From the Quasi-static to the Dynamic Maxwell’s Model in Micromagnetism

Laurence Halpern* Stéphane Labbé†

Abstract

A commonly used model for ferromagnetic materials in the quasistatic regime is the Landau-Lifshitz system coupled with the so-called quasistatic Maxwell’s equations. By an appropriate scaling, we justify this approach and we propose a new asymptotic expansion. This suggest a new numerical method.

1 The micromagnetism model

The magnetic material fills a bounded domain $\Omega$ in $\mathbb{R}^3$. The evolution of the magnetization field is governed by the Landau-Lifshitz system

\[
\frac{\partial M}{\partial T} = -\gamma\mu_0 \left( M \times H_T + \frac{\alpha}{|M|} M \times (M \times H_T) \right) \text{ in } \Omega,
\]

with initial condition $M(0)$. $M$ is the magnetization field; it vanishes outside $\Omega$, and has a prescribed length in $\Omega$

\[
|M(X,T)| = |M(X,0)| = M_S \text{ a.e in } \Omega.
\]

$\mu_0$ is the magnetic permeability, $\gamma$ the Larmor precession factor, and $\alpha$ a dimensionless damping factor. They are all positive factors. The total magnetic field $H_T$ is a linear function of $M$. It is the sum of three magnetic contributions (we consider the external field to be zero): the exchange field $H_{ex} = A\Delta M$, the anisotropy field $H_a = -K u \times (M \times u)$, where $u$ is the direction of anisotropy, and $K$ and $A$ are physical positive constants. Finally the Maxwell’s field $H$ solves the system of equations whose unknowns are the magnetic field $H$, the electric field $E$, and the electrostatic charge $\rho$, the magnetization field $M$ being given

\[
\varepsilon(X) \frac{\partial E}{\partial T} + \sigma(X)E - \text{rot } H = 0,
\]

\[
\mu_0 \frac{\partial}{\partial T}(H + M) + \text{rot } E = 0,
\]

with prescribed initial conditions. Furthermore we have for all time the following constraints

\[
\text{div } (\varepsilon(X)E) = \rho, \quad \text{div } (\mu_0(H + M)) = 0.
\]

*University of Paris 13, Département de Mathématiques, Institut Galilée,93430 Villetaneuse, France, halpern@math.univ-paris13.fr.
†University of Paris 11, Laboratoire de Mathématique, Bat. 425, 91405 Orsay, France, labbe@math.u-psud.fr.
\( \varepsilon_0 \) is the permittivity in the vacuum, \( \varepsilon_r \) the relative permittivity of the material, and the value of \( \varepsilon(x) \) is \( \varepsilon_0 \varepsilon_r \) in \( \Omega \), \( \varepsilon_0 \) in the exterior. \( \sigma(x) \) is the conductivity of the material; it vanishes outside \( \Omega \).

The total field is thus given by

\[
H_T(M) = -K u \times (M \times u) + A \Delta M + H(M).
\]

The system we consider here is composed of (1,\ldots,5), with the mandatory constraints (4) and initial conditions.

2 Two scalings for the micromagnetism model

We perform the following scaling

\[
H = \tilde{h} \hat{h}, \ E = \tilde{e} \hat{e}, \ \rho = \tilde{\rho} \hat{\rho}, \ M = \tilde{m} \hat{m},
\]

and the change of variables

\[
X = \tilde{x} x, \ T = \tilde{t} t,
\]

where \( \tilde{x} \) and \( \tilde{t} \) are the characteristic length and time. By homogeneity, we have the following relations:

\[
\tilde{m} = \tilde{\rho} \hat{m}, \ \mu_0 \tilde{t} = \tilde{\rho} \hat{\rho}, \ \tilde{\rho} = \varepsilon_0 \varepsilon_r \tilde{\rho} = \tilde{\rho}.
\]

The dimensionless Landau and Lifshitz system is

\[
\frac{\partial \tilde{m}}{\partial \tilde{t}} = -\gamma \mu_0 \tilde{m} \left( \tilde{m} \times \tilde{h}_T + \frac{\alpha}{|\tilde{m}|} \tilde{m} \times (\tilde{m} \times \tilde{h}_T) \right) \text{ in } \omega,
\]

with the constraint \(|\tilde{m}(x, t)| = \frac{M_S}{\tilde{m}} \text{ a.e in } \omega.

By homogeneity in the Landau-Lifshitz system, and linearity in \( H_T \), there appears a new scale \( \zeta = \tilde{t} \gamma \mu_0 \). Applying the new scaling to all variables,

\[
\hat{h} = \zeta \tilde{h}, \ \hat{R} = \zeta \tilde{R}, \ \hat{e} = \zeta \tilde{e}, \ \hat{m} = \zeta \tilde{m},
\]

and choosing \( \tilde{m} = \tilde{\gamma} \mu_0 M_S \), system (9) becomes

\[
\frac{\partial \hat{m}}{\partial \tilde{t}} = -\hat{m} \times h_T - \alpha \hat{m} \times (\hat{m} \times h_T) \text{ in } \omega,
\]

with the constraint

\[
|\hat{m}(x, t)| = |\hat{m}(x, 0)| = 1 \text{ a.e in } \omega.
\]

The total field \( h_T \) is given by the three contributions \( h_a = -K u \times (m \times u) \), \( h_{ex} = A \Delta m = -\frac{\Delta \Delta m}{\tilde{x}^2} \text{ and } h :\)

\[
h_T(m) = -K u \times (m \times u) + A \Delta m + h(m).
\]

We set \( \eta = \frac{\tilde{x}}{c \tilde{t}} \), where \( c \) is the speed of light. In our context, the length of \( \Omega \) is supposed to be small with respect to the wavelengths. Thus the parameter \( \eta \) is small. With these notations, the Maxwell's system of equations becomes
\[ \eta^2 \varepsilon \frac{\partial e}{\partial t} + \eta \bar{\varepsilon} e - \text{rot} \ h = 0, \]
\[ \frac{\partial}{\partial t}(h + m) + \text{rot} \ e = 0, \]
\[ \text{div} (h + m) = 0, \quad \text{div} (\bar{e} e) = R, \]

with \textit{ad hoc} initial values. The problem is now ready for asymptotic expansion. Note that it is a kind of singular perturbation for the electric and magnetic fields, in the time variable.

3 Asymptotic expansion for the Maxwell's system

We place ourselves in the linear case, where the magnetization field \( m \) is given, and we consider the system (14). For other asymptotic expansions and scaling concerning Maxwell's equations see [1] and [3]. The well-posedness can be shown using the theory of semi-groups.

We expand now \( R \) and \( m \) as functions of \( \eta \),

\[ R = \sum_{i=0}^{\infty} \eta^i R_i, \quad m = \sum_{i=0}^{\infty} \eta^i m_i, \]

and we search for \( e \) and \( h \) such that

\[ e = \sum_{i=0}^{\infty} \eta^i e_i, \quad h = \sum_{i=0}^{\infty} \eta^i h_i. \]

Inserting these expansions into the system (14), we obtain first the so-called quasi-static Maxwell's system

\[ \begin{cases} 
\text{div} (h_0 + m_0) = 0, & \text{rot} \ h_0 = 0, \\
\text{rot} \ e_0 = -\frac{\partial}{\partial t}(m_0 + h_0), & \text{div} (\bar{e} e_0) = R_0, 
\end{cases} \]

and a sequence of systems for \( k \geq 1 \)

\[ \begin{cases} 
\text{div} (h_k + m_k) = 0, & \text{rot} \ h_k = \varepsilon \frac{\partial e_{k-2}}{\partial t} + \bar{\varepsilon} e_{k-1}, \\
\text{rot} \ e_k = -\frac{\partial}{\partial t}(h_k + m_k), & \text{div} (\bar{e} e_k) = R_k, 
\end{cases} \]

(with the convention \( e_{-2} = e_{-1} = 0 \)). Using the Helmholtz decomposition in weighted Sobolev spaces, we proved

**Theorem 3.1.** Suppose \( m_k \) belongs to \( C_{V^{-k+1}}(\mathbb{R}^+, L^2(\omega)) \) and \( R_k \) belongs to \( C_{V^{-k}}(\mathbb{R}^+, L^2(\omega)) \) for \( 0 \leq k \leq p \). Then problems (17) and (18) have a unique solution \((h_k, e_k)\) in \( C_{V^{-k+1}}(\mathbb{R}^+, L^2(\mathbb{R}^3)) \times C_{V^{-k}}(\mathbb{R}^+, L^2(\mathbb{R}^3)) \) for \( 0 \leq k \leq p \).

We verify now that the asymptotic expansions really approximate the fields. Let \( \hat{h}_p \) and \( \hat{e}_p \) be the partial sums, \( \tilde{h}_p \) and \( \tilde{e}_p \) denote the errors, \textit{i.e.}

\[ \begin{align*}
\hat{e}_p &= \sum_{i=0}^{p} \eta^i e_i, \hat{e}_p = e - e_p; \quad \hat{h}_p &= \sum_{i=0}^{p} \eta^i h_i, \tilde{h}_p = h - h_p.
\end{align*} \]

The errors satisfy, for any \( p \geq 0 \), a system of the type
\[
\begin{align*}
\eta^2 \varepsilon \frac{\partial \mathbf{e}_p}{\partial t} + \eta \bar{\varepsilon} \mathbf{e}_p - \text{rot} \mathbf{h}_p = 0(\eta^{p+1}), \\
\frac{\partial \mathbf{h}_p}{\partial t} + \text{rot} \mathbf{e}_p = 0(\eta^{p+1}), \\
h_p(x, 0) = \mathbf{e}_p(x, 0) = 0.
\end{align*}
\]

We obtain hyperbolic estimates by multiplying the first equation by \( \mathbf{e} \), the second by \( \mathbf{h} \), and using Green's formula. The Gronwall lemma leads to the conclusion.

**Theorem 3.2.** For any \( p \geq 1 \), the following error estimates hold: for any positive time \( \tau \), there exists a constant \( C \) such that

\[
\begin{align*}
||\mathbf{h}_p||_{L^\infty(0, \tau; L^2(\mathbb{R}^3))} &\leq C\eta^p, \\
||\mathbf{e}_p||_{L^\infty(0, \tau; L^2(\mathbb{R}^3))} &\leq C\eta^{p-1}, \\
||\mathbf{e}_p||_{L^2(0, \tau; L^2(\omega))} &\leq C\eta^{p-\frac{1}{2}}.
\end{align*}
\]

For \( p = 0 \), the error estimates are weaker: for any positive time \( \tau \), there exists a constant \( C \) such that

\[
\begin{align*}
||\mathbf{h}_0||_{L^\infty(0, \tau; L^2(\mathbb{R}^3))} &\leq C\sqrt{\eta}, \\
||\mathbf{h}_0||_{L^\infty(0, \tau; H^1(\mathbb{R}^3))} &\leq C\eta, \\
||\text{rot} (\bar{\varepsilon}\mathbf{e}_0)||_{L^2(0, \tau; L^2(\omega))} &\leq C\eta.
\end{align*}
\]

**4 Asymptotic development for the Micromagnetism system coupled with the Maxwell’s model**

We come back now to the Landau-Lifshitz system (11). Theorems of existence and comments on uniqueness can be found in [5].

The magnetization field \( \mathbf{m} \) is now an unknown, with initial value independent of \( \eta \). Inserting expansions (15) and (16) into (11) and (14), we obtain the first term

\[
\begin{align*}
\frac{\partial \mathbf{m}_0}{\partial t} = -\mathbf{m}_0 \times \mathbf{h}_{\tau,0} - \alpha \mathbf{m}_0 \times (\mathbf{m}_0 \times \mathbf{h}_{\tau,0}) \text{ in } \omega, \\
|\mathbf{m}_0| = 1, \mathbf{m}_0(x, 0) = \mathbf{m}^{(0)}(x), \text{a.e. in } \omega,
\end{align*}
\]

and

\[
\mathbf{h}_{\tau,0} = -K \mathbf{u} \times (\mathbf{m}_0 \times \mathbf{u}) + \mathbf{A} \Delta \mathbf{m}_0 + \mathbf{h}_0
\]

where \( \mathbf{h}_0 \) is given by the quasi-static Maxwell’s system in (17).

The problem (23) is proved to be well-posed in [4]. We first give an energy estimate on the solution to (11) and (14).

**Theorem 4.1.** Let \( (\mathbf{m}, \mathbf{e}, \mathbf{h}) \) solve the equations (11) and (14). The following energy estimate holds

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} [\eta^2 \int_{\mathbb{R}^3} \varepsilon |\mathbf{e}|^2 dx + \int_{\mathbb{R}^3} |\mathbf{h}|^2 dx + \int_{\mathbb{R}^3} \mathbf{A} |\text{grad} \mathbf{m}|^2 dx + \int_{\mathbb{R}^3} K |\mathbf{u} \cdot \mathbf{m}|^2 dx] \\
+ \eta \int_{\omega} \tilde{\sigma} |\mathbf{e}|^2 dx + \int_{\omega} |\mathbf{m} \times \mathbf{h}_{\tau}|^2 dx = 0.
\end{align*}
\]

With these estimates, we can prove convergence.

**Theorem 4.2.** The solution \( (\mathbf{m}, \mathbf{h}) \) to (11) (14) converges weak-* to the solution \( (\mathbf{m}_0, \mathbf{h}_0) \) of the quasistatic model (23) in \( L^\infty(0, \tau; H^1(\mathbb{R}^3)) \) as \( \eta \) tends to 0 (modulo the extraction of a subsequence).
The proof mimics the proof by Carbou in [2] for the convergence of the complete system towards the quasistatic system as the permittivity \( \varepsilon_0 \) tends to 0. But we still do not approximate the electric field. Therefore we introduce the other terms in the expansion.

They are given for any \( n \geq 0 \) by

\[
\frac{\partial \mathbf{m}_n}{\partial t} = - \sum_{k+l=n} \mathbf{m}_{k} \times \mathbf{h}_{T,k} - \alpha \sum_{k+l=n} \sum_{i+j=k} \mathbf{m}_{i} \times (\mathbf{m}_{i} \times \mathbf{h}_{T,j}) \text{ in } \Omega, \\
\sum_{k+l=n} \mathbf{m}_{k} \cdot \mathbf{m}_{l} = 0, \mathbf{m}_n(\mathbf{x},0) = 0, \text{ a.e. in } \Omega
\]

and

\[
\mathbf{h}_{T,j} = -K \mathbf{u} \times (\mathbf{m}_j \times \mathbf{u}) + \bar{A} \Delta \mathbf{m}_j + \mathbf{h}_j
\]

where \( \mathbf{h}_j \) is given by (18).

It is a linear equation which can be shown to be well-posed. There is no proof of convergence today.

5 A dynamical method of simulation using finite volume

The idea is to compute the partial sums \( (\mathbf{m}_n, \mathbf{h}_n, \mathbf{e}_n) \). Using the fact that for all \( i \) in \( \mathbb{N} \), \((\mathbf{e}_i, \mathbf{h}_i, \mathbf{m}_i)\) depend only on \((\mathbf{e}_j, \mathbf{h}_j, \mathbf{m}_j)\) for \( j \leq i \), we compute the \((\mathbf{e}_j, \mathbf{h}_j, \mathbf{m}_j)\) successively. At each level, we use the same finite volume method in space, but a different scheme in time to compute \((\mathbf{e}^n_j, \mathbf{h}^n_j, \mathbf{m}^n_j)\), approximation of \((\mathbf{e}_j, \mathbf{h}_j, \mathbf{m}_j)\) at time \( t_n \).

For each time step \( t_n \), \((\mathbf{e}^n_0, \mathbf{h}^n_0, \mathbf{m}^n_0)\) is first computed, by an explicit second order Taylor scheme in time for the system (26). It is proved in [4] that there exists a unique time step \( \Delta t_n \) such that the scheme is stable and has optimal convergence. \( \mathbf{e}^n_0 \) and \( \mathbf{h}^n_0 \) are obtained by solving a Laplace equation in \( \Omega \).

Then \((\mathbf{e}^n_k, \mathbf{h}^n_k, \mathbf{m}^n_k)\) for \( k > 0 \) are computed successively by the following algorithm:

1. Prediction of \( \mathbf{h}^n_k \) using \( \mathbf{e}^n_{k-1}, \mathbf{e}^n_{k-2} \) and \( \mathbf{m}^{n-1}_k \) with a first order implicit scheme in (18).
2. Computation of \( \mathbf{m}^n_k \) using \( \mathbf{h}^n_k \) with a first order implicit scheme in (26).
3. Correction of \( \mathbf{h}^n_k \) using \( \mathbf{m}^n_k \) in (18).
4. Computation of \( \mathbf{e}^n_k \) by solving (18).

All the computations in space amount to solving a Poisson equation, for which we have fast solvers (see [4]). The presented algorithm provides an accurate method to compute the solutions to Landau-Lifshitz coupled with Maxwell's equations in ferromagnets.

References