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HIGHER ORDER PARAXIAL WAVE EQUATION APPROXIMATIONS IN HETEROGENEOUS MEDIA*

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Abstract. A new family of paraxial wave equation approximations is derived. These approximations are of higher order accuracy than the parabolic approximation and they can be applied to the same computational problems, e.g., in seismology, underwater acoustics and as artificial boundary conditions. The equations are written as systems which simplify computations. The support and singular support are studied; energy estimates are given which prove the well-posedness. The reflection and transmission are shown to be continuously dependent on material interfaces in heterogeneous media.

Key words. one-way wave equation, higher order approximation, migration

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1. Introduction. Paraxial wave equation approximations are used to describe wave propagation with a preferred direction. The most common paraxial approximation is the parabolic equation

$$\frac{1}{c}\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x_2 \partial t}-\frac{c}{2}\frac{\partial^2 u}{\partial x_1^2}=0 \tag{1.1}$$

approximating the scalar wave function

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}-\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)=0. \tag{1.2}$$

The solution $u(x, t)$ of (1.1) is an exact approximation of solutions to (1.2) for plane waves traveling in the positive x_2 direction, $u=f(x_2-ct)$. In [1], we studied mathematical properties of the parabolic equation (1.1) and its generalization to variable velocity $c(x)$. The paper [1] also contains references to the applications of paraxial approximations in seismology, acoustics and as artificial computational boundary conditions.

The error in the approximation above increases with increasing angle θ between the direction of propagation and the x_2 axis ($u=f(\cos\theta x_2+\sin\theta x_1-ct)$). In order to reduce this error higher order approximations other than (1.1) have been suggested. Claerbout [5] introduced a third order equation

$$\frac{1}{c}\frac{\partial^3 u}{\partial t^3}+\frac{\partial^3 u}{\partial t^2 \partial x_2}-\frac{3c}{4}\frac{\partial^3 u}{\partial t \partial x_1^2}-\frac{c^2}{4}\frac{\partial^3 u}{\partial x_2 \partial x_1^2}=0, \tag{1.3}$$

the so-called 45°-approximation for applications in seismology. Higher order paraxial approximations have also been suggested as artificial boundary conditions [8], [11].

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In this paper we shall present a new family of higher order paraxial approximations. The approximations are written as second order systems of partial differential equations. The higher order scalar equations in [8] can be written as systems of our type (see also [16]).

In § 2, the paraxial systems of equations are derived for homogeneous media. The derivation is based on rational approximations of the dispersion relation for (1.2). The section also contains analysis of the propagation properties of the equations and an error estimate. The error estimate shows that it is possible to approximate (1.2) by paraxial equation to any accuracy by choosing the order of the paraxial equation high enough.

New higher order approximations for heterogeneous media are derived in § 3. Here it is essential that the approximation be written as a system rather than a scalar higher order equation. The formulation as a system is also fundamental for the analysis. The well-posedness is established and propagation properties are analysed. The support of the fundamental solution is proved to propagate in a half-space with a finite speed. This is an essential feature for a paraxial equation. The transmission and reflection at an interface is shown to be continuously dependent on its location.

In § 4, numerical results are presented. Numerical approximations of the fundamental solution of two higher order paraxial systems were computed. Different calculations with different velocity profiles were performed by F. Collino [6].

Some of the results of this paper were announced in [9] and some technical details in the proofs are omitted here but are given in the report [3]. As in [1] we restrict ourselves to two space dimensions, but the techniques are the same in $\mathbb{R}^n$.

2. Higher order approximations in homogeneous media.

2.1. Derivation of the equations. Consider the two-dimensional wave equation (1.2). Let us recall [1] that the paraxial (or one way) approximation consists of approximating the part of the solution to the Cauchy problem propagating close to the positive x_2 direction. By using the Fourier transform, this part u_+ can be written as a sum of harmonic waves traveling in the positive x_2 direction

$$u_+(x, t) = \iint \hat{a}(k) \exp i(\omega(k)t - k \cdot x)\, dk, \tag{2.1}$$

where the amplitude $\hat{a}(k)$ depends on the initial values, and ω is defined by

$$c\frac{k_2}{\omega(k)} = \left(1 - \left(c\frac{k_1}{\omega(k)}\right)^2\right)^{1/2}. \tag{2.2}$$

The function u_+ is solution of a pseudo-differential equation, the symbol (or the dispersion relation) of which is

$$\mathscr{L} = ck_2 - \omega\left(1 - \left(c\frac{k_1}{\omega}\right)^2\right)^{1/2}. \tag{2.3}$$

Our aim is to approximate u_+ by the solution of a partial differential equation, suitable for computation. Hence we have to approximate the symbol $\mathscr{L}$ by rational functions in ω and k. This is done by approximating the function

$$f(X) = (1 - X)^{1/2} \tag{2.4}$$

by polynomial or rational functions in $X = c(k_1/\omega)$.

The term $c(k_1/\omega)$ in (2.3) represents the sine of the angle between the direction of propagation of the harmonic plane wave and the x_2 direction: $c(k_1/\omega) = \sin\theta$. Many applications are concerned with a narrow range of wave vectors, so that this angle remains small. Thus the classical approach was to approximate $f(X)$ for small values of X.

A first-Taylor approximation to f

$$f(X)=1-\tfrac{1}{2}X+O(X^2)$$

yields the following approximation to $\mathscr{L}$

$$ck_2-\omega\left(1-\frac{1}{2}\left(c\frac{k_1}{\omega}\right)^2\right).$$

Multiplication by ω leads to the quadratic polynomial

$$ck_2\omega-\omega^2+\frac{1}{2}c^2k_1^2 \tag{2.5}$$

which is the symbol of the so-called parabolic or 15°-approximation (1.1) (for details see [1]):

$$\frac{1}{c}\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial t\partial x_2}-\frac{c}{2}\frac{\partial^2 u}{\partial x_1^2}=0. \tag{2.6}$$

A first Padé approximation

$$f(X)=\frac{1-\frac{3}{4}X}{1-\frac{1}{4}X}+O(X^3)$$

leads to the symbol

$$-\omega^3+ck_2\omega^2+\tfrac{3}{4}c^2k_1^2\omega-\tfrac{1}{4}c^3k_2k_1^2 \tag{2.7}$$

of the so-called 45°-approximation (1.3)

$$\frac{1}{c}\frac{\partial^3 u}{\partial t^3}+\frac{\partial^3 u}{\partial t^2\partial x_2}-\frac{3}{4}c\frac{\partial^3 u}{\partial t\partial x_1^2}-\frac{c^2}{4}\frac{\partial^3 u}{\partial x_2\partial x_1^2}=0. \tag{2.8}$$

A generalization to any order, by means of continued fractions (corresponding to three diagonals of the Padé table), has been used in [8] for absorbing boundary conditions. They are defined by

$$g_N(X)=1-\frac{X}{1+g_{N-1}(X)},\qquad g_1(X)=1. \tag{2.9}$$

These functions are rational fractions in X, and they have been shown in [2] to have the important approximation property

$$\|g_N-f\|_{L^\infty([0,1])}=\frac{1}{N}.$$

From g_N the partial differential operator of order N can be derived. The high order of derivative makes the practical use of these equations more difficult. Another way of writing the equations has been found independently by Halpern [9] and Guan-Quan Zhang [16] and had been used previously to design absorbing boundary conditions in [11]. It is based on the remark that a rational fraction can be split up into a sum of prime fractions. For example, the first Padé approximation is also equal to

$$f(X)=1-\frac{\frac{1}{2}X}{1-\frac{1}{4}X}+O(X^3).$$

A natural generalization of the approximation (2.9) is then (see Figure 2.1)

$$f_n(X)=1-\beta X-\sum_{k=1}^{n}\frac{\beta_k X}{1-\gamma_k^2X}. \tag{2.10}$$

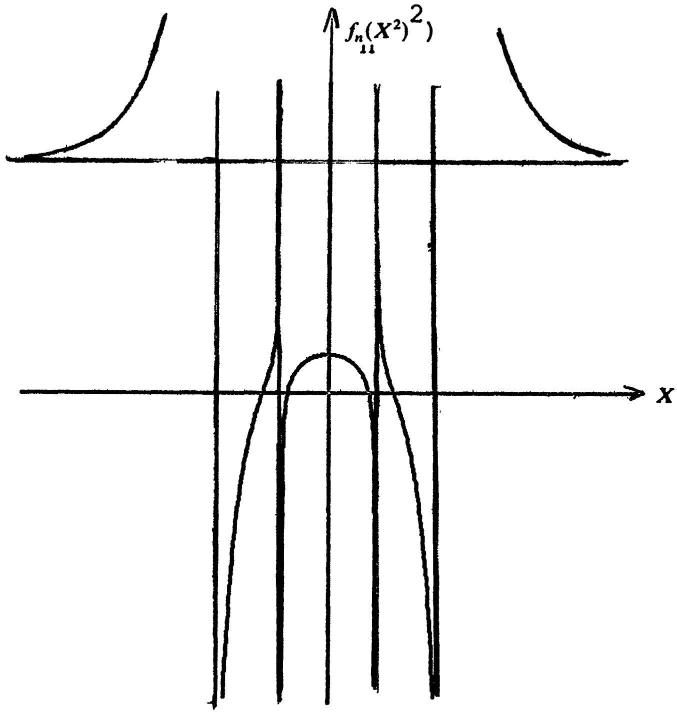


FIG. 2.1. *Variations of f_n for $\beta = 0$.*

The coefficients β, β_k and γ_k are such that

$$0 < \gamma_n < \cdots < \gamma_1 < 1, \tag{2.11}$$

$$\beta \geqq 0, \quad \beta_k > 0, \quad 1 \leqq k \leqq n. \tag{2.12}$$

This generalization is justified by the fact that the Padé approximations g_N can be written in the form (2.10). More precisely, the results are the following:

(i) $g_{2n+1}(x) = f_n(x)$

$$\text{with} \begin{cases} \beta = 0, \\ \beta_k = \dfrac{2}{2n+1} \sin^2 \dfrac{k\pi}{2n+1}, \\ \gamma_k = \cos \dfrac{k\pi}{2n+1}. \end{cases}$$

(ii) $g_{2n}(x) = f_{n-1}(x)$

$$\text{with} \begin{cases} \beta = \dfrac{1}{2n}, \\ \beta_k = \dfrac{1}{n} \sin^2 \dfrac{k\pi}{2n}, \\ \gamma_k = \cos \dfrac{k\pi}{2n}. \end{cases}$$

The constraints (2.11) are natural. They express that f_n is continuous on $[0, 1]$. The constraints (2.12) on β and β_k ensure the hyperbolicity of the corresponding operator, as will be discussed below.

Remark 2.1. The choice of the coefficients γ_k, β, β_k depends on the applications. In many areas, one is concerned with thin beams, and hence the family g_N will be chosen. For other purposes, one could, for instance, compute them to get an L^∞ approximation to f. (For a more complete discussion see [14].)

The decomposition (2.10) of the function f_n enables us to write the approximate equation in a very convenient form. If u is a solution, its Fourier transform $\hat{u}$ satisfies

$$\left[ck_2 - \omega f_n\left(\left(c\frac{k_1}{\omega}\right)^2\right)\right]\hat{u} = 0 \tag{2.13}$$

which implies

$$ck_2\hat{u} - \omega\hat{u} + \omega \sum_{k=1}^{n} \beta_k \frac{c^2k_1^2}{\omega^2 - \gamma_k^2 c^2 k_1^2}\hat{u} = 0 \tag{2.14}$$

if $\beta = 0$. We define n function φ_k, $1 \leqq k \leqq n$, by their Fourier transforms

$$\hat{\varphi}_k = \frac{c^2k_1^2}{\omega^2 - \gamma_k^2 c^2 k_1^2}\hat{u}, \tag{2.15}$$

so that equation (2.14) can be rewritten as

$$ck_2\hat{u} - \omega\hat{u} + \omega \sum_{k=1}^{n} \beta_k\hat{\varphi}_k = 0. \tag{2.16}$$

When the inverse Fourier transform is applied, (2.15) and (2.16) lead to the following system of equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x_2} - \displaystyle\sum_{k=1}^{n} \beta_k \frac{\partial \varphi_k}{\partial t} = 0, \\ \dfrac{\partial^2 \varphi_k}{\partial t^2} - c^2\gamma_k^2 \dfrac{\partial^2 \varphi_k}{\partial x_1^2} = c^2 \dfrac{\partial^2 u}{\partial x_1^2}, \qquad 1 \leqq k \leqq n. \end{cases} \tag{2.17}$$

This is a system of $(n+1)$ linear equations: one transport equation in the x_2 direction and n one-dimensional wave equation in the x_1 direction. For $\beta \neq 0$, we get the following system:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + c\dfrac{\partial^2 u}{\partial t \partial x_2} - \beta c^2 \dfrac{\partial^2 u}{\partial x_1^2} - \displaystyle\sum_{k=1}^{n} \beta_k \frac{\partial^2 \varphi_k}{\partial t^2} = 0. \\ \dfrac{\partial^2 \varphi_k}{\partial t^2} - c^2\gamma_k^2 \dfrac{\partial^2 \varphi_k}{\partial x_1^2} = c^2 \dfrac{\partial^2 u}{\partial x_1^2}. \end{cases} \tag{2.18}$$

This formulation is useful in three ways. It is easy to derive a priori estimates, it can be extended to heterogeneous media and it is convenient for numerical computations.

Remark 2.2. In order to solve the Cauchy problems for (2.17) and (2.18), additional initial data are required for the functions φ_k and $\partial\varphi_k/\partial t$. We shall return to this later.

2.2. Propagation properties of the operator. In this section, we shall only consider the case $\beta = 0$. If $\beta \neq 0$, the results are somewhat different, but the techniques are the same [3]. The propagation properties depend only on the determinant of the system, i.e., the approximate operator $\mathcal{L}_n$ the symbol of which is

$$-c\frac{k_2}{\omega}+f_n\left(\left(c\frac{k_1}{\omega}\right)^2\right)$$

or, after clearing the denominator,

$$\mathcal{L}_n(\omega, k)=(ck_2-\omega)\prod_{k=1}^{n}(\omega^2-c^2\gamma_k^2k_1^2)-c^2\omega^2k_1^2\sum_{k=1}^{n}\beta_k\prod_{\substack{j=1\\ j\neq k}}^{n}(\omega^2-c^2\gamma_j^2k_1^2). \tag{2.19}$$

$\mathcal{L}_n$ is a homogeneous operator of global order $2n+1$ and of order 1 in x_2. It is clear from (2.17) that the equation is hyperbolic. Let us make this precise.

DEFINITION [10]. *The operator $\mathcal{L}$ is hyperbolic in time if $\mathcal{L}(1,0)\neq 0$ (i.e. time is not characteristic for $\mathcal{L}$) and if $\mathcal{L}(\omega, k)$ has only real roots ω for $k\in\mathbb{R}^2-\{0,0\}$. If, in addition, the roots are simple, $\mathcal{L}$ is strictly hyperbolic.*

LEMMA 2.1. *The operator $\mathcal{L}_n$ is hyperbolic in time, but not strictly hyperbolic.*

Proof. k being fixed, the number of roots of $\mathcal{L}_n$ is the number of solutions z (finite or not) to the system:

$$y=f_n(z^2), \qquad y=-\frac{k_2}{k_1}z.$$

It is now easy to check that the number of intersection points is $2n+1$. In particular, if $k_1=0$, $\omega=-k_2$ is the only simple root of $\mathcal{L}_n$, and $\omega=0$ is a root of multiplicity $2n$.

The theory of hyperbolic operators (cf. [10]) ensures that $\mathcal{L}_n$ has an unique fundamental solution E_n, defined for $t>0$. Its support can be given explicitly. We define a subset $\mathcal{D}_n$ of $\mathbb{R}^3$ as the component of $(1,0,0)$ in the set $\{(\omega,k), \mathcal{L}_n(\omega,k)\neq 0\}$. Then [10] the support $\mathcal{E}_n$ of E_n is included in the closed convex cone with vertex at 0, dual of $\mathcal{D}_n$ in $\mathbb{R}^3$, but in no smaller closed convex cone with vertex at 0 (Fig. 2.2). In our case we can write an explicit formula as follows.

THEOREM 2.1. *The support $\mathcal{E}_n(t)$ of the fundamental solution at time t is the domain of $\mathbb{R}^n$ bounded by the x_1 axis and the curve*

$$\Gamma_n^0(t)=\begin{cases} x_1=2\lambda f_n'(\lambda^2)x_2, \\ ct=[f_n(\lambda^2)-2\lambda^2 f_n'(\lambda^2)]x_2, \end{cases} \qquad x_2\geqq 0, \quad |\lambda|\leqq 1/c\gamma_1. \tag{2.20}$$

The form of the fundamental solution provides important information concerning the propagation of the solutions of $\mathcal{L}_n$. Every solution of equation $\mathcal{L}_n u=0$ propagates in the positive x_2 direction, with a velocity $V\leqq c$. This is formulated precisely in the following theorem.

THEOREM 2.2. *If the initial values are of compact support in $\mathcal{K}$, at any time t one has*

$$\operatorname{supp} u\subset\mathcal{K}+\mathcal{E}_n(t).$$

Theorem 2.2 proceeds directly from Theorem 2.1. Theorem 2.1 is achieved by writing explicitly the equation of $\mathcal{D}_n$, and then the equation of the dual. For more details see [3].

It is more difficult to study the singular support than the support of the fundamental solution, since $\mathcal{L}_n$ is not strictly hyperbolic. Let us first recall that the singular support of a distribution u is the smallest closed domain where u is not $\mathcal{C}^\infty$. One can use the notion of wave front set (WFS) which provides additional information on the singularities [10]. We restrict ourselves here to giving the singular support of E_n (Fig. 2.3).

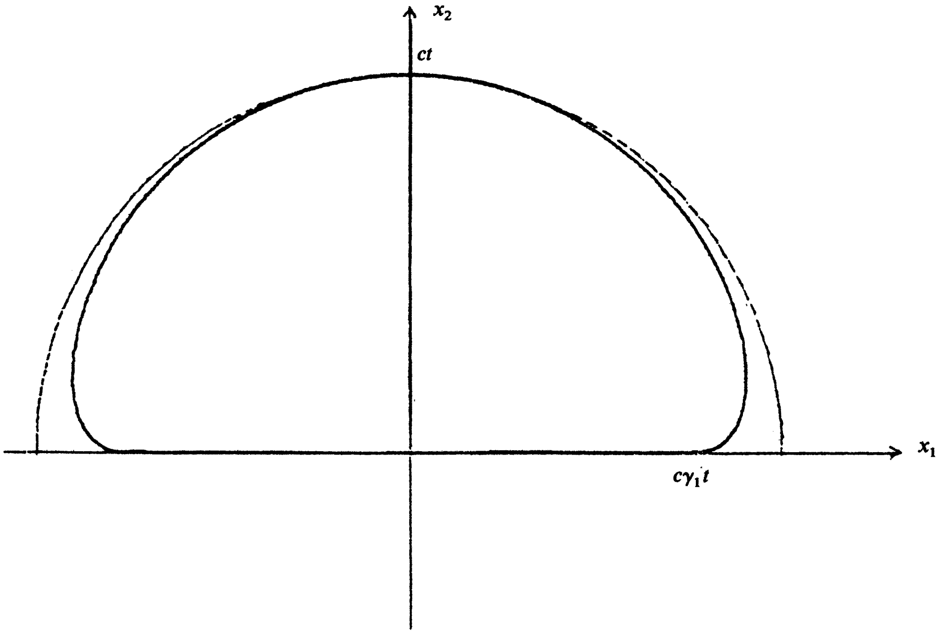


FIG. 2.2. *Support of the fundamental solution.*

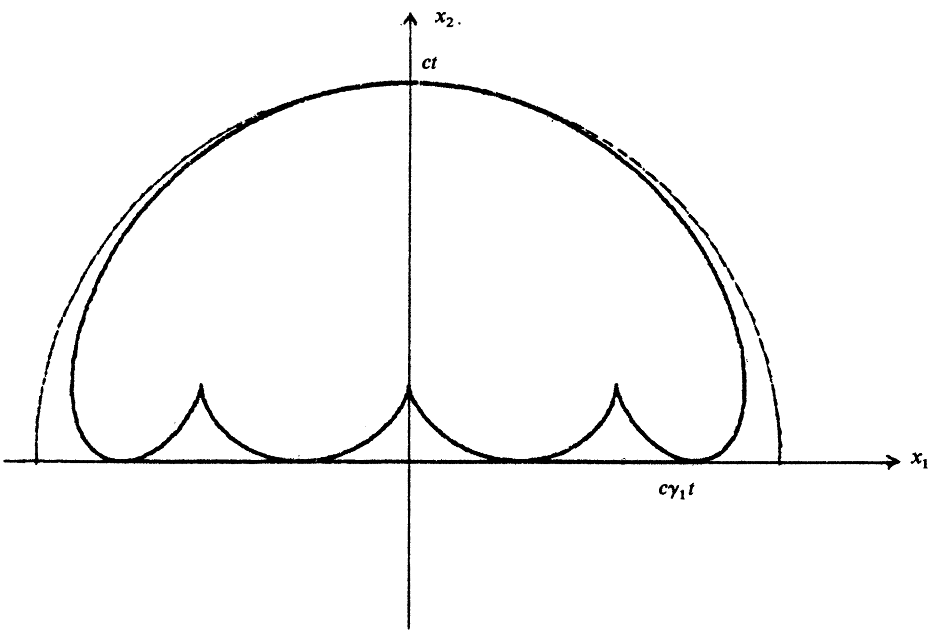


FIG. 2.3. *Singular support of the fundamental solution.*

THEOREM 2.3. *The singular support of E_n at time t consists of two parts: the curve defined by:*

$$\Gamma_n^1(t)=\begin{cases} x_1=2\lambda f_n'(\lambda^2)x_2, \\ ct=[f_n(\lambda^2)-2\lambda^2 f_n'(\lambda^2)]x_2, \end{cases} \qquad x_2\geqq 0, \quad \lambda\in\mathbb{R} \tag{2.21}$$

and the segment $\{x, |x_1|\leqq \gamma_1 ct, x_2=0\}$.

The proof is technical and will be omitted here. It is given in [3]. Let us only point out that, for a strictly hyperbolic operator, the wave front set is the set $\Gamma_n^1(t)$ of nullbicharacteristics passing through the origin. When the operator is no longer strictly hyperbolic, the wave front set contains, in addition, the lines making the set convex (see [7]).

Since the operator $\mathscr{L}_n$ is hyperbolic, it follows in standard fashion that the Cauchy problem is well posed. Moreover, the form of the support of the fundamental solutions enables us to conclude that the initial boundary value problem in the half-space $x_2>0$ is well posed (for details see [10]).

We now turn to an approximation result. All the calculations are formal, but will be fully justified by the regularity results given in § 3.

2.3. An error estimate. In practice we wish to approximate the problem in the half-space $\mathbb{R}_+^2=\{x, x_2\geqq 0\}$

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}-\Delta u=0, & x\in\mathbb{R}_+^2, \quad t\geqq 0, \\ u(t,x)=0, & x\in\mathbb{R}_+^2, \quad t\leqq 0, \\ u(t,x_1,0)=g(t,x_1), & x_1\in\mathbb{R}, \quad t\geqq 0, \end{cases} \tag{2.22}$$

by the problem,

$$\begin{cases} \mathscr{L}_n u_n=0, & x\in\mathbb{R}_+^2, \quad t\geqq 0, \\ u_n(t,x)=0, & x\in\mathbb{R}_+^2, \quad t\leqq 0, \\ u_n(t,x_1,0)=g(t,x_1), & x_1\in\mathbb{R}, \quad t\geqq 0. \end{cases} \tag{2.23}$$

We shall make the following assumptions on the boundary value g:

$$g\in L^2(\mathbb{R}_+\times\mathbb{R}), \tag{2.24}$$

$$S=\operatorname{supp}\hat{g}\subset\left\{(\omega,k), \left|c\frac{k_1}{\omega}\right|<1\right\} \tag{2.25}$$

where $\hat{g}$ is the Fourier transform of g with respect to t and x_1. The first assumption is only a smoothness assumption, and the second one ensures that g and u contain only propagating modes. This hypothesis is essential for obtaining any approximation result.

THEOREM 2.4. *If the boundary data g satisfies assumptions* (2.24) *and* (2.25) *and if $\mathscr{L}_n$ is such that f_n approximates f uniformly on* [0, 1] *then u_n converges to u in the following sense:*

$$\forall X_2\in[0,+\infty[\quad \lim_{n\to\infty}\|u-u_n\|_{L^\infty([0,X_2];L^2(\mathbb{R}^+\times\mathbb{R}))}=0. \tag{2.26}$$

Let us recall that f_n approximates f uniformly on [0, 1] means that $\|f_n-f\|_{L^\infty([0,1])}$ tends to 0 when n tends to infinity. The sequence g_N of Padé approximants introduced in (2.9) satisfies this assumption.

Proof. The analysis is similar to the one given for the absorbing boundary conditions in [8]. The Fourier transforms in t and x_1 of u and u_n can be written as

$$\tilde{u}(\omega, k_1, x_2) = \hat{g}(\omega, k_1) \exp -i\omega\left(1-\left(c\frac{k_1}{\omega}\right)^2\right)^{1/2} x_2,$$

$$\tilde{u}_n(\omega, k_1, x_2) = \hat{g}(\omega, k_1) \exp -i\omega f_n\left(\left(c\frac{k_1}{\omega}\right)^2\right) x_2.$$

Using Parseval's theorem, the L^2 norm of the error is given at every point x_2 by

$$\|u-u_n\|^2 = \iint_S |\tilde{u}-\tilde{u}_n|^2 \, d\omega \, dk_1.$$

The integral on S is handled in the following way:

$$\iint_S |\tilde{u}-\tilde{u}_n|^2 \, d\omega \, dk_1 \leqq \iint_S |\hat{g}|^2 \left| \exp -i\omega x_2 f\left(\left(c\frac{k_1}{\omega}\right)^2\right) - \exp -i\omega x_2 f_n\left(\left(c\frac{k_1}{\omega}\right)^2\right)\right| d\omega \, dk_1.$$

Lebesgue's theorem ensures this term to converge to zero if f_n tends uniformly to f.

Remark 2.3. Hypothesis (2.25) is restrictive, and it can be removed by expressing g as the solution of the wave equation on $[-L, 0]$ with g given at $x_2=-L$. L can then be chosen such that (2.26) holds for any datum at $-L$ (for details see [3]).

3. New higher order equations in heterogeneous media. The purpose of this section is the same as that for the extension of the parabolic approximation to heterogeneous media [1]. We wish to generalize the equations (2.17) and (2.18) in such a way that:

(i) The Cauchy problem and the initial boundary value problem are well posed.

(ii) It is a good approximation of the wave equation for heterogeneous media with small velocity variations.

(iii) It has good continuity properties with respect to material interfaces. The precise definitions are given in [1], and we shall come back to the last two points below.

For the parabolic approximation

$$\frac{1}{c}\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial t \partial x_2}-\frac{c}{2}\frac{\partial^2 u}{\partial x_1^2}=0,$$

we introduced unknown functions ζ, ξ, χ of c and determined them so that the equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}+\frac{1}{c\zeta(c)}\frac{\partial}{\partial x_2}\left(\zeta(c)\frac{\partial u}{\partial t}\right)-\frac{1}{2\chi(c)\xi(c)}\frac{\partial}{\partial x_1}\left(\chi(c)\frac{\partial}{\partial x_1}(\xi(c)u)\right)=0$$

satisfied (i), (ii) and (iii). The resulting equation had the form

$$\frac{1}{c}\frac{\partial^2}{\partial t^2}\left(\frac{u}{\sqrt{c}}\right)+\frac{\partial^2}{\partial t \partial x_2}\left(\frac{u}{\sqrt{c}}\right)-\frac{1}{2}\frac{\partial}{\partial x_1}\left(c\frac{\partial}{\partial x_1}\left(\frac{u}{\sqrt{c}}\right)\right)=0.$$

It is of course tempting to apply the same method to higher order equations, but it seems to be practically very difficult. We therefore restrict ourselves to transforming the terms $\partial^2/\partial x_1^2$ in (2.17) and (2.18) as we did for the parabolic approximation.

We set

$$v=c^{-1/2}u; \qquad \psi_k=c^{-1/2}\varphi_k \tag{3.1}$$

and introduce the new systems of equations,

$$\frac{1}{c}\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x_2}-\sum_{k=1}^{n}\frac{1}{c}\beta_k\frac{\partial \psi_k}{\partial t}=0, \tag{3.2a}$$

$$\frac{1}{c}\frac{\partial^2 \psi_k}{\partial t^2}-\gamma_k^2\frac{\partial}{\partial x_1}\left(c\frac{\partial \psi_k}{\partial x_1}\right)=\frac{\partial}{\partial x_1}\left(c\frac{\partial v}{\partial x_1}\right),\qquad 1\leqq k\leqq n, \tag{3.2b}$$

if $\beta=0$, and

$$\frac{1}{c}\frac{\partial^2 v}{\partial t^2}+\frac{\partial^2 v}{\partial t\partial x_2}-\beta\frac{\partial}{\partial x_1}\left(c\frac{\partial v}{\partial x_1}\right)-\sum_{k=1}^{n}\frac{1}{c}\beta_k\frac{\partial^2 \psi_k}{\partial t^2}=0, \tag{3.3a}$$

$$\frac{1}{c}\frac{\partial^2 \psi_k}{\partial t^2}-\gamma_k^2\frac{\partial}{\partial x_1}\left(c\frac{\partial \psi_k}{\partial x_1}\right)=\frac{\partial}{\partial x_1}\left(c\frac{\partial v}{\partial x_1}\right),\qquad 1\leqq k\leqq n, \tag{3.3b}$$

if $\beta\neq 0$. We shall see that these equations have the properties we expected and we begin with the well-posedness.

3.1. Analysis of the well-posedness. We shall prove the well-posedness for the system (3.2), and we only state the results for system (3.3). We define the initial value problem for system (3.2) as follows: Find $(v,\varphi_k)_{1\leqq k\leqq n}:\mathbb{R}^2\times[0,T]\to\mathbb{R}$, solutions of (3.2) with the initial data

$$\left.\begin{aligned} v(x,0)&=v^0,\\ \varphi_k(x,0)&=\varphi_k^0,\\ \frac{\partial\varphi_k}{\partial t}(x,0)&=\varphi_k^1,\end{aligned}\right\}\qquad x\in\mathbb{R}^2. \tag{3.2c}$$

THEOREM 3.1. *Assume that the initial values* (3.2c) *have the regularity*:

$$v^0,\ \psi_k^0,\ \psi_k^1,\ \frac{\partial v^0}{\partial x_2},\ \frac{\partial v^0}{\partial x_1}+\gamma_k^2\frac{\partial \psi_k^0}{\partial x_1}\in L^2(\mathbb{R}^2)\qquad 1\leqq k\leqq n.$$

The problem (3.2) *then has a unique weak solution, with the regularity*:

$$v,\ \psi_k\in W^{1,\infty}(0,T;L^2(\mathbb{R}^2))\cap W^{2,\infty}(0,T;H^{-1}(\mathbb{R}^2)),\qquad 1\leqq k\leqq n,$$

$$\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\in L^\infty(0,T;L^2(\mathbb{R}^2)),\qquad 1\leqq k\leqq n,$$

$$\frac{\partial v}{\partial x_2}\in L^\infty(0,T;L^2(\mathbb{R}^2)).$$

Moreover the following energy is constant as a function of time:

$$\begin{aligned} E(t)=&\frac{1}{2}\iint_{\mathbb{R}^2}\frac{1}{c}\left|\frac{\partial v}{\partial t}\right|^2dx+\frac{1}{2}\sum_{k=1}^{n}\beta_k\gamma_k^2\iint_{\mathbb{R}^2}\frac{1}{c}\left|\frac{\partial\psi_k}{\partial t}\right|^2dx\\ &+\frac{1}{c}\sum_{k=1}^{n}\beta_k\iint_{\mathbb{R}^2}c\left|\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\right|^2dx.\end{aligned} \tag{3.4}$$

For a definition of the Sobolev spaces H^k and $W^{m,p}$ see [12].

Proof. The well-posedness follows in standard fashion from the energy estimate using the Galerkin method. We derive the latter in four steps:

(i) We differentiate (3.2a) with respect to t, multiply by $\partial v/\partial t$ and integrate over $\mathbb{R}^2$. We then have

$$\frac{1}{2}\frac{d}{dt}\left[\iint_{\mathbb{R}^2}\frac{1}{c}\left|\frac{\partial v}{\partial t}\right|^2 dx\right]=\sum_{k=1}^{n}\beta_k\iint_{\mathbb{R}^2}\frac{1}{c}\frac{\partial^2\psi_k}{\partial t^2}\frac{\partial v}{\partial t}\,dx. \tag{3.5}$$

(ii) We multiply (3.2b) by $\partial v/\partial t+\gamma_k^2\partial\psi_k/\partial t$ and integrate over $\mathbb{R}^2$. We get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\gamma_k^2\iint_{\mathbb{R}^2}\frac{1}{c}\left|\frac{\partial\psi_k}{\partial t}\right|^2 dx+\iint_{\mathbb{R}^2}c\left|\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\right|^2 dx\right]\\&\qquad+\iint_{\mathbb{R}^2}\frac{1}{c}\frac{\partial^2\psi_k}{\partial t^2}\frac{\partial v}{\partial t}\,dx=0.\end{aligned}\tag{3.6}$$

(iii) We multiply (3.6) by β_k and add the resulting identities for $1\leqq k\leqq n$. This yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\sum_{k=1}^{n}\beta_k\gamma_k^2\iint_{\mathbb{R}^2}\frac{1}{c}\left|\frac{\partial\psi_k}{\partial t}\right|^2 dx+\sum_{k=1}^{n}\beta_k\iint_{\mathbb{R}^2}c\left|\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\right|^2 dx\right]\\&\qquad+\sum_{k=1}^{n}\beta_k\iint_{\mathbb{R}^2}\frac{1}{c}\frac{\partial^2\psi_k}{\partial t^2}\frac{\partial v}{\partial t}\,dx=0.\end{aligned}\tag{3.7}$$

(iv) We eventually add (3.5) and (3.7), and we obtain

$$\frac{d}{dt}E(t)=0.$$

Remark 3.1. Since the coefficients do not depend on time, the regularity in time of the solution only depends on the regularity of the data. The regularity in space is limited by the regularity of c, even if the data are smooth.

Remark 3.2. As a consequence of the energy estimate, a continuity result of the solution with respect to the velocity can be stated (see [3]).

For the case $\beta\neq 0$, we need to specify another initial value

$$v^1(x)=\frac{\partial v}{\partial t}(x,0). \tag{3.8}$$

The result is then the following theorem.

THEOREM 3.2. *Assume that the initial values have the regularity*

$$v^0,\frac{\partial v^0}{\partial x_1},v^1\in L^2(\mathbb{R}^2),$$

$$\psi_k^0,\frac{\partial\psi_k^0}{\partial x_1},\psi_k^1\in L^2(\mathbb{R}^2),\qquad 1\leqq k\leqq n.$$

The Cauchy problem for $\beta\neq 0$ then has a unique weak solution, with the regularity:

$$v,\psi_k\in W^{1,\infty}(0,T;L^2(\mathbb{R}^2))\cap W^{2,\infty}(0,T;H^{-1}(\mathbb{R}^2)),\qquad 1\leqq k\leqq n,$$

$$\frac{\partial v}{\partial x_1},\frac{\partial\psi_k}{\partial x_1}\in L^\infty(0,T;L^2(\mathbb{R}^2)),\qquad 1\leqq k\leqq n.$$

Moreover this solution satisfies the identity $E(t)=E(0)$ a.e. in $[0, T]$, where the energy $E(t)$ is given by

$$
\begin{aligned}
E(t)=&\frac{1}{2}\iint_{\mathbb{R}^2}\frac{1}{c}\left|\frac{\partial v}{\partial t}\right|^2 dx+\frac{\beta}{2}\iint_{\mathbb{R}^2} c\left|\frac{\partial v}{\partial x_1}\right|^2 dx \\
&+\frac{1}{2}\sum_{k=1}^{n}\beta_k\gamma_k^2\iint_{\mathbb{R}^2}\frac{1}{c}\left|\frac{\partial\psi_k}{\partial x_2}\right|^2 dx \\
&+\frac{1}{2}\sum_{k=1}^{n}\beta_k\iint_{\mathbb{R}^2} c\left|\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\right|^2 dx.
\end{aligned}
\tag{3.9}
$$

The proof is similar to the one for Theorem 3.1 above (see [3]).

Remark 3.3. Remarks 3.1 and 3.2 are still valid.

We shall now define the initial boundary value problem for (3.2) in the halfspace $\mathbb{R}^2_+$:

(3.10) Find $(v, \psi_k):\mathbb{R}^2_+\times[0, T]\to\mathbb{R}$, solutions of (3.2), with initial data $v^0, \psi_k^0, \psi_k^1:\mathbb{R}^2_+\to\mathbb{R}$ and a boundary value at $x_2=0$: $v(x_1, 0, t)=g(x_1, t)$ on $\mathbb{R}\times[0, T]$.

THEOREM 3.3. *Assuming the data have the regularity,*

- $v^0, \psi_k^0, \psi_k^1, \partial v^0/\partial x_2, \partial v^0/\partial x_1+\gamma_k^2\partial\psi_k^0/\partial x_1\in L^2(\mathbb{R}^2_+)$ $\quad 1\leqq k\leqq n$,
- $g\in H^1(0, T; L^2(\mathbb{R}))$,

the initial boundary value problem (3.2), (3.10) *has a unique weak solution with the regularity,*

$$v, \psi_k\in W^{1,\infty}(0, T; L^2(\mathbb{R}^2_+))\cap W^{2,\infty}(0, T; H^{-1}(\mathbb{R}^2_+)),\qquad 1\leqq k\leqq n,$$

$$\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\in L^\infty(0, T; L^2(\mathbb{R}^2_+)),\qquad 1\leqq k\leqq n,$$

$$\frac{\partial v}{\partial x_2}\in L^\infty(0, T; L^2(\mathbb{R}^2_+)).$$

Moreover the following energy identity holds

$$E(t)=E(0)+\frac{1}{2}\int_0^t\int_{\mathbb{R}}\left|\frac{\partial g}{\partial t}(x_1, s)\right|^2 dx_1\, ds. \tag{3.11}$$

The energy $E(t)$ is given by

$$
\begin{aligned}
E(t)=&\frac{1}{2}\iint_{\mathbb{R}^2_+}\frac{1}{c}\left|\frac{\partial v}{\partial t}\right|^2 dx+\frac{1}{2}\sum_{k=1}^{n}\beta_k\gamma_k^2\iint_{\mathbb{R}^2_+}\frac{1}{c}\left|\frac{\partial\psi_k}{\partial t}\right|^2 dx \\
&+\frac{1}{2}\sum_{k=1}^{n}\beta_k\iint_{\mathbb{R}^2_+} c\left|\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\right|^2 dx.
\end{aligned}
\tag{3.12}
$$

Proof. The energy estimate is obtained in exactly the same way as in the Cauchy problem. Only step (i) is modified since a boundary term appears. The well-posedness is proved using a Galerkin method. □

When the initial values of v and $\partial v/\partial t$ are identically zero, the initial values for ψ_k and $\partial\psi_k/\partial t$ must be chosen equal to zero. It is the important case for the applications (see for instance the migration process in geophysics). When the initial values do not vanish, the choice of ψ_k and $\partial\psi_k/\partial t$ at time zero is not so clear and remains an open question.

Remark 3.4. We required here a strong regularity for the datum on the boundary. It can actually be removed and weaker solutions can be found [3].

In the case $\beta \neq 0$, we get a similar result, with the same differences as stated in Theorem 3.2 (see [3]).

3.2. Propagation properties. We restrict ourselves to the case where $\beta = 0$, i.e., to the solution of problem (3.2). We intend to generalize the propagation properties for homogeneous medium. The results are similar to those stated in [1] for the parabolic approximation and the technique is again based on energy estimates.

Theorems 3.4 and 3.5 express in different ways that the solution propagates only in the positive x_2 direction, even in heterogeneous medium. Theorem 3.6 specifies an upper bound for the propagation speed.

THEOREM 3.4. *Under the assumptions of Theorem* 3.1 *and if*

$$\operatorname{supp} v^0 \cup \left(\bigcup_{k=1}^{n} (\operatorname{supp} \psi_k^0 \cup \operatorname{supp} \psi_k^1) \right) \subset \mathbb{R}_+^2,$$

then, at any time $t > 0$, one has

$$\operatorname{supp} v(\cdot, t) \cup \left(\bigcup_{k=1}^{n} \operatorname{supp} \psi_k(\cdot, t) \right) \subset \mathbb{R}_+^2.$$

THEOREM 3.5. *Let $(v_1^0, (\psi_k^0)_1, (\psi_k^1)_1, c_1)$ and $(v_2^0, (\psi_k^0)_2, (\psi_k^1)_2, c_2)$ be two families of data, satisfying the assumptions of Theorem* 3.1 *and defining two solutions of problem* (3.2) *$(v_1, (\psi_k)_1)$ and $(v_2, (\psi_k)_2)$. Suppose also that the data are equal in $\mathbb{R}_-^2$:*

$$\left.\begin{aligned} &v_1^0 = v_2^0 \\ &(\psi_k^0)_1 = (\psi_k^0)_2 \\ &(\psi_k^1)_1 = (\psi_k^1)_2 \\ &c_1 = c_2 \end{aligned}\right\} \quad \text{a.e. in } \mathbb{R}_-^2,$$

then, at any time $t > 0$, the solutions are equal in $\mathbb{R}_-^2$:

$$v_1(\cdot, t) = v_2(\cdot, t) \quad \text{a.e. in } \mathbb{R}_-^2,$$

$$(\psi_k)_1(\cdot, t) = (\psi_k)_2(\cdot, t) \quad \forall k, 1 \leqq k \leqq n, \quad \text{a.e. in } \mathbb{R}_-^2,$$

where $\mathbb{R}_-^2 = \{x, x_2 < 0\}$.

Proofs. As in [1], both results follow from an a priori estimate in a half-space.

LEMMA 3.1. *The solution of the problem* (3.2) *has the regularity $v(\cdot, x_2, \cdot) \in L^\infty(\mathbb{R}; H^1(0, T; L^2(\mathbb{R})))$, and it satisfies the energy estimate in the half-space $\Omega_{X_2} = \{x, x_2 \leqq X_2\}$:*

$$\begin{aligned}
&\frac{1}{2} \iint_{\Omega_{X_2}} \frac{1}{c} \left| \frac{\partial v}{\partial t} \right|^2 dx + \frac{1}{2} \sum_{k=1}^{n} \beta_k \gamma_k^2 \iint_{\Omega_{X_2}} \frac{1}{c} \left| \frac{\partial \psi_k}{\partial t} \right|^2 dx \\
&\quad + \frac{1}{2} \sum_{k=1}^{n} \beta_k \iint_{\Omega_{X_2}} c \left| \frac{\partial v}{\partial x_1} + \gamma_k^2 \frac{\partial \psi_k}{\partial x_1} \right|^2 dx \\
&\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \left| \frac{\partial v}{\partial t}(x_1, X_2, s) \right|^2 dx_1 \, ds \\
&= \frac{1}{2} \iint_{\Omega_{X_2}} \left| \frac{\partial v^0}{\partial x_1} - \sum_{k=1}^{n} \frac{\beta_k}{c} \psi_k^1 \right|^2 dx + \frac{1}{2} \sum_{k=1}^{n} \beta_k \gamma_k^2 \iint_{\Omega_{X_2}} \frac{1}{c} |\psi_k^1|^2 \, dx \\
&\quad + \frac{1}{2} \sum_{k=1}^{n} \beta_k \iint_{\Omega_{X_2}} c \left| \frac{\partial v^0}{\partial x_1} + \gamma_k^2 \frac{\partial \psi_k^0}{\partial x_1} \right|^2 dx.
\end{aligned} \tag{3.13}$$

The energy estimate is achieved by the same technique as in Theorem 3.3. To get Theorem 3.4, we apply Lemma 3.1 to (v, ψ_k), and to get Theorem 3.5, we apply it to $(v_1 - v_2, (\psi_k)_1 - (\psi_k)_2)$.

In order to describe the last result, we introduce some notation: c^* is the maximum of c on $\mathbb{R}^2$

$$c^* = \max_{x \in \mathbb{R}^2} c(x),$$

$(\mathscr{C})$ denotes the curve

$$y = f_n(x^2) = 1 - \sum_{k=1}^{n} \frac{\beta_k x^2}{1 - \gamma_k^2 x^2},$$

and $(\mathscr{C}_0)$ is the part of $(\mathscr{C})$ included in the slab $\gamma_1|x| \leqq 1$. $\mathscr{E}^*(t)$ denotes the support at time t of the fundamental solution in homogeneous medium with velocity c^*. Let us recall (§ 2) that the boundary of $\mathscr{E}^*(t)$ is the dual of $c^* t \times \mathscr{C}_0$.

For any $\theta \in]-\pi, \pi[$, D_θ is the line $x_2 = x_1 \operatorname{cotg} \theta$. $P(\theta)$ is the intersection of D_θ with $\mathscr{C}_0$, and $M(\theta)$ is the point of D_θ such that $|OP(\theta)| \cdot |OM(\theta)| = 1$. When θ varies, M varies on the set of group velocity vectors. A velocity is then defined by

$$V^*(\theta) = c^* |OM(\theta)| \tag{3.14}$$

where $|OM|$ denotes the length of the segment OM.

The solution below gives an upper bound for the propagation velocity of the solution (v, ψ_k) to problem (3.2).

THEOREM 3.6. *If the initial values for* (3.2) *are of compact support* $\mathscr{K}$

$$\mathscr{K} = \operatorname{supp} v^0 \cup \left(\bigcup_{k=1}^{n} \operatorname{supp} \psi_k^0 \right) \cup \left(\bigcup_{k=1}^{n} \operatorname{supp} \psi_k^1 \right)$$

then at any time t, (v, ψ_k) is compactly supported and

$$\operatorname{supp} v(\cdot, t) \cup \left(\bigcup_{k=1}^{n} \operatorname{supp} \psi_k(\cdot, t) \right) \subset \mathscr{K} + \mathscr{E}^*(t).$$

Proof. The proof is based on an energy estimate in a moving domain. As in [1] we first assume that the data are smooth, and $\mathscr{K}$ is the disc centered in 0 and of radius R. We define the half-plane Ω_θ^t by

$$\Omega_\theta^t = \{x \in \mathbb{R}^2, (x - (R + Vt)\vec{\theta}) \cdot \vec{\theta} > 0\}$$

for a fixed value of V. Γ_θ^t is the Ω_θ^t boundary, and $d\sigma$ is the measure on Γ_θ^t.

We shall actually prove that for any θ the energy in Ω_θ^t is a decreasing function of time if $V \geqq V^*(\theta)$. This will give the first part of the theorem. The energy is denoted by $E(v, \psi_k, \Omega_\theta^t, t)$ and is given by

$$\begin{aligned} E(v, \psi_k, \Omega_\theta^t, t) = &\frac{1}{2} \iint_{\Omega_\theta^t} \frac{1}{c} \left| \frac{\partial v}{\partial t} \right|^2 dx + \frac{1}{2} \sum_{k=1}^{n} \beta_k \gamma_k^2 \iint_{\Omega_\theta^t} \frac{1}{c} \left| \frac{\partial \psi_k}{\partial t} \right|^2 dx \\ &+ \frac{1}{2} \sum_{k=1}^{n} \beta_k \iint_{\Omega_\theta^t} \left| \frac{\partial v}{\partial x_1} + \gamma_k^2 \frac{\partial \psi_k}{\partial x_1} \right|^2 dx. \end{aligned} \tag{3.15}$$

Using a Green formula we can express the energy as

$$\frac{d}{dt} E(v, \psi_k, \Omega_\theta^t, t) + \frac{1}{2} \int_{\Gamma_\theta^t} \frac{1}{c} \Phi \, d\sigma = 0 \tag{3.16}$$

where Φ can be written as a quadratic form in $\partial v/\partial t, (\partial\psi_k/\partial t)_{1\leq k\leq n}$, $(\partial v/\partial x_1+\gamma_k^2\partial\psi_k/\partial x_1)_{1\leq k\leq n}$:

$$
\begin{aligned}
\Phi = (V-\cos\theta)\left|\frac{\partial v}{\partial t}\right|^2 &+ \sum_{k=1}^n \beta_k\gamma_k^2 V\left|\frac{\partial\psi_k}{\partial t}\right|^2 \\
&+2c^2\sin\theta\sum_{k=1}^n \beta_k\left(\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\right)\left(\frac{\partial v}{\partial t}+\gamma_k^2\frac{\partial\psi_k}{\partial t}\right) \\
&+c^2V\sum_{k=1}^n \beta_k\left|\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\right|^2.
\end{aligned}
\tag{3.17}
$$

In order to determine the sign of Φ, we now perform a Gauss decomposition of Φ:

$$
\begin{aligned}
\Phi = &\sum_{k=1}^n \beta_k\gamma_k^2 V\left[\frac{\partial\psi_k}{\partial t}+\frac{c^2}{V}\sin\theta\left(\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\right)\right]^2 \\
&+V\sum_{k=1}^n\beta_k\left(1-\gamma_k^2\frac{c^2}{V^2}\sin^2\theta\right)\left[c\left(\frac{\partial v}{\partial x_1}+\gamma_k^2\frac{\partial\psi_k}{\partial x_1}\right)+\frac{(c/V)\sin\theta}{1-\gamma_k^2(c^2/V^2)\sin^2\theta}\frac{\partial v}{\partial t}\right]^2 \\
&+V\left[f_n\left(\left(\frac{c}{V}\sin\theta\right)^2\right)-\frac{c}{V}\cos\theta\right]\left|\frac{\partial v}{\partial t}\right|^2.
\end{aligned}
$$

It is easy to see under which conditions Φ is positive, and to conclude that, for any value of V such that $V\geqq V^*(\theta)$, the energy in Ω_θ^t is decreasing as a function of time. This proves the first part of Theorem 3.6. The second part is then derived by taking the intersection of all the half-spaces Ω_θ^t for $V=V^*(\theta)$, when θ varies. By translation, linearity and continuity the result is extended to any support and to discontinuous data.

3.3. Reflection and transmission at a linear interface. As for the parabolic approximation, we consider two homogeneous half-spaces Ω^- and Ω^+, with a velocity c^- and c^+, respectively, separated by an interface $\Gamma(\alpha)$. The unit normal and tangent vectors to the interface are denoted by ν and τ, respectively:

$$
\begin{aligned}
&\tau=(\cos\alpha,\sin\alpha), \qquad \nu=(-\sin\alpha,\cos\alpha),\\
&\Gamma(\alpha)=(x, x\cdot\nu=0\},\\
&\Omega^-(\alpha)=\{x, x\cdot\nu<0\},\\
&\Omega^+(\alpha)=\{x, x\cdot\nu>0\}.
\end{aligned}
\tag{3.18}
$$

It is easy to derive the transmission conditions at the interface for the equations written on the form (3.2), (3.3).

- If $\alpha=0$,

$$[v]=0. \tag{3.19}$$

- If $\alpha\neq 0$

Equation (3.2):

$$
\begin{cases}
[v]=0, & \\
[\psi_k]=0, & 1\leqq k\leqq n,\\
\left[c\dfrac{\partial}{\partial x_1}(\gamma_k^2\psi_k+v)\right]=0, & 1\leqq k\leqq n.
\end{cases}
\tag{3.20}
$$

Equation (3.3):

$$
(3.21)\qquad \begin{cases} [v]=0, & \\ [\psi_k]=0, & 1\leqq k\leqq n, \\ \left[c\dfrac{\partial v}{\partial x}\right]=0, & \\ \left[c\dfrac{\partial \psi_k}{\partial x}\right]=0, & 1\leqq k\leqq n. \end{cases}
$$

Again the cases of oblique interfaces and horizontal interfaces are very different. The latter produces no reflected wave and one transmitted wave, while the former gives rise to n reflected waves and $(n+1)$ transmitted waves (equation (3.2)) or $(n+1)$ reflected waves and $(n+1)$ transmitted waves (equation (3.3)).

We first recall some basic definitions and notations: u_I is the incident wave in Ω^-

$$
(3.22)\qquad u_I(x,t)=\exp i(\omega t-k\cdot x),
$$

where ω and k are related by the dispersion relation in Ω^-

$$
(3.23)\qquad c^-\frac{k_2}{\omega}=f_n\left(\left(c^-\frac{k_1}{\omega}\right)^2\right).
$$

(For simplicity, the vector k defined here is such that $k_2/\omega>0$.) We define the incident slowness vector:

$$
(3.24)\qquad K=\frac{k}{\omega}
$$

and the group velocity vector:

$$
(3.25)\qquad V_G(K)=\nabla_k\omega.
$$

The group velocity vector is said to be "ingoing" (in the interface) in Ω^- if $V_G\cdot\nu>0$, and "outgoing" otherwise.

Throughout the remainder of this paper we assume that $|c^-K_1|\leqq 1/\gamma_1$, that is to say that the vector K belongs to the "parabolic" branch of the dispersion curve.

(i) REFLECTION. When the interface is horizontal, there is no reflected wave. When it is oblique, the reflected slowness vectors are defined to have the same projection as K on the interface, and such that their group velocity vectors are outgoing in Ω^-. The number of such vectors is n and they are denoted by ζ^i, $1\leqq i\leqq n$, $\zeta_1^i<\zeta_1^{i+1}$ (Fig. 3.1).

The following Lemma gives the behaviour of $\zeta^1,\cdots,\zeta^k$ when α tends to 0.

LEMMA 3.2. *When α tends to 0, the reflected slowness vectors are such that*

$$
(3.26)\qquad \begin{aligned} \zeta_1^i&=\frac{1}{c^-\gamma_i}-\mu_i\alpha+O(\alpha^2),\\ \zeta_2^i&=\frac{1}{\alpha}\left(K_1-\frac{1}{c^-\gamma_i}+\nu_i\alpha+O(\alpha^2)\right), \end{aligned}\qquad 1\leqq i\leqq n
$$

where μ_i and ν_i are given by

$$
(3.27)\qquad \begin{aligned} \mu_i&=\frac{\beta_i}{2c^{-2}\gamma_i^3(-K_1+1/(c^-\gamma_i))},\\ \nu_i&=K_2+\mu_i, \end{aligned}\qquad 1\leqq i\leqq n.
$$

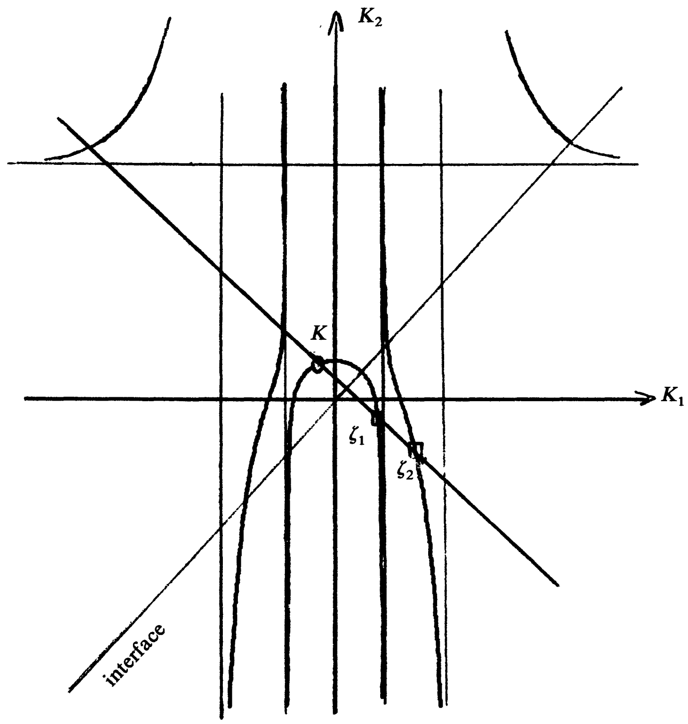


FIG. 3.1. *Reflected slowness vectors* ($n=2$).

Proof of Lemma 3.2. It is clear that ζ_1^i tends to $1/(c^-\gamma_i)$ and ζ_2^i tends to infinity when α tends to zero. We then seek the expansion of ζ^i in the form

$$\zeta_1^i=\frac{1}{c^-\gamma_i}-\mu_i\alpha+O(\alpha^2),$$

$$\zeta_2^i=\frac{1}{\alpha}(\rho_i+\nu_i\alpha+O(\alpha^2))$$

where the coefficients μ_i, ν_i, ρ_i are to be determined.

We first write that K and ζ^i have the same projection on the interface:

$$K_1\cos\alpha+K_2\sin\alpha=\zeta_1^i\cos\alpha+\zeta_2^i\sin\alpha$$

and we expand the equality in terms of α. We thus get

$$\rho_i=K_1-\frac{1}{c^-\gamma_i},\qquad \nu_i=\mu_i+K_2.$$

We then write the dispersion relation for ζ^i

$$\zeta_2^i=\frac{1}{c^-}\left(1-\sum_{k=1}^{n}\frac{\beta_k(c^-\zeta_1^i)^2}{1-\gamma_k^2(c^-\zeta_1^i)^2}\right).$$

When α tends to zero, we have

$$\zeta_2^i\sim-\frac{\beta_i}{c^-}\frac{(c^-\zeta_1^i)^2}{1-\gamma_i^2(c^-\zeta_1^i)^2}\sim-\frac{\beta_i}{c^-}\frac{\beta_i}{2c^-\gamma_i^2\mu_i}\times\frac{1}{\alpha}$$

which gives a third relation between the three unknowns

$$-\frac{\beta_i}{2c^{-2}\gamma_i^3\mu_i}=K_1-\frac{1}{c^-\gamma_i}.$$

The three equations then define μ_i, ρ_i and ν_i.

(ii) TRANSMISSION. We choose here $c^-\leqq c^+$. When the interface is horizontal (i.e., $\alpha=0$), there is one transmitted slowness vector η^* such that

$$\eta_1^*=K_1,\qquad c^+\eta_2^*=f_n((c^+\eta_1^*)^2). \tag{3.28}$$

When $\alpha\neq 0$ there are $n+1$ transmitted vectors $\eta^0,\cdots,\eta^n$, η^0 belonging to the "parabolic" branch of the dispersion curve, i.e.,

$$|\eta_1^0|\leqq\frac{1}{c^+\gamma^1}.$$

(See Fig. 3.2.)

Again we give the behaviour of η^i when α tends to 0.

LEMMA 3.3. *When α tends to 0, the transmitted slowness vectors have the expansion*:

$$\begin{aligned}
&\eta_1^0=K_I+(K_2-\eta_2^*)\alpha+O(\alpha^2),\\
&\eta_1^i=-\frac{1}{c^+\gamma_i}-\lambda_i\alpha+O(\alpha^2),\qquad 1\leqq i\leqq n,\\
&\eta_2^i=\frac{1}{\alpha}\left(K_1+\frac{1}{c^+\gamma_i}+\rho_2\alpha+O(\alpha^2)\right),\qquad 1\leqq i\leqq n,
\end{aligned}\tag{3.29}$$

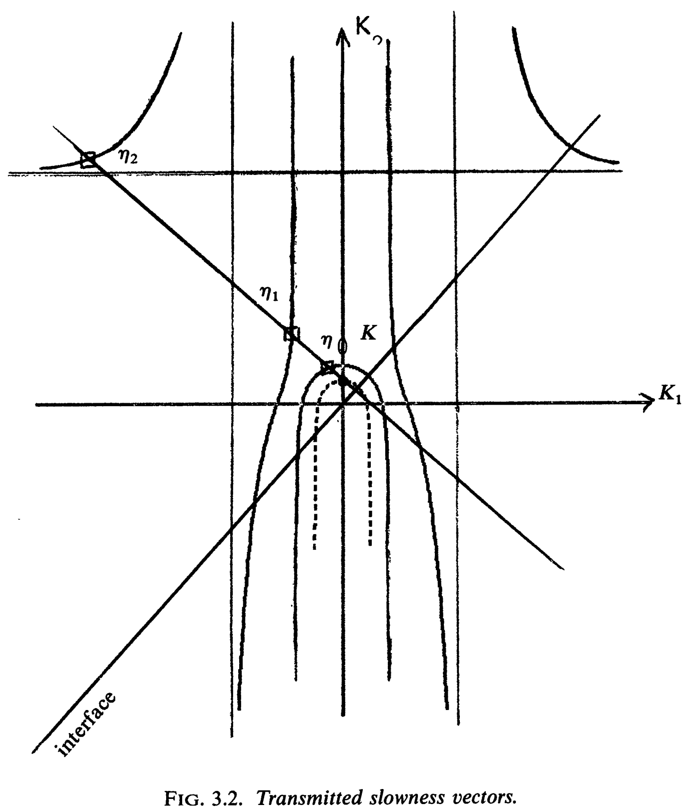


FIG. 3.2. *Transmitted slowness vectors.*

where λ_i and ρ_i are given by

$$\begin{aligned} &\lambda_i = \frac{\beta_i}{2(c^+)^2\gamma_i^3(K_1+1/(c^+\gamma_i))}, \\ &\rho_i = \lambda_i + K_2, \end{aligned} \qquad 1 \leqq i \leqq n. \tag{3.30}$$

The proof is similar to that of Lemma 3.2.

(iii) REFLECTION AND TRANSMISSION COEFFICIENTS. We start with the case of horizontal interface. There is no reflected wave and one transmitted wave

$$u_T = T^* \exp i\omega(t-\eta^* \cdot x). \tag{3.31}$$

When the interface is oblique, the reflected wave is

$$u_R = \sum_{j=1}^{n} R_j(\alpha) \exp i\omega(t-\zeta^j \cdot x) \tag{3.32}$$

and the transmitted wave is

$$u_T = \sum_{j=0}^{n} T_j(\alpha) \exp i\omega(t-\eta^j \cdot x). \tag{3.33}$$

The main result of this part is the following theorem, which shows that the equation has the properties for which we aimed.

THEOREM 3.7. *When $\alpha=0$, the transmission coefficient T^* is*

$$T^* = \sqrt{c^+/c^-}. \tag{3.34}$$

When $\alpha \neq 0$, one has

$$\begin{aligned} &\lim_{\alpha\to 0} R_j(\alpha) = \lim_{\alpha \to 0} T_j(\alpha) = 0, \qquad 1 \leqq j \leqq n, \\ &\lim_{\alpha\to 0} T_0(\alpha) = T^*. \end{aligned} \tag{3.35}$$

The coefficients have the following form

$$\begin{aligned} &R_i(\alpha) \sim \frac{\beta_i}{2} \frac{(c^+-c^-)K_1}{(1+\gamma_i c^- K_1)^2(1-\gamma_i c^+ K_1)}\alpha, \qquad 1 \leqq i \leqq n, \\ &T_i(\alpha) \sim -\frac{\beta_i}{2} \frac{(c^+-c^-)K_1}{(1+\gamma_i c^- K_1)(1+\gamma_i c^+ K_1)^2}\alpha, \qquad 1 \leqq i \leqq n. \end{aligned} \tag{3.36}$$

Proof. We seek the solution u of (3.2) in the following form

$$\begin{aligned} u &= u_I + u_R \quad \text{in } \Omega^-, \\ u &= u_T \qquad\quad \text{in } \Omega^+. \end{aligned}$$

If $\alpha=0$, T^* is easily obtained from the transmission condition $[c^{-1/2}u]=0$. If $\alpha \neq 0$, the transmission conditions (3.20) provide a $(2n+1)\times(2n+1)$ system of equations the solution of which is $(R_1, \cdots, R_n, T_0, T_1, \cdots, T_n)$:

$$\begin{aligned} &\frac{1}{\sqrt{c^-}}\left(1+\sum_{i=1}^{n} R_i\right) = \frac{1}{\sqrt{c^+}}\sum_{i=0}^{n} T_i, \\ &\frac{1}{\sqrt{c^-}}\left(\frac{1}{1-\gamma_k c^- K_1} + \sum_{i=1}^{n} \frac{R_i}{1-\gamma_k c^- \zeta_1^i}\right) = \frac{1}{\sqrt{c^+}}\sum_{i=0}^{n} \frac{T_i}{1-\gamma_k c^+ \eta_1^i}, \qquad 1 \leqq k \leqq n, \\ &\frac{1}{\sqrt{c^-}}\left(\frac{1}{1+\gamma_k c^- K_1} + \sum_{i=1}^{n} \frac{R_i}{1+\gamma_k c^- \zeta_1^i}\right) = \frac{1}{\sqrt{c^+}}\sum_{i=0}^{n} \frac{T_i}{1+\gamma_k c^+ \eta_1^i}, \qquad 1 \leqq k \leqq n. \end{aligned}$$

The solution of this system is given by the following formulae (using a Gauss determinant):

$$R_k=-\prod_{\substack{1\leq j\leq n\\ j\neq k}}\frac{\zeta_1^j-K_1}{\zeta_1^j-\zeta_1^k}\prod_{0\leq j\leq n}\frac{c^+\eta_1^j-c^-K_1}{c^+\eta_1^j-c^-\zeta_1^k}\prod_{1\leq j\leq n}\frac{1-(c^-\gamma_j\zeta_1^k)^2}{1-(c^-\gamma_jK_1)^2},$$

$$T_k=\sqrt{\frac{c^+}{c^-}}\prod_{1\leq j\leq n}\frac{c^-(\zeta_1^j-K_1)}{c^-\zeta_1^j-c^+\eta_1^k}\prod_{\substack{0\leq j\leq n\\ j\neq k}}\frac{c^+\eta_1^j-c^-K_1}{c^+(\eta_1^j-\eta_1^k)}\prod_{1\leq j\leq n}\frac{1-(c^+\gamma_j\eta_1^k)^2}{1-(c^-\gamma_jK_1)^2}.$$

We now apply Lemmas 3.2 and 3.3 to each term of the product which finishes the proof of Theorem 3.7.

Remark 3.5. η^0 appears to be the approximation to the exact transmitted wave, while $\eta^1,\cdots,\eta^n$ can be considered as "parasitic" transmitted waves.

By Theorems 3.1 and 3.7, criteria (i) and (iii) are verified. The value of T^* given by (3.34) is the same as in the parabolic case. Criterion (ii) is thus obviously satisfied. Let us just notice that this high order paraxial approximation improves the accuracy in each homogeneous part of the medium, but not the transmission coefficient at the interface.

4. Numerical experiments. In order to illustrate our theoretical results, we present here numerical experiments implemented by F. Collino at IFP. The results correspond to two equations of the family (3.2) obtained by the continued fractions expansion (2.8) for $N=3$ and $N=5$, i.e.,

$$v=c^{-1/2}u,\qquad \psi=c^{-1/2}\varphi, \tag{4.1a}$$

$$\frac{1}{c}\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x_2}-\frac{1}{2c}\frac{\partial \psi}{\partial t}=0, \tag{4.1b}$$

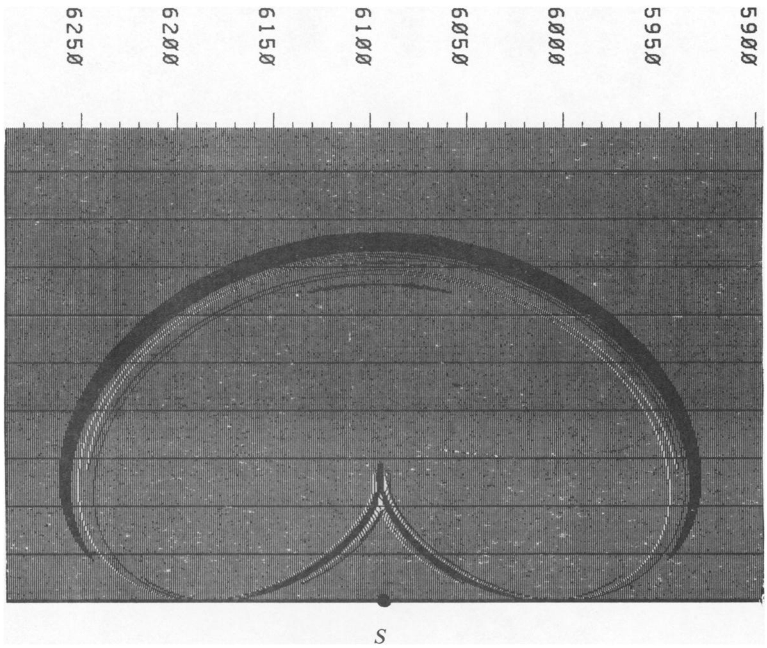
$$\frac{1}{c}\frac{\partial^2\psi}{\partial t^2}-\frac{1}{4}\frac{\partial}{\partial x_1}\left(c\frac{\partial\psi}{\partial x_1}\right)=\frac{\partial}{\partial x_1}\left(c\frac{\partial v}{\partial x_1}\right), \tag{4.1c}$$

$$v=c^{-1/2}u,\qquad \psi_1=c^{-1/2}\varphi_1,\qquad \psi_2=c^{-1/2}\varphi_2, \tag{4.2a}$$

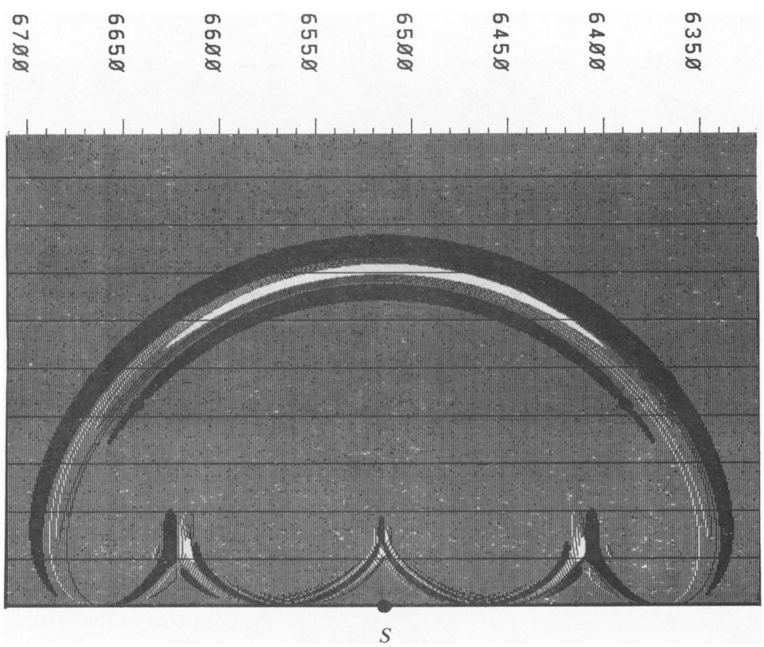
$$\frac{1}{c}\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x_2}-\frac{2}{5c}\left(\sin^2\frac{\pi}{5}\frac{\partial\psi_1}{\partial t}+\sin^2\frac{2\pi}{5}\frac{\partial\psi_2}{\partial t}\right)=0, \tag{4.2b}$$

$$\begin{aligned}&\frac{1}{c}\frac{\partial^2\psi_1}{\partial t^2}-\cos^2\frac{\pi}{5}\frac{\partial}{\partial x_1}\left(c\frac{\partial\psi_1}{\partial x_1}\right)=\frac{\partial}{\partial x_1}\left(c\frac{\partial v}{\partial x_1}\right),\\ &\frac{1}{c}\frac{\partial^2\psi_2}{\partial t^2}-\cos^2\frac{2\pi}{5}\frac{\partial}{\partial x_1}\left(c\frac{\partial\psi_2}{\partial x_1}\right)=\frac{\partial}{\partial x_1}\left(c\frac{\partial v}{\partial x_1}\right).\end{aligned} \tag{4.2c}$$

The way the equations are written enables us to use a splitting method. As in [1], the time-dependence in (4.1b) and (4.1c) is handled by Fourier transform. The equations are then semi-discretized in x_1 by P_1 finite elements. A Crank-Nicolson scheme is finally used in the x_2 direction. For further details and properties about these numerical schemes see [6]. Each of the figures we present here are snapshots of the solution at a given time. This gives an image of the solution in the (x_1, x_2) plane (this representation

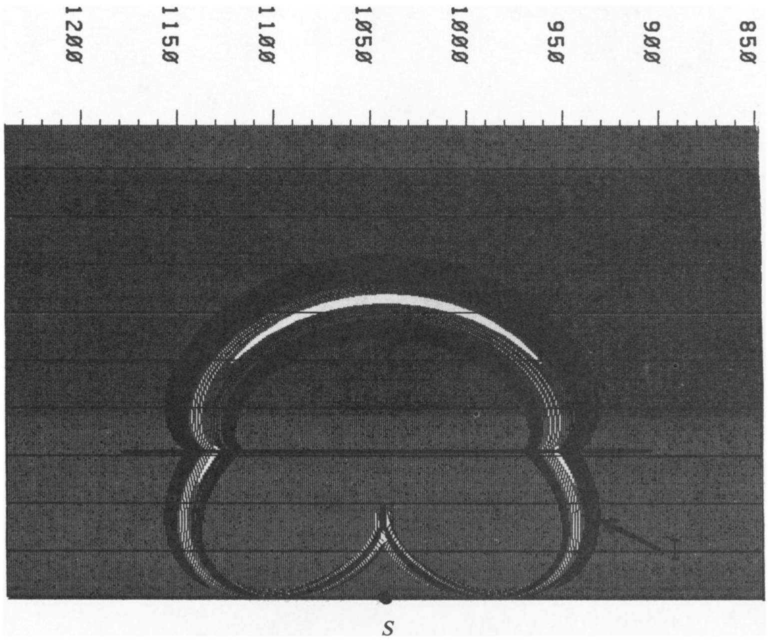


Equation (4.1)

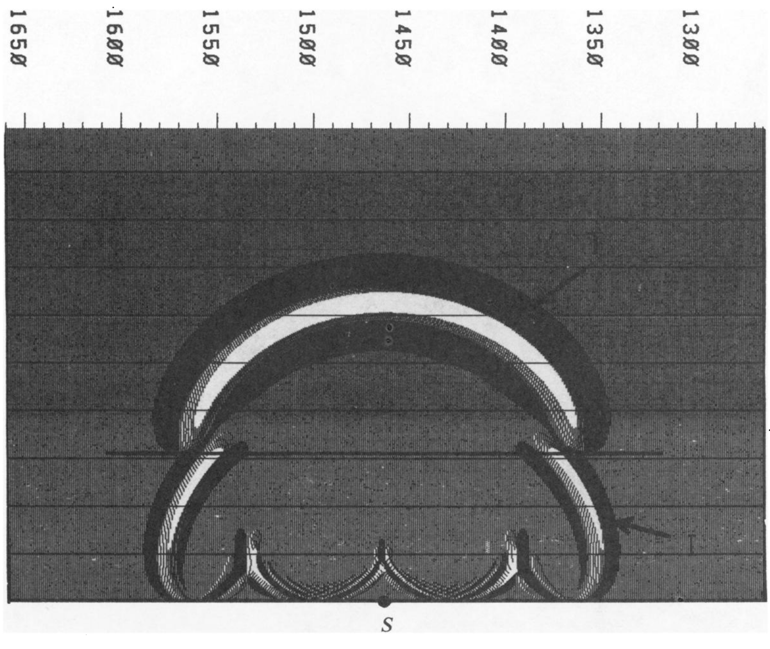


Equation (4.2)

FIG. 4.1. *Fundamental solutions.*

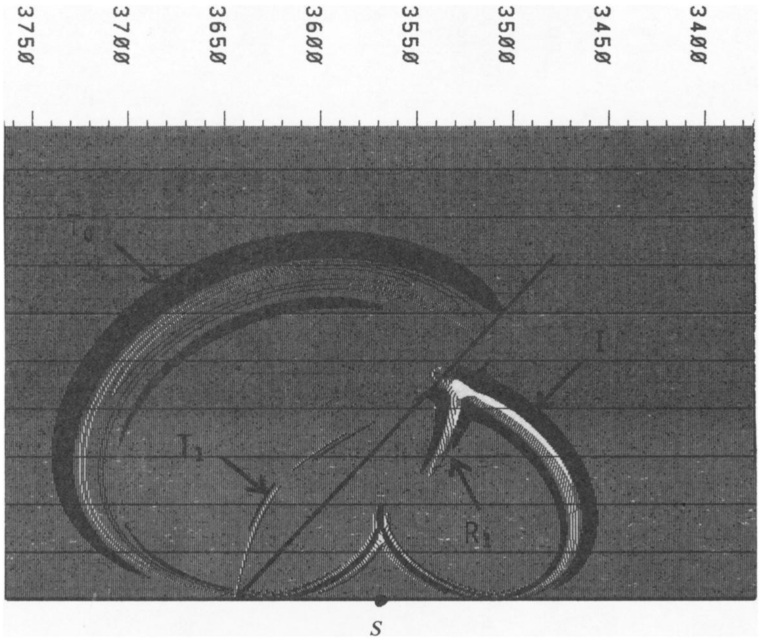


Equation (4.1)

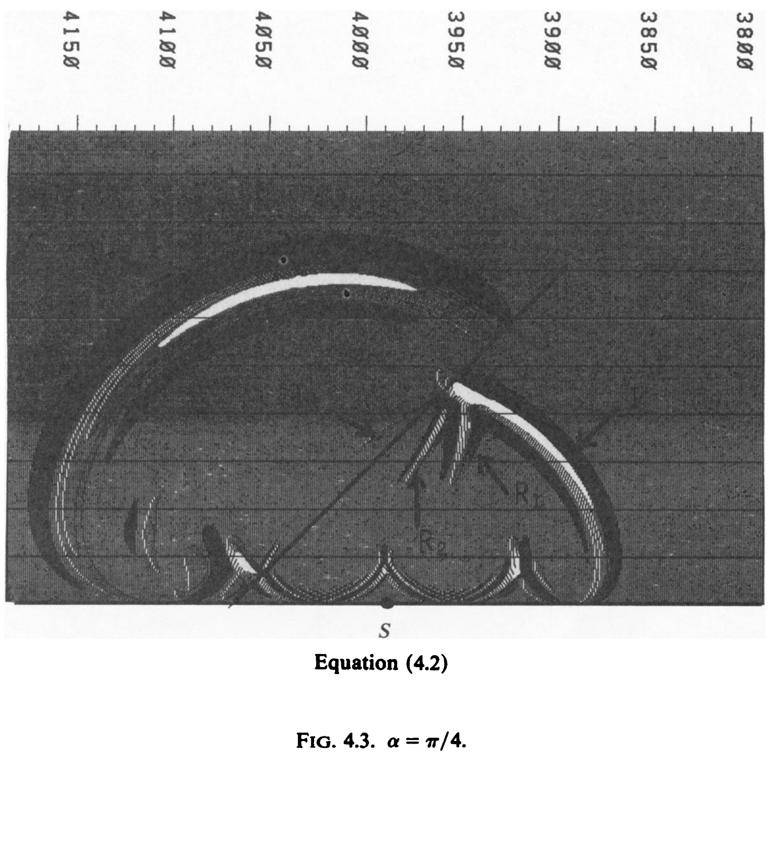


Equation (4.2)

FIG. 4.2. *Horizontal interface.*

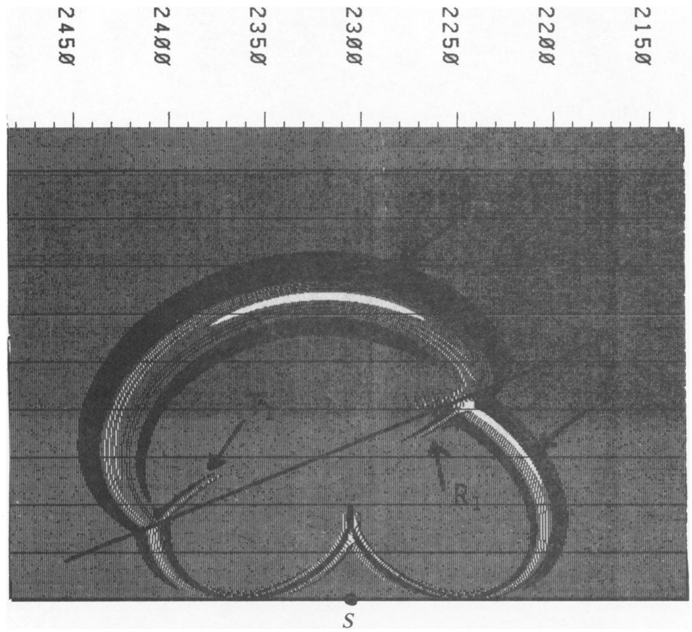


Equation (4.1)

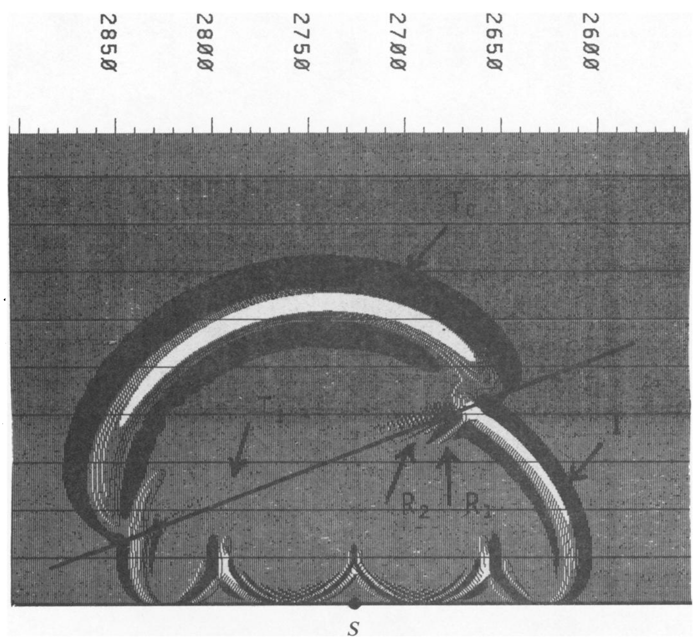


Equation (4.2)

FIG. 4.3. $\alpha = \pi/4$.



Equation (4.1)



Equation (4.2)

FIG. 4.4. $\alpha = \pi/8$.

is commonly used by geophysicists). The areas where the solution is positive are darker, the ones where it is negative are lighter.

For each simulation, the source is quasi-punctual, i.e., its support is very small. Its position is indicated on the figures by the point S. Its time dependence is given by the second derivative of a Gaussian function (Ricker source in Geophysics).

In Fig. 4.1 the fundamental solutions of the paraxial approximations are plotted for (4.1) and (4.2). We can easily see that the support of the solution tends to the ideal semi-disk and that the number of parasitic branches increases with N.

We now consider the specific heterogeneous medium we studied in § 3 (see (3.18)). This medium consists of two homogeneous half-spaces Ω^- (with velocity c^-) and Ω^+ (with velocity c^+) separated by an interface $\Gamma(\alpha)$ whose angle with the horizontal x_1 direction is equal to α. The ratio c^+/c^- is equal to 2 and the source is located in the medium Ω^-.

Figure 4.2 is a snapshot of the solution when the interface is horizontal (i.e. $\alpha = 0$). In each case we easily distinguish, as indicated, the incident wave and the unique transmitted one. Note that the parasitic waves have not yet reached the interfaces at the time we consider. Also it is interesting to remark that for equations (4.2) the wave front, although it is not, seems to be discontinuous along the interface. In fact such a discontinuity occurs for the full wave equation if one does not consider the reflected waves: the head wave connects the reflected wave and the transmitted one.

In Fig. 4.3 we give the results when $\alpha = \pi/4$. For (4.1) one clearly sees the reflected wave R_1 (whose amplitude is rather strong) and the two transmitted ones T_0 and T_1. Note the curious shape of the second transmitted wave front, which is the parasitic one. Its amplitude is much less important than the one of the first transmitted wave. For (4.2), the two reflected waves R_1 and R_2 are clearly visible but we can only distinguish two transmitted waves (denoted by T_0 and T_1 in the figure). The slowest transmitted wave (which would be T_2) is too weak to be visible. We can also remark the existence of a second family of reflected and transmitted waves which are due to the first parasitic arch of the incident wave which reached the interface.

Finally we notice in both cases the existence in the medium Ω^- of a head-wave connecting the first transmitted wave to the first reflected one.

In Fig. 4.4 the angle α is equal to $\pi/8$. The involved phenomena are qualitatively the same as for $\alpha = \pi/4$. It is moreover interesting to notice that the reflected and parasitic transmitted waves are much weaker, as the theory predicts, than for $\alpha = \pi/4$.

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