Praktise on iterative methods

Exercice 1 Let $a \in \mathbb{R}$ and $A_a = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$. For which values of a the Jacobi method for A_a is convergent ?

Exercice 2 Pour $\alpha \in \mathbb{R}$, let

$$A_\alpha = \begin{pmatrix} 2 & \alpha & 0 \\ \alpha & 2 & \alpha \\ 0 & \alpha & 2 \end{pmatrix}.$$

(1) Write the matrix $\mathcal{J}$ of Jacobi algorithm. For which values of α does it converge ?
(2) Write the matrix $\mathcal{L}_1$ of Gauss-Seidel algorithm. For which values of α does it converge ?

Exercice 3 Let

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{et} \quad B = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{pmatrix}.$$

(1)(a) Write the matrices of Jacobi and Gauss-Seidel algorithm.
(b) Compute their spectral radii. What is known in general about these ?
(2) Consider the system to be solved

$$(S) \quad \left\{ \begin{array}{rrrcl} x & +2y & -2z & = & -9 \\ x & +\ y & +\ z & = & 2 \\ 2x & +2y & +\ z & = & 1 \end{array} \right.$$

(a) What is the most appropriate method, between Jacobi and Gauss-Seidel ?
(b) Compute the 4 first iterations, starting with $X_0 = (0,0,0)^t$. What is the solution of (S) ?

Exercice 4 Let

$$A = \begin{pmatrix} 1 & 0 & -m_1 \\ -m_2 & 1 & 0 \\ 0 & -m_3 & 1 \end{pmatrix} \quad \text{et} \quad A' = \begin{pmatrix} 1 & -m_1 & 0 \\ 0 & 1 & -m_2 \\ -m_3 & 0 & 1 \end{pmatrix}$$

with $m_j \in \mathbb{R}$.
(1) By explicit computation, determine for which values of the m_j the Gauss-Seidel algorithm does converge for A, then for A'.
(2) Same questions for Jacobi. Compare the methods.

Exercice 5 Let $C \in \mathcal{M}_n(\mathbb{C})$ defined by

$$C_{i,j} = \left\{ \begin{array}{ll} 1 & i \neq j \\ 0 & i = j \end{array} \right.$$

For any $\alpha \in \mathbb{R}$, define $A(\alpha) = \alpha.I_n + C$ and $D_n(\alpha) = \det(A(\alpha))$.
(1) Show the recursion relation

$$D_n(\alpha) = (\alpha - 1)^{n-1} + (\alpha - 1)D_{n-1}(\alpha).$$

Deduce that $D_n(\alpha) = (\alpha - 1)^{n-1}(\alpha + n - 1)$.
Consider now the resolution of the system $A(\alpha)x = b$ (for $\alpha \neq 1,\ n-1$).
(2) Describe the Jacobi, Gauss-Seidel and Richardson algorithms.

(3) Give a necessary and sufficient condition on α for Jacobi algorithm to converge.
(3) Same question for Gauss-Seidel and Richardson.