma 1. *The sequence $(Y_n, n \geq 0)$ is tight with respect to the Skorokhod topology.*

Our proof is based on Aldous' tightness criterion, which we first recall.

Lemma 2 (Aldous' tightness criterion [2], Theorem 16.10). *Let $(F_n, n \geq 0)$ be a sequence of $\mathcal{D}$-valued stochastic processes and for all n denote by $\mathcal{J}(F_n)$ the set of stopping times with respect to the filtration generated by F_n. Suppose that for all fixed $t > 0$, $\varepsilon > 0$*

(i) $\displaystyle \lim_{a\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{s\in[0,t]} F_n(s) > a\Big) = 0;$

(ii) $\displaystyle \lim_{\theta_0\to 0} \limsup_{n\to\infty} \sup_{T\in\mathcal{J}(F_n), T\leq t} \sup_{0\leq\theta\leq\theta_0} \mathbb{P}\big(|F_n(T) - F_n(T+\theta)| > \varepsilon\big) = 0,$

then the sequence $(F_n, n \geq 0)$ is tight with respect to the Skorokhod topology.

Proof of Lemma 1. Part (i) of Aldous' tightness criterion is obvious since $Y_n(t) \in [0, 1]$, for every $n \in \mathbb{Z}_+$, $t \geq 0$. To check part (ii), consider some $\lambda > \max(\gamma, 1)$, where γ denotes the index of regular variation of $(a_n, n \geq 0)$. Then, on the one hand, for all $n \geq 1$, since the process Y_n is non-increasing and $\lambda \geq 1$, we have for all (possibly random) times T and all $\theta \geq 0$,

$$|Y_n(T) - Y_n(T+\theta)|^{\lambda} \leq Y_n^{\lambda}(T) - Y_n^{\lambda}(T+\theta).$$

On the other hand, let T be a bounded stopping time in $\mathcal{J}(Y_n)$. Then $\lfloor a_n T\rfloor$ is a stopping time with respect to the filtration generated by X_n. Applying Doob's optional stopping theorem to the martingale (22) yields, for every $\theta \geq 0$,

$$\mathbb{E}[Y_n^\lambda(T) - Y_n^\lambda(T+\theta)] = n^{-\lambda}\mathbb{E}\left[\sum_{i=\lfloor a_n T\rfloor}^{\lfloor a_n(T+\theta)\rfloor - 1} X_n^\lambda(i)\big(1 - G_{X_n(i)}(\lambda)\big)\right]$$
$$\leq c_1(\lambda) n^{-\lambda}\mathbb{E}\left[\sum_{i=\lfloor a_n T\rfloor}^{\lfloor a_n(T+\theta)\rfloor - 1} \frac{X_n^\lambda(i)}{a_{X_n(i)}}\right],$$

where we used (16) at the last step. Next, since $\lambda > \gamma$, $X_n^\lambda(i)/a_{X_n(i)} \leq c_2'(\lambda-\gamma)n^\lambda/a_n$ for all $n, i \geq 0$, where $c_2'(\varepsilon)$ was introduced in (19) (note that the inequality is obvious when $X_n(i) = 0$, since $a_0 > 0$). Hence, for every bounded $T \in \mathcal{J}(Y_n)$ and $\theta \geq 0$,

$$\mathbb{E}[|Y_n(T) - Y_n(T+\theta)|^\lambda] \leq \mathbb{E}[Y_n^\lambda(T) - Y_n^\lambda(T+\theta)]$$
$$\leq \frac{c_1(\lambda)c_2'(\lambda-\gamma)}{a_n}\mathbb{E}[\lfloor a_n(T+\theta)\rfloor - \lfloor a_n T\rfloor]$$
$$\leq c_1(\lambda)c_2'(\lambda-\gamma)(\theta + a_n^{-1}),$$

which immediately yields (ii) in Aldous' tightness criterion. □

4.2. Identification of the limit

We now want to prove uniqueness of the possible limits in distribution of subsequences of $Y_n, n \geq 0$. Let $(n_k, k \geq 0)$ be a strictly increasing sequence, such that the process Y_n converges in distribution to a limit Y' when n varies along (n_k). To identify the distribution of Y', recall the definition of $Z_n = (Y_n(\tau_n^{-1}(t)), t \geq 0)$ at the end of Section 1.1. From the discussion in Section 3.2, we have

$$Z_n(t) = Y_n\left(\int_0^t Z_n(r)^\gamma \,\mathrm{d}r\right), \qquad t \geq 0.$$

As in Section 3.2, let $\tau_{Y'}(u) = \int_0^u Y'(r)^{-\gamma}\,\mathrm{d}r$ if $Y'(u) > 0$ and $\tau_{Y'}(u) = \infty$ otherwise, and let Z' be the process defined by

$$Z'(t) = Y'(\tau_{Y'}^{-1}(t)),$$

where $\tau_{Y'}^{-1}(t) = \inf\{u \geq 0 : \tau_{Y'}(u) > t\}$, so that

$$Z'(t) = Y'\left(\int_0^t Z'(r)^\gamma \,\mathrm{d}r\right), \qquad t \geq 0.$$

Then, as a consequence of [6], Theorem 1.5, Chapter 6 (it is in fact a consequence of a step in the proof of this theorem rather than its exact statement), the convergence in distribution of Y_n

to Y' along (n_k) entails that of (Y_n, Z_n) to (Y', Z') in $\mathcal{D}^2$ along the same subsequence, provided the following holds:

$$\sigma_{Y'} = \inf\{s \geq 0 : Y'(s)^\gamma = 0\} = \inf\left\{s \geq 0 : \int_0^s Y'(u)^{-\gamma}\,\mathrm{d}u = \infty\right\} = \lim_{\varepsilon \to 0} \inf\{s \geq 0 : Y'(s)^\gamma < \varepsilon\},$$

which is obviously true here since Y' is a.s. càdlàg non-increasing. Therefore, the proof of Theorem 1 and Proposition 2 will be completed provided we show the following.

Lemma 3. *The process Z' has same distribution as $Z = (\exp(-\xi_t), t \geq 0)$, where ξ is a subordinator with Laplace exponent ψ.*

To see that this entails Proposition 2 (hence Theorem 1), note that $\tau_{Y'}(t) = \inf\{u \geq 0 : \int_0^u Z'(r)^\gamma\,\mathrm{d}r > t\}$ and $Y'(t) = Z'(\tau_{Y'}(t))$, for $t \geq 0$, as detailed in Section 3.2. So the previous lemma entails that the only possible limiting distribution for (Y_n) along a subsequence is that of Y as defined in (4). Since $(Y_n, n \geq 0)$ is a tight sequence, this shows that it converges in distribution to Y, and then that (Y_n, Z_n) converges to (Y, Z), entailing Proposition 2.

To prove Lemma 3, we need a pair of results on Skorokhod convergence, which are elementary and left to the reader. The first lemma is an obvious consequence of the definition of Skorokhod convergence. The second one can be proved, for example, by using Proposition 6.5 in [6], Chapter 3.

Lemma 4. *Suppose that $f_n \to f$ on $\mathcal{D}$ and that $(g_n, n \geq 0)$ is a sequence of càdlàg non-negative functions on $[0, \infty)$ converging uniformly on compacts to a continuous function g. Then $f_n g_n \to fg$ on $\mathcal{D}$.*

Lemma 5. *Suppose that f_n, f are non-increasing, non-negative functions in $\mathcal{D}$ such that $f_n \to f$. Let $\varepsilon > 0$ be such that there is at most one $x \in [0, \infty)$ such that $f(x) = \varepsilon$. Define*

$$t_{n,\varepsilon} := \inf\{t \geq 0 : f_n(t) \leq \varepsilon\} \quad \textit{and} \quad t_\varepsilon := \inf\{t \geq 0 : f(t) \leq \varepsilon\}$$

(which can be infinite). Then it holds that $t_{n,\varepsilon} \to t_\varepsilon$ as $n \to \infty$, and if $f(t_\varepsilon -) > \varepsilon$ or $f(t_\varepsilon -) = f(t_\varepsilon)$, then

$$\big(f_n(t \wedge t_{n,\varepsilon}), t \geq 0\big) \to \big(f(t \wedge t_\varepsilon), t \geq 0\big).$$

Proof of Lemma 3. Fix $\lambda > 0$ and consider the martingale $(\Upsilon_n^{(\lambda)}(k), k \geq 0)$ of Proposition 4. This is a martingale with respect to the filtration generated by X_n. Therefore, the process $(\Upsilon_n^{(\lambda)}(\lfloor a_n t \rfloor), t \geq 0)$ is a continuous-time martingale with respect to the filtration generated by Y_n. Next, note that for all $t \geq 0$, $\tau_n^{-1}(t)$ is a stopping time with respect to this filtration, which is bounded (by t). Hence, by Doob's optional stopping theorem, the process

$$M_n^{(\lambda)}(t) = Z_n(t)^\lambda \left(\prod_{i=0}^{\lfloor a_n \tau_n^{-1}(t) \rfloor - 1} G_{X_n(i)}(\lambda) \right)^{-1}, \qquad t \geq 0 \tag{24}$$

(with the usual convention $0 \cdot \infty = 0$) is a continuous-time martingale with respect to the filtration generated by Z_n.

We want to exploit the sequences of martingales $(M_n^{(\lambda)}, n \geq 0)$ in order to prove that the processes $(Z'(t)^\lambda \exp(\psi(\lambda)t), t \geq 0)$ are (càdlàg) martingales with respect to the filtration that they generate, for every $\lambda > 0$. It is then easy to check that $-\ln(Z')$ is a subordinator starting from 0 with Laplace exponent ψ.

Using the Skorokhod representation theorem, we may assume that the convergence of Z_n to Z' along (n_k) is almost sure. We consider stopped versions of the martingale $M_n^{(\lambda)}$. For all $\varepsilon > 0$ and all $n \geq 1$, let

$$T_{n,\varepsilon} := \inf\{t \geq 0 : Z_n(t) \leq \varepsilon\} \quad \text{and} \quad T_\varepsilon := \inf\{t \geq 0 : Z'(t) \leq \varepsilon\},$$

(which are possibly infinite) and note that $T_{n,\varepsilon}$ (resp. T_ε) is a stopping time with respect to the filtration generated by Z_n (resp. Z').

Let C_1 be the set of positive real numbers $\varepsilon > 0$ such that

$$\mathbb{P}\big(\exists t_1, t_2 \geq 0 : t_1 \neq t_2, Z'(t_1) = Z'(t_2) = \varepsilon\big) > 0.$$

We claim that this set is at most countable. Indeed, fix an $\varepsilon > 0$ and an integer $K > 0$, and consider the set

$$B_{\varepsilon,K} = \{\exists t_1, t_2 \in [0, K] : |t_1 - t_2| > K^{-1}, Z'(t_1) = Z'(t_2) = \varepsilon\}.$$

Let $C_{1,K}$ be the set of numbers ε such that $\mathbb{P}(B_{\varepsilon,K}) > K^{-1}$. If this set contained an infinite sequence $(\varepsilon_i, i \geq 0)$, then by the reverse Fatou lemma, we would obtain that the probability that infinitely many of the events $(B_{\varepsilon_i,K}, i \geq 0)$ occur is at least K^{-1}. Clearly, this is impossible. Therefore, $C_{1,K}$ is finite for every integer $K > 0$. Since C_1 is the increasing union

$$C_1 = \bigcup_{K \in \mathbb{N}} C_{1,K},$$

we conclude that it is at most countable. For similar reasons, the set C_2 of real numbers $\varepsilon > 0$ such that $\mathbb{P}(Z'(T_\varepsilon-) = \varepsilon > Z'(T_\varepsilon)) > 0$ is at most countable.

In the rest of this proof, although all the statements and convergences are in the almost sure sense, we omit the "a.s." in order to have a lighter presentation. Our goal is to check that for all $\lambda > 0$ and all $\varepsilon \notin C_1 \cup C_2$,

(a) as $n \to \infty$, the sequence of martingales $(M_n^{(\lambda)}(t \wedge T_{n,\varepsilon}), t \geq 0)$ converges to the process $(Z'(t \wedge T_\varepsilon)^\lambda \exp(\psi(\lambda)(t \wedge T_\varepsilon)), t \geq 0)$,
(b) the process $(Z'(t \wedge T_\varepsilon)^\lambda \exp(\psi(\lambda)(t \wedge T_\varepsilon)), t \geq 0)$ is a martingale with respect to its natural filtration,
(c) the process $(Z'(t)^\lambda \exp(\psi(\lambda)t)), t \geq 0)$ is a martingale with respect to its natural filtration.

We start with the proof of (a). Fix $\lambda > 0$, a positive $\varepsilon \notin C_1 \cup C_2$ and recall the definition of $n_0(\lambda)$ in (17). Let $n \geq n_0(\lambda)/\varepsilon$. When $Z_n(t \wedge T_{n,\varepsilon}) > 0$, we can rewrite

$$M_n^{(\lambda)}(t \wedge T_{n,\varepsilon}) = \big(Z_n(t \wedge T_{n,\varepsilon})\big)^\lambda \exp\left(\int_0^{\lfloor a_n \tau_n^{-1}(t \wedge T_{n,\varepsilon})\rfloor} -\ln\big(G_{X_n(\lfloor r \rfloor)}(\lambda)\big)\,\mathrm{d}r\right). \tag{25}$$

This identity is still true when $Z_n(t \wedge T_{n,\varepsilon}) = 0$. Indeed, even when $Z_n(t \wedge T_{n,\varepsilon}) = 0$, for $r < \lfloor a_n \tau_n^{-1}(t \wedge T_{n,\varepsilon}) \rfloor$, it holds that $X_n(\lfloor r \rfloor) \geq n\varepsilon \geq n_0(\lambda)$. Therefore, by (17), the integral involved in (25) is well defined and finite. Hence (25) is valid for all $t \geq 0$.

More precisely, as soon as $r < \lfloor a_n \tau_n^{-1}(t \wedge T_{n,\varepsilon}) \rfloor$, we have by (17) that

$$-\ln G_{X_n(\lfloor r \rfloor)}(\lambda) \leq \frac{c_2(\lambda)}{a_{X_n(\lfloor r \rfloor)}},$$

which, together with the change of variable identity (20), implies that

$$\int_0^{\lfloor a_n \tau_n^{-1}(t \wedge T_{n,\varepsilon}) \rfloor} -\ln\bigl(G_{X_n(\lfloor r \rfloor)}(\lambda)\bigr)\,\mathrm{d}r \leq c_2(\lambda) \int_0^{t \wedge T_{n,\varepsilon}} \frac{a_n}{a_{nZ_n(r)}} Z_n(r)^\gamma\,\mathrm{d}r.$$

Potter's bounds (19) and the fact that $Z_n(r) > \varepsilon$ for $r < t \wedge T_{n,\varepsilon}$ lead to the existence of a finite constant $c_{\lambda,\varepsilon}$ such that for every $r < t \wedge T_{n,\varepsilon}$,

$$\frac{c_2(\lambda) a_n Z_n(r)^\gamma}{a_{nZ_n(r)}} \leq c_{\lambda,\varepsilon}.$$

Therefore, for every $t \geq 0$,

$$\int_0^{\lfloor a_n \tau_n^{-1}(t \wedge T_{n,\varepsilon}) \rfloor} -\ln\bigl(G_{X_n(\lfloor r \rfloor)}(\lambda)\bigr)\,\mathrm{d}r \leq c_{\lambda,\varepsilon}(t \wedge T_{n,\varepsilon}) \leq c_{\lambda,\varepsilon} t.$$

In particular,

$$M_n^{(\lambda)}(t \wedge T_{n,\varepsilon}) \leq \exp(c_{\lambda,\varepsilon} t) \qquad \forall t \geq 0. \tag{26}$$

Now we let $n \to \infty$. Since $\varepsilon \notin C_1 \cup C_2$, we have, by Lemma 5, with probability 1,

$$T_{n,\varepsilon} \to T_\varepsilon \quad \text{and} \quad \bigl(Z_n(t \wedge T_{n,\varepsilon}), t \geq 0\bigr) \to \bigl(Z'(t \wedge T_\varepsilon), t \geq 0\bigr).$$

Using (25) and Lemma 4, we see that it is sufficient to prove that

$$\left(\exp\left(\int_0^{\lfloor a_n \tau_n^{-1}(t \wedge T_{n,\varepsilon}) \rfloor} -\ln\bigl(G_{X_n(\lfloor r \rfloor)}(\lambda)\bigr)\,\mathrm{d}r\right), t \geq 0\right) \underset{n\to\infty}{\to} \bigl(\exp\bigl(\psi(\lambda)(t \wedge T_\varepsilon)\bigr), t \geq 0\bigr) \tag{27}$$

uniformly on compacts to get the convergence of martingales stated in (a).

Since we are dealing with non-decreasing processes and the limit is continuous, it is sufficient to check the pointwise convergence by Dini's theorem. Fix $t \geq 0$. It is well known (see [6], Proposition 5.2, Chapter 3) that the Skorokhod convergence implies that $Z_n(r) \to Z'(r)$ for all r that is not a jump time of Z', hence for a.e. r. For such an r, if $Z'(r) > 0$, we have $nZ_n(r) \to \infty$. Hence if $r < t \wedge T_{n,\varepsilon}$, we have $Z_n(r) \geq \varepsilon$, so that

$$-\ln\bigl(G_{nZ_n(r)}(\lambda)\bigr) a_n Z_n(r)^\gamma \underset{n\to\infty}{\sim} \frac{a_n}{a_{nZ_n(r)}} Z_n(r)^\gamma \psi(\lambda) \underset{n\to\infty}{\to} \psi(\lambda),$$

using the uniform convergence theorem for slowly varying functions ([3], Theorem 1.2.1). Moreover, as explained above, the left-hand side of this expression is bounded from above by $c_{\lambda,\varepsilon}$ as soon as $n \geq n_0(\lambda)/\varepsilon$. This implies, using (20), that

$$\int_0^{a_n \tau_n^{-1}(t\wedge T_{n,\varepsilon})} -\ln\bigl(G_{X_n(\lfloor r \rfloor)}(\lambda)\bigr)\,\mathrm{d}r = \int_0^{t\wedge T_{n,\varepsilon}} -\ln\bigl(G_{nZ_n(r)}(\lambda)\bigr)a_n Z_n(r)^\gamma\,\mathrm{d}r$$

converges to $\psi(\lambda)(t\wedge T_\varepsilon)$ by dominated convergence. Last, note that

$$\begin{aligned}\int_{\lfloor a_n\tau_n^{-1}(t\wedge T_{n,\varepsilon})\rfloor}^{a_n \tau_n^{-1}(t\wedge T_{n,\varepsilon})} -\ln\bigl(G_{X_n(\lfloor r \rfloor)}(\lambda)\bigr)\,\mathrm{d}r &\leq -\ln\bigl(G_{X_n(\lfloor a_n\tau_n^{-1}(t)\rfloor)}(\lambda)\bigr)\mathbf{1}_{\{t<T_{n,\varepsilon}\}}\\ &= -\ln\bigl(G_{nZ_n(t)}(\lambda)\bigr)\mathbf{1}_{\{t<T_{n,\varepsilon}\}},\end{aligned}$$

since $X_n(\lfloor r\rfloor)$ is constant on the integration interval and $a_n\tau_n^{-1}(t\wedge T_{n,\varepsilon})$ is an integer when $t\geq T_{n,\varepsilon}$. The right-hand side in the inequality above converges to 0 as $n\to\infty$ since $nZ_n(t)>n\varepsilon$ when $t<T_{n,\varepsilon}$ and $G_n(\lambda)\to 1$ as $n\to\infty$. Finally, we have proved the convergence (27), hence (a).

The assertion (b) follows as a simple consequence of (a). By (26) we have that, for each $t\geq 0$, $(M_n^{(\lambda)}(t\wedge T_{n,\varepsilon}), n\geq n_0(\lambda)/\varepsilon)$ is uniformly integrable. Together with the convergence of (a), this is sufficient to deduce that the limit process $(Z'(t\wedge T_\varepsilon)^\lambda\exp(\psi(\lambda)(t\wedge T_\varepsilon)), t\geq 0)$ is a martingale with respect to its natural filtration. See [6], Example 7, page 362.

We finally prove (c). Note that

$$\bigl(Z'(t\wedge T_\varepsilon)^\lambda \exp\bigl(\psi(\lambda)(t\wedge T_\varepsilon)\bigr), t\geq 0\bigr) \underset{\varepsilon\to 0}{\longrightarrow} \bigl(Z'(t)^\lambda\exp(\psi(\lambda)t), t\geq 0\bigr)$$

for the Skorokhod topology. Besides, for each $t\geq 0$ and $\varepsilon>0$, we have

$$Z'(t\wedge T_\varepsilon)^\lambda\exp(\psi(\lambda)(t\wedge T_\varepsilon)) \leq \exp(\psi(\lambda)t).$$

As before, we can use an argument of uniform integrability to conclude that $(Z'(t)^\lambda\exp(\psi(\lambda)t), t\geq 0)$ is a martingale. □

4.3. Absorption times

Recall that A_n denotes the first time at which X_n reaches the set of absorbing states $\mathcal{A}$. To start with, we point out that there is no loss of generality in assuming that $\mathcal{A}=\{0\}$. Indeed, let $a_{\max}$ be the largest element of $\mathcal{A}$. If $a_{\max}\geq 1$, one can build a Markov chain $\tilde{X}_n$ starting from n and with transition probabilities $\tilde{p}_{i,j}=p_{i,j}$ for $i\notin\mathcal{A}$ and all $j\geq 0$, $\tilde{p}_{i,0}=1$ for $i\in\mathcal{A}$, so that

$$\tilde{X}_n(k)=X_n(k) \qquad \text{for } k\leq A_n \quad\text{and}\quad \tilde{X}_n(k)=0 \qquad \text{if } k>A_n.$$

Clearly, this modified chain has a unique absorbing state, which is 0, and the transition probabilities $(\tilde{p}_{n,k})$ satisfy (H) if and only if $(p_{n,k})$ do. Besides, the first time $\tilde{A}_n$ at which $\tilde{X}_n$ reaches 0 is clearly either equal to A_n or to $A_n + 1$. Moreover, constructing $\tilde{Y}_n$ from $\tilde{X}_n$ as Y_n is defined from X_n in Section 1.1, we see that $\sup_{t\geq 0} |\tilde{Y}_n(t) - \tilde{Y}_n(t)| \leq a_{\max}/n$. This is enough to see that the convergence in distribution as $n \to \infty$ of $(\tilde{A}_n/a_n, \tilde{Y}_n)$ entails that of $(A_n/a_n, Y_n)$ towards the same limit. This in turn entails the convergence in distribution of $(A_n/a_n, Y_n, Z_n)$ to the required limit, using a part of the proof of [6], Theorem 1.5, Chapter 6, as already mentioned at the beginning of Section 4.2. In conclusion, if the convergence of Theorem 2 is proved for the sequence $(\tilde{A}_n/a_n, \tilde{Y}_n, \tilde{Z}_n)$, $n \geq 0$, it will also hold for $(A_n/a_n, Y_n, Z_n)$, $n \geq 0$, with the same distribution limit. In the following, we will therefore additionally suppose that $a_{\max} = 0$, that is,

$$\mathcal{A} = \{0\}, \quad \text{or equivalently,} \quad p_{n,n} < 1 \qquad \text{for every } n \geq 1. \tag{28}$$

We now set out a preliminary lemma that we will use for the proof of Theorem 2.

Lemma 6. *For every $\lambda > 0$ and $\beta > 0$, there exists some finite constant $c_{\lambda,\beta} > 0$ such that for all $n \in \mathbb{Z}_+$ and all $t \geq 0$,*

$$Z_n(t)^\lambda \leq \frac{c_{\lambda,\beta} M_n^{(\lambda)}(t) + 1}{t^\beta}, \tag{29}$$

where the processes $M_n^{(\lambda)}$ are the martingales defined in (24). Consequently,

$$\mathbb{E}[Z_n(t)^\lambda] \leq \frac{c_{\lambda,\beta} + 1}{t^\beta}.$$

In the cases where $n^{-\gamma} a_n \to \ell \in (0, \infty)$, our proof can be adapted to get the following stronger result: There exists some finite constant c_λ such that for all $n \in \mathbb{Z}_+$ and all $t \geq 0$, $Z_n(t)^\lambda \leq M_n^{(\lambda)}(t) \exp(c_\lambda(1-t))$, and consequently, $\mathbb{E}[Z_n(t)^\lambda] \leq \exp(c_\lambda(1-t))$.

Proof. Fix $\lambda > 0$ and $\beta > 0$. For a given n, t, if $Z_n(t)^\lambda \leq t^{-\beta}$, then obviously (29) is satisfied, irrespective of any choice of $c_{\lambda,\beta}$. So we assume that $Z_n(t)^\lambda > t^{-\beta}$, and in particular, $Z_n(t) > 0$. By (24), we have

$$Z_n(t)^\lambda = M_n^{(\lambda)}(t) \exp\left(\int_0^{\lfloor a_n \tau_n^{-1}(t) \rfloor} \ln\bigl(G_{X_n(\lfloor r \rfloor)}(\lambda)\bigr)\, \mathrm{d}r\right).$$

Note that $X_n(\lfloor r \rfloor) \geq 1$ as soon as $r \leq \lfloor a_n \tau_n^{-1}(t) \rfloor$ and $Z_n(t) > 0$. Moreover, under the assumption (28), we have $\ln(G_n(\lambda)) \leq -c_4(\lambda)/a_n < 0$ for every $n \geq 1$ by (18). Hence, for all $\varepsilon > 0$,

$$\begin{aligned}
&\int_0^{\lfloor a_n \tau_n^{-1}(t) \rfloor} \ln\bigl(G_{X_n(\lfloor r \rfloor)}(\lambda)\bigr)\, \mathrm{d}r \\
&\quad \leq \int_0^{\lfloor a_n \tau_n^{-1}(t) \rfloor} \frac{-c_4(\lambda)}{a_{X_n(\lfloor r \rfloor)}}\, \mathrm{d}r = \int_0^{a_n \tau_n^{-1}(t)} \frac{-c_4(\lambda)}{a_{X_n(\lfloor r \rfloor)}}\, \mathrm{d}r - \int_{\lfloor a_n \tau_n^{-1}(t) \rfloor}^{a_n \tau_n^{-1}(t)} \frac{-c_4(\lambda)}{a_{X_n(\lfloor r \rfloor)}}\, \mathrm{d}r
\end{aligned}$$

$$\underset{\text{by (20)}}{\leq} -c_4(\lambda)\int_0^t \frac{a_n}{a_{nZ_n(r)}} Z_n(r)^\gamma \, \mathrm{d}r + \frac{c_4(\lambda)}{\inf_{k\geq 0} a_k}$$

$$\underset{\text{by (19)}}{\leq} -c_4(\lambda)c_1'(\varepsilon)\int_0^t Z_n(r)^\varepsilon \, \mathrm{d}r + \frac{c_4(\lambda)}{\inf_{k\geq 0} a_k} \leq -c_4(\lambda)c_1'(\varepsilon) t Z_n(t)^\varepsilon + \frac{c_4(\lambda)}{\inf_{k\geq 0} a_k}.$$

Since $Z_n(t)^\lambda > t^{-\beta}$, we have, taking $\varepsilon = \lambda/2\beta$, the existence of a finite constant $c_{\lambda,\beta}$, independent of n and t, such that

$$Z_n(t)^\lambda \leq M_n^{(\lambda)}(t)\exp\Bigl(-c_4(\lambda)c_1'(\lambda/2\beta)t^{1/2} + c_4(\lambda)/\inf_{k\geq 0} a_k\Bigr) \leq c_{\lambda,\beta} M_n^{(\lambda)}(t)/t^\beta,$$

giving the result. □

Proof of Theorem 2. Notice that the first time at which Y_n reaches 0 is

$$\int_0^\infty Z_n(r)^\gamma \, \mathrm{d}r = \sigma_n = A_n/a_n$$

using (21) for the first equality and (28) for the second equality. The previous lemma ensures that $\sup_{n\geq 1} \mathbb{E}[\sigma_n] < \infty$, which implies that the sequence $(\sigma_n, n \geq 1)$ is tight. In turn, this implies that the sequence $((Y_n, Z_n, \sigma_n), n \geq 1)$ is tight.

The proof of Theorem 2 will therefore be completed if we prove the uniqueness of possible limiting distributions of $((Y_n, Z_n, \sigma_n), n \geq 1)$ along a subsequence. In that aim, consider a strictly increasing sequence of integers $(n_k, k \geq 0)$ such that the sequence $((Y_n, Z_n, \sigma_n), n \geq 1)$ converges in distribution along (n_k) to a limit (Y', Z', σ'). By Proposition 2, (Y', Z') has same distribution as (Y, Z), so by abuse of notations, for simplicity, we write (Y, Z) instead of (Y', Z'). Our goal is to show that σ' is the extinction time $\sigma = \sigma_Y = \int_0^\infty Z(r)^\gamma \, \mathrm{d}r$, with the notations of Section 3.2.

By the Skorokhod representation theorem, we may suppose that the convergence of (Y_n, Z_n, σ_n) to (Y, Z, σ') is almost sure. It is then immediately checked that a.s.,

$$\sigma' = \liminf_{n\to\infty} \inf\{t \geq 0 : Y_n(t) = 0\} \geq \inf\{t \geq 0 : Y(t) = 0\} = \sigma,$$

so in order to show that $\sigma = \sigma'$ a.s., it suffices to check that $\mathbb{E}[\sigma'] \leq \mathbb{E}[\sigma]$. To see this, note that the convergence in the Skorokhod sense implies that a.s., for a.e. t, $Z_n(t) \to Z(t)$ and therefore, by Fubini's theorem, that for a.e. t, $Z_n(t) \to Z(t)$ a.s. We then obtain that for a.e. t, $Z_n(t)^\gamma \to Z(t)^\gamma$ a.s., and since all these quantities are bounded by 1, we have, by dominated convergence, that for a.e. t, $\mathbb{E}[Z_n(t)^\gamma] \to \mathbb{E}[Z(t)^\gamma]$. Then, again by dominated convergence, using Lemma 6, we get $\int_0^\infty \mathbb{E}[Z_n(r)^\gamma] \, \mathrm{d}r \to \int_0^\infty \mathbb{E}[Z(r)^\gamma] \, \mathrm{d}r < \infty$. Hence, by Fubini's theorem

$$\mathbb{E}\biggl[\int_0^\infty Z_n(r)^\gamma \, \mathrm{d}r\biggr] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\biggl[\int_0^\infty Z(r)^\gamma \, \mathrm{d}r\biggr].$$

But, by Fatou's lemma,

$$\mathbb{E}[\sigma'] \leq \liminf_n \mathbb{E}\biggl[\int_0^\infty Z_n(r)^\gamma \, \mathrm{d}r\biggr] = \mathbb{E}\biggl[\int_0^\infty Z(r)^\gamma \, \mathrm{d}r\biggr] = \mathbb{E}[\sigma],$$

as wanted. This shows that (Y_n, Z_n, σ_n) converges in distribution (without having to take a subsequence) to (Y, Z, σ), which gives the first statement of Theorem 2.

Since (Y_n, Z_n, σ_n) converges in distribution to (Y, Z, σ), by using Skorokhod's representation theorem, we assume that the convergence is almost-sure. Note that the above proof actually entails the convergence of moments of order 1, $\mathbb{E}[\sigma_n] \to \mathbb{E}[\sigma]$. We now want to prove the convergence of moments of orders $u \geq 0$. It is well known (see [4], Proposition 3.3) that the random variable σ has positive moments of all orders and that its moment of order $p \in \mathbb{N}$ is equal to $p!/\prod_{j=1}^{p} \psi(\gamma j)$. Let $u \geq 0$. Since $\sigma_n \to \sigma$ a.s., if we show that $\sup_{n\geq 1} \mathbb{E}[\sigma_n^p] < \infty$ for some $p > u$, then $((\sigma_n - \sigma)^u, n \geq 0)$ will be uniformly integrable, entailing the convergence of $\mathbb{E}[|\sigma_n - \sigma|^u]$ to 0. So fix $p > 1$, consider q such that $p^{-1} + q^{-1} = 1$ and use Hölder's inequality to get

$$\int_1^\infty Z_n(r)^\gamma \,\mathrm{d}r \leq \left(\int_1^\infty Z_n(r)^{\gamma p} r^{2p/q} \,\mathrm{d}r\right)^{1/p} \left(\int_1^\infty r^{-2} \,\mathrm{d}r\right)^{1/q}.$$

Together with Lemma 6 this implies that

$$\sup_{n\geq 1} \mathbb{E}\left[\left(\int_1^\infty Z_n(r)^\gamma \,\mathrm{d}r\right)^p\right] < \infty,$$

which, clearly, leads to the required $\sup_{n\geq 1} \mathbb{E}[\sigma_n^p] < \infty$. □

4.4. Proof of Proposition 1

Consider a probability measure μ on $[0, 1]$, a real number $\gamma > 0$ and a function $\ell : \mathbb{R}_+ \to (0, \infty)$ slowly varying at ∞. Then set $a_n = n^\gamma \ell(n)$, let γ' be such that $\max(1, \gamma) < \gamma' < \gamma + 1$ and assume that n is large enough so that $n^{\gamma'-1} < a_n \leq n^{\gamma'}$. For such an n and $0 \leq k \leq n - 1$, set

$$\begin{aligned} p_{n,k} &= a_n^{-1} \int_{[0,1-a_n^{-1})} \binom{n}{k} x^k (1-x)^{n-k-1} \mu(\mathrm{d}x) \\ &\quad + n^{1-\gamma'} \mu(\{1\}) \mathbf{1}_{\{k = n - \lfloor n^{\gamma'}/a_n \rfloor\}}, \qquad 0 \leq k \leq n-1. \end{aligned}$$

Clearly, these quantities are non-negative and

$$\sum_{k=0}^{n-1} p_{n,k} = a_n^{-1} \int_{[0,1-a_n^{-1})} (1-x^n)(1-x)^{-1} \mu(\mathrm{d}x) + n^{1-\gamma'} \mu(\{1\}) \leq \mu([0, 1)) + \mu(\{1\}) = 1.$$

Let $p_{n,n} = 1 - \sum_{k=0}^{n-1} p_{n,k}$, in order to define a probability vector $(p_{n,k}, 0 \leq k \leq n)$ on $\{0, 1, \ldots, n\}$. Now, for any continuous test function $f : [0, 1] \to \mathbb{R}_+$,

$$\begin{aligned} a_n \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\left(1 - \frac{k}{n}\right) p_{n,k} &= \int_{[0,1-a_n^{-1})} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\left(1 - \frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k-1} \mu(\mathrm{d}x) \\ &\quad + a_n n^{-\gamma'} \lfloor n^{\gamma'}/a_n \rfloor f\left(1 - n^{-1} \lfloor n^{\gamma'}/a_n \rfloor\right) \mu(\{1\}). \end{aligned}$$

The term involving $\mu(\{1\})$ clearly converges to $f(1)\mu(\{1\})$ since f is continuous. For the other term, note that for $B_{n,x}$ a binomial random variable with parameters n, x,

$$\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\left(1-\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} = \mathbb{E}\left[f\left(\frac{B_{n,x}}{n}\right)\left(1-\frac{B_{n,x}}{n}\right)\right],$$

which converges to $f(x)(1-x)$ as $n \to \infty$ and is bounded on $[0, 1]$ by a constant times $(1-x)$ since f is bounded. Hence by dominated convergence,

$$a_n \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\left(1-\frac{k}{n}\right) p_{n,k} \to \int_{[0,1]} f(x)\mu(\mathrm{d}x).$$

5. Scaling limits of random walks with a barrier

Recall that $(q_k, k \geq 0)$ is a probability distribution satisfying $q_0 < 1$, as well as the definition of the random walk with a barrier model $X_n = n - S^{(n)}$ and notation from Section 2.1. In the following, n will always be implicitly assumed to be large enough so that $\overline{q}_n < 1$.

Proof of Theorem 3. Let us first prove (i). We assume that $\overline{q}_n = n^{-\gamma}\ell(n)$, where $(\ell(x), x \geq 0)$ is slowly varying at ∞. We want to show that (H) is satisfied, with $a_n = 1/\overline{q}_n$ and $\mu(\mathrm{d}x) = \gamma(1-x)^{-\gamma}\,\mathrm{d}x\,\mathbf{1}_{\{0<x<1\}}$. From this, the conclusion follows immediately.

Using the particular form of the transition probabilities (6), it is sufficient to show that for every function f that is continuously differentiable on $[0, 1]$,

$$\frac{1}{\overline{q}_n} \sum_{k=0}^{n} \frac{q_{n-k}}{\sum_{i=0}^{n} q_i}\left(1-\frac{k}{n}\right) f\left(\frac{k}{n}\right) \longrightarrow \gamma \int_0^1 f(x)(1-x)^{-\gamma}\,\mathrm{d}x. \tag{30}$$

Let $g(x) = xf(1-x)$. By Taylor's expansion, we have, for every $x \in (0, 1)$, $g((x+\frac{1}{n}) \wedge 1) - g(x) = g'(x)/n + \varepsilon_n(x)/n$, where $\sup_{x\in[0,1]} \varepsilon_n(x)$ converges to 0 as $n \to \infty$. Therefore, since $g(0) = 0$,

$$\begin{aligned}
\frac{1}{\overline{q}_n} \sum_{k=0}^{n} \frac{q_{n-k}}{\sum_{i=0}^{n} q_i}\left(1-\frac{k}{n}\right) f\left(\frac{k}{n}\right) &= \frac{1}{\overline{q}_n(1-\overline{q}_n)} \sum_{k=0}^{n} q_k g\left(\frac{k}{n}\right) \\
&= \frac{1}{\overline{q}_n(1-\overline{q}_n)} \sum_{k=0}^{n-1} \overline{q}_k \left(g\left(\frac{k+1}{n}\right) - g\left(\frac{k}{n}\right)\right) - \frac{g(1)}{1-\overline{q}_n} \\
&= \frac{1}{n\overline{q}_n(1-\overline{q}_n)} \sum_{k=0}^{n-1} \overline{q}_k \left(g'\left(\frac{k}{n}\right) + \varepsilon_n\left(\frac{k}{n}\right)\right) - \frac{g(1)}{1-\overline{q}_n}.
\end{aligned}$$

Because of the uniform convergence of ε_n to 0, this is equivalent as $n \to \infty$ to

$$\frac{1}{n(1-\overline{q}_n)} \sum_{k=0}^{n-1} \frac{\overline{q}_k}{\overline{q}_n} g'\left(\frac{k}{n}\right) - \frac{g(1)}{1-\overline{q}_n} \underset{n\to\infty}{\longrightarrow} \int_0^1 x^{-\gamma} g'(x)\, \mathrm{d}x - g(1)$$

by a simple use of the uniform convergence theorem for regularly varying functions ([3], Theorem 1.5.2). Integrating by parts, the latter integral is the right-hand side of (30).

Statement (ii) is even simpler. Fix $\lambda > 0$. For all $k \geq 1$, it holds that $n(1-(1-k/n)^\lambda)$ converges to λk as $n \to \infty$. Moreover, $n(1-(1-k/n)^\lambda) \leq \max(1,\lambda)k$. Hence, when $m = \sum_{k=0}^\infty kq_k < \infty$, we have by dominated convergence that

$$n\left(1 - \sum_{k=0}^{n} p_{n,k}\left(\frac{k}{n}\right)^\lambda\right) \underset{n\to\infty}{\longrightarrow} \lambda m.$$

We conclude by Theorems 1 and 2 and Proposition 3. □

Let us now consider some variants of the random walk with a barrier. The results below recover and generalize results of [14]. Let $(\zeta_i, i \geq 1)$ be an i.i.d. sequence with distribution $(q_n, n \geq 0)$. Set $S_0 = 0$ and

$$S_k = \sum_{i=1}^{k} \zeta_i, \qquad k \geq 1,$$

for the random walk associated with $(\zeta_i, i \geq 1)$. We let $\widetilde{S}_k^{(n)} = n \wedge S_k, k \geq 0$ be the walk truncated at level n. We also define $\widehat{S}_k^{(n)}$ recursively as follows: $\widehat{S}_0^{(n)} = 0$, and given $\widehat{S}_k^{(n)}$ has been defined, we let

$$\widehat{S}_{k+1}^{(n)} = \widehat{S}_k^{(n)} + \zeta_{k+1} \mathbf{1}_{\{\widehat{S}_k^{(n)} + \zeta_{k+1} \leq n\}}.$$

In other words, the process $\widehat{S}^{(n)}$ evolves as S, but ignores the jumps that would bring it to a level higher than n. This is what is called the random walk with a barrier in [14]. However, in the latter reference, the authors assume that $q_0 = 0$ and therefore really consider the variable A_n associated with X_n as defined above, as they are interested in the number of *strictly positive* jumps that $\widehat{S}^{(n)}$ accomplishes before attaining its absorbing state. See the forthcoming Lemma 7 for a proof of the identity in distribution between A_n and the number of strictly positive jumps of $\widehat{S}^{(n)}$ when $q_0 = 0$.

The processes $\widetilde{X}_n = n - \widetilde{S}^{(n)}$ and $\widehat{X}_n = n - \widehat{S}^{(n)}$ are non-increasing Markov chains with transition probabilities given by

$$\widetilde{p}_{i,j} = q_{i-j} + \mathbf{1}_{\{j=0\}} \overline{q}_i, \qquad \widehat{p}_{i,j} = q_{i-j} + \mathbf{1}_{\{j=i\}} \overline{q}_i, \qquad 0 \leq j \leq i.$$

We let $\widetilde{A}_n$ and $\widehat{A}_n$ be the respective absorption times. By an argument similar to that in the above proof, it is easy to show that when $\overline{q}_n$ is of the form $n^{-\gamma}\ell(n)$ for some $\gamma \in (0,1)$ and slowly varying function ℓ, then (H) is satisfied for these two models, with sequence $a_n = 1/\overline{q}_n$ and measures

$$\widetilde{\mu} = \delta_0 + \mu = \delta_0 + \gamma(1-x)^{-\gamma}\,\mathrm{d}x\,\mathbf{1}_{\{0<x<1\}}, \qquad \widehat{\mu} = \mu = \gamma(1-x)^{-\gamma}\,\mathrm{d}x\,\mathbf{1}_{\{0<x<1\}}.$$

Consequently, we obtain the joint convergence of $\widehat{Y}_n = (\widehat{X}_n(\lfloor t/\overline{q}_n\rfloor)/n, t \geq 0)$ and $\overline{q}_n \widehat{A}_n$ to the same distributional limit as $(Y_n, \overline{q}_n A_n)$ as in (i), Theorem 3, with the obvious notation for Y_n. In the same way, $(\widetilde{Y}_n(\lfloor t/\overline{q}_n\rfloor), t \geq 0)$ and $\widetilde{A}_n$ converge to the limits involved in Theorems 1 and 2, but this time, using a killed subordinator $\xi^{(\mathrm{k})}$ with Laplace exponent

$$\psi^{(\mathrm{k})}(\lambda) = \psi(\lambda) + 1 = 1 + \int_0^\infty (1-\mathrm{e}^{-\lambda y})\frac{\gamma \mathrm{e}^{-y}\,\mathrm{d}y}{(1-\mathrm{e}^{-y})^{\gamma+1}}, \qquad \lambda \geq 0.$$

If ξ is a subordinator with Laplace exponent ψ, and if $\mathbf{e}$ is an exponential random variable with mean 1, independent of ξ, then $\xi^{(\mathrm{k})}(t) = \xi(t) + \infty\mathbf{1}_{\{t\geq \mathbf{e}\}}, t \geq 0$ is a killed subordinator with Laplace exponent $\psi^{(\mathrm{k})}$.

In fact, we have a joint convergence linking the processes $X_n, \widetilde{X}_n, \widehat{X}_n$ together. Note that the three can be joined together in a very natural way, by building them with the same variables $(\zeta_i, i \geq 1)$. This is obvious for $\widetilde{X}_n$ and $\widehat{X}_n$, by construction. Now, a process with the same distribution as X_n can be constructed simultaneously with $\widehat{X}_n$ by a simple time change, as follows.

Lemma 7. *Let $T_0^{(n)} = 0$, and recursively, let*

$$T_{k+1}^{(n)} = \inf\bigl\{i > T_k^{(n)} : \widehat{S}_{i-1}^{(n)} + \zeta_i \leq n\bigr\}.$$

Then the process $(\widehat{X}_n(T_k^{(n)}), k \geq 0)$ has same distribution as X_n, with the convention that $\widehat{X}_n(\infty) = \lim_{k\to\infty} \widehat{X}_n(k)$.

Proof. We observe that the sequence $(\zeta_{T_k^{(n)}}, k \geq 1)$ is constructed by rejecting elements ζ_i such that $\widehat{S}_{i-1}^{(n)} + \zeta_i > n$, so by a simple recursive argument, given $\widehat{X}_n(T_k^{(n)})$, the random variable $\zeta_{T_{k+1}^{(n)}}$ has the same distribution as a random variable ζ with distribution q conditioned on $\widehat{X}_n(T_k^{(n)}) + \zeta \leq n$. This is exactly the definition of $S^{(n)}$. □

In the following statement, we assume that $X_n, \widetilde{X}_n, \widehat{X}_n$ are constructed jointly as above. We let ξ be a subordinator with Laplace exponent ψ as in the statement of (i) in Theorem 3. Let $\xi^{(\mathrm{k})}$ be defined as above, using an independent exponential variable $\mathbf{e}$. Let τ be the time change defined as in the Theorem 3, and let $\tau^{(\mathrm{k})}$ be defined similarly from $\xi^{(\mathrm{k})}$. Let

$$Y = \bigl(\exp\bigl(-\xi_{\tau(t)}\bigr), t \geq 0\bigr), \qquad \widetilde{Y} = \bigl(\exp\bigl(-\xi^{(\mathrm{k})}_{\tau^{(\mathrm{k})}(t)}\bigr), t \geq 0\bigr).$$

Proposition 5. *Under the same hypotheses as in* (i), *Theorem* 3, *the following convergence in distribution holds in* $\mathcal{D}^3$:

$$(Y_n, \widetilde{Y}_n, \widehat{Y}_n) \xrightarrow[n\to\infty]{(d)} (Y, \widetilde{Y}, Y),$$

and jointly,

$$\overline{q}_n(A_n, \widetilde{A}_n, \widehat{A}_n) \xrightarrow[n\to\infty]{(d)} \left(\int_0^\infty \mathrm{e}^{-\gamma\xi_t}\,\mathrm{d}t, \int_0^{\mathbf{e}} \mathrm{e}^{-\gamma\xi_t}\,\mathrm{d}t, \int_0^\infty \mathrm{e}^{-\gamma\xi_t}\,\mathrm{d}t\right).$$

Proof (sketch). The convergence of one-dimensional marginals holds by the above discussion. Let $(Y^{(1)}, Y^{(2)}, Y^{(3)}, \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)})$ be a limit in distribution of the properly rescaled 6-tuple $(X_n, \widetilde{X}_n, \widehat{X}_n, A_n, \widetilde{A}_n, \widehat{A}_n)$ along some subsequence. These variables are constructed by three subordinators, $\xi^{(1)}, \xi^{(2)}, \xi^{(3)}$, with the same law as $\xi, \xi^{(\mathrm{k})}, \xi$, respectively. Now, we use the obvious fact that $\widetilde{X}_n \le X_n \le \widehat{X}_n$. Taking limits, we have $Y^{(2)} \le Y^{(1)} \le Y^{(3)}$ a.s. Taking expectations, using the fact that $Y^{(1)}$ and $Y^{(3)}$ have the same distribution and using the fact that these processes are càdlàg, we obtain that $Y^{(1)} = Y^{(3)}$ a.s. Similarly, $\sigma^{(1)} = \sigma^{(3)} \ge \sigma^{(2)}$ a.s., and $\sigma^{(2)}$ is the first time where $Y^{(2)}$ attains 0 (which is done by accomplishing a negative jump). Moreover, we have $\widetilde{X}_n(k) = X_n(k) = \widehat{X}_n(k)$ for every $k < \widetilde{A}_n$. By passing to the limit, we obtain that $Y^{(1)} = Y^{(2)} = Y^{(3)}$ a.s. on the interval $[0, \sigma^{(2)}]$. This shows that $\xi^{(1)} = \xi^{(3)}$ and that $\xi^{(1)} = \xi^{(2)} = \xi^{(3)}$ on the interval where $\xi^{(2)}$ is finite. Since $\xi^{(2)}$ is a killed subordinator, this completely characterizes the distribution of $(\xi^{(1)}, \xi^{(2)}, \xi^{(3)})$ as that of $(\xi, \xi^{(\mathrm{k})}, \xi)$, and this allows us to conclude. Details are left to the reader. □

6. Collisions in Λ-coalescents

We now prove Theorem 4. Using Theorems 1 and 2, all we have to check is that the hypothesis (H) is satisfied with the parameters $a_n = \int_{[1/n,1]} x^{-2}\Lambda(\mathrm{d}x)$, $n \ge 1$, and ψ defined by (9). This is an easy consequence of the following Lemmas 8 and 9. We recall that the transition probabilities of the Markov chain $(X_n(k), k \ge 0)$, where $X_n(k)$ is the number of blocks after k coalescing events when starting with n blocks, are given by (7).

Lemma 8. *Assume that* $\Lambda(\{0\}) = 0$ *and that* $u \to \int_{[u,1]} x^{-2}\Lambda(\mathrm{d}x)$ *varies regularly at* 0 *with index* $-\gamma$, $\gamma \in (0,1)$. *Then,*

$$g_n \sim \Gamma(2-\gamma) \int_{[1/n,1]} x^{-2}\Lambda(\mathrm{d}x) \qquad \text{as } n \to \infty.$$

Proof. First note that

$$\begin{aligned} g_n &= \int_{(0,1]} \bigl(1 - (1-x)^n - n(1-x)^{n-1}x\bigr) x^{-2}\Lambda(\mathrm{d}x) \\ &= I_n - J_n, \end{aligned}$$

where, defining by $\tilde{\Lambda}$ the push-forward of Λ by the mapping $x \mapsto -\log(1-x)$,

$$I_n = \int_{(0,\infty]} \bigl(1 - \exp(-nx) - n\exp(-xn)x\bigr)\bigl(1-\exp(-x)\bigr)^{-2}\tilde{\Lambda}(\mathrm{d}x),$$

$$J_n = \int_{(0,\infty]} \bigl(n\exp\bigl(-x(n-1)\bigr)\bigl(1-\exp(-x) - x\exp(-x)\bigr)\bigr)\bigl(1-\exp(-x)\bigr)^{-2}\tilde{\Lambda}(\mathrm{d}x).$$

The integrand in the integral J_n converges to 0 as $n \to \infty$, for all $x \in (0, \infty]$. And, clearly, there exists some finite constant C such that for all $x \in (0, \infty]$, and all $n \geq 1$,

$$\begin{aligned}&\left|n \exp\bigl(-x(n-1)\bigr)\bigl(1-\exp(-x)-x\exp(-x)\bigr)\bigl(1-\exp(-x)\bigr)^{-2}\right|\\&\quad\leq C\bigl(1-\exp(-x)\bigr)^{-1}.\end{aligned}$$

Hence, by dominated convergence, $J_n \to 0$ as $n \to \infty$. Next, I_n can be rewritten as

$$\begin{aligned}I_n &= \int_{(0,\infty]} \left(\int_{(0,x]} n^2 u \exp(-nu)\,\mathrm{d}u\right)\bigl(1-\exp(-x)\bigr)^{-2}\tilde{\Lambda}(\mathrm{d}x)\\&= n^2 \int_{(0,\infty)} \exp(-nu)u\left(\int_{[1-\exp(-u),1]} x^{-2}\Lambda(\mathrm{d}x)\right)\mathrm{d}u.\end{aligned}$$

Since $\int_{[u,1]} x^{-2}\Lambda(\mathrm{d}x)$ varies regularly as $u \to 0$ with index $-\gamma$,

$$u \int_{[1-\exp(-u),1]} x^{-2}\Lambda(\mathrm{d}x) \underset{u\to 0}{\sim} u \int_{[u,1]} x^{-2}\Lambda(\mathrm{d}x)$$

and these functions vary regularly at 0 with index $1-\gamma$. It is then standard that

$$\int_{[0,t]} u\left(\int_{[1-\exp(-u),1]} x^{-2}\Lambda(\mathrm{d}x)\right)\mathrm{d}u \underset{t\to 0}{\sim} \frac{t^2}{2-\gamma}\int_{[1-\exp(-t),1]} x^{-2}\Lambda(\mathrm{d}x)$$

and then, applying Karamata's Tauberian theorem (cf. [3], Theorem 1.7.1′), that

$$\begin{aligned}&\int_{(0,\infty)} \exp(-nu)u\left(\int_{[1-\exp(-u),1]} x^{-2}\Lambda(\mathrm{d}x)\right)\mathrm{d}u\\&\quad\underset{n\to\infty}{\sim} \frac{\Gamma(3-\gamma)}{(2-\gamma)n^2}\int_{[1/n,1]} x^{-2}\Lambda(\mathrm{d}x).\end{aligned}$$

Using $\Gamma(3-\gamma) = \Gamma(2-\gamma)(2-\gamma)$, we therefore have, as $n \to \infty$

$$g_n \sim I_n \sim \Gamma(2-\gamma)\int_{[1/n,1]} x^{-2}\Lambda(\mathrm{d}x).$$

□

Lemma 9. *For all measures* Λ *such that* $\int_{[0,1]} x^{-1}\Lambda(\mathrm{d}x) < \infty$, *and all* $\lambda \geq 0$

$$\sum_{k=1}^{n-1} g_{n,k}\left(1-\left(\frac{k}{n}\right)^{\lambda}\right) \underset{n\to\infty}{\to} \int_{[0,1]} \bigl(1-(1-x)^{\lambda}\bigr)x^{-2}\Lambda(\mathrm{d}x).$$

Proof. Note that

$$
\begin{aligned}
&\sum_{k=1}^{n-1} g_{n,k}\left(1-\left(\frac{k}{n}\right)^{\lambda}\right)\\
&\quad=\int_{[0,1]}\left(\sum_{k=0}^{n-2}\binom{n}{k}x^{n-k}(1-x)^{k}\left(1-\left(\frac{k+1}{n}\right)^{\lambda}\right)\right)x^{-2}\Lambda(\mathrm{d}x)\\
&\quad=\int_{[0,1]}\left(\mathbb{E}\left[1-\left(\frac{B^{(n,x)}+1}{n}\right)^{\lambda}\right]-(1-x)^{n}\left(1-\left(\frac{n+1}{n}\right)^{\lambda}\right)\right)x^{-2}\Lambda(\mathrm{d}x),
\end{aligned}
$$

where $B^{(n,x)}$ denotes a binomial random variable with parameters $n, 1-x$. By the strong law of large numbers and dominated convergence ($0 \le (B^{(n,x)}+1)/n \le 2$), we have that

$$
\mathbb{E}\left[1-\left(\frac{B^{(n,x)}+1}{n}\right)^{\lambda}\right] \underset{n\to\infty}{\to} 1-(1-x)^{\lambda} \qquad \forall x \in [0,1].
$$

Moreover, $(1-x)^{n}(1-(\frac{n+1}{n})^{\lambda}) \to 0$, for every $x \in [0,1]$. Besides, since $1-y^{\lambda} \le \max(1,\lambda)(1-y)$ for $y \in [0,1]$,

$$
\begin{aligned}
\sum_{k=0}^{n-2}\binom{n}{k}x^{n-k}(1-x)^{k}\left(1-\left(\frac{k+1}{n}\right)^{\lambda}\right) &\le \max(1,\lambda)\sum_{k=0}^{n-2}\binom{n}{k}x^{n-k}(1-x)^{k}\left(1-\left(\frac{k+1}{n}\right)\right)\\
&\le \max(1,\lambda)\big(1-(1-x+1/n)+(1-x)^{n}/n\big)\\
&\le \max(1,\lambda)x.
\end{aligned}
$$

Using that $\int_{[0,1]} x^{-1}\Lambda(\mathrm{d}x) < \infty$, we conclude by dominated convergence. □

Proof of Theorem 4. Under the assumptions of Theorem 4, by Lemmas 8 and 9,

$$
1-\sum_{k=0}^{n} p_{n,k}\left(\frac{k}{n}\right)^{\lambda} = \frac{1}{g_n}\left(\sum_{k=1}^{n-1} g_{n,k}\left(1-\left(\frac{k}{n}\right)^{\lambda}\right)\right) \sim \frac{\int_{[0,1]}(1-(1-x)^{\lambda})x^{-2}\Lambda(\mathrm{d}x)}{\Gamma(2-\gamma)\int_{[1/n,1]} x^{-2}\Lambda(\mathrm{d}x)}
$$

as $n \to \infty$ and for all $\lambda \ge 0$. Hence (H) holds by Proposition 3 and Theorem 4 is proved. □

Acknowledgement

This work is partially supported by the Agence Nationale de la Recherche, ANR-08-BLAN-0190 and ANR-08-BLAN-0220-01.

References

[1] Bertoin, J. and Yor, M. (2001). On subordinators, self-similar Markov processes and some factorizations of the exponential variable. *Electron. Comm. Probab.* **6** 95–106 (electronic). MR1871698

[2] Billingsley, P. (1999). *Convergence of Probability Measures*, 2nd ed. New York: Wiley. MR1700749

[3] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1989). *Regular Variation. Encyclopedia of Mathematics and its Applications* **27** Cambridge: Cambridge Univ. Press. MR1015093

[4] Carmona, P., Petit, F. and Yor, M. (1997). On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential Functionals and Principal Values Related to Brownian Motion. Bibl. Rev. Mat. Iberoamericana* 73–130. Madrid: Rev. Mat. Iberoamericana. MR1648657

[5] Delmas, J.-F., Dhersin, J.-S. and Siri-Jegousse, A. (2008). Asymptotic results on the length of coalescent trees. *Ann. Appl. Probab.* **18** 997–1025. MR2418236

[6] Ethier, S.N. and Kurtz, T.G. (1986). *Markov Processes*. New York: Wiley. MR0838085

[7] Gnedin, A., Iksanov, A. and Möhle, M. (2008). On asymptotics of exchangeable coalescents with multiple collisions. *J. Appl. Probab.* **45** 1186–1195. MR2484170

[8] Gnedin, A. and Pitman, J. (2005). Regenerative composition structures. *Ann. Probab.* **33** 445–479. MR2122798

[9] Gnedin, A., Pitman, J. and Yor, M. (2006). Asymptotic laws for compositions derived from transformed subordinators. *Ann. Probab.* **34** 468–492. MR2223948

[10] Gnedin, A. and Yakubovich, Y. (2007). On the number of collisions in Λ-coalescents. *Electron. J. Probab.* **12** 1547–1567 (electronic). MR2365877

[11] Haas, B. and Miermont, G. (2010). Scaling limits of Markov branching trees, with applications to Galton–Watson and random unordered trees. Available at ArXiv:1003.3632.

[12] Haas, B., Pitman, J. and Winkel, M. (2009). Spinal partitions and invariance under re-rooting of continuum random trees. *Ann. Probab.* **37** 1381–1411. MR2546748

[13] Iksanov, A., Marynych, A. and Möhle, M. (2009). On the number of collisions in beta(2,b)-coalescents. *Bernoulli*. **15** 829–845. MR2555201

[14] Iksanov, A. and Möhle, M. (2008). On the number of jumps of random walks with a barrier. *Adv. in Appl. Probab.* **40** 206–228. MR2411821

[15] Lamperti, J. (1962). Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* **104** 62–78. MR0138128

[16] Lamperti, J. (1972). Semi-stable Markov processes. I. *Z. Wahrsch. Verw. Gebiete* **22** 205–225. MR0307358

[17] Pitman, J. (1999). Coalescents with multiple collisions. *Ann. Probab.* **27** 1870–1902. MR1742892

[18] Sagitov, S. (1999). The general coalescent with asynchronous mergers of ancestral lines. *J. Appl. Probab.* **36** 1116–1125. MR1742154

[19] Schweinsberg, J. (2000). A necessary and sufficient condition for the Λ-coalescent to come down from infinity. *Electron. Comm. Probab.* **5** 1–11 (electronic). MR1736720

[20] Stone, C. (1963). Limit theorems for random walks, birth and death processes, and diffusion processes. *Illinois J. Math.* **7** 638–660. MR0158440

Received September 2009 and revised June 2010