Continuous Analysis of the Additive Schwarz Method: a Stable Decomposition in H^1 with Explicit Constants

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Abstract

The classical convergence result for the additive Schwarz preconditioner with coarse grid is based on a stable decomposition. The result holds for discrete versions of the Schwarz preconditioner, and states that the preconditioned operator has a uniformly bounded condition number that depends only on the number of colors of the domain decomposition, the ratio between the average diameter of the subdomains and the overlap width, and on the shape regularity of the domain decomposition.

The Schwarz method was however defined at the continuous level, and similarly, the additive Schwarz preconditioner can also be defined at the continuous level. We present in this paper a continuous analysis of the additive Schwarz preconditioned operator with coarse grid in two dimensions. We show that the classical condition number estimate also holds for the continuous formulation, and as in the discrete case, the result is based on a stable decomposition, but now of the Sobolev space H^1. The advantage of such a continuous result is that it is independent of the type of fine grid discretization, and thus does the more natural continuous formulation of the Schwarz method justice. The upper bound we provide for the classical condition number is also explicit, which gives us the quantitative dependence of the upper bound on the shape regularity of the domain decomposition.

1 Introduction

With the generalization of parallelism in today's computers, parallelizable algorithms are of increasing importance. Domain decomposition methods make it possible to perform numerical simulations in parallel, see for example the books [33, 31, 35], or the monographs [37, 8], and references therein. Consider a partial

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differential equation to be solved on a big domain Ω. In domain decomposition methods, an iterative approach introduced by Schwarz [32] is to decompose the big domain Ω into several smaller overlapping subdomains Ω_i, $\Omega = \bigcup_{i=1}^{n} \Omega_i$, and then to compute approximations u_i^k defined by

$$\begin{array}{rcll} \mathscr{L} u_i^k & = & f & \text{in } \Omega_i, \\ u_i^k & = & u_j^{k-1} & \text{on } \Gamma_{ij}, \end{array} \tag{1}$$

where Γ_{ij} denote the interfaces. In practice, it is more efficient to use the general algorithm (1) as a preconditioner for a Krylov subspace method, like GMRES or conjugate gradients, see for example [19, 20] for a more detailed explanation. The Additive Schwarz operator defines one such preconditioned operator, related to (1). For a domain decomposition with both an overlap and a coarse mesh, Dryja and Widlund [14] proved that the condition number of the discrete Additive Schwarz operator is uniformly bounded, *i.e.* it does not depend on the number of subdomains. However, it depends on the number of colors of the domain decomposition, on the ratio between the diameter of the subdomain and the thickness of the overlaps, and on the shape regularity of the domain decomposition, see also Toselli and Widlund [35, Chap. 2]. Schwarz preconditioners have then mostly been analyzed at the discrete level, see for example [7, 29, 30, 23] for spectral discretizations, [3] for the non-selfadjoint case, [4] for parabolic problems, [6] for some non-symmetric and indefinite problems, [5] for multiplicative versions of the algorithm, [10] for discretizations on unstructured meshes, [9] when also the coarse grid is non-matching, [16, 12, 27, 28] for mixed finite element discretizations, [13] for mortar finite element problems, [17] for discontinuous Galerkin discretizations, and [18] for numerical linear algebra techniques. For lower bounds on the convergence of Schwarz methods, see [2].

Schwarz domain decomposition methods are however naturally defined and analyzed at the continuous level, like in (1), see for example [24, 25, 26]. Schwarz methods were also invented by Schwarz at the continuous level [32], and the more recent class of optimized Schwarz methods was formulated and analyzed at the continuous level, for an introduction see [19] and references therein. It is however much less clear how to analyze a two level method at the continuous level. In a recent review on coarse space components [36], we find the comment:

> Early on, coarse spaces were not used and only continuous problems were considered; in fact it is unclear what a coarse problem then might be.

The purpose of our paper is to present an analysis of the two level Additive Schwarz operator in a continuous setting, and to prove that its condition number is bounded independently of the number of subdomains. The proof succeeds by establishing the existence of a stable decomposition of every function in $H_0^1(\Omega)$ as a sum of functions belonging to the $H_0^1(\Omega_i)$ plus a coarse function belonging to the space of continuous piecewise linear functions $P_1(\mathscr{T})$ where $\mathscr{T}$ is our coarse triangular mesh.

Our goal in this paper is to obtain at each step of the analysis explicit estimates also for the constants involved. To do so, we have to give a precise and quantitative definition of the notion of shape regularity. The upper bounds we obtain for the condition number for the Additive Schwarz operator at the continuous level contains explicit expressions for all the constants. Such precise estimates are useful when studying the Additive Schwarz operator for non-shape regular domain decompositions, *i.e.*, when some subdomains are very small, while others are very large. In this case, the classical result would give us a condition number linear in $\max(H(\boldsymbol{x}))/\min(\delta(\boldsymbol{x}))$. Using the methods developed in this paper, we prove in [21], that the condition number is actually linear in $\max(H(\boldsymbol{x})/\delta(\boldsymbol{x}))$ which is a much better estimate for non shape regular domain decompositions, see also [11]. Our continuous analysis can also be helpful to prove properties of the Additive Schwarz preconditioned operator when discretized by various consistent numerical methods for partial differential equations, as soon as the discretization error is small enough. The condition number estimate should then not depend on the fine discretization.

First, we recall in section 2 the definition of the preconditioned additive Schwarz operator, and the abstract results giving an estimate of the condition number of the Additive Schwarz operator as soon as three assumptions hold. The rest of the paper is then devoted to showing that these assumptions hold for a decomposition at the continuous level, the key assumption being the existence of a stable decomposition. After specifying in section 3 the geometric parameters of the domain decomposition, we prove in section 4 the existence of a stable decomposition in the continuous case in the absence of a coarse mesh albeit with a constant that depends on the number of subdomains. Section 5 is dedicated to proving our main theorem, Theorem 5.12, which establishes that, in the presence of a coarse mesh, there exists a uniformly stable decomposition with an explicit upper bound that does not depend on the number of subdomains. Using this result, we prove in section 6 that the condition number of the additive Schwarz operator has a uniformly bounded condition number in the continuous case when there is a coarse P_1 mesh.

2 The Additive Schwarz operator

In this section, we recall the abstract results in Toselli and Widlund [35, chap. 2]. Let $(V_i)_{0\le i\le N}$ be Hilbert spaces, with V_0 being a coarse space. Let $V=\sum_{i=0}^{n} R_i^T V_i$, where the R_i^T are linear extension operators. Let $a(\cdot,\cdot)$ be a symmetric, positive definite bilinear form on V. We wish to find the unique u in V satisfying

$$a(u,v)=(f,v) \text{ for all } v \text{ in } V.$$

Let $\widetilde{a}_i(\cdot,\cdot)$ be symmetric positive definite bilinear forms on the V_i. We define $\widetilde{P}_i : V\to V_i$ by

$$\widetilde{a}_i(\widetilde{P}_i u,v)=a(u,R_i^T v) \qquad \text{for all } v \text{ in } V_i.$$

Let $P_i = R_i^T \widetilde{P}_i$. The additive Schwarz operator is defined by

$$P_{ad} := \sum_{i=0}^{N} P_i. \tag{2}$$

This is an a-symmetric a-positive operator. We are interested in bounding the condition number (with respect to the bilinear form a) of the preconditioned operator P_{ad}.

Definition 2.1. *Let a be a symmetric, positive bilinear form on a vector space V. Let P be a continuous linear application from V to V. We call*

$$\kappa(P) = \frac{\max_{\substack{\boldsymbol{u}\in V\\ a(\boldsymbol{u},\boldsymbol{u})=1}} a(Pu,u)}{\min_{\substack{\boldsymbol{u}\in V\\ a(\boldsymbol{u},\boldsymbol{u})=1}} a(Pu,u)}$$

the a-condition number of P.

Assumption 2.2 (Stable decomposition)**.** *There exists a constant C_0 such that all u in V admit the decomposition*

$$u = \sum_{i=0}^{N} R_i^T u_i, \text{ with } u_i \in V_i \text{ for } i = 0 \ldots N, \quad \text{and} \quad \sum_{i=0}^{N} \widetilde{a}_i(u_i,u_i) \le C_0^2 a(u,u). \tag{3}$$

Assumption 2.3 (Strengthened Cauchy Schwarz inequality)**.** *For all $i, j \ge 1$, there exist constants $0 \le \varepsilon_{ij} \le 1$ such that for all $u_i \in V_i$ and $u_j \in V_j$ we have*

$$|a(R_i^T u_i, R_j^T u_j)| \le \varepsilon_{ij} a(R_i^T u_i, R_i^T u_i)^{\frac{1}{2}} a(R_j^T u_j, R_j^T u_j)^{\frac{1}{2}}. \tag{4}$$

We denote by $\rho(\mathscr{E})$ the spectral radius of the matrix $\mathscr{E} = \{\varepsilon_{ij}\}$.

Assumption 2.4 (Local stability)**.** *There exists $\omega > 0$ such that $\forall i \ge 0$ and $\forall u_i \in$ range$(\widetilde{P}_i)$ we have*

$$a(R_i^T u_i, R_i^T u_i) \le \omega \widetilde{a}_i(u_i,u_i). \tag{5}$$

The following fundamental result can be found in Toselli and Widlund [35], see Theorem 2.7.

Theorem 2.5. *Let Assumptions 2.2, 2.3 and 2.4, be satisfied. Then the a-condition number $\kappa(P_{ad})$ of the additive Schwarz operator satisfies*

$$\kappa(P_{ad}) \le C_0^2 \omega(\rho(\mathscr{E}) + 1). \tag{6}$$

Proof. The proof of Theorem 2.7 in Toselli and Widlund [35] also holds if the V_i have infinite dimension. □

In order to get a more concrete estimate, the strengthened Cauchy-Schwarz Assumption 2.3 is often replaced in the literature by an assumption on the number of colors of the decomposition. The number of colors is defined as follows:

Definition 2.6 (Number of colors). *In an abstract domain decomposition into the fine spaces $(V_i)_{1\le i\le N}$, the number of colors is the smallest integer N_c such that there exists a partition of $\{1,\dots,N\}$ into N_c sets $(I_k)_{1\le k\le N_c}$ such that $R_i^{\mathrm{T}}V_i$ is a-orthogonal with $R_j^{\mathrm{T}}V_j$ whenever i and j are distinct indices that belong to the same I_k. The fine spaces V_i and V_j are said to have the same color when i and j belong to the same I_k.*

Then we can use the number of colors in estimate (6) instead of relying on the spectral radius of the strengthened Cauchy-Schwarz matrix.

Theorem 2.7. *Let Assumptions 2.2, and 2.4, be satisfied. Suppose that the fine decomposition V_i has N_c colors. Then the a-condition number $\kappa(P_{ad})$ satisfies*

$$\kappa(P_{ad}) \le C_0^2\omega(N_c+1). \tag{7}$$

Before proving the result we make the following remark:

Remark 2.8. In the literature, three distinct integers are used in estimate (7), and these constants can be defined both in the concrete geometric setting of domain decomposition, and in the abstract setting:

- In the concrete setting of domain decomposition, one can define N_k as the maximum number N_k of neighbors, including itself, a subdomain can have. This integer is the connectivity of the domain decomposition. This number can replace $\rho(\mathscr{E})$ in Theorem 2.5, since we always have $\rho(\mathscr{E}) \le N_k$, see [35, Lemma 2.10] (where N^c is used as the name for this constant). In the abstract setting, one could define N_k as the maximum over i in $\{1,\dots,N\}$ of the number of $R_j^T V_j$, j in $\{1,\dots,N\}$, which are not a-orthogonal to $R_i^T V_i$.

- The number of colors N_c we defined in the abstract setting, see Definition 2.6, can also be defined in a transparent way in the concrete geometric setting of domain decomposition, see Definition 3.6. We always have $N_c \le N_k$ in both the concrete and abstract setting, and thus proving a result with the constant N_c implies the result with the constant N_k.

- In the concrete setting of domain decomposition, one can define $\hat{N}$ as the maximum number of subdomains a point can belong to. In the abstract setting, one can define $\hat{N}$ as the largest integer for which there exist $\hat{N}$ distinct $R_i^T V_i$ whose intersection is not $\{0\}$. We always have $\hat{N} \le N_c$ in both the abstract setting and the concrete setting, so a result with the constant $\hat{N}$ is the most accurate. In the concrete case, when the $\widetilde{a}_i$ are defined as integrals over a subdomain, it is possible to replace N_c with $\hat{N}$ in (7), see the original proof of [15, Th. 4.1]. It is unknown to the authors if the result with $\hat{N}$ can be generalized to an abstract domain decomposition.

In the remainder of this paper, we always work with the number of colors N_c.

We now proceed with the proof of Theorem 2.7.

Proof. We only need to change part of the proof of Theorem 2.7 in Toselli and Widlund [35]. We already know that a lower bound for the smallest eigenvalue is $1/C_0^2$, see [35, Lemma 2.5]. To get the estimate on the largest eigenvalue, we follow the ideas of [35, Lemma 2.6] but additionally group the V_i by color. For each color k in $\{1,\dots,N_c\}$, we get:

$$a(\sum_{i\in I_k} P_i u, \sum_{j\in I_k} P_j u) = \sum_{i\in I_k}\sum_{j\in I_k} a(P_i u, P_j u) = \sum_{i\in I_k} a(P_i u, P_i u) \leq \omega \sum_{i\in I_k} \widetilde{a}_i(\widetilde{P}_i u, \widetilde{P}_i u)$$
$$\leq \omega \sum_{i\in I_k} a(P_i u, u) \leq \omega a(\sum_{i\in I_k} P_i u, u) \leq \omega a(\sum_{i\in I_k} P_i u, \sum_{j\in I_k} P_j u)^{\frac{1}{2}} a(u,u)^{\frac{1}{2}}.$$

Dividing by $a(\sum_{i\in I_k} P_i u, \sum_{j\in I_k} P_j u)^{\frac{1}{2}}$, we therefore get

$$a(\sum_{i\in I_k} P_i u, \sum_{j\in I_k} P_j u)^{\frac{1}{2}} \leq \omega a(u,u)^{\frac{1}{2}},$$

and thus can estimate using again the Cauchy-Schwarz inequality

$$a(\sum_{i\in I_k} P_i u, u) \leq a(\sum_{i\in I_k} P_i u, \sum_{j\in I_k} P_j u)^{\frac{1}{2}} a(u,u)^{\frac{1}{2}} \leq \omega a(u,u).$$

We also know that $a(P_0 u, u) \leq \omega a(u,u)$, see [35, Lemma 2.6]. Therefore, summing over all colors and P_0, we get $a(P_{ad} u, u) \leq (N_c + 1)\omega a(u,u)$ for all u in V. $\square$

While the local stability and the strengthened Cauchy-Schwarz inequality can naturally be extended to the continuous case, the stable decomposition result is traditionally shown using properties of the fine discretization of the problem, see for example Toselli and Widlund [35]. For a continuous formulation, we need to use other techniques, which is the purpose of this paper.

3 Geometry and decomposition into subdomains

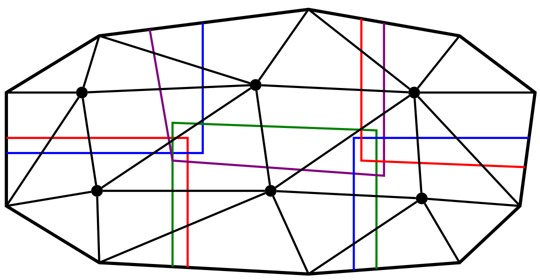

Figure 1: Domain decomposition with a coarse mesh

First, we recall the definition of a domain:

Definition 3.1. *A domain of $\mathbb{R}^2$ is an open connected set of $\mathbb{R}^2$ whose boundary $\partial\Omega$ is of null Lebesgue measure*[1]*. We denote by $|\Omega|$ the Lebesgue measure of the domain Ω.*

We recall the definition of a non overlapping and an overlapping domain decomposition:

Definition 3.2 (Non Overlapping Decomposition). *Let Ω be a bounded domain of $\mathbb{R}^2$. A collection of domains $(U_i)_{1\le i\le N}$, is a non overlapping domain decomposition of Ω if*

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{U}_i, \quad U_i \cap U_j = \emptyset \qquad \text{for all } i \neq j. \tag{8}$$

Definition 3.3 (Overlapping Decomposition). *Let Ω be a bounded domain of $\mathbb{R}^2$. A collection of domains $(\Omega_i)_{1\le i\le N}$ is an overlapping domain decomposition of Ω if*

$$\Omega = \bigcup_{i=1}^{N} \Omega_i.$$

In this article, we use the parameter H to represent the average size of subdomains. For the definition of H, we use the concept of diameter, which we recall here:

Definition 3.4 (Domain Diameter). *Let U be a bounded subset of $\mathbb{R}^2$. We define the diameter of U to be*

$$\operatorname{diam}(U) = \sup_{\substack{\boldsymbol{x}\in U\\ \boldsymbol{y}\in U}} \|\boldsymbol{x}-\boldsymbol{y}\|.$$

The concept of an overlapping domain decomposition raises the question on how to define the overlap width of the decomposition. We use the following definition:

Definition 3.5 (Overlap of the Decomposition). *A domain decomposition $(\Omega_i)_{1\le i\le N}$ is said to have overlap width*[2] *$\delta > 0$, if there exists a non overlapping domain decomposition $(U_i)_{1\le i\le N}$ of Ω such that for all i, $1 \le i \le N$, $U_i \subset \Omega_i$ and*

$$\{\boldsymbol{x} \in \Omega \mid \operatorname{dist}(\boldsymbol{x}, U_i) < \delta\} \subset \Omega_i.$$

Definition 3.6 (Colors of the Decomposition). *The number of colors of an overlapping domain decomposition $(\Omega_i)_{1\le i\le N}$ of domain Ω is the smallest integer N_c such that there exists a partition of $\{1,\dots,N\}$ into N_c sets $(I_k)_{1\le k\le N_c}$ such that*

$$\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset,$$

whenever $i \neq j$ and i, j both belong to the same color I_k.

[1] It is possible for a pathological open connected set of $\mathbb{R}^2$ to have a boundary with strictly positive measure. For example $(0,1)\times(1/4,3/4)\cup\bigcup_{j=1}^{\infty}\bigcup_{k=0}^{2^{j-1}-1}(\frac{2k+1}{2^j}-2^{-4j}, \frac{2k+1}{2^j}+2^{-4j})\times(0,1)$ is open, connected and dense in $(0,1)\times(0,1)$ but has a measure smaller than $9/14$.

[2] Geometrically, the parameter δ corresponds to half the overlap of the subdomains

Remark 3.7. The geometric Definition 3.6 for the number of colors is equivalent to the algebraic Definition 2.6 for the bilinear forms implied by the geometric domain decomposition, like in this paper, where $R_i^T V_i$ contains all functions that are H_0^1 in Ω and null outside Ω_i and a is an integral over Ω.

4 Stable decomposition without a coarse solver

To understand the importance of the coarse solver, we begin by proving the existence of a stable decomposition without a coarse mesh. In that case, the constant C_0 in (7) stemming from the stable decomposition depends on the number of subdomains. We consider a bounded domain Ω being decomposed into N overlapping subdomains Ω_i with overlap width δ. We make the following assumptions on the domain decomposition:

Assumption 4.1. *The overlapping domain decomposition* $(\Omega_i)_{1\le i\le N}$ *is derived from a non overlapping one by* $\Omega_i = \{\boldsymbol{x} \in \Omega \mid \mathrm{dist}(\boldsymbol{x}, U_i) < \delta\}$, *and we refer to it by* $((U_i)_{1\le i\le N}, (\Omega_i)_{1\le i\le N})$.

Assumption 4.2. *Let* H *be the smallest diameter among the diameters of the subdomains* U_i. *We suppose there exist uniform parameters* $C_d > 0$, $c_a > 0$ *and* $C_a > 0$ *such that for all* i *in* $\{1,\dots,N\}$

$$H \le \mathrm{diam}(U_i) \le C_d H, \quad c_a H^2 \le |U_i| \le C_a H^2, \tag{9}$$

where $|U_i|$ *is the Lebesgue measure of the subdomain* U_i.

To construct the stable decomposition, we use a partition of unity.

Lemma 4.3 (Partition of Unity)**.** *Let* Ω *be an open domain of* $\mathbb{R}^2$, $N > 0$ *be the number of subdomains, and* $(U_i)_{1\le i\le N}$ *be domains of* $\mathbb{R}^2$ *satisfying* (8). *With* $\delta > 0$ *the overlap width, we define*

$$\widetilde{\Omega}_i = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \mathrm{dist}(\boldsymbol{x}, U_i) < \delta\},$$

and denote by N_c *the number of colors of this domain decomposition*[3]. *Then, there exists a universal*[4] *constant* λ_2, $0 < \lambda_2 \le 6$, *and* N *functions* $(\psi_i)_{1\le i\le N}$ *in* $\mathscr{C}^\infty(\mathbb{R}^2)$ *having the following properties:*

1. *For all* i *in* $\{1,\dots,N\}$, ψ_i *vanishes outside of* $\widetilde{\Omega}_i$.
2. *For all* $\boldsymbol{x}$ *in* $\mathbb{R}^2$, $0 \le \psi_i(\boldsymbol{x}) \le 1$.
3. *For all* $\boldsymbol{x}$ *in* Ω, $\sum_i \psi_i(\boldsymbol{x}) = 1$.
4. *For all* $\boldsymbol{x}$ *in* Ω, $\sum_{i=1}^N \|\nabla \psi_i(\boldsymbol{x})\|^2 \le 2\lambda_2^2 \frac{(N_c-1)^2}{\delta^2}$.

[3]The $\tilde{\Omega}_i$ can extend beyond the domain Ω, in contrast to the Ω_i defined earlier

[4]It depends only on the dimension but we have restricted ourselves to two-dimensional domains

Proof. The result is classical and well known, see [1, Th. 3.15], we only show how to obtain the explicit constant in the bound given in 4. We start with a function ρ in $\mathscr{C}^\infty(\mathbb{R}^2)$ which vanishes outside the unit ball, and satisfies for all $\boldsymbol{x}$ in $\mathbb{R}^2$ that $0 \leq \rho(\boldsymbol{x}) \leq 1$, and the integral $\int_{\mathbb{R}^2} \rho(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 1$. For all $\varepsilon > 0$, we then set $\rho_\varepsilon(\boldsymbol{x}) = \frac{1}{\varepsilon^2}\rho(\frac{\boldsymbol{x}}{\varepsilon})$, and we define for all i in $\{1, \ldots, N\}$ the function

$$h_i(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \operatorname{dist}(\boldsymbol{x}, U_i) < \frac{\delta}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

We now regularize the functions h_i using a convolution,

$$\phi_i := \rho_{\delta/2} * h_i.$$

The functions ϕ_i vanish outside of $\widetilde{\Omega}_i$, are identically equal to 1 in $\overline{U}_i$, and for all $\boldsymbol{x}$ in $\mathbb{R}^2$ we have $0 \leq \phi_i \leq 1$. Moreover, since $\|\nabla \phi_i\|_{L^\infty(\mathbb{R}^2)} \leq \|\nabla \rho_{\frac{\delta}{2}}\|_{L^1(\mathbb{R}^2)} \|h_i\|_{L^\infty(\mathbb{R}^2)}$,

$$\|\nabla \phi_i\|_{L^\infty(\mathbb{R}^2)} \leq \frac{2\|\nabla \rho\|_{L^1(\mathbb{R}^2)}}{\delta}.$$

We then set

$$\psi_i = \phi_i \prod_{k=1}^{i-1} (1 - \phi_k),$$

and the $(\psi_i)_{1 \leq i \leq N}$ are then a partition of unity. Moreover

$$\nabla \psi_i = \nabla \phi_i \prod_{k=1}^{i-1} (1 - \phi_k) - \sum_{j=1}^{i-1} \phi_i \nabla \phi_j \prod_{\substack{k=1 \\ k \neq j}}^{i-1} (1 - \phi_k).$$

At a given point $\boldsymbol{x}$, at most $N_c - 1$ terms of the above sum may be non zero, therefore, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\sum_{i=1}^{N} \|\nabla \psi_i(\boldsymbol{x})\|^2 &\leq (N_c - 1) \left(\sum_{i=1}^{N} \|\nabla \phi_i(\boldsymbol{x})\|^2 \prod_{k=1}^{i-1} (1 - \phi_k)^2 + \sum_{i=1}^{N} \sum_{j=1}^{i-1} |\phi_i|^2 \|\nabla \phi_j(\boldsymbol{x})\|^2 \prod_{k=1, k \neq j}^{i-1} (1 - \phi_k)^2 \right) \\
&\leq (N_c - 1) \sum_{i=1}^{N} \|\nabla \phi_i(\boldsymbol{x})\|^2 \prod_{k=1}^{i-1} (1 - \phi_k)^2 \left(1 + \sum_{j=i+1}^{N} |\phi_j|^2 \prod_{k=i+1}^{j-1} (1 - \phi_k)^2 \right) \\
&\leq (N_c - 1) \sum_{i=1}^{N} \|\nabla \phi_i(\boldsymbol{x})\|^2 \prod_{k=1}^{i-1} (1 - \phi_k)^2 \left(2 - \prod_{k=i}^{N} (1 - \phi_k) \right) \\
&\leq 2(N_c - 1) \sum_{i=1}^{N} \|\nabla \phi_i(\boldsymbol{x})\|^2.
\end{aligned}$$

Moreover, each term is bounded by $\max_{1 \leq j \leq i} \|\nabla \phi_j\|^2_{L^\infty(\Omega)}$, and at no point $\boldsymbol{x}$ in Ω, there may be more than $N_c - 1$ nonzero terms in the sum. Hence, for all $\boldsymbol{x}$ in Ω

$$\sum_{i=1}^{N} \|\nabla \psi_i(\boldsymbol{x})\|^2 \leq \frac{8(N_c - 1)^2 \|\nabla \rho\|^2_{L^1(\mathbb{R}^2)}}{\delta^2}.$$

Setting $\lambda_2 := 2\|\nabla\rho\|^2_{L^1(\mathbb{R}^2)}$, the result follows. Note that here $\|\nabla\rho\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2}(|\partial_x\rho|^2+|\partial_y\rho|^2)^{1/2}\mathrm{d}\boldsymbol{x}$. Using the $W^{1,1}(\mathbb{R}^2)$ function $\rho(\boldsymbol{x}) = 1-\|\boldsymbol{x}\|_2$, we obtain the estimate $\lambda_2 = 6$. □

It is easy to build a stable decomposition using a partition of unity, however to get an estimate in H/δ instead of an estimate in H^2/δ^2 we need more assumptions on the regularity of the U_i, specifically we must control the curvature of the boundary of the U_i. Unfortunately, the subdomains of a non overlapping domain decomposition are at best piecewise $\mathscr{C}^\infty$: there will always be corners at cross points. For this reason, we introduce the notions of pseudo normal and pseudo curvature:

Assumption 4.4. *Let U be a bounded domain of $\mathbb{R}^2$. We suppose there exist an open layer L containing ∂U and a vector field $\boldsymbol{X}$ continuous on $L\cap\overline{U}$, $\mathscr{C}^\infty$ on $L\cap U$ such that:*

$$\mathrm{D}\boldsymbol{X}(\boldsymbol{x})(\boldsymbol{X}(\boldsymbol{x})) = 0, \qquad\qquad \|\boldsymbol{X}(\boldsymbol{x})\| = 1$$

and such that there exists $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$ and for all $\hat{\boldsymbol{x}}$ in ∂U:

$$\hat{\boldsymbol{x}}+\varepsilon\boldsymbol{X}(\hat{\boldsymbol{x}}) \in U, \qquad\qquad \hat{\boldsymbol{x}}-\varepsilon\boldsymbol{X}(\hat{\boldsymbol{x}}) \notin U.$$

The vector field $\boldsymbol{X}$ is called an interior pseudo normal.

Setting for all positive δ

$$\begin{aligned} U^\delta &= \{\boldsymbol{x}\in U \text{ s.t. } \mathrm{dist}(\boldsymbol{x},\partial U) < \delta\},\\ V^\delta &= \{\hat{\boldsymbol{x}}+sX(\hat{\boldsymbol{x}}), \hat{\boldsymbol{x}}\in\partial U, 0<s<\delta\},\end{aligned}$$

we assume there exist $\hat{R} > 0$, $\theta_{\boldsymbol{X}}$, $0 < \theta_{\boldsymbol{X}} \le \pi/2$ and δ_0, $0 < \delta_0 \le \hat{R}\sin\theta_{\boldsymbol{X}}$ such that

$$\begin{aligned} &V^{\hat{R}} \subset L\cap U,\\ &U^\delta \subset V^{\delta/\sin\theta_{\boldsymbol{X}}} \quad \text{for all positive } \delta \le \delta_0.\end{aligned}$$

The parameter $\theta_{\boldsymbol{X}}$ formally represents the smallest angle between the pseudo normal and the tangents. We finally set

$$\tilde{R} := \frac{1}{\|\mathrm{div}\boldsymbol{X}\|_{L^\infty(L)}}.$$

We call $\tilde{R}$ the $\boldsymbol{X}$-pseudo curvature of U.

When the boundary of the domain U is $\mathscr{C}^1$, $\boldsymbol{X}$ is the interior normal. Unfortunately, as the U_i form a non overlapping domain decomposition of Ω, they cannot be supposed to be $\mathscr{C}^1$. It is perfectly reasonable to assume the existence of a pseudo normal for Lipschitz domains, see [22, §1.5].

Using these assumptions, we can prove the following lemma:

Lemma 4.5. *Let U be an open domain that satisfies assumptions 4.4, then for all $\delta \leq \delta_0$, we have*

$$\|u\|_{L^2(U^\delta)}^2 \leq 2\left(1+\frac{\hat{R}}{\tilde{R}}\right)\frac{\delta\hat{R}}{\sin\theta_{\boldsymbol{X}}}\|\nabla u\|_{L^2(U)}^2 + 2\left(1+\frac{\hat{R}}{\tilde{R}}\right)\frac{\delta}{\hat{R}\sin\theta_{\boldsymbol{X}}}\|u\|_{L^2(U)}^2.$$

Proof. We have $\|u\|_{L^2(U^\delta)} \leq \|u\|_{L^2(V^{\delta/\sin\theta_{\boldsymbol{X}}})}$. For all $\boldsymbol{x}$ in $V^{\hat{R}}$, we define $d(\boldsymbol{x}) = \inf\{s, \boldsymbol{x} - s\boldsymbol{X}(\boldsymbol{x}) \notin L\cap U\}$. The function d is lower semicontinuous. Note that $d(\boldsymbol{x}+s\boldsymbol{X}(\boldsymbol{x})) = d(\boldsymbol{x}+s\boldsymbol{X})$ provided the whole segment $[\boldsymbol{x}, \boldsymbol{x}+s\boldsymbol{X}(\boldsymbol{x})]$ belongs to $L\cap U$. Also note that for all $\delta < \hat{R}$, $V^\delta = \{\boldsymbol{x}\in V^{\hat{R}} \text{ s.t. } d(\boldsymbol{x}) < \delta\}$. Define function ψ by

$$\psi(\boldsymbol{x}) = \boldsymbol{x} + (\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta} - 1)d(\boldsymbol{x})\boldsymbol{X}(\boldsymbol{x}).$$

for all $\boldsymbol{x}$ in $V^{\delta/\sin\theta_{\boldsymbol{X}}}$. We have $d(\psi(\boldsymbol{x}) = \frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}d(\boldsymbol{x})$ and

$$\int_{V^{\delta/\sin\theta_{\boldsymbol{X}}}}|u(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} \leq 2\underbrace{\int_{V^{\delta/\sin\theta_{\boldsymbol{X}}}}|u(\psi(\boldsymbol{x}))|^2\mathrm{d}\boldsymbol{x}}_{I}$$
$$+2\underbrace{\int_{V^{\delta/\sin\theta_{\boldsymbol{X}}}} d(\boldsymbol{x})(\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)\int_0^{d(\boldsymbol{x})(\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)}|\nabla u(\boldsymbol{x}+s\boldsymbol{X}(\boldsymbol{x}))|^2\mathrm{d}s\mathrm{d}\boldsymbol{x}}_{II}. \tag{10}$$

To further estimate the term I, we need to compute the Jacobian of ψ: let us first suppose that d is $\mathscr{C}^1$. In the orthonormal basis $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ where $\boldsymbol{\tau}_1 = \boldsymbol{X}(\boldsymbol{x})$ and $\boldsymbol{\tau}_2$ is orthogonal to $\boldsymbol{\tau}_1$, we have

$$J\psi(\boldsymbol{x}) = \begin{bmatrix} \frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta} & (\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)\frac{\partial d}{\partial\boldsymbol{\tau}_2} \\ 0 & 1+(\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)d(\boldsymbol{x})\,\mathrm{div}\,\boldsymbol{X}(\boldsymbol{x}) \end{bmatrix}.$$

Therefore, since $\psi(V^{\delta/\sin\theta_{\boldsymbol{X}}}) = V^{\hat{R}}$, we get

$$\det(J\psi(\boldsymbol{x})) = \frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}(1+(\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)d(\boldsymbol{x})\,\mathrm{div}\,\boldsymbol{X}(\boldsymbol{x})).$$

This does not depend on the derivatives of d. Besides, one can prove that for all s in $\mathbb{R}$ such that the segment $[\boldsymbol{x}, \boldsymbol{x}+s\boldsymbol{X}(\boldsymbol{x})]$ is included in $L\cap U$:

$$(1+s\,\mathrm{div}\,\boldsymbol{X}(\boldsymbol{x}))(1-s\,\mathrm{div}\,\boldsymbol{X}(\boldsymbol{x}+s\boldsymbol{X})) = 1.$$

Therefore, setting $\boldsymbol{y} = \psi(\boldsymbol{x})$, we get

$$\begin{aligned} I &= \int_{V^{\delta/\sin\theta_{\boldsymbol{X}}}}|u(\psi(\boldsymbol{x}))|^2\mathrm{d}\boldsymbol{x} \\ &= \frac{\delta}{\hat{R}\sin\theta_{\boldsymbol{X}}}\int_{V^{\hat{R}}}|u(\boldsymbol{y})|^2(1-(1-\frac{\delta}{\hat{R}\sin\theta_{\boldsymbol{X}}})d(\boldsymbol{y})\,\mathrm{div}\,\boldsymbol{X}(\boldsymbol{y}))\mathrm{d}\boldsymbol{y} \\ &\leq (1+\frac{\hat{R}}{\tilde{R}})\frac{\delta}{\hat{R}\sin\theta_{\boldsymbol{X}}}\int_{V^{\hat{R}}}|u(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}. \end{aligned}$$

This formula holds even when d is not $\mathscr{C}^1$: the idea is to prove by Fubini that the formula holds on open subsets of the form $V_{\boldsymbol{x}} = \{\boldsymbol{x}+r\boldsymbol{\tau}_2 + s\boldsymbol{X}(\boldsymbol{x}+r\boldsymbol{\tau}_2), 0 < r,s < \varepsilon\}$ where $\boldsymbol{\tau}_2$ is orthogonal to $\boldsymbol{X}(\boldsymbol{x})$, and then to proceed by way of a partition of unity. Therefore we have

$$|I| \leq (1+\frac{\hat{R}}{\tilde{R}})\frac{\delta}{\hat{R}\sin\theta_{\boldsymbol{X}}}\|u\|^2_{L^2(L\cap U)}. \tag{11}$$

We now deal with the term II: we compute

$$\begin{aligned}
II &= (\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)\int_{V^{\delta/\sin\theta_{\boldsymbol{X}}}} d(\boldsymbol{x})\int_0^{d(\boldsymbol{x})(\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)} |\nabla u(\boldsymbol{x}+s\boldsymbol{X}(\boldsymbol{x}))|^2 \mathrm{d}s\mathrm{d}\boldsymbol{x}\\
&= (\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)\int_0^{\hat{R}-\frac{\delta}{\sin\theta_{\boldsymbol{X}}}}\int_{V^{\hat{R}}}\chi\left\{\frac{s}{\hat{R}\sin\theta_{\boldsymbol{X}}/\delta-1} < d(\boldsymbol{x}) < \frac{\delta}{\sin\theta_{\boldsymbol{X}}}\right\} d(\boldsymbol{x})|\nabla u(\boldsymbol{x}+s\boldsymbol{X}(\boldsymbol{x}))|^2\mathrm{d}\boldsymbol{x}\mathrm{d}s,
\end{aligned}$$

and then using the change of variables $\boldsymbol{y} = \boldsymbol{x}+s\boldsymbol{X}(\boldsymbol{x})$ we obtain

$$\begin{aligned}
II &= (\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)\int_0^{\hat{R}-\frac{\delta}{\sin\theta_{\boldsymbol{X}}}}\int_{V^{\hat{R}}}\chi\left\{\frac{s\hat{R}\sin\theta_{\boldsymbol{X}}}{\hat{R}\sin\theta_{\boldsymbol{X}}-\delta} < d(\boldsymbol{y}) < \frac{\delta}{\sin\theta_{\boldsymbol{X}}}+s\right\}\\
&\qquad\qquad (d(\boldsymbol{y})-s)|\nabla u(\boldsymbol{y})|^2(1-s\operatorname{div}\boldsymbol{X}(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\mathrm{d}s\\
&= (\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)\int_{V^{\hat{R}}}\int_{(d(\boldsymbol{y})-\delta/\sin\theta_{\boldsymbol{X}})_+}^{\frac{d(\boldsymbol{y})}{\hat{R}}(\hat{R}-\frac{\delta}{\sin\theta_{\boldsymbol{X}}})} (d(\boldsymbol{y})-s)|\nabla u(\boldsymbol{y})|^2(1-s\operatorname{div}\boldsymbol{X}(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\mathrm{d}s\\
&\leq (\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)\int_{V^{\hat{R}}}|\nabla u(\boldsymbol{y})|^2\int_{(d(\boldsymbol{y})-\delta/\sin\theta_{\boldsymbol{X}})_+}^{\frac{d(\boldsymbol{y})}{\hat{R}}(\hat{R}-\delta/\sin\theta_{\boldsymbol{X}})}(d(\boldsymbol{y})-s)(1-s\operatorname{div}\boldsymbol{X}(\boldsymbol{y}))\mathrm{d}s\mathrm{d}\boldsymbol{y}\\
&\leq (\frac{\hat{R}\sin\theta_{\boldsymbol{X}}}{\delta}-1)(1+\frac{\hat{R}-\delta/\sin\theta_{\boldsymbol{X}}}{\tilde{R}})\frac{\delta^2}{\sin^2\theta_{\boldsymbol{X}}}\int_{V^{\hat{R}}}\left(1-\frac{d(\boldsymbol{y})}{\hat{R}}\right)|\nabla u(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}\\
&\leq (1+\frac{\hat{R}}{\tilde{R}})(\hat{R}-\frac{\delta}{\sin\theta_{\boldsymbol{X}}})\frac{\delta}{\sin\theta_{\boldsymbol{X}}}\int_{V^{\hat{R}}}|\nabla u(\boldsymbol{y})|^2\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

We thus obtain the estimate

$$|II| \leq \left(1+\frac{\hat{R}}{\tilde{R}}\right)\hat{R}\frac{\delta}{\sin\theta_{\boldsymbol{X}}}\|\nabla u\|^2_{L^2(V^{\hat{R}})}. \tag{12}$$

Combining inequalities (11) and (12) with inequality (10) concludes the proof. $\square$

Theorem 4.6 (Stable Decomposition without Coarse Grid)**.** *Let Ω be a bounded domain of $\mathbb{R}^2$, and $(U_i)_{1\leq i\leq N}$ be a non overlapping domain decomposition of Ω, satisfying Assumption 4.4 with uniform $\hat{R}$, $\tilde{R}$, δ_0 and $1/\sin\theta_{\boldsymbol{X}}$. Let $\delta < \delta_0$ be positive and $(\Omega_i)_{1\leq i\leq N}$ be an overlapping domain decomposition defined from the U_i and δ as in Assumption 4.1.*

Then, if u is in $H_0^1(\Omega)$, there exist $(u_i)_{1\le i\le N}$ such that for all i, $1\le i\le N$, u_i is in $H_0^1(\Omega_i)$ and

$$u=\sum_{i=1}^N u_i, \quad \textit{with} \tag{13}$$

$$\sum_{i=1}^N \|u_i\|^2_{L^2(\Omega_i)} \le \|u\|^2_{L^2(\Omega)}, \tag{14}$$

$$\sum_{i=1}^N \|\nabla u_i\|^2_{L^2(\Omega_i)} \le 2\|\nabla u\|^2_{L^2(\Omega)} + \frac{4\lambda_2^2(N_c-1)^2}{\delta^2}\|u\|^2_{L^2(\Omega^\delta)}, \tag{15}$$

where λ_2 is the universal constant of Lemma 4.3 and where $\Omega^\delta=\bigcup_{i\neq j}\Omega_i\cap\Omega_j$. We further have:

$$\begin{aligned}\sum_{i=1}^N \|\nabla u_i\|^2_{L^2(\Omega_i)} \le \left(2+8\lambda_2^2(N_c-1)^2\left(1+\frac{\hat{R}}{\tilde{R}}\right)\frac{\hat{R}}{\delta\sin\theta_{\boldsymbol{X}}}\right)\|\nabla u\|^2_{L^2(\Omega)} \\ +8\lambda_2^2(N_c-1)^2\left(1+\frac{\hat{R}}{\tilde{R}}\right)\frac{1}{\hat{R}\delta\sin\theta_{\boldsymbol{X}}}\|u\|^2_{L^2(\Omega)}.\end{aligned} \tag{16}$$

Proof. We use Lemma 4.3 and set $u_i:=\psi_i u$, which satisfies already (13). We then estimate

$$\sum_{i=1}^N\int_\Omega |u_i(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} = \int_\Omega\sum_{i=1}^N|\psi_i(\boldsymbol{x})u(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x} = \int_\Omega |u(\boldsymbol{x})|^2\sum_{i=1}^N(\psi_i(\boldsymbol{x}))^2\mathrm{d}\boldsymbol{x}\le\int_\Omega|u(\boldsymbol{x})|^2,$$

since $\sum_{i=1}^N(\psi_i(\boldsymbol{x}))^2\le 1$, which shows (14). We finally need to estimate the derivative term. We have $\nabla u_i=\psi_i\nabla u+u\nabla\psi_i$, and therefore

$$\sum_{i=1}^N\int_\Omega|\nabla u_i(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\le 2\int_\Omega|\nabla u(\boldsymbol{x})|^2\sum_{i=1}^N|\psi_i|^2\mathrm{d}\boldsymbol{x}+2\int_\Omega|u(\boldsymbol{x})|^2\sum_{i=1}^N|\nabla\psi_i|^2\mathrm{d}\boldsymbol{x}. \tag{17}$$

The first term on the right in (17) can be bounded as above. To bound the second term, we use result 4 in Lemma 4.3:

$$\int_\Omega|u(\boldsymbol{x})|^2\sum_{i=1}^N|\nabla\psi_i(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\le\|\sum_{i=1}^N|\nabla\psi_i|^2\|_{L^\infty(\mathbb{R}^2)}\int_\Omega|u(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}\le\frac{2\lambda_2^2(N_c-1)^2}{\delta^2}\|u\|^2_{L^2(\Omega^\delta)}. \tag{18}$$

Combining these estimates leads to (15). To get (16), one first notice that

$$\|u\|^2_{L^2(\Omega^\delta)}\le\sum_{i=1}^N\|u\|^2_{L^2(U_i^\delta)}$$

then apply Lemma 4.5 on each U_i^δ. □

The lone $1/\delta^2$ factor in estimate (15) can further be treated using the Poincaré inequality on Ω, see [1, Th. 6.30], which then explicitly reveals the dependence on the number of subdomains:

Corrollary 4.7. *Let Ω be a bounded domain. Let U_i, Ω_i and $(u_i)_{1\le i\le N}$ be as in Theorem 4.6. Then we have*

$$\sum_{i=1}^{N}\|\nabla u_i\|^2_{L^2(\Omega_i)} \le \Bigg(2+8\lambda_2^2(N_c-1)^2\Big(1+\frac{\hat{R}}{\tilde{R}}\Big)\frac{\hat{R}}{\delta\sin\theta_{\boldsymbol{X}}} \\ +\lambda_2^2 C_a N(N_c-1)^2\Big(1+\frac{\hat{R}}{\tilde{R}}\Big)\frac{(\mathrm{diam}(\Omega))^2}{|\Omega|}\frac{H}{\hat{R}\sin\theta_{\boldsymbol{X}}}\frac{H}{\delta}\Bigg)\|\nabla u\|^2_{L^2(\Omega)}, \tag{19}$$

where C_a is the constant of Assumption 4.2.

Proof. We start with (15), and use Poincaré's inequality on $H_0^1(\Omega)$, *i.e.*

$$\|u\|^2_{L^2(\Omega)} \le C\|\nabla u\|^2_{L^2(\Omega)},$$

but we need an estimate of the constant C. Since Ω is, up to a rotation, a subset of $(0,\mathrm{diam}(\Omega))\times\mathbb{R}$, the constant C is bounded by the Poincaré constant for $(0,\mathrm{diam}(\Omega))\times\mathbb{R}$, which is smaller than $1/8(\mathrm{diam}(\Omega))^2$. We therefore obtain

$$\sum_{i=1}^{N}\|\nabla u_i\|^2_{L^2(\Omega_i)} \le \Bigg(2+8\lambda_2^2(N_c-1)^2\Big(1+\frac{\hat{R}}{\tilde{R}}\Big)\frac{\hat{R}}{\delta\sin\theta_{\boldsymbol{X}}} \\ +\lambda_2^2(N_c-1)^2\Big(1+\frac{\hat{R}}{\tilde{R}}\Big)\frac{(\mathrm{diam}(\Omega))^2}{\hat{R}\delta\sin\theta_{\boldsymbol{X}}}\Bigg)\|\nabla u\|^2_{L^2(\Omega)}.$$

But we also have

$$(\mathrm{diam}(\Omega))^2 = \frac{(\mathrm{diam}(\Omega))^2}{|\Omega|}\frac{|\Omega|}{H^2}H^2 \le C_a N\frac{(\mathrm{diam}(\Omega))^2}{|\Omega|}H^2$$

because $\frac{|\Omega|}{H^2} \le C_a N$ by Assumption 4.2, which concludes the proof. □

The dependence on the number of subdomains N in estimate (19) is undesirable for domain decomposition methods, since these methods should be scalable, which means their convergence behavior should not deteriorate as one uses more and more subdomains (which corresponds to more and more processors). In the next section, we show how to establish a better estimate with the use of a coarse mesh.

5 Stable decomposition with a coarse solver

We now introduce a discrete structure into our continuous analysis, namely a coarse mesh over the entire domain, in order to remove the dependence on the number of subdomains in estimate (19), see Figure 1. We present the general idea of the continuous proof in the presence of a discrete, coarse mesh first in subsection 5.1. We then show the details of the proof in the next three subsections. In subsection 5.2, we construct the coarse component of the stable decomposition. In subsection 5.3, we construct the non coarse components. Finally, we conclude by stating our main theorem in subsection 5.4.

5.1 General idea

The main idea is to use the following classical lemma [34, chap. II §1.4 pp. 51]:

Lemma 5.1 (Generalized Poincaré's inequality)**.** *Let O be a bounded open set satisfying the cone condition*[5]*. Let ℓ be a continuous linear form on $H^1(O)$ such that* $\mathrm{Ker}(\ell) \cap \mathbb{R} = \{0\}$ *Then, there exists a constant $C > 0$ such that*

$$\|u\|^2_{L^2(O)} \leq C(\|\nabla u\|^2_{L^2(O)} + |\ell(u)|^2)$$

for all u in $H^1(O)$.

For our purposes, we need estimates for the constants. Unfortunately the proof of the classical lemma is by contradiction, is not constructive, and does not allow us to estimate the constant C when the domain O varies. However for convex and star shaped domains, the constants can be estimated, as we will show later in Lemma 5.10.

We return to the stable decomposition problem with a coarse mesh. How can we use the coarse mesh to prevent the constant to depend on the number of subdomains? The basic idea is to define N linear forms ℓ_i on $H^1(\Omega_i)$ such that for all u in $H^1_0(\Omega)$ there exists $(u_i)_{1 \leq i \leq N}$, such that for all i, $1 \leq i \leq N$, u_i is in $H^1_0(\Omega_i)$ and

$$u = \sum_{i=1}^{N} u_i \tag{20}$$

$$\sum_{i=1}^{N} \|\nabla u_i\|^2_{L^2(\Omega_i)} \leq C(\frac{\delta}{H}, N_c)\|\nabla u\|^2_{L^2(\Omega)} + C(\frac{\delta}{H}, N)\sum_{i=1}^{N} |\ell_i(u)|^2, \tag{21}$$

where by extension $\ell_i(u)$ means $\ell_i(u_{|\Omega_i})$, effectively replacing the L^2 square norm in (15) with $\sum_{i=1}^{N} |\ell_i(u)|^2$. We propose here to take $\ell_i(u) := \frac{1}{|A_i|}\int_{A_i} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ with $A_i \subset \Omega_i$. We then search for u_0 in the space of continuous, piecewise linear functions $P_1(\mathscr{T})$, where $\mathscr{T}$ is a coarse triangular grid, such that $\ell_i(u_0) = \ell_i(u)$ for all i, $1 \leq i \leq N$ and $\|\nabla u_0\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$. Then, we apply (21) to $u - u_0$. The second term vanishes and the constant of the stable decomposition does not depend on the number of subdomains in the decomposition any longer. This idea implies that the coarse mesh should be able to control at least one constant in each subdomain, *i.e.*, for the coarse mesh to prevent the dependence of the condition number on

[5] See [1, §4.6].

the number of subdomains, it only needs to be able to subtract one constant per subdomain! Intuitively, this means that the coarse mesh must have at least one node in each subdomain.

5.2 Projection of H_0^1 into $\mathscr{P}_1(\mathscr{T})$

In this subsection, we will consider a family of triangular meshes $\mathscr{T}$ of domain Ω with the following uniform properties:

Assumption 5.2 (Geometric Properties of the Coarse Grid)**.** *1. All angles θ for all cells in the mesh $\mathscr{T}$ are bounded by $0 < \theta_{min} \leq \theta \leq \theta_{max} < \pi$ where θ_{min} and θ_{max} do not depend on H.*

2. *The length of any edge in mesh $\mathscr{T}$ lies between $c_p H$ and $C_p H$ where $c_p > 0$ and $C_p > 0$ depend neither on the cell nor on H.*

3. *No node has more than K neighbors.*

In order to simplify our analysis, we make the following assumption:

Assumption 5.3. *We assume that the coarse mesh $\mathscr{T}$ has precisely one node per subdomain, $\boldsymbol{x}_i \in \Omega_i$.*

Even though it should be possible to derive mesh independent estimates for the norm of the coarse component without Assumption 5.3, this could be rather cumbersome, since it leads to a rectangular instead of a square matrix, see the analysis below. In addition, in practical situations, one node for the coarse mesh per subdomain is a common choice.

Given a mesh $\mathscr{T}$, and given $r > 0$, we introduce the linear forms

$$\begin{aligned} \ell_i : H_0^1(\Omega) &\to \mathbb{R}, \\ u &\mapsto \frac{1}{\pi r^2}\int_{B(\boldsymbol{x}_i,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{22}$$

where i belongs to $\{1,\ldots,N\}$ and where $\boldsymbol{x}_i$ is the position of the i-th node in mesh $\mathscr{T}$. We also define

$$\begin{aligned} \ell : H_0^1(\Omega) &\to \mathbb{R}^N, \\ u &\mapsto (\ell_i(u))_{1\leq i\leq N}. \end{aligned}$$

Theorem 5.4. *Let Ω be a bounded domain of $\mathbb{R}^2$. Let $\mathscr{T}$ be a coarse mesh on Ω satisfying Assumption 5.2, with H_h the shortest height of all triangles in $\mathscr{T}$, K the maximum number of neighbors of any node in $\mathscr{T}$, and let r be smaller than $\frac{H_h}{4K+1}$. Then, for all u in $H_0^1(\Omega)$, there exists u_H in $P_1(\mathscr{T})\cap H_0^1(\Omega)$ such that*

$$\ell_i(u_H) = \ell_i(u) \qquad \textit{for all } i \textit{ in } \{1,\ldots,N\},$$

$$\|\nabla u_H\|^2_{L^2(\Omega)} \leq \frac{1}{\tan\theta_{min}}\frac{1+2r/H_h}{1-\big((2K+1)+(4K+1)r/H_h\big)r/H_h}2K(\frac{2C_pH}{\pi r}+\pi)\|\nabla u\|^2_{L^2(\Omega)}.$$

Note here that $r \leq \frac{H_h}{4K+1}$ ensures that $1-\big((2K+1)+(4K+1)r/H_h\big)r/H_h$ is positive. The remainder of this subsection is dedicated to the proof of this theorem.

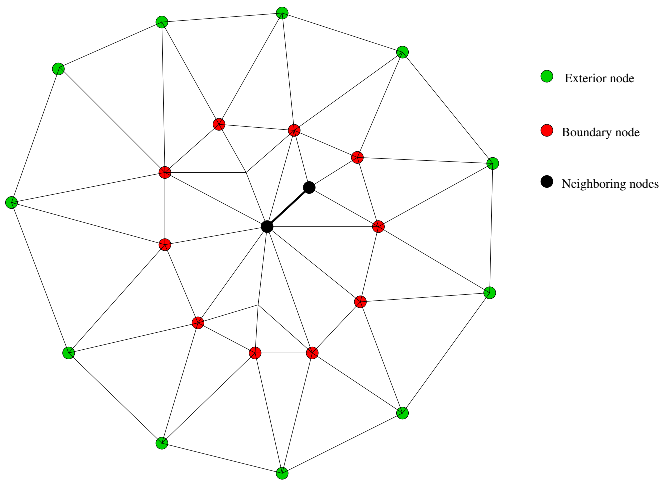

Figure 2: Boundary and exterior nodes in mesh $\mathscr{T}$

5.2.1 An equivalent norm

Our goal is to construct a convenient equivalent norm to the $H_0^1(\Omega)$ norm for functions in $P_1(\mathscr{T})$. Let $\mathscr{T}$ be a mesh of Ω having N nodes. As a convention, nodes of mesh $\mathscr{T}$ located exactly on $\partial\Omega$ will be called exterior nodes and are not counted among the numbered nodes. This choice is motivated by the homogeneous Dirichlet condition. We denote by $\mathscr{V}$ the set of all (i,j) in $\{1,\dots,N\}^2$ that are indices of neighboring nodes. We also denote by $\mathscr{B}$ the set of all nodes i in $\{1,\dots,N\}$ who are neighbor to an exterior node, see Figure 2.

Definition 5.5. *Let $\mathscr{T}$ be a mesh of domain Ω. Let $\mathscr{V}$ and $\mathscr{B}$ be the neighbor and the boundary set of mesh $\mathscr{T}$. We define*

$$\begin{aligned}\|\cdot\|_{\mathscr{V},\mathscr{B}} : \mathbb{R}^N &\to \mathbb{R}^+,\\ \boldsymbol{y} &\mapsto \sqrt{\sum_{(i,j)\in\mathscr{V}} |y_i - y_j|^2 + \sum_{i\in\mathscr{B}} |y_i|^2}.\end{aligned}$$

When u is in $P_1(\mathscr{T})\cap H_0^1(\Omega)$, we define

$$\|u\|_{\mathscr{V},\mathscr{B}} := \|(u(\boldsymbol{x}_i))_{1\le i\le N}\|_{\mathscr{V},\mathscr{B}},$$

where the $\boldsymbol{x}_i$ are the interior nodes of mesh $\mathscr{T}$.

Lemma 5.6. *Let u_H belong to $P_1(\mathscr{T})\cap H_0^1(\Omega)$, then the norms $u_H \mapsto \|\nabla u_H\|_{L^2(\Omega)}$ and $\|\cdot\|_{\mathscr{V},\mathscr{B}}$ are equivalent. Moreover, the equivalence constants depend only on the constants of Assumption 5.2,*

$$\frac{2}{3}\frac{\min_{ABC\in\mathscr{T}}|ABC|}{C_p^2 H^2}\|u_H\|^2_{\mathscr{V},\mathscr{B}} \le \|\nabla u_H\|^2_{L^2(\Omega)} \le \frac{1}{\tan\theta_{min}}\|u_H\|^2_{\mathscr{V},\mathscr{B}}. \qquad (23)$$

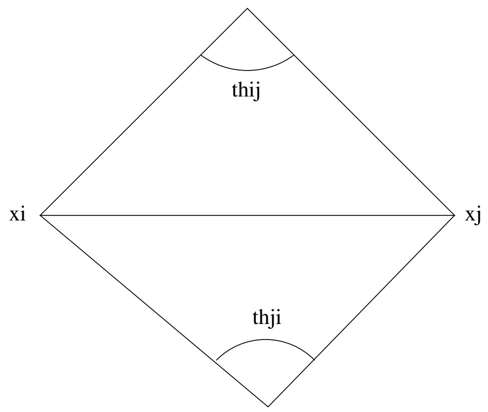

Figure 3: Angles and gradient norm in $P_1(\mathscr{T})$

Proof. It is easy to compute the norm, see Appendix A for details. For all u_H in $P_1(\mathscr{T})\cap H_0^1(\Omega)$, we then have

$$\|\nabla u_H\|^2_{L^2(\Omega)} = \frac{1}{2}\sum_{(i,j)\in\mathscr{V}}\left(\frac{1}{\tan(\theta_{ij})}+\frac{1}{\tan(\theta_{ji})}\right)|u_i-u_j|^2 \\ +\frac{1}{2}\sum_{i\in\mathscr{B}}\sum_{i'\in\mathscr{V}_i'}\left(\frac{1}{\tan(\theta_{ii'})}+\frac{1}{\tan(\theta_{i'i})}\right)|u_i|^2,$$

where θ_{ij} and θ_{ji} are the angles opposite to edge $[\boldsymbol{x}_i\boldsymbol{x}_j]$, see Figure 3, and where $\mathscr{V}_i'$ is the set of all exterior nodes located on the boundary of Ω that are neighbors of node i. The problem is that the $\tan(\theta_{ij})$ can be negative when $\theta_{ij}>\frac{\pi}{2}$. This is not a problem for the right-hand side of inequality (23), but to establish the left-hand side of inequality (23), when there are obtuse angles in the mesh, we need to estimate

$$\begin{aligned}
\|u_H\|^2_{\mathscr{V},\mathscr{B}} &= \frac{1}{2}\sum_{ABC\in\mathscr{T}}\left(|u_H(A)-u_H(B)|^2+|u_H(B)-u_H(C)|^2+|u_H(C)-u_H(A)|^2\right)\\
&= \frac{1}{2}\sum_{ABC\in\mathscr{T}}\left(|\nabla u_H(ABC)\cdot(\boldsymbol{x}_A-\boldsymbol{x}_B)|^2+|\nabla u_H(ABC)\cdot(\boldsymbol{x}_B-\boldsymbol{x}_C)|^2+|\nabla u_H(ABC)\cdot(\boldsymbol{x}_C-\boldsymbol{x}_A)|^2\right)\\
&\le \frac{1}{2}\sum_{ABC\in\mathscr{T}}\|\nabla u_H(ABC)\|^2_{\mathbb{R}^2}\left(\|\boldsymbol{x}_A-\boldsymbol{x}_B\|^2+\|\boldsymbol{x}_B-\boldsymbol{x}_C\|^2+\|\boldsymbol{x}_C-\boldsymbol{x}_A\|^2\right)\\
&\le \frac{3}{2}C_p^2H^2\sum_{ABC\in\mathscr{T}}\|\nabla u_H(ABC)\|^2_{\mathbb{R}^2} \le \frac{3}{2}C_p^2H^2\sum_{ABC\in\mathscr{T}}\frac{\|\nabla u_H\|^2_{L^2(ABC)}}{|ABC|}\\
&\le \frac{3}{2}\frac{C_p^2H^2}{\min_{ABC\in\mathscr{T}}|ABC|}\|\nabla u_H\|^2_{L^2(\Omega)},
\end{aligned}$$

where the sum is taken over all triangles ABC in mesh $\mathscr{T}$. □

5.2.2 Boundedness

Our goal now is to estimate $\|\ell(u)\|_{\mathscr{V},\mathscr{B}}$ as function of $\|\nabla u\|_{L^2(\Omega)}$ when u is in $H_0^1(\Omega)$.

Lemma 5.7. *Let $\mathscr{T}$ be a coarse mesh on Ω, and let $r > 0$ be such that $2r$ is smaller than the smallest height of any triangle in $\mathscr{T}$. Then, for all u in $H_0^1(\Omega)$, we have*

$$\sum_{(i,j)\in\mathscr{V}} \left|\ell_i(u)-\ell_j(u)\right|^2 + \sum_{i\in\mathscr{B}} |\ell_i(u)|^2 \leq 2(\frac{2C_pH}{\pi r}+\pi)K\|\nabla u\|^2_{L^2(\Omega)}. \tag{24}$$

Proof. By density, we only need to prove the result for u in $\mathscr{C}_c^\infty$. Dealing with the second term of (24) is possible but cumbersome. It would be much easier to estimate this term if the sum was over the exterior nodes that are physically on the boundary of Ω. Let $\mathscr{B}'$ be the set of the indices of the exterior nodes of $\mathscr{T}$ located on the boundary: their indices are outside of $\{1,\dots,N\}$. Let $\mathscr{V}'$ be the set of all pairs of indices of neighboring nodes including exterior nodes (these nodes were excluded in $\mathscr{V}$). Note that i belongs to $\mathscr{B}$ if and only if there exists at least one index j in $\mathscr{B}'$ such that (i,j) belongs to $\mathscr{V}'$. We have

$$\begin{aligned}\sum_{i\in\mathscr{B}}|\ell_i(u)|^2 &\leq \sum_{i\in\mathscr{B}}\sum_{\substack{j\in\mathscr{B}'\\(i,j)\in\mathscr{V}'}} \left|\ell_i(u)-\ell_j(u)+\ell_j(u)\right|^2\\ &\leq 2\sum_{\substack{(i,j)\in\mathscr{V}'\\ j\in\mathscr{B}'}}\left|\ell_i(u)-\ell_j(u)\right|^2 + 2K\sum_{j\in\mathscr{B}'}\left|\ell_j(u)\right|^2,\end{aligned}$$

where the first sum has been dropped, since the indices i can only vary in $\mathscr{B}$ due to the constraints on the second sum. We thus obtain

$$\sum_{(i,j)\in\mathscr{V}}\left|\ell_i(u)-\ell_j(u)\right|^2+\sum_{i\in\mathscr{B}}|\ell_i(u)|^2 \leq 2\sum_{(i,j)\in\mathscr{V}'}\left|\ell_i(u)-\ell_j(u)\right|^2+2K\sum_{i\in\mathscr{B}'}|\ell_i(u)|^2. \tag{25}$$

We start by estimating the first term. Let (i,j) be in $\mathscr{V}'$, *i.e.* be neighbor nodes. We have

$$\begin{aligned}\frac{1}{\pi^2r^4}\left|\int_{B(\boldsymbol{x}_i,r)}u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}-\int_{B(\boldsymbol{x}_j,r)}u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2 &= \frac{1}{\pi^2r^4}\left|\int_{B(0,r)}u(\boldsymbol{x}+\boldsymbol{x}_i)-u(\boldsymbol{x}+\boldsymbol{x}_j)\mathrm{d}\boldsymbol{x}\right|^2\\ &= \frac{1}{\pi^2r^4}\left|\int_{B(0,r)}\int_0^1\nabla u(\boldsymbol{x}+(1-t)\boldsymbol{x}_j+t\boldsymbol{x}_i)\cdot(\boldsymbol{x}_i-\boldsymbol{x}_j)\mathrm{d}t\mathrm{d}\boldsymbol{x}\right|^2\\ &\leq \frac{1}{\pi r^2}\int_{B(0,r)}\int_0^1\left\|\nabla u(\boldsymbol{x}+(1-t)\boldsymbol{x}_j+t\boldsymbol{x}_i)\right\|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^2\\ &\leq \frac{d^2}{\pi r^2}\int_{B(0,r)}\int_0^1\left\|\nabla u(\boldsymbol{x}+(1-t)\boldsymbol{x}_j+t\boldsymbol{x}_i)\right\|^2\mathrm{d}t\mathrm{d}\boldsymbol{x}.\end{aligned}$$

where $d := \|\boldsymbol{x}_i-\boldsymbol{x}_j\| \leq C_pH$. We define $\boldsymbol{v} := \frac{\boldsymbol{x}_i-\boldsymbol{x}_j}{d}$, and let $\boldsymbol{w}$ be a unit vector orthogonal to $\boldsymbol{v}$. Then using the equality $\boldsymbol{x}_i-\boldsymbol{x}_j = d\boldsymbol{v}$ and the change of variables

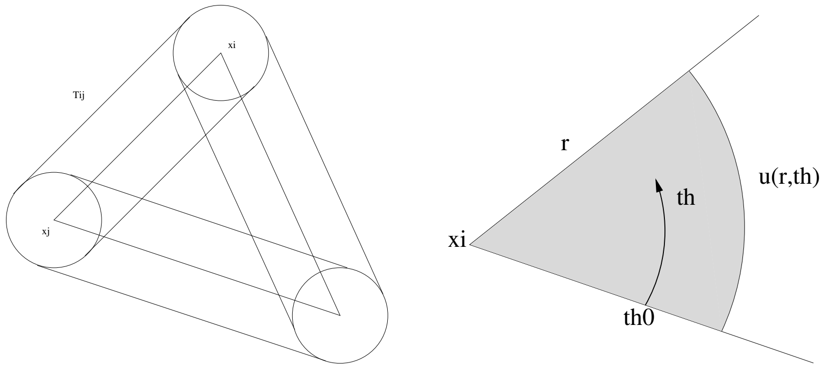

Figure 4: Tubes and their overlaps on the left, and estimate of the mean on a ball centered on an exterior edge on the right

$\boldsymbol{x} = s\boldsymbol{v} + \sigma\boldsymbol{w}$, we get

$$\begin{aligned}
&\int_{B(0,r)} \int_0^1 \left\|\nabla u(\boldsymbol{x} + (1-t)\boldsymbol{x}_j + t\boldsymbol{x}_i)\right\|^2 \mathrm{d}t\mathrm{d}\boldsymbol{x} \\
&= \int_{-r}^{r} \int_{-\sqrt{r^2-\sigma^2}}^{+\sqrt{r^2-\sigma^2}} \int_0^1 \left\|\nabla u(\boldsymbol{x}_j + \sigma\boldsymbol{w} + (s+td)\boldsymbol{v})\right\|^2 \mathrm{d}t\mathrm{d}s\mathrm{d}\sigma \\
&= \int_{-r}^{r} \int_0^1 \int_{-\sqrt{r^2-\sigma^2}}^{+\sqrt{r^2-\sigma^2}} \left\|\nabla u(\boldsymbol{x}_j + \sigma\boldsymbol{w} + (s+td)\boldsymbol{v})\right\|^2 \mathrm{d}s\mathrm{d}t\mathrm{d}\sigma \\
&= \int_{-r}^{r} \int_0^1 \int_{-\sqrt{r^2-\sigma^2}+td}^{+\sqrt{r^2-\sigma^2}+td} \left\|\nabla u(\boldsymbol{x}_j + \sigma\boldsymbol{w} + \tilde{s}\boldsymbol{v})\right\|^2 \mathrm{d}\tilde{s}\mathrm{d}t\mathrm{d}\sigma \\
&= \int_{-r}^{r} \int_{-\sqrt{r^2-\sigma^2}}^{+\sqrt{r^2-\sigma^2}+d} \left\|\nabla u(\boldsymbol{x}_j + \sigma\boldsymbol{w} + \tilde{s}\boldsymbol{v})\right\|^2 \left(\int_0^1 \chi_{\left[\frac{\tilde{s}-\sqrt{r^2-\sigma^2}}{d}, \frac{\tilde{s}+\sqrt{r^2-\sigma^2}}{d}\right]}(t)\mathrm{d}t\right) \mathrm{d}\tilde{s}\mathrm{d}\sigma \\
&\leq \frac{2r}{d} \int_{-r}^{r} \int_{-\sqrt{r^2-\sigma^2}}^{+\sqrt{r^2-\sigma^2}+d} \left\|\nabla u(\boldsymbol{x}_j + \sigma\boldsymbol{w} + \tilde{s}\boldsymbol{v})\right\|^2 \mathrm{d}\tilde{s}\mathrm{d}\sigma,
\end{aligned}$$

which leads to the estimate

$$\frac{1}{\pi^2 r^4} \left|\int_{B(\boldsymbol{x}_i,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \int_{B(\boldsymbol{x}_j,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2 \leq \frac{2C_pH}{\pi r} \int_{T_{i,j}} \|\nabla u(\boldsymbol{x})\|^2 \mathrm{d}\boldsymbol{x},$$

where $T_{i,j}$ is the set of all points $\boldsymbol{x}$ whose distance to the segment $[\boldsymbol{x}_i, \boldsymbol{x}_j]$ is smaller than r. Since $2r$ is smaller than the height of any triangle in the mesh, no point $\boldsymbol{x}$ may belong to more than K tubes $T_{i,j}$, see Figure 4 on the left. Therefore, we have

$$\sum_{(i,j)\in\mathscr{V}'} \left|\frac{1}{\pi r^2} \int_{B(\boldsymbol{x}_i,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \frac{1}{\pi r^2} \int_{B(\boldsymbol{x}_j,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2 \leq K\frac{2C_pH}{\pi r} \int_{\Omega} \|\nabla u(\boldsymbol{x})\|^2 \mathrm{d}\boldsymbol{x}. \tag{26}$$

We now estimate the second term of the right-hand side of (25). Let i be in $\mathscr{B}'$, *i.e.* i is the index of a node located exactly on the boundary of domain Ω, then u

vanishes on at least two radii. Let θ_1 be the angle between the horizontal and one of the radii on which u is zero, see Figure 4 on the right. With $\boldsymbol{e}_\rho(\theta) := (\cos\theta, \sin\theta)$ and $\boldsymbol{e}_\theta(\theta) := (-\sin\theta, \cos\theta)$, we obtain

$$
\begin{aligned}
&\frac{1}{\pi^2 r^4}\left|\int_{B(\boldsymbol{x}_i,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2 \le \frac{1}{\pi r^2}\int_{B(\boldsymbol{x}_i,r)} |u(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} = \frac{1}{\pi r^2}\int_0^r \int_{\theta_0-\pi}^{\theta_0+\pi} \left|u(\boldsymbol{x}_i + \rho\boldsymbol{e}_\rho(\theta))\right|^2 \mathrm{d}\theta\rho\mathrm{d}\rho \\
&= \frac{1}{\pi r^2}\int_0^r \rho \int_{\theta_0-\pi}^{\theta_0+\pi} \left|\int_{\theta_0}^{\theta} \nabla u(\boldsymbol{x}_i + \rho\boldsymbol{e}_\rho(t))\cdot(\rho\boldsymbol{e}_\theta(t))\mathrm{d}t\right|^2 \mathrm{d}\theta\rho\mathrm{d}\rho \\
&\le \frac{1}{\pi r^2}\int_0^r \rho^2 \int_{\theta_0-\pi}^{\theta_0+\pi} |\theta-\theta_0| \int_{\min(\theta_0,\theta)}^{\max(\theta_0,\theta)} \left\|\nabla u(\boldsymbol{x}_i + \rho\boldsymbol{e}_\rho(t))\right\|^2 \mathrm{d}t\mathrm{d}\theta\rho\mathrm{d}\rho \\
&\le \int_0^r \int_{\theta_0-\pi}^{\theta_0+\pi} \int_{\min(\theta_0,\theta)}^{\max(\theta_0,\theta)} \left\|\nabla u(\boldsymbol{x}_i + \rho\boldsymbol{e}_\rho(t))\right\|^2 \mathrm{d}t\mathrm{d}\theta\rho\mathrm{d}\rho \\
&= \int_0^r \int_{\theta_0-\pi}^{\theta_0+\pi} \left\|\nabla u(\boldsymbol{x}_i + \rho\boldsymbol{e}_\rho(t))\right\|^2 \left(\int_{\theta_0-\pi}^{\theta_0+\pi} \chi_{(\min(\theta_0,\theta),\max(\theta_0,\theta))}(t)\mathrm{d}\theta\right) \mathrm{d}t\rho\mathrm{d}\rho \\
&\le \pi \int_0^r \int_{\theta_0-\pi}^{\theta_0+\pi} \left\|\nabla u(\boldsymbol{x}_i + \rho\boldsymbol{e}_\rho(t))\right\|^2 \mathrm{d}t\rho\mathrm{d}\rho = \pi\int_{B(\boldsymbol{x}_i,r)} \|\nabla u(\boldsymbol{x})\|^2\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

No point[6] $\boldsymbol{x}$ in Ω can be in more than one ball $B(\boldsymbol{x}_i, r)$, therefore summing this inequality over i in $\mathscr{B}'$, we get

$$
\sum_{i\in\mathscr{B}'} |\ell_i(u)|^2 \le \pi\int_\Omega \|\nabla u(\boldsymbol{x})\|^2 \mathrm{d}\boldsymbol{x}. \tag{27}
$$

Combining (25) with Inequalities (26) and (27), we finally obtain

$$
\|u\|^2_{\mathscr{V},\mathscr{B}} \le 2\left(\frac{2C_p H}{\pi r} + \pi\right) K \int_\Omega \|\nabla u(\boldsymbol{x})\|^2 \mathrm{d}\boldsymbol{x}. \qquad \square
$$

5.2.3 Continuity of the linear form ℓ^{-1}

Let $\varepsilon \in \mathbb{R}$, with $0 < \varepsilon < \frac{1}{2}$, and choose $r := \varepsilon H_h$, where H_h is the smallest triangle height among all the triangles in the coarse mesh $\mathscr{T}$. Let $L := [l_{ij}]$ be the matrix associated with the linear function ℓ, *i.e.* the matrix such that $L \cdot (u_H(\boldsymbol{x}_i))_{1\le i\le N} = \ell(u_H)$ for all u_H in $P_1(\Omega)$. This is a square matrix, by Assumption 5.3, of size $N \times N$, and satisfies the following properties:

- For all i, j in $\{1, \ldots, N\}$, we have $l_{ij} \ge 0$.
- For all i, j not belonging to $\mathscr{V}$, $l_{i,j} = 0$, which implies that for any given i, there are at most K integers j such that $l_{ij} \ne 0$.

[6] One can construct pathological meshes in non pathological cases where two exterior nodes A and B that are **not** neighbors are closer than H_h. However, in that case, one can easily avoid that problem by redefining $\ell_A(u)$ whenever A is in $\mathscr{B}'$ to be $\frac{1}{\pi r^2}\int_{V_A\cap B(\boldsymbol{x}_A,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ where V_A is the union of all triangles in mesh $\mathscr{T}$ that have node A as a vertex.

- For all i in $\{1,\dots,N\}$, we have $l_{ii} \geq 1-\varepsilon$.
- For all i in $\{1,\dots,N\}$, we have $\sum_{j=1}^N l_{ij} = 1$ if $i \notin \mathscr{B}$, and $\sum_{j=1}^N l_{ij} \leq 1$ if $i \in \mathscr{B}$.

Lemma 5.8. *If $\varepsilon \leq \frac{1}{4K+1}$, then the matrix L is invertible, and for all $\boldsymbol{u}$ in $\mathbb{R}^n$, we have, with $1-\big((2K+1)+(4K+1)\varepsilon\big)\varepsilon \geq 0$ that*

$$\frac{1-\big((2K+1)+(4K+1)\varepsilon\big)\varepsilon}{1+2\varepsilon}\|\boldsymbol{u}\|_{\mathscr{V},\mathscr{B}} \leq \|L\boldsymbol{u}\|_{\mathscr{V},\mathscr{B}} \leq (1+(2K+3)\varepsilon+(4K+1)\varepsilon^2)\|\boldsymbol{u}\|_{\mathscr{V},\mathscr{B}}. \tag{28}$$

Proof. For all integers i, $1 \leq i \leq N$, we have $l_{ii} \geq 1-\varepsilon$ and $\sum_j |l_{ij}| \leq 1$. Since $\varepsilon < \frac{1}{2}$, L is a strictly diagonally dominant matrix, hence invertible. For the remainder of this proof, we will denote by $\mathscr{V}_i$ the set of all integer j such that (i,j) belongs to $\mathscr{V}$. We also define $l_i^* := 1-\sum_{j=1}^n l_{ij}$, and note that l_i^* is always non negative and smaller than ε, and it vanishes if i does not belong to $\mathscr{B}$.

We start by estimating the first term of the norm $\|\cdot\|_{\mathscr{V},\mathscr{B}}$, see Definition 5.5. We have

$$\begin{aligned}
&\sum_{(i,j)\in\mathscr{V}} |\sum_{k=1}^N (l_{ik}-l_{jk})u_k|^2 \\
&= \sum_{(i,j)\in\mathscr{V}} \left| \sum_{\substack{k=1\\k\neq i}}^N l_{ik}(u_k-u_i) - \sum_{\substack{k=1\\k\neq j}}^N l_{jk}(u_k-u_j) + (u_i-u_j) - l_i^* u_i + l_j^* u_j \right|^2,
\end{aligned}$$

and using now the Cauchy-Schwarz inequality for $\sqrt{l_{ik}} \times \sqrt{l_{ik}}(u_k-u_i)$, we obtain

$$\begin{aligned}
&\leq \sum_{(i,j)\in\mathscr{V}} \left(\sum_{\substack{k=1\\k\neq i}}^N l_{ik} + \sum_{\substack{k=1\\k\neq j}}^N l_{jk} + 1 + l_i^* + l_j^* \right) \times \\
&\qquad \times \left(\sum_{\substack{k=1\\k\neq i}}^N l_{ik}|u_k-u_i|^2 + \sum_{\substack{k=1\\k\neq j}}^N l_{jk}|u_k-u_j|^2 + |u_i-u_j|^2 + l_i^*|u_i|^2 + l_j^*|u_j|^2 \right) \\
&\leq (1+2\varepsilon) \sum_{(i,j)\in\mathscr{V}} \left(\sum_{\substack{k=1\\k\neq i}}^N l_{ik}|u_k-u_i|^2 + \sum_{\substack{k=1\\k\neq j}}^N l_{jk}|u_k-u_j|^2 + |u_i-u_j|^2 + l_i^*|u_i|^2 + l_j^*|u_j|^2 \right) \\
&\leq (1+2\varepsilon) \left(\sum_{(i,j)\in\mathscr{V}} |u_i-u_j|^2 + 2K \max_{i\neq j}|l_{ij}| \sum_{(i,j)\in\mathscr{V}} |u_j-u_i|^2 + 2K \max_{i\in\mathscr{B}} |l_i^*| \sum_{i\in\mathscr{B}}^N |u_i|^2 \right) \\
&\leq (1+2\varepsilon) \left((1+2K\varepsilon) \sum_{(i,j)\in\mathscr{V}} |u_i-u_j|^2 + 2K\varepsilon \sum_{i\in\mathscr{B}}^N |u_i|^2 \right),
\end{aligned}$$

which yields the inequality

$$\sum_{(i,j)\in\mathscr{V}} |\sum_{k=1}^{N}(l_{ik}-l_{jk})u_k|^2 \leq (1+2\varepsilon)\left((1+2K\varepsilon)\sum_{(i,j)\in\mathscr{V}}|u_i-u_j|^2+2K\varepsilon\sum_{i\in\mathscr{B}}^{N}|u_i|^2\right). \tag{29}$$

We now estimate the second term of the norm in Definition 5.5,

$$\sum_{i\in\mathscr{B}}\left|\sum_{k=1}^{N} l_{ik}u_k\right|^2 = \sum_{i\in\mathscr{B}}\left|\sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}(u_k-u_i)+(1-l_i^*)u_i\right|^2$$

and again using the Cauchy-Schwarz inequality on $\sqrt{l_{ik}}\times\sqrt{l_{ik}}(u_k-u_i)$, we obtain

$$\begin{aligned}
&\leq \sum_{i\in\mathscr{B}}\left(\sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}+(1-l_i^*)\right)\left(\sum_{k=1}^{N} l_{ik}|u_k-u_i|^2+(1-l_i^*)|u_i|^2\right),\\
&\leq (1+\varepsilon)\sum_{i\in\mathscr{B}}\left(\sum_{k=1}^{N} l_{ik}|u_k-u_i|^2+(1-l_i^*)|u_i|^2\right)\\
&\leq (1+\varepsilon)\max_{i\neq k}|l_{ik}|\sum_{(i,k)\in\mathscr{V}}|u_k-u_i|^2+(1+\varepsilon)\sum_{i\in\mathscr{B}}|u_i|^2\\
&\leq (1+\varepsilon)\varepsilon\sum_{(i,k)\in\mathscr{V}}|u_k-u_i|^2+(1+\varepsilon)\sum_{i\in\mathscr{B}}|u_i|^2,
\end{aligned}$$

which proves the inequality

$$\sum_{i\in\mathscr{B}}\left|\sum_{k=1}^{N} l_{ik}u_k\right|^2 \leq (1+\varepsilon)\varepsilon\sum_{(i,k)\in\mathscr{V}}|u_k-u_i|^2+(1+\varepsilon)\sum_{i\in\mathscr{B}}|u_i|^2. \tag{30}$$

Now combining inequalities (29) and (30), we establish the right part of inequality (28).

Proving the left part of inequality (28) is a little more difficult. We start by estimating the first term of the norm $\|\cdot\|_{\mathscr{V},\mathscr{B}}$. To establish (29), we used the equality

$$\sum_{k=1}^{N}(l_{ik}-l_{jk})u_k = \sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}(u_k-u_i)-\sum_{\substack{k=1\\k\neq j}}^{N} l_{jk}(u_k-u_j)+(u_i-u_j)-l_i^*u_i+l_j^*u_j.$$

Putting the (u_i-u_j) term onto the left-hand side of the equation and all the other terms onto the right-hand side, we get

$$
\begin{aligned}
&\sum_{(i,j)\in\mathscr{V}} |u_i - u_j|^2 \\
&= \sum_{(i,j)\in\mathscr{V}} \left| \sum_{k=1}^{N} (l_{ik} - l_{jk}) u_k - \sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}(u_k - u_i) + \sum_{\substack{k=1\\k\neq j}}^{N} l_{jk}(u_k - u_j) + l_i^* u_i - l_j^* u_j \right|^2 \\
&= \sum_{(i,j)\in\mathscr{V}} \left| \big((\boldsymbol{Lu})_i - (\boldsymbol{Lu})_j\big) - \sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}(u_k - u_i) + \sum_{\substack{k=1\\k\neq j}}^{N} l_{jk}(u_k - u_j) + l_i^* u_i - l_j^* u_j \right|^2,
\end{aligned}
$$

and using again the Cauchy-Schwarz inequality, as we did earlier, we find

$$
\begin{aligned}
&\leq \sum_{(i,j)\in\mathscr{V}} \big(1 + (1 - l_i^* - l_{ii}) + (1 - l_j^* - l_{jj}) + l_i^* + l_j^*\big) \times \\
&\qquad \times \left(|(\boldsymbol{Lu})_i - (\boldsymbol{Lu})_j|^2 + \sum_{\substack{k=1\\k\neq i}}^{N} l_{ik} |u_k - u_i|^2 + \sum_{\substack{k=1\\k\neq j}}^{N} l_{jk} |u_k - u_j|^2 + l_i^* |u_i|^2 + l_j^* |u_j|^2 \right) \\
&\leq (1 + 2\varepsilon) \sum_{(i,j)\in\mathscr{V}} \Bigg(|(\boldsymbol{Lu})_i - (\boldsymbol{Lu})_j|^2 + \max_{k\neq i} |l_{ik}| \sum_{k\in\mathscr{V}_i} |u_k - u_i|^2 \\
&\qquad + \max_{k\neq j} |l_{jk}| \sum_{k\in\mathscr{V}_j} |u_k - u_j|^2 + l_i^* |u_i|^2 + l_j^* |u_j|^2 \Bigg) \\
&\leq (1 + 2\varepsilon) \sum_{(i,j)\in\mathscr{V}} \left(|(\boldsymbol{Lu})_i - (\boldsymbol{Lu})_j|^2 + \varepsilon \sum_{k\in\mathscr{V}_i} |u_k - u_i|^2 + \varepsilon \sum_{k\in\mathscr{V}_j} |u_k - u_j|^2 + l_i^* |u_i|^2 + l_j^* |u_j|^2 \right) \\
&\leq (1 + 2\varepsilon) \sum_{(i,j)\in\mathscr{V}} |(\boldsymbol{Lu})_i - (\boldsymbol{Lu})_j|^2 + 2(1 + 2\varepsilon) K\varepsilon \sum_{(i,j)\in\mathscr{V}} |u_j - u_i|^2 + 2(1 + 2\varepsilon) K\varepsilon \sum_{i\in\mathscr{B}} |u_i|^2.
\end{aligned}
$$

The ε terms will be absorbed by the left-hand side, provided we choose ε small enough. To absorb the third term, we must first estimate the second term in norm $\|\cdot\|_{\mathscr{V},\mathscr{B}}$. To establish (30), we used

$$
\sum_{k=1}^{N} l_{ik} u_k = \sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}(u_k - u_i) + (1 - l_i^*) u_i.
$$

We put u_i onto the left-hand side of the equality and all the other terms onto the right-hand side to obtain

$$\sum_{i\in\mathscr{B}}|u_i|^2 = \sum_{i\in\mathscr{B}}|\sum_{k=1}^{N} l_{ik}u_k - \sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}(u_k-u_i)+l_i^*u_i|^2 = \sum_{i\in\mathscr{B}}|(\boldsymbol{Lu})_i - \sum_{k=1}^{N} l_{ik}(u_k-u_i)+l_i^*u_i|^2$$
$$\leq \sum_{i\in\mathscr{B}}\left(1+\sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}+l_i^*\right)\left(|(\boldsymbol{Lu})_i|^2+\sum_{\substack{k=1\\k\neq i}}^{N} l_{ik}|u_k-u_i|^2+l_i^*|u_i|^2\right)$$
$$\leq (1+\varepsilon)\sum_{i\in\mathscr{B}}\left(|(\boldsymbol{Lu})_i|^2+\max_{k\neq i}|l_{ik}|\sum_{k\in\mathscr{V}_i}|u_k-u_i|^2+\max_{j\in\mathscr{B}}|l_j^*||u_i|^2\right)$$
$$\leq (1+\varepsilon)\sum_{i\in\mathscr{B}}|(\boldsymbol{Lu})_i|^2+(1+\varepsilon)\max_{i\neq j}|l_{ij}|\sum_{(i,j)\in\mathscr{V}}|u_j-u_i|^2+(1+\varepsilon)\max_{i\in\mathscr{B}}|l_i^*|\sum_{i\in\mathscr{B}}|u_i|^2$$
$$\leq (1+\varepsilon)\sum_{i\in\mathscr{B}}|(\boldsymbol{Lu})_i|^2+(1+\varepsilon)\varepsilon\sum_{(i,j)\in\mathscr{V}}|u_j-u_i|^2+(1+\varepsilon)\varepsilon\sum_{i\in\mathscr{B}}|u_i|^2.$$

We add now the last two estimates to get

$$\|\boldsymbol{u}\|^2_{\mathscr{V},\mathscr{B}} \leq (1+2\varepsilon)\|\boldsymbol{Lu}\|^2_{\mathscr{V},\mathscr{B}}+\big((2K+1)+(4K+1)\varepsilon\big)\varepsilon\|\boldsymbol{u}\|^2_{\mathscr{V},\mathscr{B}}.$$

If $\varepsilon \leq \frac{1}{4K+1}$ then $\big((2K+1)+(4K+1)\varepsilon\big)\varepsilon < 1$, which concludes the proof. □

5.2.4 End of the proof of the Theorem 5.4

We just combine Lemma 5.6, Lemma 5.7, and Lemma 5.8, and we have successively the existence and uniqueness of u_H (since the matrix L is invertible), and the estimates

$$\|\nabla u_H\|^2_{L^2(\Omega)} \leq C_1\|u_H\|^2_{\mathscr{V},\mathscr{B}} \leq C_2C_1\|\ell(u)\|^2_{\mathscr{V},\mathscr{B}} \leq C_3C_2C_1\|\nabla u\|^2_{L^2(\Omega)},$$

where $C_1 = \frac{1}{\tan\theta_{min}}$, $C_2 = \frac{1+2r/H_h}{1-\big((2K+1)+(4K+1)r/H_h\big)r/H_h}$ and $C_3 = 2K(\frac{2C_pH}{\pi r}+\pi)$. To apply these inequalities, it is sufficient for the ratio r/H_h to be smaller than $1/((4K+1))$, where H_h is the length of the shortest height of any triangle in the mesh $\mathscr{T}$.

5.3 Non coarse elements

In this subsection, we construct the non coarse elements of the stable decomposition. We make the following assumption on the U_i:

Assumption 5.9 (Star shape of U_i). *We assume that there exists a uniform ε such that for all the domain decompositions we consider for Ω, U_i is star shaped with respect to any point in the ball $B(\boldsymbol{x}_i, r)$, where $r = \varepsilon H_h$ and where the $\boldsymbol{x}_i$ are the nodes of the coarse mesh $\mathscr{T}$ and where H_h is the length of the shortest height of any triangle in mesh $\mathscr{T}$.*

First we improve Lemma 5.1 in order to obtain estimates for the constants involved.

Lemma 5.10. *Let ω be an open domain of $\mathbb{R}^2$ with a diameter smaller than H. Let $r < H$. We suppose there exists $\boldsymbol{x}_O$ in ω such that*

- *The ball $B(\boldsymbol{x}_O, 2r)$ is included in ω.*
- *The set ω is star-shaped with respect to all $\boldsymbol{x}$ in the ball $B(\boldsymbol{x}, r)$.*

Then for all u in $H^1(\omega)$, and for all $\eta > 0$, we have the estimate:

$$\int_\omega |u(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y} \le \frac{(1+\eta)r^2}{3}\left(\left(\left(\frac{H^2}{r^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{H}{\sqrt[4]{2}r}\right)^4 - \frac{1}{2} - \frac{H^2}{r^2} - \frac{H^4}{2r^4}\right)\int_\omega \|\nabla u(\boldsymbol{x})\|^2 \mathrm{d}\boldsymbol{x}$$
$$+\left(1+\frac{1}{\eta}\right)\frac{H^2}{\pi r^4}\left|\int_{B(\boldsymbol{x}_O,r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2.$$

Proof. Without loss of generality, we can suppose that $\boldsymbol{x}_O = \boldsymbol{0}$. Then, for all $\eta > 0$:

$$\begin{aligned}
&\int_\omega |u(\boldsymbol{y})|^2 \mathrm{d}\boldsymbol{y} \\
&= \int_\omega \left| u(\boldsymbol{y}) - \frac{1}{\pi r^2}\int_{B(\boldsymbol{0},r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \frac{1}{\pi r^2}\int_{B(\boldsymbol{0},r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2 \mathrm{d}\boldsymbol{y} \\
&\le \frac{1+\eta}{\pi^2 r^4}\int_\omega \left|\int_{B(\boldsymbol{0},r)} (u(\boldsymbol{y})-u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right|^2 \mathrm{d}\boldsymbol{y} + \left(1+\frac{1}{\eta}\right)\frac{|\omega|}{\pi^2 r^4}\left|\int_{B(\boldsymbol{0},r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2 \\
&\le \frac{1+\eta}{\pi^2 r^4}\int_\omega \left|\int_{B(\boldsymbol{0},r)} (u(\boldsymbol{y})-u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right|^2 \mathrm{d}\boldsymbol{y} + \left(1+\frac{1}{\eta}\right)\frac{H^2}{\pi r^4}\left|\int_{B(\boldsymbol{0},r)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|^2,
\end{aligned}$$

and it remains to estimate the first term in the sum on the right,

$$\begin{aligned}
I &:= \frac{1}{\pi^2 r^4}\int_\omega \left|\int_{B(\boldsymbol{0},r)} (u(\boldsymbol{y})-u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right|^2 \mathrm{d}\boldsymbol{y} \\
&= \frac{1}{\pi^2 r^4}\int_\omega \left|\int_{B(\boldsymbol{0},r)}\int_0^1 \nabla u((1-t)\boldsymbol{x}+t\boldsymbol{y})\cdot(\boldsymbol{y}-\boldsymbol{x})\mathrm{d}t\mathrm{d}\boldsymbol{x}\right|^2 \mathrm{d}\boldsymbol{y} \\
&\le \frac{1}{\pi r^2}\int_\omega\int_{B(\boldsymbol{0},r)}\int_0^1 \|\nabla u((1-t)\boldsymbol{x}+t\boldsymbol{y})\|^2\|\boldsymbol{y}-\boldsymbol{x}\|^2 \mathrm{d}t\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

Now using the change of variables $\boldsymbol{x}' = (1-t)\boldsymbol{x}+t\boldsymbol{y}$, we get

$$\begin{aligned}
I &\le \frac{1}{\pi r^2}\int_\omega\int_0^1\int_{B(t\boldsymbol{y},(1-t)r)} \|\nabla u(\boldsymbol{x}')\|^2\|\boldsymbol{y}-\boldsymbol{x}'\|^2 \mathrm{d}\boldsymbol{x}'\frac{\mathrm{d}t}{(1-t)^4}\mathrm{d}\boldsymbol{y} \\
&= \frac{1}{\pi r^2}\int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \int_0^1\int_\omega \|\boldsymbol{y}-\boldsymbol{x}'\|^2 \chi\{\|\boldsymbol{x}'-t\boldsymbol{y}\| \le (1-t)r\}\mathrm{d}\boldsymbol{y}\frac{\mathrm{d}t}{(1-t)^4}\mathrm{d}\boldsymbol{x}'.
\end{aligned}$$

Using the further change of variables $\boldsymbol{y}' = \boldsymbol{y} - \boldsymbol{x}'$ yields

$$
\begin{aligned}
I &\leq \frac{1}{\pi r^2} \int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \int_0^1 \int_{\omega - \boldsymbol{x}'} \|\boldsymbol{y}'\|^2 \chi\{\|\boldsymbol{x}' - \frac{t}{1-t}\boldsymbol{y}'\| \leq r\} \mathrm{d}\boldsymbol{y}' \frac{\mathrm{d}t}{(1-t)^4} \mathrm{d}\boldsymbol{x}' \\
&\leq \frac{1}{\pi r^2} \int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \int_0^1 \int_{B(0,H)} \|\boldsymbol{y}'\|^2 \chi\{\|\boldsymbol{x}' - \frac{t}{1-t}\boldsymbol{y}'\| \leq r\} \mathrm{d}\boldsymbol{y}' \frac{\mathrm{d}t}{(1-t)^4} \mathrm{d}\boldsymbol{x}'
\end{aligned}
$$

and a final change of variables $\boldsymbol{y}'' = \frac{t}{1-t}\boldsymbol{y}'$ gives

$$
\begin{aligned}
I &\leq \frac{1}{\pi r^2} \int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \int_0^1 \int_{B(0,\frac{tH}{1-t}) \cap B(\boldsymbol{x}',r)} \|\boldsymbol{y}''\|^2 \mathrm{d}\boldsymbol{y}'' \frac{\mathrm{d}t}{t^4} \mathrm{d}\boldsymbol{x}' \\
&\leq \frac{1}{\pi r^2} \int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \int_0^1 \min\Big(\int_{B(0,\frac{tH}{1-t})} \|\boldsymbol{y}''\|^2 \mathrm{d}\boldsymbol{y}'', \int_{B(\boldsymbol{x}',r)} \|\boldsymbol{y}''\|^2 \mathrm{d}\boldsymbol{y}''\Big) \frac{\mathrm{d}t}{t^4} \mathrm{d}\boldsymbol{x}' \\
&= \frac{1}{r^2} \int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \int_0^1 \min\Big(\frac{t^4 H^4}{2(1-t)^4}, r^2\big(\frac{r^2}{2} + \|\boldsymbol{x}'\|^2\big)\Big) \frac{\mathrm{d}t}{t^4} \mathrm{d}\boldsymbol{x}' \\
&\leq \frac{1}{r^2} \left(\int_0^1 \min\Big(\frac{t^4 H^4}{2(1-t)^4}, r^2\big(\frac{r^2}{2} + H^2\big)\Big) \frac{\mathrm{d}t}{t^4} \right) \int_\omega \|\nabla u(\boldsymbol{x}')\|^2 \mathrm{d}\boldsymbol{x}' \\
&= \frac{r^2}{3} \left(\left(\left(\frac{H^2}{r^2} + \frac{1}{2} \right)^{\frac{1}{4}} + \frac{H}{\sqrt[4]{2}r} \right)^4 - \frac{1}{2} - \frac{H^2}{r^2} - \frac{H^4}{2r^4} \right) \int_\omega \|\nabla u(\boldsymbol{x})\|^2 \mathrm{d}\boldsymbol{x},
\end{aligned}
$$

which is the desired result. $\square$

Lemma 5.11. *Let Ω be a bounded domain of $\mathbb{R}^2$, and $(U_i, \Omega_i)_{1 \leq i \leq N}$ be an associated domain decomposition with overlap width $\delta > 0$. Let $\mathscr{T}$ be a coarse mesh on Ω, and assume that Assumptions 5.3, 5.2 and 5.9 are verified. We also assume the U_i satisfy Assumption 4.4 with uniform $\hat{R}$, $\tilde{R}$ and $1/\sin\theta_{\boldsymbol{X}}$. Then for any u in $H_0^1(\Omega)$, there exists $(u_i)_{1 \leq i \leq N}$ in $H_0^1(\Omega_i)$, such that for all i, $1 \leq i \leq N$, u_i is in $H_0^1(\Omega_i)$, $u = \sum_{i=1}^N u_i$ and for all $\eta > 0$,*

$$
\begin{aligned}
\sum_{i=1}^N \|\nabla u_i\|^2_{L^2(\Omega_i)} \leq \Bigg(& 2 + 8\lambda_2^2 (N_c - 1)^2 \Big(1 + \frac{\hat{R}}{\tilde{R}}\Big) \frac{\hat{R}}{\delta \sin\theta_{\boldsymbol{X}}} \\
& + \frac{8(1+\eta)}{3} \lambda_2^2 (N_c - 1)^2 \Big(1 + \frac{\hat{R}}{\tilde{R}}\Big) \frac{r^2}{\hat{R}\delta \sin\theta_{\boldsymbol{X}}} \times \\
& \times \left(\left(\left(\frac{C_d^2 H^2}{r^2} + \frac{1}{2} \right)^{\frac{1}{4}} + \frac{C_d H}{\sqrt[4]{2}r} \right)^4 - \frac{C_d^4 H^4}{2r^4} - \frac{1}{2} - \frac{C_d^2 H^2}{r^2} \right) \Bigg) \|\nabla u\|^2_{L^2(\Omega)} \\
& + 8\left(1 + \frac{1}{\eta}\right) \lambda_2^2 (N_c - 1)^2 \Big(1 + \frac{\hat{R}}{\tilde{R}}\Big) \pi \frac{C_d^2 H^2}{\hat{R}\delta \sin\theta_{\boldsymbol{X}}} \sum_{i=1}^N |\ell_i(u)|^2,
\end{aligned}
$$

where λ_2 is the universal constant of Lemma 4.3, and $\ell_i(u) = \frac{1}{\pi r^2} \int_{B(\boldsymbol{x}_i, r)} u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$.

Proof. We use the same u_i as in the proof of Theorem 4.6. Since $\operatorname{diam}(U_i) \leq C_d H$, we have, for all $\eta > 0$,

$$\begin{aligned}
\|u\|^2_{L^2(\Omega)} &= \sum_{i=1}^{N} \|u\|^2_{L^2(U_i)} \\
&\leq \frac{(1+\eta)r^2}{3} \left(\left(\left(\frac{C_d^2 H^2}{r^2} + \frac{1}{2} \right)^{\frac{1}{4}} + \frac{C_d H}{\sqrt[4]{2} r} \right)^4 - \frac{C_d^4 H^4}{2r^4} - \frac{1}{2} - \frac{C_d^2 H^2}{r^2} \right) \sum_{i=1}^{N} \|\nabla u\|^2_{L^2(U_i)} \\
&\quad + \left(1 + \frac{1}{\eta}\right) \pi C_d^2 H^2 \sum_{i=1}^{N} |\ell_i(u)|^2 \\
&= \frac{(1+\eta)r^2}{3} \left(\left(\left(\frac{C_d^2 H^2}{r^2} + \frac{1}{2} \right)^{\frac{1}{4}} + \frac{C_d H}{\sqrt[4]{2} r} \right)^4 - \frac{C_d^4 H^4}{2r^4} - \frac{1}{2} - \frac{C_d^2 H^2}{r^2} \right) \|\nabla u\|^2_{L^2(\Omega)} \\
&\quad + \left(1 + \frac{1}{\eta}\right) \pi C_d^2 H^2 \sum_{i=1}^{N} |\ell_i(u)|^2.
\end{aligned}$$

Inserting this estimate into estimate (16). concludes the proof. $\square$

5.4 Stable Decomposition with Coarse Mesh

Combining our previous results, we obtain now our main theorem on the existence of a stable decomposition with a coarse mesh. We provide this theorem with all assumptions in order for it to be self contained.

Theorem 5.12 (Stable Decomposition of H_0^1 with Coarse Mesh)**.** *Let Ω be a bounded domain of $\mathbb{R}^2$, and $(U_i)_{1\leq i\leq N}$ be a non overlapping domain decomposition of Ω, satisfying Assumption 4.4 with uniform $\hat{R}$, $\tilde{R}$, δ_0 and $1/\sin\theta_{\boldsymbol{X}}$. Let $\delta < \delta_0$ be positive and $(\Omega_i)_{1\leq i\leq N}$ be an overlapping domain decomposition defined from the U_i and δ as in Assumption 4.1. Let H be the smallest diameter among all U_i. We suppose the existence of uniform C_d, c_a and C_a such that the U_i satisfy Assumption 4.2. Let N_c be the number of colors of this decomposition. Let $\mathscr{T}$ be a triangular coarse mesh of the domain Ω with N nodes $(\boldsymbol{x}_i)_{1\leq i\leq N}$, satisfying Assumptions 4.2 and 5.3 with uniforms θ_{min}, θ_{max}, C_p, c_p and K parameters. Let H_h be the length of the shortest height of any triangle in $\mathscr{T}$. We suppose the existence of a uniform parameter r, $r \leq \frac{H_h}{4K+1}$ such that the (U_i) satisfy Assumption 5.9.*

Then, there exists a stable decomposition of $H_0^1(\Omega)$ in $P_1(\mathscr{T}) \cap H_0^1(\Omega) + \sum_{i=1}^{N} H_0^1(\Omega_i)$, i.e. for all u in $H_0^1(\Omega)$, there exists u_0 in $P_1(\mathscr{T}) \cap H_0^1(\Omega)$ and $(u_i)_{1\leq i\leq N}$, $u_i \in H_0^1(\Omega_i)$, such that

$$u = \sum_{i=0}^{N} u_i, \qquad \sum_{i=0}^{N} \|\nabla u_i\|^2_{L^2(\Omega_i)} \leq C \|\nabla u\|^2_{L^2(\Omega)},$$

where $C = C_1 + 2(1+C_1)C_2$ *and*[7]

$$C_1 = \frac{1}{\tan\theta_{min}} \frac{1+2r/H_h}{1-\big((2K+1)+(4K+1)r/H_h\big)r/H_h} 2K(\frac{2C_pH}{\pi r}+\pi), \tag{31}$$

$$\begin{aligned} C_2 = {} & 2 + 8\lambda_2^2(N_c-1)^2\Big(1+\frac{\hat{R}}{\tilde{R}}\Big)\frac{\hat{R}}{\delta\sin\theta_{\boldsymbol{X}}} \\ & + \frac{8}{3}\lambda_2^2(N_c-1)^2\Big(1+\frac{\hat{R}}{\tilde{R}}\Big)\frac{r^2}{\hat{R}\delta\sin\theta_{\boldsymbol{X}}}\times \\ & \times\left(\left(\left(\frac{C_d^2H^2}{r^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{C_dH}{\sqrt[4]{2}r}\right)^4 - \frac{C_d^4H^4}{2r^4} - \frac{1}{2} - \frac{C_d^2H^2}{r^2}\right), \end{aligned} \tag{32}$$

where λ_2 *is the universal constant of Lemma 4.3.*

Proof. We take $u_0 = u_H$ from Theorem 5.4, and we apply Lemma 5.11 to $u - u_0$. The term in $1 + 1/(4\eta)$ disappears. We let go η tend to 0 and obtain the stable decomposition with the given constant. □

6 Condition number bound at the continuous level

We can now use the stable decomposition established in Theorem 5.12 to bound the condition number of the continuous Additive Schwarz operator, which leads to the following result:

Theorem 6.1 (Condition Number Estimate at the Continuous Level)**.** *Let* Ω *be a bounded domain of* $\mathbb{R}^2$. *Let* A *be a continuous function from* $\overline{\Omega}$ *to the set of* 2×2 *symmetric positive definite matrices. We suppose that* $A(\boldsymbol{x})$ *is uniformly coercive and uniformly bounded: there exist* $\alpha > 0$ *and* $\beta > 0$ *such that for all* $\boldsymbol{x}$ *in* Ω, *and for all* $\boldsymbol{\xi}$ *in* $\mathbb{R}^2$

$$\alpha\|\boldsymbol{\xi}\|_2^2 \leq \boldsymbol{\xi}^{\mathrm{T}}A(\boldsymbol{x})\boldsymbol{\xi} \leq \beta\|\boldsymbol{\xi}\|_2^2.$$

Let $a(\cdot,\cdot)$ *be the continuous bilinear form on* $H_0^1(\Omega)$ *defined by*

$$a(u,v) = \int_\Omega \nabla u(\boldsymbol{x})\cdot A(\boldsymbol{x})\nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

We use the same notation and the same hypotheses as in Theorem 5.12 to define the U_i, *the* Ω_i, *the mesh* $\mathscr{T}$ *and all the geometric parameters on which the constants depend.*

Let $V_0 = P_1(\mathscr{T})$. *Let* $V_i = H_0^1(\Omega_i)$ *for* $1 \leq i \leq N$. *Let* R_i^T *be defined by*

$$\begin{aligned} R_i^T : H_0^1(\Omega_i) &\to H_0^1(\Omega), \\ u &\mapsto \left(\boldsymbol{x}\mapsto\begin{cases} u(\boldsymbol{x}) & \textit{if } \boldsymbol{x}\in\Omega_i, \\ 0 & \textit{otherwise}\end{cases}\right). \end{aligned}$$

[7] Note that $r \leq \frac{H_h}{4K+1}$ ensures that $1-\big((2K+1)+(4K+1)r/H_h\big)r/H_h$ is positive.

For all $1 \leq i \leq N$, let $\widetilde{a}_i$ be the bilinear forms on $H^1_0(\Omega_i)$ defined by $\widetilde{a}_i(u,v) = a(R_i^T u, R_i^T v)$, i.e.

$$\widetilde{a}_i(u,v) = \int_{\Omega_i} \nabla u(\boldsymbol{x}) \cdot A(\boldsymbol{x}) \nabla v(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

Let P_{ad} be the preconditioned Additive Schwarz operator defined by equation (2)*. Then the a-condition number of P_{ad} is bounded by*

$$\kappa(P_{ad}) \leq \frac{\beta^2}{\alpha^2} C(N_c + 1),$$

where $C = C_1 + 2(1 + C_1)C_2$, with C_1 and C_2 given by (31) *and* (32)*, and with λ_2 being the universal constant of Lemma 4.3.*

Proof. Assumption 2.4 is satisfied by definition with the local stability parameter $\omega = 1$, and Assumption 2.2 is satisfied by Theorem 5.12, since A is uniformly coercive and uniformly bounded. Therefore, we have a stable decomposition whose constant is the C of Theorem 5.12 multiplied by $\frac{\beta^2}{\alpha^2}$. We apply then Theorem 2.7 to conclude. □

The bound of the condition does not depend on the number of subdomains and the lengths in the formulas always come in ratios, which means that the condition number stays bounded.

7 Conclusion

We have analyzed the Additive Schwarz preconditioned operator with a coarse mesh at the continuous level. We provided explicit estimates which show that the condition number is independent of the number of subdomains. The explicit dependence of our constants on the shape regularity of the domain decomposition allowed us already to prove sharper convergence estimates than the classical ones for only locally shape regular decompositions. We are currently also working on a general convergence result for the two level additive Schwarz Method which does not depend on the particular discretization chosen.

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A The L^2 norm of the gradient in $P_1(\mathscr{T})$

Let ABC be a triangle, and let v_a, v_b, v_c in $\mathbb{R}$ be the values at the corners. There exists a unique affine mapping u defined over ABC, such that $u(A) = v_A$, $u(B) = v_B$

and $u(C) = v_C$. We want to compute $\int_{ABC} \|\nabla u\|^2$. Inside ABC, ∇u is a constant that satisfies the two equations

$$\nabla u \cdot (\boldsymbol{AB}) = v_B - v_A, \qquad\qquad \nabla u \cdot (\boldsymbol{AC}) = v_C - v_A.$$

Hence, in a matrix formulation, we have

$$\begin{bmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{bmatrix} \nabla u = \begin{bmatrix} v_B - v_A \\ v_C - v_A \end{bmatrix}.$$

The inverse of this matrix is readily computed, and we obtain

$$\begin{aligned} \nabla u &= \frac{\begin{bmatrix} y_C - y_A & -(y_B - y_A) \\ -(x_C - x_A) & x_B - x_A \end{bmatrix} \begin{bmatrix} v_B - v_A \\ v_C - v_A \end{bmatrix}}{\begin{vmatrix} x_B - x_A & y_B - y_A \\ x_C - x_A & y_C - y_A \end{vmatrix}} \\ &= \frac{1}{2\mathscr{S}(ABC)} \begin{bmatrix} (y_C - y_A)(v_B - v_A) - (y_B - y_A)(v_C - v_A) \\ -(x_C - x_A)(v_B - v_A) + (x_B - x_A)(v_C - v_A) \end{bmatrix}, \end{aligned}$$

where $\mathscr{S}(ABC)$ is the area of triangle ABC. Therefore, we obtain

$$\begin{aligned} \|\nabla u\|^2_{\mathbb{R}^2} &= \frac{\|\boldsymbol{AC}\|^2 (v_B - v_A)^2 + \|\boldsymbol{AB}\|^2 (v_C - v_A)^2 - 2(\boldsymbol{AB}, \boldsymbol{AC})(v_B - v_A)(v_C - v_A)}{4\mathscr{S}(ABC)^2} \\ &= \frac{(\boldsymbol{CB}, \boldsymbol{CA})(v_B - v_A)^2 + (\boldsymbol{BA}, \boldsymbol{BC})(v_C - v_A)^2 + (\boldsymbol{AB}, \boldsymbol{AC})(v_B - v_C)^2}{4\mathscr{S}(ABC)^2}, \end{aligned}$$

since $2(v_B - v_A)(v_c - v_A) = (v_B - v_A)^2 + (v_C - v_A)^2 - (v_B - v_C)^2$. We thus have

$$\|\nabla u\|^2_{L^2(ABC)} = \mathscr{S}(ABC) \|\nabla u\|^2_{\mathbb{R}^2} = \frac{(v_B - v_A)^2}{2\tan(\theta_C)} + \frac{(v_A - v_C)^2}{2\tan(\theta_B)} + \frac{(v_C - v_B)^2}{2\tan(\theta_A)}. \quad (33)$$

References

[1] Robert A. Adams and John F. Fournier. *Sobolev Spaces*, volume 140 of *Pure and Applied Mathematics Series*. Academic Press, second edition, 2003.

[2] Susanne C. Brenner. Lower bounds for two-level additive Schwarz preconditioners with small overlaps. *SIAM J. Sci. Comput.*, 21(5):1657–1669, 2000.

[3] Xiao-Chuan Cai. An additive Schwarz algorithm for nonselfadjoint elliptic equations. In Tony Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 232–244. SIAM, Philadelphia, PA, 1990.

[4] Xiao-Chuan Cai. Additive Schwarz algorithms for parabolic convection-diffusion equations. *Numer. Math.*, 60(1):41–61, 1991.

[5] Xiao-Chuan Cai. Multiplicative Schwarz methods for parabolic problems. *SIAM J. Sci Comput.*, 15(3):587–603, 1994.

[6] Xiao-Chuan Cai and Olof Widlund. Multiplicative Schwarz algorithms for some nonsymmetric and indefinite problems. *SIAM J. Numer. Anal.*, 30(4):936–952, August 1993.

[7] Claudio Canuto and Daniele Funaro. The Schwarz algorithm for spectral methods. *SIAM J. Numer. Anal.*, 25(1):24–40, 1988.

[8] Tony F. Chan and Tarek P. Mathew. Domain decomposition algorithms. In *Acta Numerica 1994*, pages 61–143. Cambridge University Press, 1994.

[9] Tony F. Chan, Barry F. Smith, and Jun Zou. Overlapping Schwarz methods on unstructured meshes using non-matching coarse grids. *Numer. Math.*, 73(2):149–167, 1996.

[10] Tony F. Chan and Jun Zou. Additive Schwarz domain decomposition methods for elliptic problems on unstructured meshes. *Numer. Algorithms*, 8(2–4):329–346, 1994.

[11] Tony F. Chan and Jun Zou. A convergence theory of multilevel additive Schwarz methods on unstructured meshes. *Numerical Algorithms*, 13(2):365–398, feb 1996.

[12] Lawrence C. Cowsar. Dual variable Schwarz methods for mixed finite elements. Technical Report TR93-09, Department of Mathematical Sciences, Rice University, March 1993.

[13] Maksymilian Dryja. Additive Schwarz methods for elliptic mortar finite element problems. In Małgorzata Peszyńska Kazimierz Malanowski, Zbigniew Nahorski, editor, *Modeling and Optimization of Distributed Parameter Systems with Applications to Engineering*, pages 31–50. IFIP, Chapman & Hall, London, 1996.

[14] Maksymilian Dryja and Olof B. Widlund. An additive variant of the Schwarz alternating method for the case of many subregions. Technical Report 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute, 1987.

[15] Maksymilian Dryja and Olof B. Widlund. Domain decomposition algorithms with small overlap. *SIAM J. Sci.Comput.*, 15(3):604–620, May 1994.

[16] Richard E. Ewing and Junping Wang. Analysis of the Schwarz algorithm for mixed finite element methods. *RAIRO Mathematical Modelling and Numerical Analysis*, 26(6):739–756, 1992.

[17] Xiaobing Feng and Ohannes A. Karakashian. Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.*, 39(4):1343–1365, 2002.

[18] Andreas Frommer and Daniel B. Szyld. Weighted max norms, splittings, and overlapping additive Schwarz iterations. *Numerische Mathematik*, 83:259–278, 1999.

[19] Martin J. Gander. Optimized Schwarz methods. *SIAM J. Numer. Anal.*, 44(2):699–731, 2006.

[20] Martin J. Gander. Schwarz methods over the course of time. *Electronic Transactions on Numerical Analysis*, 31:228–255, 2008.

[21] Martin J. Gander, Laurence Halpern, and Kévin Santugini-Repiquet. Non shape regular domain decompositions: An analysis using a stable decomposition in H_0^1. In Randolph Bank, Michael Holst, Olof Widlund, and Jinchao Xu, editors, *Domain Decomposition Methods in Science and Engineering XX*, number 91 in Lecture Notes in Computational Science and Engineering, pages 485–492, Berlin, 2013. Springer.

[22] Pierre Grisvard. *Elliptic problems in nonsmooth domains*. Pitman Publishing, Boston, 1985.

[23] Mario Arlindo Casarin Jr. Quasi-optimal Schwarz methods for the conforming spectral element discretization. *SIAM J. Numer. Anal.*, 34(6):2482–2502, 1997.

[24] Pierre-Louis Lions. On the Schwarz alternating method. I. In Roland Glowinski, Gene H. Golub, Gérard A. Meurant, and Jacques Périaux, editors, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 1–42, Philadelphia, PA, 1988. SIAM.

[25] Pierre-Louis Lions. On the Schwarz alternating method. II. In Tony Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Domain Decomposition Methods*, pages 47–70, Philadelphia, PA, 1989. SIAM.

[26] Pierre-Louis Lions. On the Schwarz alternating method. III: a variant for nonoverlapping subdomains. In Tony F. Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, *Third International Symposium on Domain Decomposition Methods for Partial Differential Equations , held in Houston, Texas, March 20-22, 1989*, pages 202–223, Philadelphia, PA, 1990. SIAM.

[27] Tarek P. Mathew. Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part I: Algorithms and numerical results. *Numer. Math.*, 65(4):445–468, 1993.

[28] Tarek P. Mathew. Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part II: Theory. *Numer. Math.*, 65(4):469–492, 1993.

[29] Luca F. Pavarino. Additive Schwarz methods for the p-version finite element method. *Numer. Math.*, 66(4):493–515, 1994.

[30] Luca F. Pavarino. Schwarz methods with local refinement for the p-version finite element method. *Numer. Math.*, 69(2):185–211, 1994.

[31] Alfio Quarteroni and Alberto Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford Science Publications, 1999.

[32] H. A. Schwarz. Über einen Grenzübergang durch alternierendes Verfahren. *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, 15:272–286, May 1870.

[33] Barry F. Smith, Petter E. Bjørstad, and William Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*. Cambridge University Press, 1996.

[34] Roger Temam. *Infinite Dimensional Dynamical Systems*, volume 68 of *Applied Mathematical Sciences*. Springer Verlag, second edition, 1997.

[35] Andrea Toselli and Olof Widlund. *Domain Decomposition Methods - Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer, 2004.

[36] Olof B. Widlund. The development of coarse spaces for domain decomposition algorithms. In M. Bercovier, M.J. Gander, R. Kornhuber, and O. Widlund, editors, *Domain Decomposition Methods in Science and Engineering XVIII*, volume 70 of *Lecture Notes in Computational Science and Engineering*, pages 241–248, May 2009.

[37] Jinchao Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review*, 34(4):581–613, December 1992.