Local Space-Time Refinement for the One Dimensional Wave Equation

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Abstract. *We presented recently a new method to design a non-conforming space-time scheme for the wave equation [10]. We introduce here a new concept of stability for domain decomposition, including perturbations on the boundaries of the numerical subdomains. We prove that our scheme is stable in that strong sense, and overall second order in time and space, for a constant velocity.*

Keywords. ***Non conforming mesh; Schwarz Waveform Relaxation; Domain Decomposition***

1 Introduction

When solving a partial differential equation in domains where small scales and large scales are present, it is often desirable to use small meshes in some parts of the domain. For stationary problems, domain decomposition methods with non-matching grids have been developed, in connection with finite volumes or finite elements discretization methods [3, 4, 13]. More recently, mortar element methods have been coupled with optimized Schwarz algorithms [1].

When dealing with the wave equation, refinement in space implies a refinement in time, as the coefficient $c\Delta t/\Delta x$ has to be, for example for the leapfrog scheme, close to 1 [15]. The key point is how to connect the schemes on the interfaces of the numerical domains. This is often done by interpolation of the node values on the interface [2]. A complete analysis of this procedure in the case of the leap-frog scheme has been done, and new transmission conditions using discrete energy estimates have been proposed [5, 6], which are stable, in the energy norm, with respect to the right-hand side in the equation and the initial conditions. As far as we know, no evaluation of the order of convergence is available . For non linear conservation law, a refinement strategy leads to a convergence result [14].

Our approach is different and relies on the use of Schwarz waveform relaxation algorithms [10]. The wave equation in $\mathbf{R} \times (0, T)$ is first written as a collection of wave equations in $\Omega_i \times (0, T)$ with perfectly transmitting conditions on the boundaries between neighboring subdomains. We then discretize in time and space using finite volumes, which allows us to naturally take the transmission conditions into account. In the interior of the subdomains, finite volumes produce the usual leap-frog scheme. The transport operators on the interfaces are naturally discretized by a Lax-Wendroff scheme. The space and time steps are chosen independently and optimally in each subdomain, and the solution is transmitted to the neighbours by a projection procedure. We proved the procedure to be stable for $\gamma \leq 1$ in the L^2 norm, for a constant time step [11] . Furthermore, we showed numerical evidence that, for any type of mesh refinement, the scheme is overall second order in time and space. This paper is devoted to proving this result.
We introduce here a new stability concept, including a perturbation on the interface. In the case where the velocity is constant throughout the domain, we prove the scheme to be stable in that sense. This relies on energy estimates, and an extension lemma. As a consequence, we prove the scheme to be second order in time and space on a sufficiently short time interval. This, together with the use of time windows [10], produces on any time interval a second order scheme.
In Section 2, we introduce the continuous problem, and the concept of stability. We prove the well-posedness through an extension lemma and energy estimates.
In Section 3, we study the discrete problem. We set the scheme, and give the a priori estimates, which will be useful to study the well-posedness. We write the transmission conditions, and prove the well-posedness, assuming an extension result, which is proved at the end of the Section. The order of convergence follows

from the stability result.
So far, the proof is valid only for constant velocity, but the results should extend to variable coefficients.

2 The continuous problem

2.1 Definitions

We consider the one dimensional wave equation with wave speed c,

$$\Box\, u := \frac{1}{c^2(x)}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f, \tag{1}$$

on the domain $\mathbb{R} \times (0,T)$, with initial conditions $u(\cdot,0) = p$ and $\frac{\partial u}{\partial t}(\cdot,0) = q$. If p is in $H^1(\Omega)$, q is in $L^2(\Omega)$, and f is in $L^2(\Omega\times(0,T))$, there exists a unique solution u in $\mathcal{C}^0(0,T;H^1(\Omega))\cap\mathcal{C}^1(0,T;L^2(\Omega))$. If furthermore p is in $H^2(\Omega)$, q is in $H^1(\Omega)$, and f is in $H^1(0,T;L^2(\Omega))$, u is in $V(\Omega) := \mathcal{C}^0(0,T;H^2(\Omega))\cap\mathcal{C}^1(0,T;H^1(\Omega))\cap\mathcal{C}^2(0,T;L^2(\Omega))$ [7]. All throughout the paper, we will assume that it is the case.

2.2 The transmission problem

Suppose $\Omega = \mathbb{R}$ is divided into connected subdomains $\Omega_i = (a_i, a_{i+1})$, $i = 1, \cdots, I$, $a_j < a_i$ for $j < i$, and $a_1 = -\infty$, $a_{I+1} = \infty$. Solving equation (1) in $\mathbb{R}\times(0,T)$ amounts to solving the equation in each subdomain Ω_i with transmission conditions on the interfaces $\Gamma_i = \{a_i\}\times(0,T)$ given by the continuity of u and its normal derivative. We define the transmission operators

$$\mathcal{B}_i^- = \frac{1}{c_-(a_i)}\partial_t - \partial_x, \quad \mathcal{B}_i^+ = \frac{1}{c_+(a_{i+1})}\partial_t + \partial_x, \tag{2}$$

where $c_\pm(x) = \lim_{\epsilon\to 0} c(x\pm\epsilon)$, and solve in each subdomain Ω_i

$$\Box\, u_i = f \text{ in } \Omega_i \times (0,T), \tag{3}$$

with transmission conditions on Γ_i,

$$\begin{array}{rcll} \mathcal{B}_i^-\, u_i\,(a_i,\cdot) &=& \mathcal{B}_i^-\, u_{i-1}\,(a_i,\cdot) & \text{in } (0,T), \\ \mathcal{B}_i^+\, u_i\,(a_{i+1},\cdot) &=& \mathcal{B}_i^+\, u_{i+1}\,(a_{i+1},\cdot) & \text{in } (0,T), \end{array} \tag{4}$$

and initial conditions

$$u_i(\cdot,0) = p, \quad \frac{\partial u_i}{\partial t}(\cdot,0) = q \text{ in } \Omega_i. \tag{5}$$

The transmission conditions together with initial conditions are equivalent to enforcing the continuity of the function and its normal derivative, so that for any i we have $u_i = u/\Omega_i$.

2.3 The Schwarz waveform relaxation algorithm

We now define a Schwarz waveform relaxation algorithm by

$$\begin{array}{rcll} \Box\, u_i^k & = & f & \text{in } \Omega_i \times (0,T), \\ \mathcal{B}_i^- \, u_i^k\,(a_i,\cdot) & = & \mathcal{B}_i^- \, u_{i-1}^{k-1}\,(a_i,\cdot) & \text{in } (0,T), \\ \mathcal{B}_i^+ \, u_i^k\,(a_{i+1},\cdot) & = & \mathcal{B}_i^+ \, u_{i+1}^{k-1}\,(a_{i+1},\cdot) & \text{in } (0,T), \\ u_i^k(\cdot,0) & = & p & \text{in } \Omega_i, \\ \frac{\partial u_i^k}{\partial t}(\cdot,0) & = & q & \text{in } \Omega_i, \end{array} \tag{6}$$

where by convention $\mathcal{B}_i^-(u_{i-1}^0)(a_i,\cdot) = d_i^-$ and $\mathcal{B}_i^+(u_{i+1}^0)(a_{i+1},\cdot) = d_i^+$ are arbitrary initial guesses. For ease of notation, we defined here $u_0^k := 0$ and $u_{I+1}^k := 0$, so that the index i in (6) ranges from $i = 1$ to $i = I$. In order to obtain sufficiently smooth solutions, we introduce compatiblity conditions on the initial guess

$$\frac{1}{c_-(a_i)}q(a_i) - \frac{\partial p}{\partial x}(a_i) = d_i^-; \quad \frac{1}{c_+(a_{i+1})}q(a_{i+1}) + \frac{\partial p}{\partial x}(a_{i+1}) = d_i^+; \quad 2 \le i \le I. \tag{7}$$

Theorem 2.1 *Problem (3), (4), (5) has a unique solution $\{u_i\}_{1\le i\le I}$ in $\cup_i V(\Omega_i)$. For any data $d_i^\pm$ in $H^1(0,T)$ satisfying the compatibility conditions (7), algorithm (6) is well-posed in $\cup_i V(\Omega_i)$. If c is constant in each subdomain, equal to c_i, the algorithm converges in two iterations for $T < T_0 = \min_{1\le i\le I} \frac{a_{i+1}-a_i}{c_i}$, i.e. we have $u_i^2 \equiv u_i$ in each subdomain. If c is continuous at the interfaces, u_i^k converges to u_i in the energy norm. Furthermore we have the energy estimate*

$$\sum_i E_{\Omega_i}(u_i)(t) \le \frac{1}{2}e^T(\|c\|^2_{L^\infty(\Omega)}\|f\|^2_{L^2(\Omega\times(0,T))} + \|\partial_x p\|^2_{L^2(\Omega)} + \|\frac{1}{c}q\|^2_{L^2(\Omega)}), \tag{8}$$

where $E_\Omega(u)(t)$ is the energy of u in Ω at time t,

$$E_\Omega(u)(t) = \frac{1}{2}[\,\|\frac{1}{c}\partial_t u(\cdot,t)\|^2_{L^2(\Omega)} + \|\partial_x u(\cdot,t)\|^2_{L^2(\Omega)}\,]. \tag{9}$$

Proof The existence and uniqueness for problem (3,4,5) follows from the equivalence with problem (1). The well-posedness for the algorithm comes from energy estimates and a Galerkin method [16]. The proof of the convergence in the norm of energy is inspired by the analysis for Helmholtz equation [8], and we show it here for completeness, although it was first given in our study on Schwarz Waveform Relaxation [10]. For u_i, solution of the wave equation in each subdomain Ω_i, the following equality holds:

$$\begin{array}{rcl} E_{\Omega_i}(u_i)(t) & + & \frac{1}{4}\int_0^t [c(a_i)(\mathcal{B}_{i-1}^+ u_i(a_i,s))^2 + c(a_{i+1})(\mathcal{B}_{i+1}^- u_i(a_{i+1},s))^2]\,ds \\ & = & \frac{1}{4}\int_0^t [c(a_i)(\mathcal{B}_i^- u_i(a_i,s))^2 + c(a_{i+1})(\mathcal{B}_i^+ u_i(a_{i+1},s))^2]\,ds \\ & + & \int_0^t (f(\cdot,s),\partial_t u_i(\cdot,s))_{L^2(\Omega_i)} + E_{\Omega_i}(u_i)(0). \end{array} \tag{10}$$

Consider first the convergence of the Schwarz waveform relaxation algorithm (6): we define $e_i^k = u_i^k - u_i$, and set the data to zero. We use the transmission conditions and sum up on the subdomains. We obtain by a translation of the indices

$$\begin{aligned}\sum_{i=1}^{I} E_{\Omega_i}(e_i^k)(t) \quad + \quad & \frac{1}{4}\sum_{i=2}^{I} c(a_i)\left[\int_0^t [(\mathcal{B}_{i-1}^+ e_i^k(a_i,s))^2 + (\mathcal{B}_i^- e_{i-1}^k(a_i,s))^2]\,ds\right] \\ = \quad & \frac{1}{4}\sum_{i=2}^{I} c(a_i)\left[\int_0^t [(\mathcal{B}_i^- e_{i-1}^{k-1}(a_i,s))^2 + (\mathcal{B}_{i-1}^+ e_i^{k-1}(a_i,s))^2]\,ds\right].\end{aligned}$$

This proves that, for any t, the sequence

$$\alpha_k = \sum_{i=2}^{I} c(a_i)\left[\int_0^t [(\mathcal{B}_{i-1}^+ e_i^k(a_i,s))^2 + (\mathcal{B}_i^- e_{i-1}^k(a_i,s))^2]\,ds\right]$$

is decreasing, and since it is positive, it converges. Then $\alpha_k - \alpha_{k-1}$ tends to zero, and the energy $\sum_{i=1}^{I} E_{\Omega_i}(u_i^k)$ tends to zero in $L^\infty(0,T)$. Therefore, the Schwarz waveform relaxation algorithm (6) converges. Turning back to the solution of the coupled problem, we obtain in the same way,

$$\sum_i E_{\Omega_i}(u_i)(t) = \sum_i \int_0^t (f(\cdot,s), \partial_t u_i(\cdot,s))_{L^2(\Omega_i)} + \sum_i E_{\Omega_i}(u_i)(0).$$

Using the Cauchy Schwarz inequality and the Gronwall lemma, we get (8), which completes the proof. ■

2.4 Well-posedness

We now study the stability of problem (3), (4), (5) with respect to the right-hand side, initial conditions, and transmission conditions: we add a perturbation g_i^- on the interface a_i, and g_i^+ on the interface a_{i+1}, and use the transmission conditions

$$\begin{aligned}\mathcal{B}_i^-\, u_i\,(a_i,\cdot) &= \mathcal{B}_i^-\, u_{i-1}\,(a_i,\cdot) + g_i^- \quad \text{in } (0,T), \\ \mathcal{B}_i^+\, u_i\,(a_{i+1},\cdot) &= \mathcal{B}_i^+\, u_{i+1}\,(a_{i+1},\cdot) + g_i^+ \quad \text{in } (0,T).\end{aligned} \tag{11}$$

In order to obtain an energy estimate, we now build a special solution of the wave equation:

Proposition 2.1 *Suppose the velocity is constant in $\Omega = \mathbb{R}$. For $T < T_0 = \min_{1\le i\le I} \frac{a_{i+1}-a_i}{c}$, for any g_i^- in $L^2(0,T)$, there exists w_{i-1}^- solution of the homogeneous wave equation in $\Omega_{i-1}\times(0,T)$, with zero final values, supported in the cone $\mathcal{C}_i^- = \{(x,t)\in\Omega_{i-1}\times(0,T), c(T-t)+(x-a_i)\ge 0\}$, such that $\mathcal{B}_i^-\, w_{i-1}^-\,(a_i,\cdot) = g_i^-$.*

Symmetrically, for any g_{i-1}^+ in $L^2(0,T)$, there exists w_i^+ solution of the homogeneous wave equation in $\Omega_i \times (0,T)$, with zero final values, supported in the cone $\mathcal{C}_i^+ = \{(x,t) \in \Omega_i \times (0,T), -c(T-t)+(x-a_i) \geq 0\}$, such that $\mathcal{B}_{i-1}^+ w_i^+ (a_i,\cdot) = g_{i-1}^+$. Furthermore we have

$$E_{\Omega_{i-1}}(w_{i-1}^-)(t) \leq \frac{c}{4} \, \|g_i^-\|^2_{L^2(0,T)}, \quad E_{\Omega_i}(w_i^+)(t) \leq \frac{c}{4} \, \|g_{i-1}^+\|^2_{L^2(0,T)}. \tag{12}$$

Proof Since the velocity is constant, w_i^+ is a function of $x - c(T-t)$, and w_{i-1}^- is a function of $x + c(T-t)$. They can be given explicitly, for instance

$$w_i^+(x,t) = \frac{c}{2} \int_{\frac{x-a_i}{c}+t}^T g_{i-1}^+(s)\, ds.$$

Energy estimates analogous to (10), but backward in time, give for instance for w_i^+

$$\begin{aligned} E_{\Omega_i}(w_i^+)(t) \quad + & \quad \frac{c}{4}\int_t^T [(\mathcal{B}_i^- w_i^+(a_i,s))^2 + (\mathcal{B}_i^+ w_i^+(a_{i+1},s))^2]\, ds \\ = & \quad \frac{c}{4}\int_t^T [(\mathcal{B}_{i-1}^+ w_i^+(a_i,s))^2 + (\mathcal{B}_{i+1}^- w_i^+(a_{i+1},s))^2]\, ds. \end{aligned}$$

Since w_i^+ vanishes identically on $a_{i+1} \times (0,T)$, and since $\mathcal{B}_i^- w_i^+(a_i,\cdot) = 0$, we obtain

$$E_{\Omega_i}(w_i^+)(t) = \frac{c}{4}\int_t^T |g_{i-1}^+(s)|^2\, ds,$$

which has (12) as a consequence. ∎

We deduce the well-posedness:

Theorem 2.2 *For constant velocity, the transmission problem (3) with perturbed transmission conditions (11) has a unique solution $\{u_i\}_{1\leq i\leq I}$, and there exists a positive constant α such that the following estimate holds*

$$\sum_i E_{\Omega_i}(u_i)(t) \leq \alpha e^T [\, \|f\|^2_{L^2(\Omega\times(0,T))} + \|\partial_x p\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)} + \sum_{\pm}\sum_{i=1}^I \|g_i^\pm\|^2_{L^2(0,T)}]. \tag{13}$$

Proof Using the construction in Proposition 2.1, we rewrite (11) as

$$\begin{aligned} \mathcal{B}_i^- \, u_i\,(a_i,\cdot) \quad &= \quad \mathcal{B}_i^- \,(u_{i-1} + w_{i-1}^-)\,(a_i,\cdot) \qquad \text{in } (0,T), \\ \mathcal{B}_i^+ \, u_i\,(a_{i+1},\cdot) \quad &= \quad \mathcal{B}_i^+ \,(u_{i+1} + w_{i+1}^+)\,(a_{i+1},\cdot) \quad \text{in } (0,T). \end{aligned}$$

Since w_i^- and $\mathcal{B}_i^- \, w_i^+$ vanish identically on $\{a_i\} \times (0,T)$, we have

$$\mathcal{B}_i^- \,(u_i + w_i^- + w_i^+)\,(a_i,\cdot) \quad = \quad \mathcal{B}_i^- \,(u_{i-1} + w_{i-1}^- + w_{i-1}^+)\,(a_i,\cdot) \quad \text{in } (0,T),$$

and symmetrically at a_{i+1}. We thus define $w_i = w_i^- + w_i^+$ in Ω_i, and $v_i = u_i + w_i$. The transmission conditions on the v_i are homogeneous, which gives the existence of the u_i. Furthermore we have, by estimates (8),

$$\sum_i E_{\Omega_i}(v_i)(t) \leq \frac{1}{2} e^T (c^2 \|f\|^2_{L^2(\Omega\times(0,T))} + \sum_i E_{\Omega_i}(w_i)(0) + \|\partial_x p\|^2_{L^2(\Omega)} + \|\frac{1}{c} q\|^2_{L^2(\Omega)}),$$

and using the estimate in Proposition 2.1, we obtain (13), from which the uniqueness follows.

■

Remark 2.3 *The proof of the theorem could seem unnecessarily complicated, since solutions in closed form are at hand. However, it is a first step in the understanding of mesh refinement: the stability of the coupling involves perturbations on the boundary. Furthermore, the proof in the discrete case will follow the same steps.*

3 The discrete problem

3.1 The numerical scheme

Let $\Omega = (a, b)$ be discretized with a mesh Δx. The mesh points x_j, are numbered from 0 to $J + 1$. The time interval is discretized with a mesh Δt, and the time steps t_n are numbered from 0 to $N + 1$. The discrete value in domain Ω, at point j and time n is denoted by $U(j, n)$.

In order to handle more easily the boundary conditions, we choose a "vertex centered" finite volume scheme [9]. We denote by $\mathcal{I}(x)$ the interval $(x - \Delta x/2, x + \Delta x/2)$ if x is in the interior of Ω, $\mathcal{I}(x) = (x, x + \Delta x/2)$ if $x = a$, and $\mathcal{I}(x) = (x - \Delta x/2, x)$ if $x = b$. Accordingly the intervals in time are called $\mathcal{J}(t)$. The displacement u is considered to be constant on the cell or finite volume $\mathcal{S}(x, t) := \mathcal{I}(x) \times \mathcal{J}(t)$, and the scheme is obtained by integrating equation (1) on $\mathcal{S}_{j,n} = \mathcal{S}(x_j, t_n)$. In the interior of Ω, this produces the usual leapfrog scheme

$$\Box_d U(j, n) := (\frac{1}{C^2(j)} D_t^+ D_t^- - D_x^+ D_x^-) U(j, n) = F(j, n), \quad 1 \leq j \leq J, 1 \leq n \leq N, \tag{14}$$

where $\frac{1}{C^2(j)}$ is the mean value of $\frac{1}{c^2(x)}$ in $\mathcal{I}(x_j)$, and $F(j, n)$ is the mean value of f in the cell $\mathcal{S}_{j,n}$ (the mean value of a function φ on a domain D in $\mathbb{R}^d$ is

$\frac{1}{|D|}\int_D \varphi(\xi)\, d\xi)$. We use the notations for finite difference operators

$$\begin{aligned}
D_t^+ U(j,n) &:= \frac{U(j,n+1)-U(j,n)}{\Delta t}, \quad & D_t^- U(j,n) &:= \frac{U(j,n)-U(j,n-1)}{\Delta t},\\
D_x^+ U(j,n) &:= \frac{U(j+1,n)-U(j,n)}{\Delta x}, \quad & D_x^- U(j,n) &:= \frac{U(j,n)-U(j-1,n)}{\Delta x},\\
D_t^0 U(j,n) &:= \frac{U(j,n+1)-U(j,n-1)}{2\Delta t}, \quad & D_x^0 U(j,n) &:= \frac{U(j+1,n)-U(j-1,n)}{2\Delta x}.
\end{aligned} \tag{15}$$

The initial conditions are

$$\begin{gathered}
U(j,0) = P(j), 0 \le j \le J+1,\\
\left(\frac{1}{C^2(j)}D_t^+ - \frac{\Delta t}{2}D_x^+ D_x^-\right)U(j,0) = \frac{1}{C^2(j)}Q(j) + \frac{\Delta t}{2}F(j,0), 1 \le j \le J,
\end{gathered} \tag{16}$$

where $P(j)$ and $Q(j)$ are the mean values of p and q on $\mathcal{I}(x_j)$, and the second equation is obtained by integrating equation (1) on the half-cell $\mathcal{S}(x_j,0)$ and using the initial values.

3.2 A first energy estimate

We define a discrete energy.We consider sequences of the form $V = \{V(j)\}_{0\le j\le J+1}$ in $\mathbb{R}^{J+2}$, and we define a bilinear form on $\mathbb{R}^{J+2}$ by

$$a(V,W) = \Delta x \sum_{j=1}^{J+1} D_x^- (V)(j) \cdot D_x^- (W)(j). \tag{17}$$

We also define a particular sum denoted by $\sum'$, which is given in space by

$$\sum_{j=0}^{J+1}{}' V(j) = \frac{1}{2}V(0) + \sum_{j=1}^{J} V(j) + \frac{1}{2}V(J+1),$$

and analogously in time. For a mesh function V of time and space, the discrete energy $E(V)(n)$ at time step n, is defined as the sum of a discrete kinetic energy $E_K(V)(n)$, and a discrete potential energy $E_P(V)(n)$, and given by

$$\begin{aligned}
E_K(V)(n) &= \Delta x \sum_{j=0}^{J+1}{}' \frac{1}{C^2(j)}(D_t^+ V(j,n))^2,\\
E_P(V)(n) &= a(V(\cdot,n), V(\cdot,n-1)).\\
E &= E_K + E_P.
\end{aligned} \tag{18}$$

The quantity E_K is clearly a discrete kinetic energy. It is less evident to identify E_P as an energy. The following lemma gives a lower bound for E under a CFL condition, and hence shows that E is then indeed an energy.

Lemma 3.1 *For any $n \geq 1$, we have*

$$E(V)(n) \geq \left(1 - \left(C\frac{\Delta t}{\Delta x}\right)^2\right) E_K(V)(n), \tag{19}$$

where C is defined by $C = \sup_{1 \leq j \leq J+1} C(j)$. Hence, under the CFL condition

$$C\frac{\Delta t}{\Delta x} < 1, \tag{20}$$

E is bounded from below by an energy.

Proof The proof is classical [10]. ■

We now obtain the basic energy estimate:

Lemma 3.2 *Suppose the wave speed is constant. For any U solution of (14) in Ω, we have for any $n \geq 1$,*

$$\begin{aligned} E(U)(n) \quad &- \quad E(U)(n-1) + c\frac{\Delta t}{2}[(\widetilde{B}^+ U(0,n))^2 + (\widetilde{B}^- U(J+1,n))^2] \\ &= \quad c\frac{\Delta t}{2}[(B^- U(0,n))^2 + (B^+ U(J+1,n))^2] + 2\Delta t \Delta x \sum_{j=1}^{J} F(j,n) D_t^0 U(j,n), \end{aligned} \tag{21}$$

and for $n = 0$,

$$\begin{aligned} E_K(U)(0) \quad &+ E(U)(0) + c\frac{\Delta t}{4}[(\widetilde{B}^+ U(0,0))^2 + (\widetilde{B}^- U(J+1,0))^2] \\ &= c\frac{\Delta t}{4}[(B^- U(0,0))^2 + (B^+ U(J+1,0))^2] + a(P,P) + 2\frac{\Delta x}{c^2}\sum_{j=1}^{J} Q(j) D_t^+ U(j,0). \end{aligned} \tag{22}$$

with the boundary operators defined for $n \geq 1$ by

$$\begin{aligned} B^- \quad &:= \quad \frac{1}{c}D_t^0 - D_x^+ + \frac{\Delta x}{2c^2}D_t^+ D_t^-, \qquad & \widetilde{B}^- \quad &:= \quad \frac{1}{c}D_t^0 - D_x^- - \frac{\Delta x}{2c^2}D_t^+ D_t^-, \\ B^+ \quad &:= \quad \frac{1}{c}D_t^0 + D_x^- + \frac{\Delta x}{2c^2}D_t^+ D_t^-, \qquad & \widetilde{B}^+ \quad &:= \quad \frac{1}{c}D_t^0 + D_x^+ - \frac{\Delta x}{2c^2}D_t^+ D_t^-, \end{aligned} \tag{23}$$

and for $n = 0$ by

$$\begin{aligned} B^- \quad &:= \quad \frac{1}{c}D_t^+ - D_x^+ + \frac{\Delta x}{c^2 \Delta t}D_t^+, \qquad & \widetilde{B}^- \quad &:= \quad \frac{1}{c}D_t^+ - D_x^- - \frac{\Delta x}{c^2 \Delta t}D_t^+, \\ B^+ \quad &:= \quad \frac{1}{c}D_t^+ + D_x^- + \frac{\Delta x}{c^2 \Delta t}D_t^+, \qquad & \widetilde{B}^+ \quad &:= \quad \frac{1}{c}D_t^+ + D_x^+ - \frac{\Delta x}{c^2 \Delta t}D_t^+. \end{aligned} \tag{24}$$

Proof The first estimate is obtained by multiplying (14) by $D_t^0 U(j,n)$ and discrete integration by parts, whereas the second one is obtained by multiplying (16) by $D_t^+ U(j,0)$ and integrating by part [10]. ■

3.3 The boundary value problem: stability and order

We now introduce the boundary value problem in Ω, with boundary conditions given by

$$\mathcal{B}^- u(a,\cdot) = g^- \text{ in } (0,T), \quad \mathcal{B}^+ u(b,\cdot) = g^+ \text{ in } (0,T). \tag{25}$$

By the same construction as before, integrating on the half-cells $\mathcal{S}(a,t_n)$ and $\mathcal{S}(b,t_n)$, we obtain the discrete boundary conditions

$$\begin{aligned} B^- U(0,n) &= \frac{\Delta x}{2} F(0,n) + G^-(n) = H^-(n), \\ B^+ U(J+1,n) &= \frac{\Delta x}{2} F(J+1,\cdot) + G^+(n) = H^+(n), \end{aligned} \quad \text{for n} \geq 1, \tag{26}$$

$$\begin{aligned} B^- U(0,0) &= G^-(0) + \frac{\Delta x}{c^2 \Delta t} Q(0) + \frac{\Delta x}{4} F(0,0) = H^-(0), \\ B^+ U(J+1,0) &= G^+(0) - \frac{\Delta x}{c^2 \Delta t} Q(J+1) + \frac{\Delta x}{4} F(J+1,0) = H^+(0), \end{aligned} \tag{27}$$

where $G^\pm(n)$ are the mean-values of $g^\pm$ on the interval $\mathcal{J}(t_n)$. Comparing formulas (2) and (23), we note that the transport operators $\mathcal{B}^\pm$ are approximated by a Lax-Wendroff scheme [12]. We define the discrete ℓ^2 norms and the associated scalar products by

$$\|\Phi\|^2_{\ell^2_x} = \Delta x \sum_{j=0}^{J+1} \Phi^2(j); \ \|\Phi\|^2_{\ell^2_t} = \Delta t \sum_{n=0}^{N+1} \Phi^2(n); \ \|\Phi\|^2_{\ell^2_{x,t}} = \Delta x \Delta t \sum_{j=0}^{J+1} \sum_{n=0}^{N+1} \Phi^2(j,n). \tag{28}$$

We will also make use of the equivalent norms

$$|||\Phi|||^2_{\ell^2_x} = \Delta x \sum_{j=0}^{J+1}{}' \, \Phi^2(j); |||\Phi|||^2_{\ell^2_t} = \Delta t \sum_{n=0}^{N+1}{}' \, \Phi^2(n); |||\Phi|||^2_{\ell^2_{x,t}} = \Delta x \Delta t \sum_{j=0}^{J+1}{}' \sum_{n=0}^{N+1}{}' \, \Phi^2(j,n). \tag{29}$$

Since we have an explicit scheme, problem (14), (16), (26), (27) has obviously a unique solution. We prove below that this solution depends continuously of the data.

Theorem 3.3 *Let U be the solution of the discrete wave equation (14) with initial conditions (16) and boundary conditions (26), (27). For constant wave speed c, if $\gamma = c\Delta t/\Delta x < 1$, there exist positive constants $\alpha(T,\gamma)$ and Δt_0, such that for $\Delta t \leq \Delta t_0$, for any n, $1 \leq n \leq N$, the following estimate holds:*

$$\begin{aligned} |||D_t^+ U(\cdot,n)|||^2_{\ell^2_x} \; &+ \; \frac{c\Delta t}{2} \sum_{p=0}^{n}{}' \, [\, |\widetilde{B}^+ U(0,p)|^2 + |\widetilde{B}^- U(J+1,p)|^2] \\ &\leq \; \alpha \, (\, \|H^-\|^2_{\ell^2_t} + \|H^+\|^2_{\ell^2_t} + \|D_x^+ P\|^2_{\ell^2_x} + \|Q\|^2_{\ell^2_x} \; + \|F\|^2_{\ell^2_{x,t}}). \end{aligned} \tag{30}$$

Proof Summing equation (23) in Lemma 3.2 in n, adding (24) and using Lemma 3.1, we get

$$\begin{aligned}(1-\gamma^2) \quad & \frac{\Delta x}{c^2} \sum_{j=0}^{J+1}{}' |D_t^+ U(\cdot,n)|^2 + \frac{\Delta x}{c^2} \sum_{j=0}^{J+1}{}' |D_t^+ U(\cdot,0)|^2 \\ + \quad & \frac{c\Delta t}{2} \sum_{p=0}^{n}{}' [\, |\widetilde{B}^+ U(0,p)|^2 + |\widetilde{B}^- U(J+1,p)|^2] \\ \leq \quad & \frac{c\Delta t}{2} \sum_{p=0}^{n}{}' [\, |H^-(p)|^2 + |H^+(p)|^2] + 2 \sum_{p=1}^{n} \sum_{j=1}^{J} F(j,n) D_t^0 U(j,n) \\ + \quad & a(P,P) + 2\frac{\Delta x}{c^2} \sum_{j=1}^{J} Q(j) D_t^+ U(j,0).\end{aligned}$$

Now applying the discrete Cauchy-Schwarz Lemma in the last two terms, together with the notations introduced in (28) and (29), leads to

$$\begin{aligned}\frac{1-\gamma^2}{c^2} \, |||D_t^+ U(\cdot,n)|||_{\ell_x^2}^2 + \frac{1}{2c^2} \, |||D_t^+ U(\cdot,0)|||_{\ell_x^2}^2 + \frac{c\Delta t}{2} \sum_{p=0}^{n}{}' [\, |\widetilde{B}^+ U(0,p)|^2 + |\widetilde{B}^- U(J+1,p)|^2] \\ \leq \quad \Phi + \Delta t \frac{1-\gamma^2}{c^2} \sum_{p=0}^{n} |||D_t^+ U(\cdot,p)|||_{\ell_x^2}^2,\end{aligned}$$

with

$$\Phi = \frac{c}{2} \, (|||H^-|||_{\ell_t^2}^2 + |||H^+|||_{\ell_t^2}^2) + |||D_x^+ P\,|||_{\ell_x^2}^2 + \frac{1}{2c^2} \, |||Q|||_{\ell_x^2}^2 \; + \frac{c^2}{1-\gamma^2} \, \|F\|_{\ell_{x,t}^2}^2.$$

We now use a discrete Gronwall Lemma. We define $\Psi(n) = \dfrac{1-\gamma^2}{c^2} \displaystyle\sum_{p=0}^{n} |||D_t^+ U(\cdot,p)|||_{\ell_x^2}^2$, and rewrite the last inequality as

$$(1-\Delta t)\Psi(n) \leq \Phi + \Psi(n-1),$$

where the recurrence can be solved,

$$\Psi(n) \leq \frac{1}{(1-\Delta t)^n} (\frac{\Phi}{\Delta t} + \Psi(0)).$$

We now use the inequality

$$\frac{1}{1-X} \leq e^{2X} \text{ for sufficiently small } X,$$

and obtain

$$\begin{aligned}\frac{1-\gamma^2}{c^2} \, |||D_t^+ U(\cdot,n)|||_{\ell_x^2}^2 \quad + \frac{1}{2c^2} \, |||D_t^+ U(\cdot,0)|||_{\ell_x^2}^2 + \frac{c\Delta t}{2} \sum_{p=0}^{n}{}' [\, |\widetilde{B}^+ U(0,p)|^2 + |\widetilde{B}^- U(J+1,p)|^2] \\ \leq \quad \Phi + e^{2n\Delta t}(\Phi + \Delta t \, \frac{1-\gamma^2}{c^2} |||D_t^+ U(\cdot,0)|||_{\ell_x^2}^2) \\ \leq \quad \Phi + e^{2T}(\Phi + \Delta t \, \frac{1-\gamma^2}{c^2} |||D_t^+ U(\cdot,0)|||_{\ell_x^2}^2).\end{aligned}$$

We can choose Δt small enough, so that $(1-\gamma^2)\Delta t\; e^{2T} \leq \frac{1}{4}$, and hence obtain

$$\begin{aligned}\frac{1-\gamma^2}{c^2}\; |||D_t^+\, U(\cdot,n)|||^2_{\ell^2_x} \quad &+ \frac{1}{4c^2}\; |||D_t^+\, U(\cdot,0)|||^2_{\ell^2_x} + \frac{c\Delta t}{2}\sum_{p=0}^{n}{}'\, [\, |\widetilde{B}^+ U(0,p)|^2 + |\widetilde{B}^- U(J+1,p)|^2] \\ &\leq \quad (1+e^{2T})\Phi,\end{aligned}$$

which leads to (30). ■

This result gives us the order of convergence for the scheme.

Theorem 3.4 *Suppose the wave speed is constant, and $\gamma = c\Delta t/\Delta x < 1$. Let u be the solution in $V(\Omega)$ of the wave equation (1) on the domain $\Omega \times (0,T)$, with initial conditions $u(\cdot,0) = p$ and $\frac{\partial u}{\partial t}(\cdot,0) = q$. Let $U_d(j,n)$ be the mean value of u on the cell $\mathcal{S}_{j,n}$, and U be the solution of the discrete wave equation (14) with initial conditions (16) and boundary conditions (26,27). Then there exist positive constants α and Δt_0, such that for $\Delta t \leq \Delta t_0$ and for any n, $0 \leq n \leq N$, the following estimate holds:*

$$\begin{aligned} \|D_t^+\,(U-U_d)(\cdot,n)\|_{\ell^2_x} \leq \alpha M \Delta t^2, \\ \max(\|\widetilde{B}^+(U-U_d)(0,\cdot)\|_{\ell^2_t},\ \|\widetilde{B}^-(U-U_d)(J+1,\cdot)\|_{\ell^2_t}) \leq \alpha M\Delta t^2, \end{aligned} \tag{31}$$

with

$$M^2 = \|\partial_t^4 u\|^2_{L^2((a,b)\times(0,T))} + \|\partial_t^3 u(x,\cdot)\|^2_{L^2(\Gamma\times(0,T))} + \|\partial_t^2 u(\cdot,0)\|^2_{L^2(a,b)}.$$

Therefore, the scheme is second order in time and space.

Proof Let $U_d(j,n)$ be the mean value of u on the finite volume $\mathcal{S}_{j,n}$. By construction, the error $e(j,n) = U_d(j,n) - U(j,n)$ is solution of the discrete wave equation with data which is the truncation errors. These errors can be easily estimated by Taylor expansions, we obtain

$$\begin{aligned} |\,\Box_d\, e(j,n)| = |\,(\Box_d\, U_d - F)(j,n)| \leq \beta\Delta t^2 \sup_{\mathcal{S}(x_j,t_n)} (|\partial_t^4 u|), \\ e(j,0) = 0; \quad |(\frac{1}{C^2(j)}D_t^+ \; - \frac{\Delta t}{2} D_x^+\, D_x^-\,)e(j,0)| \leq \beta\Delta x^2 \sup_{\mathcal{S}(x_j,0)} (|\partial_t^2 u|), \\ |B^\pm e(0,n)| \leq \beta\Delta t^2 \sup_{\mathcal{S}(a,t_n)} (|\partial_t^3 u|). \end{aligned}$$

Inserting these estimates into (30), we conclude the proof. ■

Remark 3.5 *The case $\gamma = 1$ is excluded in Theorem 3.4. However in this case, the scheme is exact in the interior and on the boundary.*

3.4 The discrete transmission problem: definition

Each domain Ω_i is discretized with a mesh Δx_i, the points x_i, are numbered from 0 to J_i+1. The time interval is discretized in Ω_i with a mesh Δt_i, and the time steps t_n are numbered from 0 to N_i+1. The discrete value in domain Ω_i, at point j in space and n in time, is denoted by $U_i(j,n)$. The discrete transmission problem corresponding to (3), (4, (5) is given by

$$\begin{aligned} \left(\frac{1}{C_i^2(j)} D_t^+ D_t^- - D_x^+ D_x^-\right) U_i(j,n) &= F(j,n), \quad 1 \le j \le J_i,\, 1 \le n \le N_i \quad (32)\\ B_i^- U_i(0,\cdot) &= \mathbf{P}_{i,i-1}\, \widetilde{B}_i^- U_{i-1}(J_{i-1}+1,\cdot), \quad (33)\\ B_i^+ U_i(J_i+1,\cdot) &= \mathbf{P}_{i,i+1}\, \widetilde{B}_i^+ U_{i+1}(0,\cdot), \quad (34) \end{aligned}$$

with the discrete operators for $n \ge 1$ given by

$$\begin{aligned} B_i^- U_i(0,n) &= \left(\tfrac{1}{C_{i-1}} D_t^0 - D_x^+ + \tfrac{\Delta x_i}{2C_i^2} D_t^+ D_t^-\right) U_i(0,n),\\ \widetilde{B}_i^- U_{i-1}(J_{i-1}+1,n) &= \left(\tfrac{1}{C_{i-1}} D_t^0 - D_x^- - \tfrac{\Delta x_{i-1}}{2C_{i-1}^2} D_t^+ D_t^-\right) U_{i-1}(J_{i-1}+1,n), \end{aligned} \quad (35)$$

$$\begin{aligned} B_i^+ U_i(J_i+1,n) &= \left(\tfrac{1}{C_{i+1}} D_t^0 + D_x^- + \tfrac{\Delta x_i}{2C_i^2} D_t^+ D_t^-\right) U_i(J_i+1,n),\\ \widetilde{B}_i^+ U_{i+1}(0,n) &= \left(\tfrac{1}{C_{i+1}} D_t^0 + D_x^+ - \tfrac{\Delta x_{i+1}}{2C_{i+1}^2} D_t^+ D_t^-\right) U_{i+1}(0,n), \end{aligned} \quad (36)$$

and for $n = 0$ by

$$\begin{aligned} B_i^- U_i(0,n) &= \left(\tfrac{1}{C_{i-1}} D_t^+ - D_x^+ + \tfrac{\Delta x_i}{C_i^2 \Delta t_i} D_t^+\right) U_i(0,n),\\ \widetilde{B}_i^- U_{i-1}(J_{i-1}+1,n) &= \left(\tfrac{1}{C_{i-1}} D_t^+ - D_x^- - \tfrac{\Delta x_{i-1}}{C_{i-1}^2 \Delta t_{i-1}} D_t^+\right) U_{i-1}(J_{i-1}+1,n), \end{aligned} \quad (37)$$

$$\begin{aligned} B_i^+ U_i(J_i+1,n) &= \left(\tfrac{1}{C_{i+1}} D_t^+ + D_x^- + \tfrac{\Delta x_i}{C_i^2 \Delta t_i} D_t^+\right) U_i(J_i+1,n),\\ \widetilde{B}_i^+ U_{i+1}(0,n) &= \left(\tfrac{1}{C_{i+1}} D_t^+ + D_x^+ - \tfrac{\Delta x_{i+1}}{C_{i+1}^2 \Delta t_{i+1}} D_t^+\right) U_{i+1}(0,n). \end{aligned} \quad (38)$$

In the previous formulas, we used the conventions $C_{i-1} := C_{i-1}(J_{i-1}+1), C_i := C_i(0)$, and $C_{i+1} := C_{i+1}(0)$. We now define the projection operators $\mathbf{P}_{i,j}$. When applied to U_i, the divided difference operators operate with the meshes Δt_i and Δx_i. Thus, for instance in $B_i^-(U_i)(0,n)$, $D_t^0\, U_i(0,n) = \frac{U_i(0,n+1) - U_i(0,n-1)}{2\Delta t_i}$, whereas in $\widetilde{B}_i^-(U_{i-1})(J_{i-1}+1,n)$, $D_t^0\, U_{i-1}(J_{i-1}+1,n) = \frac{U_{i-1}(J_{i-1}+1,n+1) - U_{i-1}(J_{i-1}+1,n-1)}{2\Delta t_{i-1}}$. The vectors $B_i^-(U_i)(0,\cdot)$ and $B_i^+(U_i)(J_i+1,\cdot)$ are in $\mathbb{R}^{N_i+2}$, $\widetilde{B}_i^-(U_{i-1})(J_{i-1}+1,\cdot)$ is in $\mathbb{R}^{N_{i-1}+2}$ and $\widetilde{B}_i^+(U_{i+1})(0,\cdot)$ is in $\mathbb{R}^{N_{i+1}+2}$. Therefore, we need to introduce a projection step in (33), (34) which we describe now. We denote by V_i the subspace of $L^2(0,T)$,whose elements are continuous on $(0,T)$ and affine on each interval $(n\Delta t_i, (n+1)\Delta t_i)$ for $0 \le n \le N_i$. The orthogonal projection in $L^2(0,T)$ on V_i is called $\mathbf{Q}_i$. The restriction of $\mathbf{Q}_i$ to V_j is called $\mathbf{Q}_{i,j}$. $\mathbf{F}_i$ is the canonical isomorphism from $\mathbb{R}^{N_i+2}$ onto V_i, which maps Ψ to the affine function which takes the value $\Psi(n)$ at point $n\Delta t_i$. If $\mathbb{R}^{N_i+2}$ is equipped with the norm

$$|||\Psi|||_i^2 = \Delta t \sum_{n=0}^{N_i+2}{}' \, \Psi^2(n),$$

we have for any Ψ in $\mathbb{R}^{N_i+2}$, $\|\mathbf{F}_i\Psi\|_{L^2(0,T)} = |||\Psi|||_{\mathbb{R}^{N_i+2}}$. The operator $\mathbf{P}_{i,j}$ is now given by

$$\mathbf{P}_{i,j} = (\mathbf{F}_i)^{-1} \circ \mathbf{Q}_{i,j} \circ \mathbf{F}_j.$$

Since $\mathbf{Q}_{i,j}$ is a projection operator, we have $|||\mathbf{P}_{i,j}||| \leq 1$. To actually compute the solution of this problem, we need to introduce the Schwarz waveform relaxation algorithm [10] given by

$$\begin{aligned}
\left(\frac{1}{C_i^2(j)} D_t^+ D_t^- - D_x^+ D_x^- \right) U_i^k(j,n) &= F(j,n), \quad 1 \leq j \leq J_i,\ 1 \leq n \leq N_i, && (39)\\
B_i^- \, U_i^k \, (0,\cdot) &= \mathbf{P}_{i,i-1} \, \widetilde{B}_i^- \, U_{i-1}^{k-1} \, (J_{i-1}+1,\cdot), && (40)\\
B_i^+ \, U_i^k \, (J_i+1,\cdot) &= \mathbf{P}_{i,i+1} \, \widetilde{B}_i^+ \, U_{i+1}^{k-1} \, (0,\cdot), && (41)
\end{aligned}$$

with initial guesses $\widetilde{B}_i^- \, U_{i-1}^0 \, (J_{i-1}+1,\cdot) = d_i^-$, and $\widetilde{B}_i^+ \, U_{i+1}^0 = d_i^+$.

3.5 The discrete transmission problem: well-posedness and order

Theorem 3.6 *Suppose the velocity is constant in $\mathbb{R}$. Suppose that, for $1 \leq i \leq I$, $\gamma_i = c\Delta t_i/\Delta x_i < 1$ and $2N_i \leq J_i + 1$. For any initial values P and Q, problem (32), (33), (34) has a unique solution, which is the limit of the sequence U_i^k. Furthermore there exists a positive α, depending only on T and the γ_i, such that*

$$\sum_i \|D_t^+ \, U_i(\cdot, N_{i+1})\|_{\ell_x^2}^2 \leq \alpha(\|D_x^+ \, P\,\|_{\ell_x^2}^2 + \|Q\|_{\ell_x^2}^2 + \|F\|_{\ell_{x,t}^2}^2). \tag{42}$$

Proof We first introduce the energy estimate we need in the proof: for U_i solution of the discrete wave equation (32) in each Ω_i, we use (21), (22) in each domain Ω_i, and sum up on the time indices. We obtain

$$\begin{aligned}
E(U_i)(N_{i+1}) &+ E_K(U_i)(0) + c\frac{\Delta t_i}{2} \sum_{n=0}^{N_i+1}{}'[(\widetilde{B}_{i-1}^+ U_i(0,n))^2 + (\widetilde{B}_{i+1}^- U_i(J_i+1,n))^2]\\
&= c\frac{\Delta t_i}{2} \sum_{n=0}^{N_i+1}{}'[(B_i^- U_i(0,n))^2 + (B_i^+ U_i(J_i+1,n))^2].
\end{aligned} \tag{43}$$

Since we deal with a finite dimensional problem, which has been constructed to be square, proving existence and uniqueness reduces to proving uniqueness. Supposing vanishing initial values and right-hand side, and using the boundary conditions (33), (34), we obtain

$$|||B_i^- U_i(0,\cdot)|||_i \leq |||\widetilde{B}_i^- \, U_{i-1}\,(J_{i-1}{+}1,\cdot)|||_{i-1}, \quad |||B_i^+ U_i(J_i{+}1,\cdot)|||_i \leq |||\widetilde{B}_i^+ \, U_{i+1}\,(0,\cdot)|||_{i+1}.$$

We now sum up on the subdomains, translate the indices in the right-hand side and obtain

$$\begin{aligned}\sum_i E(U_i)(N_{i+1}) \quad &+ \quad \frac{c}{2}\sum_i [|||\widetilde{B}_{i-1}^+ U_i(0,\cdot)|||_i^2 + |||\widetilde{B}_{i+1}^- U_i(J_i+1,\cdot)|||_i^2] \\ &\leq \quad \frac{c}{2}\sum_i [|||\widetilde{B}_i^- \, U_{i-1}\,(J_{i-1}+1,\cdot)|||_{i-1}^2 + |||\widetilde{B}_i^+ \, U_{i+1}\,(0,\cdot)|||_{i+1}^2].\end{aligned}$$

By a shift of indices, the boundary terms cancel out and we deduce that $E(U_i)(N_{i+1})$ vanishes for each i. By Lemma 3.1, this implies that $D_t^+ U_i(N_{i+1},\cdot) = 0$. With the assumption $2N_i \leq J_i + 1$, since the initial values vanish, there exists a j, such that $U_i(j, N_{i+1}) = 0$, with $U_i(j-1, N_i) = U_i(j, N_i) = U_i(j+1, N_i) = 0$. This in turn implies that $U_i(j-1, N_{i+1}) = U_i(j, N_{i+1}) = U_i(j+1, N_{i+1}) = 0$. Using the equation, we then have $U_i(j-2, N_i) = U_i(j+2, N_i) = 0$. We carry on the process down and up until we have filled the grid with zero values. This proves the uniqueness.

To prove the convergence of the sequence, we apply (43) to $V_i^k = U_i^k - U_i$, with vanishing data:

$$\begin{aligned}E(V_i^k)(N_{i+1}) \quad &+ \quad c\frac{\Delta t_i}{2}\sum_{n=0}^{N_i+1}{}'[(\widetilde{B}_{i-1}^+ V_i^k(0,n))^2 + (\widetilde{B}_{i+1}^- V_i^k(J_i+1,n))^2] \\ &= \quad c\frac{\Delta t_i}{2}\sum_{n=0}^{N_i+1}{}'[(B_i^- V_i^k(0,n))^2 + (B_i^+ V_i^k(J_i+1,n))^2].\end{aligned}$$

By (40), (41), we can estimate

$$|||B_i^- V_i^k(0,\cdot)|||_i \leq |||\widetilde{B}_i^- \, V_{i-1}^{k-1}\,(J_{i-1}{+}1,\cdot)|||_{i-1}, \quad |||B_i^+ V_i^k(J_i{+}1,\cdot)|||_i \leq |||\widetilde{B}_i^+ \, V_{i+1}^{k-1}\,(0,\cdot)|||_{i+1}.$$

We now sum up on the subdomains and obtain

$$\begin{aligned}\sum_i E(V_i^k)(N_{i+1}) \quad &+ \quad \frac{c}{2}\sum_i [|||\widetilde{B}_{i-1}^+ V_i^k(0,\cdot)|||_i^2 + |||\widetilde{B}_{i+1}^- V_i^k(J_i+1,\cdot)|||_i^2] \\ &\leq \quad \frac{c}{2}\sum_i [|||\widetilde{B}_i^- \, V_{i-1}^{k-1}\,(J_{i-1}+1,\cdot)|||_{i-1}^2 + |||\widetilde{B}_i^+ \, V_{i+1}^{k-1}\,(0,\cdot)|||_{i+1}^2].\end{aligned}$$

Translating the indices in the right-hand side, we obtain

$$\begin{aligned}\sum_i E(V_i^k)(N_{i+1}) \quad &+\frac{c}{2}\sum_i [|||\widetilde{B}_{i-1}^+ V_i^k(0,\cdot)|||_i^2 + |||\widetilde{B}_{i+1}^- V_i^k(J_i+1,\cdot)|||_i^2], \\ &\leq \frac{c}{2}\sum_i [|||\widetilde{B}_{i+1}^- \, V_i^{k-1}\,(J_i+1,\cdot)|||_i^2 + |||\widetilde{B}_i^+ \, V_{i-1}^{k-1}\,(0,\cdot)|||_i^2].\end{aligned}$$

The same argument as in the continuous case proves that $\sum_i E(V_i^k)(N_i+1)$ tends to 0 as k tends to infinity, and we conclude as in the uniqueness proof that U_i^k tends to a solution of problem (32), (33), (34). To prove energy estimate (42), we do the same calculations as before, but including the data, and we conclude as in Theorem 3.3. ■

Remark 3.7 *Assumption $2N_i \leq J_i + 1$ is equivalent to $2T \leq \gamma_i \frac{a_{i+1}-a_i}{c}$.*

We now consider the question of stability. By the previous results, we know that the scheme is stable in each subdomain. There remains the most delicate question of stability with respect to the transmission conditions. We consider zero initial data and right-hand side, and boundary conditions of the form

$$\begin{aligned} B_i^- \, U_i\,(0,\cdot) \;&=\; \mathbf{P}_{i,i-1}\, \widetilde{B}_i^- \, U_{i-1}\,(J_{i-1}+1,\cdot) + G_i^-\,, \\ {}' \; B_i^+ \, U_i\,(J_i+1,\cdot) \;&=\; \mathbf{P}_{i,i+1}\, \widetilde{B}_i^+ \, U_{i+1}\,(0,\cdot) + G_i^+\,. \end{aligned} \tag{44}$$

To prove stability, we first need an extension result, analogous to Proposition 2.1 in the continous case:

Proposition 3.1 *Suppose that for any i, $\Delta t_i/\Delta t_{i-1}$ or $\Delta t_{i-1}/\Delta t_i$ is an integer. Suppose c to be constant, $\gamma_i < 1$ and $T < \inf_i \frac{\gamma_i}{c}(a_{i+1}-a_i)$. For any (G_i^-, G_{i-1}^+) in $\mathbb{R}^{N_i+2} \times \mathbb{R}^{N_{i-1}+2}$, there exists W_{i-1}^- solution of the discrete wave equation in Ω_{i-1} with final values $W_{i-1}^-(\cdot, N_{i-1}+1) = W_{i-1}^-(\cdot, N_{i-1}) = 0$, supported in the cone $C_i^- = \{(j,n),\, 0 \leq j \leq J_{i-1}+1, 0 \leq n \leq N_{i-1}+1, J_{i-1}-j+n \leq N_{i-1}\}$ and W_i^+ solution of the discrete wave equation in Ω_i with final values $W_i^+(\cdot, N_i+1) = W_i^+(\cdot, N_i) = 0$, supported in the cone $C_i^+ = \{(j,n),\, 0 \leq j \leq J_i+1, 0 \leq n \leq N_{i-1}+1, N_i+1-n+(J_i+1)-j \leq 0\}$, such that*

$$\begin{aligned} B_i^- \, W_i^+\,(0,\cdot) \;-\; \mathbf{P}_{i,i-1}\, \widetilde{B}_i^- \, W_{i-1}^-\,(J_{i-1}+1,\cdot) - G_i^- \;&=\; R_{i,i-1}, \\ B_{i-1}^+ \, W_{i-1}^-\,(J_{i-1}+1,\cdot) \;-\; \mathbf{P}_{i-1,i}\, \widetilde{B}_{i-1}^+ \, W_i^+\,(0,\cdot) - G_{i-1}^+ &= R_{i-1,i}, \end{aligned} \tag{45}$$

where $R_{i,i-1}$ is in the orthogonal of Im $\mathbf{P}_{i,i-1}$ in V_i, and $R_{i-1,i}$ is in the orthogonal of Im $\mathbf{P}_{i-1,i}$ in V_{i-1}. Furtheremore there exists a positive constant α, independent of the N_i, such that

$$\begin{aligned} E(W_{i-1}^-)(0) + E(W_i^+)(0) \;&\leq\; \alpha(|||G_i^-|||_i + |||G_{i-1}^+|||_{i-1}), \\ |||R_{i,i-1}|||_i + |||R_{i-1,i}|||_{i-1} \;&\leq\; \alpha(|||G_i^-|||_i + |||G_{i-1}^+|||_{i-1}). \end{aligned} \tag{46}$$

Before proving the proposition, we state the stability result which follows from Proposition 3.1.

Theorem 3.8 *Suppose that for any i, $\Delta t_i/\Delta t_{i-1}$ or $\Delta t_{i-1}/\Delta t_i$ is an integer. Suppose the velocity to be constant in $\mathbb{R}$, and, for each i, suppose $\gamma_i < 1$ and $2T \leq \gamma_i \frac{a_{i+1}-a_i}{c}$. For any initial values P and Q, problem (32), (44) has a unique* *solution. Furthermore, there exists a positive α, depending only on T and the γ_i, such that*

$$\sum_i \|D_t^+ U_i(\cdot, N_{i+1})\|^2_{\ell^2_x} \leq \alpha(\sum_i (|||G_i^-|||_i^2 + |||G_i^+|||_i^2) + \|D_x^+ P\|^2_{\ell^2_x} + \|Q\|^2_{\ell^2_x} + \|F\|^2_{\ell^2_{x,t}}). \tag{47}$$

Proof To simplify the notation, the proof is given for $F \equiv 0$. It extends to general F through Cauchy -Schwarz and the Gronwall lemma as in Theorem 3.3. We split V_i and V_{i-1} into orthogonal sums:

$$V_i = \text{Im } \mathbf{P}_{i,i-1} \oplus H_{i,i-1}, \quad V_{i-1} = \text{Im } \mathbf{P}_{i-1,i} \oplus H_{i-1,i}. \tag{48}$$

In each domain Ω_i, Proposition 3.1 provides two retropropagating solutions of the discrete wave equation W_i^+ and W_i^-. We define $W_i := W_i^+ + W_i^-$, and $\tilde{U}_i := U_i - W_i$. In Ω_i, $\tilde{U}_i$ is a solution of the discrete wave equation, with transmission conditions

$$\begin{aligned} B_i^- \tilde{U}_i(0,\cdot) &= \mathbf{P}_{i-1,i} \widetilde{B}_i^- \tilde{U}_{i-1}(J_{i-1}+1,\cdot) + R_{i,i-1}, \\ B_{i-1}^+ \tilde{U}_{i-1}(J_{i-1}+1,\cdot) &= \mathbf{P}_{i,i-1} \widetilde{B}_{i-1}^+ \tilde{U}_i(0,\cdot) + R_{i-1,i}, \end{aligned}$$

where $R_{i,i-1} \in H_{i,i-1}$ and $R_{i-1,i} \in H_{i-1,i}$. The initial conditions are $\tilde{U}_i(\cdot,0) = -W_i(\cdot,0)$ and $\tilde{U}_i(\cdot,1) = -W_i(\cdot,1)$. We use energy estimate (21) in each subdomain, and sum up in time in order to obtain

$$\begin{aligned} E(\tilde{U}_i\ )(N_i+1) - E(\tilde{U}_i)(0) + \frac{c}{2}[|||\widetilde{B}_{i-1}^+ U_i(0,\cdot)|||_i^2 + |||\widetilde{B}_{i+1}^- U_i(J_i+1,\cdot)|||_i^2] \\ = \frac{c}{2}[|||\mathbf{P}_{i-1,i} \widetilde{B}_i^- \tilde{U}_{i-1}(J_{i-1}+1,\cdot) + R_{i,i-1}|||_i^2 + |||\mathbf{P}_{i,i-1} \widetilde{B}_{i-1}^+ \tilde{U}_i(0,\cdot) + R_{i-1,i}|||_i^2], \end{aligned}$$

and by projection, using the Pythagorean Theorem,

$$\begin{aligned} E(\tilde{U}_i\ )(N_i+1) - E(\tilde{U}_i)(0) + \frac{c}{2}[|||\widetilde{B}_{i-1}^+ U_i(0,\cdot)|||_i^2 + |||\widetilde{B}_{i+1}^- U_i(J_i+1,\cdot)|||_i^2] \\ \leq \frac{c}{2}[|||\widetilde{B}_i^- \tilde{U}_{i-1}(J_{i-1}+1,\cdot)|||_{i-1}^2 + |||R_{i,i-1}|||_i^2 + |||\widetilde{B}_{i-1}^+ \tilde{U}_i(0,\cdot)|||_{i+1}^2 + |||R_{i-1,i}|||_i^2]. \end{aligned}$$

Suming up in i, the boundary operators cancel, and only the $R_{i,i-1}$ and $R_{i-1,i}$ terms remain:

$$\sum_i E(\tilde{U}_i)(N_i+1) \leq E(W_i)(0) + \sum_i [|||R_{i,i-1}|||_i^2 + |||R_{i-1,i}|||_i^2].$$

Using now Proposition 3.1, we obtain estimate (47). ∎

This gives us the final error estimates:

Theorem 3.9 *Suppose that for any i, $\Delta t_i/\Delta t_{i-1}$ or $\Delta t_{i-1}/\Delta t_i$ is an integer. Suppose the velocity is constant in $\mathbb{R}$, and, for each i, suppose $\gamma_i < 1$ and $2T \leq \gamma_i \frac{a_{i+1}-a_i}{c}$. Let u be a smooth solution of (1) with constant velocity c and initial* *conditions p and q. Let $\{U_i\}$ be the discrete solution of the scheme (32), (33), (34) with initial conditions (16). There exist positive constants α and Δt_0, depending on T and u, such that for $\Delta t \leq \Delta t_0$, the following estimate holds:*

$$\sum_i \|D_t^+ (U_i - U_{d,i})(\cdot, N_i + 1)\|_{\ell_x^2}^2 \leq \alpha \Delta t^2, \tag{49}$$

with $U_{d,i}(j,n) = u(a_i + j\Delta x_i, n\Delta t_i)$, and $\Delta t = max(\Delta t_i)_{1\leq i \leq I}$. Therefore, for $\gamma_i < 1$, the scheme (32), (33), (34) is an overall second order approximation of the wave equation.

Proof Since we know by Subsection 3.3 that the scheme is of order 2, we only need to prove that the projection step is of order two. We apply Theorem 3.8 to the error. In this case, $G_i^{\pm}$ represents the truncation error on the boundary, *i.e.*

$$\begin{aligned} G_i^- &= (B_i^- U_{d,i} - \mathbf{P}_{i,i-1} \widetilde{B}_i^- U_{d,i-1})(0, \cdot), \\ G_i^+ &= (B_i^+ U_{d,i} - \mathbf{P}_{i,i+1} \widetilde{B}_i^+ U_{d,i+1})(J_i, \cdot). \end{aligned}$$

Since in each subdomain $u_i = u/\Omega_i$, we have, like in Theorem 3.4, the truncation errors for $n \geq 1$

$$|B_i^- U_{d,i}(0,n) - \mathcal{B}_i^-(a_i, t_n)| \leq \beta \Delta t^2 \sup_{\mathcal{S}(a_i,t_n)} (|\partial_t^3 u|),$$

$$|\widetilde{B}_i^- U_{d,i-1}(J_{i-1}+1,n) - \mathcal{B}_i^-(a_i, t_n)| \leq \beta \Delta t^2 \sup_{\mathcal{S}(a_i,t_n)} (|\partial_t^3 u|),$$

$$|B_i^+ U_{d,i}(J_i+1,n) - \mathcal{B}_i^+(a_{i+1}, t_n)| \leq \beta \Delta t^2 \sup_{\mathcal{S}(a_{i+1},t_n)} (|\partial_t^3 u|),$$

$$|\widetilde{B}_i^+ U_{d,i+1}(0,n) - \mathcal{B}_i^+(a_{i+1}, t_n)| \leq \beta \Delta t^2 \sup_{\mathcal{S}(a_{i+1},t_n)} (|\partial_t^3 u|).$$

We treat now G_i^-. Since the projection is a contraction, we obtain by the triangle inequality

$$\begin{aligned} \|G_i^-\| \; &\leq \|\widetilde{B}_i^- U_{d,i-1}(J_{i-1}+1, \cdot) - \mathcal{B}_i^-(a_i, \cdot\Delta t_{i-1})\| \\ &+ \|\mathcal{B}_i^- u(a_i, \cdot\Delta t_{i-1}) - \mathbf{P}_{i,i-1}\mathcal{B}_i^- u(a_i, \cdot\Delta t_i)\| \\ &+ \|\mathcal{B}_i^- u(a_i, \cdot\Delta t_i) - \widetilde{B}_i^- U_{d,i-1})(J_{i-1}+1, \cdot)\|. \end{aligned}$$

The first and last terms on the right-hand side are treated by the truncation estimates, the second one is estimated by the following lemma:

Lemma 3.10 *For any regular function ϕ , we have*

$$\|\mathbf{F}_i \Phi_i - \mathbf{Q}_{i,j} \mathbf{F}_j \Phi_j\|_{L^2(0,T)} \leq \sqrt{2T} max(\Delta t_i^2, \Delta t_j^2) \; \|\phi''\|_{L^\infty(0,T)},$$

with $\Phi_i = \{\phi(n\Delta t_i)\}_{0 \leq n \leq N_i+1}$.

Proof By the triangle inequality, we have

$$\|\mathbf{F}_i\Phi_i - \mathbf{Q}_{i,j}\mathbf{F}_j\Phi_j\| \le \|\mathbf{F}_i\Phi_i - \phi\| + \|\mathbf{Q}_i\,\phi - \phi\| + \|\mathbf{Q}_i\,\phi - \mathbf{Q}_{i,j}\mathbf{F}_j\Phi_j\|,$$

$$\|\mathbf{F}_i\Phi_i - \mathbf{Q}_{i,j}\mathbf{F}_j\Phi_j\| \le \|\mathbf{F}_i\Phi_i - \phi\| + \|\mathbf{Q}_i\phi - \phi\| + \|\phi - \mathbf{F}_j\Phi_j\|.$$

Since $\mathbf{Q}_i\,\phi$ is the projection of ϕ on V_i, we have

$$\|\mathbf{Q}_i\,\phi - \phi\| \le \|\mathbf{F}_i\Phi_i - \phi\|,$$

and it only remains to estimate $\|\mathbf{F}_i\Phi_i - \phi\|$. This is classical, through the identity

$$(\phi - \mathbf{F}_i\Phi_i)(t) = \frac{(t-t_n)(t-t_{n+1})}{2}\phi''(\xi), \quad \xi \in (t_n, t_{n+1}).$$

We write

$$\|\mathbf{F}_i\Phi_i - \phi\|^2 = \sum_{n=0}^{N_i}\int_{t_n}^{t_{n+1}} (\mathbf{F}_i\Phi_i - \phi)^2(t)\;dt,$$

$$\|\mathbf{F}_i\Phi_i - \phi\|^2 \le \|\phi''\|^2_{L^\infty(0,T)}\sum_{n=0}^{N_i}(\Delta t_i)^5\int_0^1 s^2(1-s)^2\;ds \le T(\Delta t_i)^4\|\phi''\|^2_{L^\infty(0,T)}.$$

■

We now conclude the proof of the Theorem, applying Lemma 3.10 with $\phi = \mathcal{B}^- u(a_i,\cdot)$ and $\phi = \mathcal{B}^- u(a_{i+1},\cdot)$. We obtain for $G_i^{\pm}$ the estimates

$$\|G_i^{\pm}\|_{L^2(0,T)} \le \alpha\;\max\;\;(\Delta t_i^2, \Delta t_{i-1}^2).$$

Inserting these estimates into (47), we get (49).

■

3.6 Proof of the extension result (Proposition 3.1)

In this section, since the analysis is local, we omit the index i, and we define the operators $\mathbf{Q}^- := \mathbf{Q}_{i,i-1}$, $\mathbf{Q}^+ := \mathbf{Q}_{i-1,i}$. We denote by $\mathbf{\Pi}^-$ the orthogonal projection in V_i on Im $\mathbf{Q}^-$, and $\mathbf{\Pi}^+$ the orthogonal projection in V_{i-1} on Im $\mathbf{Q}^+$. We now define a right inverse for the operators $\mathbf{Q}^{\pm}$.

Lemma 3.11 *Suppose $\Delta t_i/\Delta t_{i-1}$ or $\Delta t_{i-1}/\Delta t_i$ is an integer. Then there exist linear operators $\widetilde{\mathbf{Q}}^-$ from V_i into V_{i-1} (resp. $\widetilde{\mathbf{Q}}^+$ from V_{i-1} into V_i) , such that $\mathbf{Q}^-\widetilde{\mathbf{Q}}^- = Id_{V_i}$ on Im $\mathbf{Q}^-$ (resp. $\mathbf{Q}^+\widetilde{\mathbf{Q}}^+ = Id_{V_{i-1}}$ on Im $\mathbf{Q}^+$) , and for any W in Im $\mathbf{Q}^-$, we have $|||\widetilde{\mathbf{Q}}^-W|||_{i-1} = |||W|||_i$ (resp. for any W in Im $\mathbf{Q}^+$, we have $|||\widetilde{\mathbf{Q}}^+W|||_i = |||W|||_{i-1}$).*

Proof Suppose $V_{i-1} \subset V_i$. In this case we have $\mathbf{Q}^- = Id_{V_i}$ and Im $\mathbf{Q}^- = V_{i-1}$, so that $\mathbf{\Pi}^- = \mathbf{Q}^+$. So $\widetilde{\mathbf{Q}}^-$ can be defined by $\widetilde{\mathbf{Q}}^- = Id_{V_{i-1}}$. Then, for W in Im $\mathbf{Q}^- = V_{i-1}$, $\mathbf{Q}^-\widetilde{\mathbf{Q}}^-W = \mathbf{Q}^-W = W$ and $|||\widetilde{\mathbf{Q}}^-W|||_{i-1} = |||W|||_{i-1}$. On the other hand, we can choose $\widetilde{\mathbf{Q}}^+ = \mathbf{Q}^+$, and for any W in Im $Q^+ \subset V_{i-1}$, we have $\widetilde{\mathbf{Q}}^+W = W$, so that $|||\widetilde{\mathbf{Q}}^+W|||_i = |||W|||_{i-1}$, and $\mathbf{Q}^+\widetilde{\mathbf{Q}}^+W = \mathbf{Q}^+W = W$. Reversing the roles of V_i and V_{i-1} gives the lemma.

■

We set $\widetilde{G}_i^- = \widetilde{\mathbf{Q}}^-\mathbf{\Pi}^-G_i^- \in V_{i-1}$, and $\widetilde{G}_i^+ = \widetilde{\mathbf{Q}}^+\mathbf{\Pi}^+G_{i-1}^+ \in V_i$. We now define two local discrete extension operators $A^\pm$: A^- maps $\widetilde{G}^-$ in V_{i-1} to the discrete left propagating solution W^- supported in C_i^- of

$$\begin{array}{rcl} \Box_d \, W^-(j,n) & = & 0, \quad 1 \leq j \leq J_{i-1},\, 1 \leq n \leq N_{i-1}, \\ \widetilde{B}_i^- \, W^- \, (J_{i-1}+1,\cdot) & = & \widetilde{G}^-, \end{array} \tag{50}$$

with vanishing final data. A^+ maps $\widetilde{G}^+$ in V_i to the discrete right propagating solution W^+ supported in C_i^- of

$$\begin{array}{rcl} \Box_d \, W^+(j,n) & = & 0, \quad 1 \leq j \leq J_i,\, 1 \leq n \leq N_i, \\ \widetilde{B}_{i-1}^+ \, W^+ \, (0,\cdot) & = & \widetilde{G}^+, \end{array} \tag{51}$$

with vanishing final data. Since W^- vanishes on the left boundary of Ω_{i-1}, and W^+ vanishes on the right boundary of Ω_i, we have the backward in time energy estimates

$$E(W^-)(0) + \frac{c}{2}|||B_{i-1}^+ W^-(J_{i-1}+1,\cdot)|||^2_{\ell^2_t} = \frac{c}{2}|||\widetilde{B}_i^- W^-(J_{i-1}+1,\cdot)|||^2_{\ell^2_t}, \tag{52}$$

$$E(W^+)(0) + \frac{c}{2}|||B_i^- W^+(0,\cdot)|||^2_{\ell^2_t} = \frac{c}{2}|||\widetilde{B}_{i-1}^+ W^+(0,\cdot)|||^2_{\ell^2_t}. \tag{53}$$

Lemma 3.12 *There exists a positive constant α depending only on T and γ, such that*

$$\begin{array}{lll} \forall \widetilde{G}^- \in V_{i-1}, & |||B_{i-1}^+ A^- \widetilde{G}^-|||_{\ell^2_t} & \leq \alpha \Delta t \quad |||\widetilde{G}^-|||_{\ell^2_t}, \\ \forall \widetilde{G}^+ \in V_i, & |||B_i^- A^+ \widetilde{G}^+|||_{\ell^2_t} & \leq \alpha \Delta t \quad |||\widetilde{G}^+|||_{\ell^2_t}. \end{array} \tag{54}$$

Proof We prove the result for W^+, the proof for W^- is similar . Let g^+ be the piecewise affine function $g^+ = \mathbf{F}_i\widetilde{G}^+$. Let w^+ be the continuous solution of the wave equation described in Proposition 2.1, associated to g^+. It satisfies $\mathcal{B}_i^- w^+(a_i,\cdot) = 0$. Let $\widetilde{W}^+(j,n)$ be the mean-value of w^+ on the finite volume $\mathcal{S}_{j,n}$. $\widetilde{W}^+(j,n)$ is a solution of the discrete wave equation with right-hand side $\epsilon_1(j,n)$, with zero final values, and boundary conditions $\widetilde{B}_{i-1}^+ \, \widetilde{W}^+ \, (0,\cdot) = \widetilde{G}^+ + \epsilon_2(n)$, where ϵ_1 and ϵ_2 are truncation errors, which will be evaluated below. We now use the estimates (30) in Theorem 3.3. The boundary data g^+ is not sufficiently regular to obtain the estimates (31). We get instead a weaker bound,

$$\|D_t^+ \, (\widetilde{W}^+ - W^+)(\cdot,n)\|_{\ell^2_x} \leq \alpha \Delta t \|g^+\|_{L^2(0,T)},$$

$$|||B_i^- \, (\widetilde{W}^+ - W^+)(0,\cdot)|||_{\ell^2_t} \leq \alpha \Delta t \|g^+\|_{L^2(0,T)},$$

and since $\mathcal{B}_i^- w^+(a_i,\cdot)=0$, we have

$$|||B_i^-\widetilde{W}^+(0,\cdot)|||_{\ell_t^2}\le \alpha\Delta t\ \|g^+\|_{L^2(0,T)},$$

which proves that

$$|||B_i^- W^+(0,\cdot)|||_{\ell_t^2}\le \alpha\Delta t\ \|g^+\|_{L^2(0,T)},$$

which is equivalent to the second inequality in the lemma.

■

Let now $(\widetilde{G}_{i,0}^-,\widetilde{G}_{i,0}^+)=(\widetilde{G}_i^-,\widetilde{G}_i^+)$ be given in $V_{i-1}\times V_i$, both non identically zero. We describe an iterative procedure to define the $W_i^\pm$ in Proposition 3.1. For Δt sufficiently small, we have $\alpha\Delta t\le\mu<1$. We define $W_{i,0}^-=A^-\widetilde{G}_{i,0}^-$ and $W_{i,0}^+=A^+\widetilde{G}_{i,0}^+$. By energy estimates (52) and (53), we have

$$|||B_{i-1}^+W_{i,0}^-(J_{i-1}+1,\cdot)|||_{\ell_t^2}\le\mu\ |||\widetilde{G}_{i,0}^-|||_{\ell_t^2},\quad |||B_i^-W_{i,0}^+(0,\cdot)|||_{\ell_t^2}\le\mu\ \|\widetilde{G}_{i,0}^+\|_{\ell_t^2}. \tag{55}$$

According to (48), we split $B_{i-1}^+W_{i,0}^-$ and $B_i^+W_{i,0}^+$ on Im $\mathbf{Q}^+$ and Im $\mathbf{Q}^-$ respectively,

$$B_{i-1}^+W_{i,0}^-=\mathbf{\Pi}^+B_{i-1}^+A^-\widetilde{G}_{i,0}^-+R_{i,0}^+,\quad B_i^-W_{i,0}^+=\mathbf{\Pi}^-B_i^-A^+\widetilde{G}_{i,0}^++R_{i,0}^-,$$

with $R_{i,0}^+$ in $H_{i-1,i}$ and $R_{i,0}^-$ in $H_{i,i-1}$. We now define $\widetilde{G}_{i,1}^-=\widetilde{\mathbf{Q}}^-\mathbf{\Pi}^-B_i^-A^+\widetilde{G}_{i,0}^+$ and $\widetilde{G}_{i,1}^+=\widetilde{\mathbf{Q}}^+\mathbf{\Pi}^+B_{i-1}^+A^-\widetilde{G}_{i,0}^-$. The new extensions are $W_{i,1}^\pm=A^\pm\widetilde{G}_{i,1}^\pm$. Since all operators are contractions, we have by Lemma 3.12

$$\begin{aligned}|||\widetilde{G}_{i,1}^-|||_{\ell_t^2}&=|||\widetilde{\mathbf{Q}}^-\mathbf{\Pi}^-B_i^-A^+\widetilde{G}_{i,0}^+|||_{\ell_t^2}\le\mu\ |||\widetilde{G}_{i,0}^+|||_{\ell_t^2},\\ |||\widetilde{G}_{i,1}^+|||_{\ell_t^2}&=|||\widetilde{\mathbf{Q}}^+\mathbf{\Pi}^+B_{i-1}^+A^-\widetilde{G}_{i,0}^-|||_{\ell_t^2}\le\mu\ |||\widetilde{G}_{i,0}^-|||_{\ell_t^2},\end{aligned}$$

which gives

$$\max(|||\widetilde{G}_{i,1}^-|||_{\ell_t^2},|||\widetilde{G}_{i,1}^+|||_{\ell_t^2})\le\mu\max(|||\widetilde{G}_{i,0}^-|||_{\ell_t^2},|||\widetilde{G}_{i,0}^+|||_{\ell_t^2}).$$

Furthermore, we have by (53) that $E(W_{i,1}^\pm)(0)\le\frac{c}{2}\|\widetilde{G}_{i,1}^\pm\|_{\ell_t^2}$. This leads to the recursion

$$\begin{aligned}&W_{i,k}^-=A^-\widetilde{G}_{i,k}^-,\quad W_{i,k}^+=A^+\widetilde{G}_{i,k}^+,\\ &B_{i-1}^+W_{i,k}^-=\mathbf{\Pi}^-B_{i-1}^+A^-\widetilde{G}_{i,k}^-+R_{i,k}^+,\quad B_i^-W_{i,k}^+=\mathbf{\Pi}^+B_i^-A^+\widetilde{G}_{i,k}^++R_{i,k}^-,\\ &\widetilde{G}_{i,k+1}^-=\widetilde{\mathbf{Q}}^-\mathbf{\Pi}^-B_i^-A^+\widetilde{G}_{i,k}^+,\quad \widetilde{G}_{i,k+1}^+=\widetilde{\mathbf{Q}}^+\mathbf{\Pi}^+B_{i-1}^+A^-\widetilde{G}_{i,k}^-,\end{aligned} \tag{56}$$

with the properties

$$|||\widetilde{G}_{i,k}^\pm|||_{\ell_t^2}\le\mu^k\sum_\pm|||\widetilde{G}_{i,0}^\pm|||_{\ell_t^2};E(W_{i,k}^\pm)(0)\le\frac{c}{2}|||\widetilde{G}_{i,k}^\pm|||_{\ell_t^2};|||R_{i,k}^\pm|||_{\ell_t^2}\le\mu^k\sum_\pm|||\widetilde{G}_{i,0}^\pm|||_{\ell_t^2}. \tag{57}$$

By the estimates (57), the series $\sum \widetilde{G}_{i,k}^{\pm}$ converges in ℓ_t^2 . The series $\sum W_{i,k}^{\pm}$ converges in the energy norm, and the series $\sum R_{i,k}^{\pm}$ converges in ℓ_t^2. We set $\widetilde{G}_i^{\pm} = \sum_{k=0}^{+\infty} \widetilde{G}_{i,k}^{\pm}$, $R_i^{\pm} = \sum_{k=0}^{+\infty} R_{i,k}^{\pm}$, $W_{i-1}^{-} = \sum_{k=0}^{+\infty} W_{i,k}^{-}$, and $W_i^{+} = \sum_{k=0}^{+\infty} W_{i,k}^{+}$. Then W_{i-1}^{-} is a solution of the discrete wave equation in Ω_{i-1} with support in the cone C_i^-, and W_i^+ is a solution of the discrete wave equation in Ω_i with support in the cone C_i^+. By construction, we have

$$\begin{aligned}
\mathbf{Q}^{-}\widetilde{B}_i^{-}W_{i-1}^{-} &= \sum_{k=0}^{+\infty} \mathbf{Q}^{-}\widetilde{G}_{i,k}^{-}, \\
\mathbf{Q}^{-}(\widetilde{B}_i^{-}W_{i-1}^{-} - \widetilde{G}_i^{-}) &= \sum_{k=1}^{+\infty} \mathbf{Q}^{-}\widetilde{G}_{i,k}^{-}, \\
B_i^{-}\, W_i^{+} &= \sum_{k=0}^{+\infty} (\mathbf{\Pi}^{+} B_i^{-} A^{+} \widetilde{G}_{i,k}^{+} + R_{i,k}^{-}),
\end{aligned}$$

but by (56) and the definition of $\widetilde{\mathbf{Q}}^{-}$, we have $\mathbf{\Pi}^{+} B_i^{-} A^{+} \widetilde{G}_{i,k}^{+} = \mathbf{Q}^{-}\widetilde{G}_{i,k+1}^{+}$, and the last equation can be rewritten as

$$B_i^{-}\, W_i^{-} = \sum_{k=1}^{+\infty} \mathbf{Q}^{-}\widetilde{G}_{i,k}^{+} + R_i^{-}.$$

Thus we have

$$B_i^{-}\, W_i^{+} - \mathbf{Q}^{-}\widetilde{B}_i^{-}W_{i-1}^{-} - \widetilde{G}_i^{-} = R_i^{-} + \mathbf{Q}^{-}\widetilde{G}_i^{-} - \widetilde{G}_i^{-} = \tilde{R}_{i,i-1} \in H_{i,i-1}.$$

In the same fashion, we obtain

$$B_{i-1}^{+}\, W_{i-1}^{-} - \mathbf{Q}^{+}\widetilde{B}_{i-1}^{+}W_i^{+} - \widetilde{G}_{i-1}^{+} = R_i^{+} + \mathbf{Q}^{+}\widetilde{G}_{i-1}^{+} - \widetilde{G}_{i-1}^{+} = \tilde{R}_{i-1,i} \in H_{i-1,i}.$$

We have $\|D_t^{+}\, W_{i-1}^{-}(\cdot,0)\|_{\ell_x^2}^2 \leq \frac{c}{2(1-\mu)}(|||\widetilde{G}_i^{-}|||_{\ell_t^2} + |||\widetilde{G}_{i-1}^{+}|||_{\ell_t^2})$, and the same for W_i^{+}, and
$\|\tilde{R}_{i,i-1}\|_{\ell_t^2} \leq (\frac{1}{1-\mu} + 2)(|||\widetilde{G}_i^{-}|||_{\ell_t^2} + |||\widetilde{G}_{i-1}^{+}|||_{\ell_t^2})$, and the same for $\tilde{R}_{i-1,i}$, which gives estimate (46).
This concludes the proof of Proposition 3.1, which is the cornerstone of the well-posedness theory.

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