Journal of Computational and Applied Mathematics 344 (2018) 904–924

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Multiscale analysis of heterogeneous domain decomposition methods for time-dependent advection–reaction–diffusion problems

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ARTICLE INFO

Article history:
Received 21 February 2017
Received in revised form 26 May 2018

MSC:
65M55
65M15

Keywords:
Heterogeneous domain decomposition
Multiscale analysis
Viscous problems with inviscid approximations

ABSTRACT

Domain decomposition methods which use different models in different subdomains are called heterogeneous domain decomposition methods. We are interested here in the case where there is an accurate but expensive model one should use in the entire domain, but for computational savings we want to use a cheaper model in parts of the domain where expensive features of the accurate model can be neglected. For the model problem of a time dependent advection–reaction–diffusion equation in one spatial dimension, we study approximate solutions of three different heterogeneous domain decomposition methods with pure advection reaction approximation in parts of the domain. Using for the first time a multiscale analysis to compare the approximate solutions to the solution of the accurate expensive model in the entire domain, we show that a recent heterogeneous domain decomposition method based on factorization of the underlying differential operator has better approximation properties than more classical variational or non-variational heterogeneous domain decomposition methods. We show with numerical experiments in two spatial dimensions that the performance of the algorithms we study is well predicted by our one dimensional multiscale analysis, and that our theoretical results can serve as a guideline to compare the expected accuracy of heterogeneous domain decomposition methods already for moderate values of the viscosity.

1. Introduction

Heterogeneous domain decomposition methods are domain decomposition methods where different models are solved in different subdomains. Models can be different because problems are heterogeneous, i.e. there are connected components with different physical properties, see for example [1–4], or because one wants to approximate a homogeneous object with different approximations, depending on their validity and cost, see for example [5–12]. In this second situation, there is in general a complex, expensive model which would give the best possible solution, and the heterogeneous domain decomposition methods try to give a good approximation to this best possible solution at a lower computational cost. It is therefore possible in this second situation to quantify the quality of heterogeneous domain decomposition approximations in a rigorous mathematical way, by comparing them to the expensive solution on the entire domain, as it was proposed in [13], see also the earlier publication [14]. Using for the first time multiscale analysis, we compare in this paper three heterogeneous

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https://doi.org/10.1016/j.cam.2018.05.058
0377-0427/

domain decomposition methods to solve time dependent advection–reaction–diffusion equations, with advection reaction approximations in parts of the domain: the method using variational and non-variational coupling conditions from [15,16], see also [17] and [18], and the factorization method, which has its roots in [17], but was only fully developed in [19] for one dimensional steady advection–reaction–diffusion problems. It was proved in [19] that the factorization method can give approximate solutions in the viscous region which can be exponentially close to the monodomain viscous solution for one dimensional steady problems. A factorization method for time dependent advection–reaction–diffusion problems was proposed in [20], and its performance was studied using a priori error estimates. We present here for the first time a multiscale analysis of the factorization method, together with the variational and non-variational ones, and we show with numerical experiments that the results of this multiscale analysis also describe the behavior of the coupling algorithms very well in higher spatial dimensions.

We present in Section 2 the three heterogeneous domain decomposition methods we will study in this paper for time dependent advection–reaction–diffusion problems. In Section 3, we perform a multiscale analysis of the factorization method, and give sharp error estimates as the viscosity goes to zero. In Section 4, we present the corresponding multiscale analysis for the variational heterogeneous domain decomposition method, and in Section 5 the one for the non-variational heterogeneous domain decomposition method. The error estimates we obtain allow us to compare the quality of the coupled solutions obtained by these three methods, and the results differ, depending on the advection direction at the interface. We then test in Section 6 the three heterogeneous domain decomposition algorithms numerically in a two dimensional setting that goes beyond our theoretical analysis. Our results show that the one dimensional multiscale analysis predicts nevertheless the performance very well also in two dimensions, and this already for moderate values of the viscosity parameter. Our theoretical results are thus really useful to guide people in the choice of coupling conditions for heterogeneous domain decomposition. We finally compare the numerical cost of the algorithms and summarize our findings in Section 7.

2. Heterogeneous domain decomposition methods

We define the time dependent advection–reaction–diffusion operator $\mathcal{L}_{ad} := \partial_t - \nu\partial_x^2 + a\partial_x + c$, $\nu > 0$ and $c \geq 0$, its non-diffusive approximation $\mathcal{L}_a := \partial_t + a\partial_x + c$, and consider two model problems: for positive advection $a > 0$, we want to approximate

$$\begin{array}{rcll} \mathcal{L}_{ad}u & = & f & \text{in } (-L_1, L_2) \times (0, T), \\ u(-L_1, \cdot) & = & g_1 & \text{on } (0, T), \\ \mathcal{L}_a u(L_2, \cdot) & = & 0 & \text{on } (0, T), \\ u(\cdot, 0) & = & h & \text{in } (-L_1, L_2), \end{array} \tag{2.1}$$

which represents the outflow from a region where viscosity is important into an area where it is not. The boundary condition at outflow is absorbing, see [21]. For negative advection, $a < 0$, we want to approximate

$$\begin{array}{rcll} \mathcal{L}_{ad}u & = & f & \text{in } (-L_1, L_2) \times (0, T), \\ u(-L_1, \cdot) & = & g_1 & \text{on } (0, T), \\ u(L_2, \cdot) & = & g_2 & \text{on } (0, T), \\ u(\cdot, 0) & = & h & \text{in } (-L_1, L_2), \end{array} \tag{2.2}$$

which represents the inflow from a region where the viscosity is not important into an area where it is, i.e. a boundary layer which is forming on the left. In both model problems (2.1) and (2.2), we want to approximate the solution by solving an advection–reaction–diffusion equation in the domain $\Omega_1 := (-L_1, 0)$, and only an advection reaction equation in $\Omega_2 := (0, L_2)$.

2.1. Variational coupling conditions

A heterogeneous domain decomposition method using variational coupling conditions was introduced in [15,16] for stationary problems. The method was obtained in a variational framework, by fixing the viscosity in a subregion, and then letting the viscosity go to zero in the remaining domain. The method is non-iterative, and when extended to our time dependent setting, it consists for $a > 0$ in solving first the advection–reaction–diffusion problem

$$\begin{array}{rcll} \mathcal{L}_{ad}u_{ad}^V & = & f & \text{in } \Omega_1 \times (0, T), \\ u_{ad}^V(-L_1, \cdot) & = & g_1 & \text{on } (0, T), \\ \partial_x u_{ad}^V(0, \cdot) & = & 0 & \text{on } (0, T), \\ u_{ad}^V(\cdot, 0) & = & h & \text{in } \Omega_1, \end{array} \tag{2.3}$$

followed by solving the advection reaction problem

$$\begin{array}{rcll} \mathcal{L}_a u_a^V & = & f & \text{in } \Omega_2 \times (0, T), \\ u_a^V(0, \cdot) & = & u_{ad}^V(0, \cdot) & \text{on } (0, T), \\ u_a^V(\cdot, 0) & = & h & \text{in } \Omega_2. \end{array} \tag{2.4}$$

If $a < 0$, one first solves an advection reaction problem,

$$\begin{aligned} \mathcal{L}_a u_a^V &= f && \text{in } \Omega_2 \times (0,T), \\ u_a^V(L_2,\cdot) &= g_2 && \text{on } (0,T), \\ u_a^V(\cdot,0) &= h && \text{in } \Omega_2, \end{aligned} \tag{2.5}$$

followed by the solution of an advection–reaction–diffusion problem

$$\begin{aligned} \mathcal{L}_{ad} u_{ad}^V &= f && \text{in } \Omega_1 \times (0,T), \\ u_{ad}^V(-L_1,\cdot) &= g_1 && \text{on } (0,T), \\ -\nu\partial_x u_{ad}^V(0,\cdot) + a u_{ad}^V(0,\cdot) &= a u_a^V(0,\cdot) && \text{on } (0,T), \\ u_{ad}^V(\cdot,0) &= h && \text{in } \Omega_1. \end{aligned} \tag{2.6}$$

2.2. Non-variational coupling conditions

Non-variational coupling conditions were also considered in [15,16] for steady problems. The idea is to put transmission conditions which lead to coupled solutions with good continuity across the interface, see also [17]. In our time dependent setting, one can, for $a > 0$, enforce both continuity of the traces and the fluxes, which leads to the heterogeneous domain decomposition method

$$\begin{aligned} \mathcal{L}_{ad} u_{ad}^{NV} &= f && \text{in } \Omega_1 \times (0,T), \\ u_{ad}^{NV}(-L_1,\cdot) &= g_1 && \text{on } (0,T), \\ \partial_x u_{ad}^{NV}(0,\cdot) &= \partial_x u_a^{NV}(0,\cdot) && \text{on } (0,T), \\ u_{ad}^{NV}(\cdot,0) &= h && \text{in } \Omega_1, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \mathcal{L}_a u_a^{NV} &= f && \text{in } \Omega_2 \times (0,T), \\ u_a^{NV}(0,\cdot) &= u_{ad}^{NV}(0,\cdot) && \text{on } (0,T), \\ u_a^{NV}(\cdot,0) &= h && \text{in } \Omega_2. \end{aligned} \tag{2.8}$$

The coupled solution defined by (2.7) and (2.8) is in general computed by an iteration, see [15,16], and Section 6, where we also propose a heuristic for an optimal choice of the relaxation parameter in the iteration to obtain convergence.

If $a < 0$, one can only enforce continuity of the traces, and one first solves an advection reaction problem,

$$\begin{aligned} \mathcal{L}_a u_a^{NV} &= f && \text{in } \Omega_2 \times (0,T), \\ u_a^{NV}(L_2,\cdot) &= g_2 && \text{on } (0,T), \\ u_a^{NV}(\cdot,0) &= h && \text{in } \Omega_2, \end{aligned} \tag{2.9}$$

followed by the solution of an advection–reaction–diffusion problem

$$\begin{aligned} \mathcal{L}_{ad} u_{ad}^{NV} &= f && \text{in } \Omega_1 \times (0,T), \\ u_{ad}^{NV}(-L_1,\cdot) &= g_1 && \text{on } (0,T), \\ u_{ad}^{NV}(0,\cdot) &= u_a^{NV}(0,\cdot) && \text{on } (0,T), \\ u_{ad}^{NV}(\cdot,0) &= h && \text{in } \Omega_1. \end{aligned} \tag{2.10}$$

2.3. The factorization algorithm

The idea of the factorization algorithm has its roots in the PhD thesis of Dubach [17], who was trying to find better transmission conditions than the variational ones from Section 2.1 and the non-variational ones from Section 2.2. This led him to study absorbing boundary conditions in this context. It is however a modified advection equation which becomes key to improve the coupling, as it was pointed out in [19], and for steady one dimensional advection–reaction–diffusion problems, exponentially small errors can be achieved in the viscosity ν, whereas the other methods only lead to algebraically small errors in ν. For time dependent problems, the factorization algorithm below was developed in [20], and uses a modified advection reaction operator, $\mathcal{L}_{ma} := \partial_t - a\partial_x + c + \frac{a^2}{\nu}$. For positive advection, $a > 0$, the algorithm is also iterative: starting with a given initial guess $u_{ad}^0(0,\cdot) = g_{ad}^0$, each iteration consists of three steps: first we solve a transport problem into the positive x direction in Ω_2,

$$\begin{aligned} \mathcal{L}_a u_a^k &= f && \text{in } \Omega_2 \times (0,T), \\ u_a^k(0,\cdot) &= u_{ad}^{k-1}(0,\cdot) && \text{on } (0,T), \\ u_a^k(\cdot,0) &= h && \text{in } \Omega_2, \end{aligned} \tag{2.11}$$

followed by a modified transport problem into the negative x direction in Ω_2 with the adapted source defined using the operator $\mathcal{R} := (\partial_t + c)^2$,

$$\begin{aligned} \mathcal{L}_{ma} u_{ma}^k &= \frac{a^2}{\nu} f + \mathcal{R}\, u_a^k && \text{in } \Omega_2 \times (0,T), \\ u_{ma}^k(L_2,\cdot) &= 0 && \text{on } (0,T), \\ u_{ma}^k(\cdot,0) &= f(\cdot,0) + \nu d_x^2 h && \text{in } \Omega_2, \end{aligned} \tag{2.12}$$

and finally an advection–reaction–diffusion problem in Ω_1,

$$\begin{aligned} \mathcal{L}_{ad} u_{ad}^k &= f && \text{in } \Omega_1 \times (0,T), \\ u_{ad}^k(-L_1,\cdot) &= g_1 && \text{on } (0,T), \\ \mathcal{L}_a u_{ad}^k(0,\cdot) &= u_{ma}^k(0,\cdot) && \text{on } (0,T), \\ u_{ad}^k(\cdot,0) &= h && \text{in } \Omega_1. \end{aligned} \tag{2.13}$$

If the advection is negative, $a < 0$, the factorization algorithm is non-iterative. It starts with an advection reaction problem in Ω_2,

$$\begin{aligned} \mathcal{L}_a u_a^1 &= f && \text{in } \Omega_2 \times (0,T), \\ u_a^1(L_2,\cdot) &= g_2 && \text{on } (0,T), \\ u_a^1(\cdot,0) &= h && \text{in } \Omega_2, \end{aligned} \tag{2.14}$$

followed by another advection reaction problem in the same domain,

$$\begin{aligned} \mathcal{L}_a u_a^2 &= \frac{a^2}{\nu} f + \mathcal{R}\, u_a^1 && \text{in } \Omega_2 \times (0,T), \\ u_a^2(L_2,\cdot) &= \mathcal{L}_{ma} u_a^1(L_2,\cdot) && \text{on } (0,T), \\ u_a^2(\cdot,0) &= \mathcal{L}_{ma} u_a^1(\cdot,0) && \text{in } \Omega_2, \end{aligned} \tag{2.15}$$

and finally an advection–reaction–diffusion problem in Ω_1,

$$\begin{aligned} \mathcal{L}_{ad} u_{ad} &= f && \text{in } \Omega_1 \times (0,T), \\ u_{ad}(-L_1,\cdot) &= g_1 && \text{on } (0,T), \\ \mathcal{L}_{ma} u_{ad}(0,\cdot) &= u_a^2(0,\cdot) && \text{on } (0,T), \\ u_{ad}(\cdot,0) &= h && \text{in } \Omega_1. \end{aligned} \tag{2.16}$$

3. Multiscale analysis of the factorization algorithm

The multiscale behavior of the advection–reaction–diffusion equation is well understood, see for example [22]; boundary layers can be created near Dirichlet walls, and there can also be characteristic boundary layers if the data lacks compatibility, see [23]. We consider here only regular and compatible data, and assume that the forcing term f is compactly supported in $(-L_1, L_2) \times (0,T]$, and the boundary data g_1 and g_2 is compactly supported in $(0,T]$. Then, all the problems defined above are well-posed, with $\mathcal{C}^\infty$ solutions, see [20]. The formal expansions we will obtain are fully justified by the a priori estimates in [20].

3.1. The case of positive advection

We start with $a > 0$ and first perform a multiscale analysis of the advection–reaction–diffusion equation (2.1), before studying the factorization algorithm in detail. When $a > 0$, we assume in addition that the initial data h is compactly supported in Ω_1.

3.1.1. Multiscale solution of the advection–reaction–diffusion equation

We seek a multiscale expansion of the solution u of (2.1) in the form

$$u(x,t) \approx \sum_{j\geq 0} \nu^j U_j(x, \frac{L_2 - x}{\nu}, t) = \sum_{j\geq 0} \nu^j u_j(x,t) + \sum_{j\geq 0} \nu^j U_j^*(\frac{L_2 - x}{\nu}, t),$$

where the functions $U_j(x,y,t)$ belong to the space of functions split in the form

$$V(x,y,t) = v(x,t) + V^*(y,t),$$

with smooth functions $v \in \mathcal{C}^\infty((0,T) \times \Omega)$ and $V^* \in e^{-\delta y}\mathcal{C}^\infty((0,T) \times \Omega)$ for some positive δ. The first series is the *outer expansion*, which satisfies the equation and the boundary condition on the left. The second one is the *inner expansion*, which is the corrector for the boundary condition on the right to be fulfilled.

Lemma 3.1. *There is a unique formal multiscale solution of the mixed Cauchy problem (2.1) in $\Omega \times (0, T)$, of the form*

$$u^{out}(x,t) + u^{in}(x,t) = \sum_{j\geq 0} \nu^j u_j(x,t) + \sum_{j\geq 2} \nu^j U_j^*(\frac{L_2 - x}{\nu}, t). \tag{3.1}$$

Each term in the outer expansion u^{out} *is solution of a transport equation,*

$$\mathcal{L}_a u_0 = f, \qquad u_0(x,0) = h(x), \qquad u_0(-L_1,t) = g_1(t), \tag{3.2}$$

$$\mathcal{L}_a u_j = \partial_x^2 u_{j-1}, \qquad u_j(x,0) = 0, \qquad u_j(-L_1,t) = 0, \quad j \geq 1. \tag{3.3}$$

The first non vanishing term in the inner expansion u^{in} *is*

$$U_2^*(y,t) = -\frac{1}{a^2}\partial_x^2 u_0(L_2,t)e^{-ay}. \tag{3.4}$$

Proof. In the sense of formal series, since $\partial_x\partial_y U_j = 0$ for any j, we have with obvious notations

$$\mathcal{L}_a u \approx -\frac{a}{\nu}\partial_y U_0 + \sum_{j\geq 0}\nu^j(\mathcal{L}_a U_j - a\,\partial_y U_{j+1}),$$

$$\mathcal{L}_{ad} u \approx -\frac{1}{\nu}(a\partial_y + \partial_y^2)U_0 - (a\partial_y + \partial_y^2)U_1 + \mathcal{L}_a U_0$$
$$+ \sum_{j\geq 1}\nu^j\left(-(a\partial_y + \partial_y^2)U_{j+1} + \mathcal{L}_a U_j - \partial_x^2 U_{j-1}\right).$$

Defining the operator in the y variable $\mathcal{L} := -(a\partial_y + \partial_y^2)$, and collecting terms in ν, we have a formal solution of the advection–reaction–diffusion equation if and only if

$$\begin{aligned} F_{-1} &:= \mathcal{L}U_0 = 0, \\ F_0 &:= \mathcal{L}U_1 + \mathcal{L}_a U_0 - f = 0, \\ F_j &:= \mathcal{L}U_{j+1} + \mathcal{L}_a U_j - \partial_x^2 U_{j-1} = 0, \quad j \geq 1. \end{aligned} \tag{3.5}$$

Since the initial data h in (2.1) does not depend explicitly on ν, the expansion of the initial condition simply is

$$\forall x \in (-L_1, L_2),\ \forall y \in (0, +\infty),\ U_0(x,y,0) = h(x),\ U_j(x,y,0) = 0, \quad \text{for } j \geq 1. \tag{3.6}$$

The boundary condition on the right in (2.1) is satisfied if and only if

$$\begin{aligned} G_{-1} &:= -a\partial_y U_0(L_2, 0, \cdot) = 0, \\ G_j &:= -a\partial_y U_{j+1}(L_2, 0, \cdot) + \mathcal{L}_a U_j(L_2, 0, \cdot) = 0, \quad j \geq 0. \end{aligned} \tag{3.7}$$

We start with the *zeroth order term* U_0. From $F_{-1} = 0$ we deduce that

$$\partial_y U_0^*(y,t) = \alpha_0(t)e^{-ay},$$

and from $G_{-1} = 0$, that $\partial_y U_0^*(y,t) \equiv 0$. Hence in the zeroth order term, we only have $U_0(x,y,t) = u_0(x,t)$.

We next split the higher order terms $F_j = 0$ into $\underline{F}_j = 0$ and $\underline{F}_j^* = 0$, where $\underline{F}_j := \lim_{y\to\infty} F_j$ and $\underline{F}_j^* := F_j - \underline{F}_j$, and obtain the equations

$$\begin{aligned} &\underline{F}_0 := \mathcal{L}_a u_0 - f = 0, \quad &&\underline{F}_j := \mathcal{L}_a u_j - \partial_x^2 u_{j-1} = 0, \\ &\underline{F}_0^* := \mathcal{L}U_1^* = 0, &&\underline{F}_j^* := \mathcal{L}U_{j+1}^* + (\partial_t + c)U_j^* = 0. \end{aligned} \tag{3.8}$$

Similarly, we split the initial data into

$$u_0(x,0) = h(x), \quad u_j(x,0) = 0, \quad \text{for } j \geq 1, \quad U_j^*(y,0) = 0, \quad \text{for } j \geq 0,$$

and the boundary data on the left as

$$u_0(-L_1,t) = g_1(t), \quad u_j(-L_1,t) = 0, \quad \text{for } j \geq 1.$$

The terms u_j in the outer expansion are determined recursively by $\underline{F}_j = 0$, yielding the recursive sequence of well-posed transport problems

$$\mathcal{L}_a u_0 = f, \qquad u_0(x,0) = h(x), \qquad u_0(-L_1,t) = g_1(t),$$
$$\mathcal{L}_a u_j = \partial_x^2 u_{j-1}, \qquad u_j(x,0) = 0, \qquad u_j(-L_1,t) = 0, \quad j \geq 1.$$

We now compute the correction given by the inner expansion: For the *first order term*, we obtain from $\underline{F}_0^* = 0$ in (3.8) the general solution

$$\partial_y U_1^*(y,t) = \alpha_1(t)e^{-ay},$$

and the boundary condition is given by $G_0 = 0$ in (3.7),

$$-a\partial_y U_1^*(0,t) + f(L_2,t) = 0 \text{ for all } t.$$

Therefore we can determine $\alpha_1(t)$ and obtain after integration

$$U_1^*(y,t) = -\frac{f(L_2,t)}{a^2}\, e^{-ay},$$

where because of the initial condition the integration constant is zero. Since f is compactly supported in $\Omega \times (0,T]$, U_1^* actually vanishes identically, and we compute the second order term: solving the corresponding equation $\underline{F}_1^* = 0$, we get

$$\partial_y U_2^*(y,t) = \alpha_2(t)e^{-ay},$$

with the boundary condition given by $G_1 = 0$,

$$a\partial_y U_2^*(0,t) = \mathcal{L}_a U_1(L_2,0,t) = \mathcal{L}_a u_1(L_2,t) = \partial_x^2 u_0(L_2,t).$$

We thus obtain again by integration, and using the homogeneous initial condition,

$$U_2^*(y,t) = -\frac{1}{a^2}\partial_x^2 u_0(L_2,t)e^{-ay}. \quad \square$$

3.1.2. Analysis of the factorization algorithm

In the first iteration of the factorization algorithm, the first step defined by (2.11) with suitable initial data gives an infinitely smooth solution u_a^1 in Ω_2 that does not depend on ν. We thus start with the expansion of the modified advection solution u_{ma}^1 in Ω_2, defined in (2.12), which is propagating to the left. We expect a boundary layer at $x = L_2$, due to the lack of compatibility.

Lemma 3.2. *There is a unique formal multiscale approximation to u_{ma}^1 in $\Omega_2 \times (0,T)$, defined by*

$$u_{ma}^{1,out}(x,t) + u_{ma}^{1,in}(x,t) = \sum_{j\geq 0} \nu^j u_{ma,j}^1(x,t) + \sum_{j\geq 1} \nu^j U_{ma,j}^{1,*}\left(\frac{L_2 - x}{\nu}, t\right). \tag{3.9}$$

Each term in the outer expansion $u_{ma}^{1,out}$ *is given by*

$$u_{ma,0}^1 = f, \quad u_{ma,j}^1 = \left(-\frac{\mathcal{L}_{ma}^0}{a^2}\right)^{j-1} \partial_x^2 u_a^1 \quad \text{for } j \geq 1, \tag{3.10}$$

with $\mathcal{L}_{ma}^0 = \partial_t + c - a\partial_x$.

Remark 3.1. The factorization algorithm uses u_{ma}^1 only at $x = 0$, and the inner expansion is exponentially decaying away from $x = L_2$. We therefore do not need to compute the inner expansion to study the factorization algorithm.

Proof. We split the ν-dependent operator $\mathcal{L}_{ma}$ into $\mathcal{L}_{ma} = \mathcal{L}_{ma}^0 + \frac{a^2}{\nu}$. The *outer expansion*, which is valid in the entire domain Ω_2, is of the form

$$u_{ma}^{1,out}(x,t) = \sum_{j\geq 0} \nu^j u_{ma,j}^1(x,t). \tag{3.11}$$

Inserting (3.11) into the differential equation, we obtain

$$\frac{a^2}{\nu} u_{ma,0}^1 + \sum_{j\geq 0} \nu^j (\mathcal{L}_{ma}^0 u_{ma,j}^1 + a^2 u_{ma,j+1}^1) = \frac{a^2}{\nu} f + \mathcal{R}\, u_a^1,$$

where $\mathcal{R} = (\partial_t + c)^2$. This yields, when collecting terms,

$$u_{ma,0}^1 = f, \quad \mathcal{L}_{ma}^0 u_{ma,0}^1 + a^2 u_{ma,1}^1 = \mathcal{R}\, u_a^1, \quad \mathcal{L}_{ma}^0 u_{ma,j}^1 + a^2 u_{ma,j+1}^1 = 0,\ j \geq 1. \tag{3.12}$$

By induction, we can thus determine $u_{ma,j}^1$ in $\Omega_2 \times (0,T)$. Start with

$$\mathcal{R}\, u_a^1 - \mathcal{L}_{ma}^0 u_{ma,0}^1 = \mathcal{R}\, u_a^1 - \mathcal{L}_{ma}^0 f = \mathcal{R}\, u_a^1 - \mathcal{L}_{ma}^0 \mathcal{L}_a u_a^1 = (a\partial_x)^2 u_a^1.$$

For the last equality, we have used the operator identity

$$\mathcal{L}_{ma}^0 \mathcal{L}_a = \mathcal{R} - (a\partial_x)^2, \tag{3.13}$$

which will be useful several times in what follows. Therefore $u^1_{ma,1} = \partial_x^2 u^1_a$, and we thus get the terms of the outer expansion,

$$u^1_{ma,0} = f, \quad u^1_{ma,j} = \left(-\frac{\mathcal{L}^0_{ma}}{a^2}\right)^{j-1} \partial_x^2 u^1_a \quad \text{for } j \geq 1. \tag{3.14}$$

However, the initial condition needs also to be satisfied in Ω_2,

$$u^1_{ma}(\cdot, 0) = f(\cdot, 0) + \nu d_x^2 h,$$

which is equivalent to

$$u^1_{ma,0}(\cdot, 0) = f(\cdot, 0), \quad u^1_{ma,1}(\cdot, 0) = d_x^2 h, \quad u^1_{ma,j}(\cdot, 0) = 0 \text{ for } j \geq 2.$$

Since h vanishes in Ω_2 and f vanishes for $t \leq 0$, the initial conditions are satisfied by the functions defined in (3.14). The boundary condition at inflow, $u^1_{ma}(L_2, \cdot) = 0$, is satisfied if and only if

$$f(L_2, \cdot) = 0, \quad \left(-\frac{\mathcal{L}^0_{ma}}{a^2}\right)^{j} \partial_x^2 u^1_a(L_2, \cdot) = 0 \text{ for } j \geq 0.$$

The first equality is the trivial statement $0 = 0$, but the second one is not satisfied, since $\partial_x^2 u^1_a(L_2, \cdot)$ has no reason to vanish. Therefore there is a boundary layer of order 1 at $x = L_2$. □

We now study the third step (2.13) in the first iteration of the factorization algorithm, which provides u^1_{ad} in Ω_1.

Lemma 3.3. *There is a unique formal multiscale approximation to u^1_{ad} in $\Omega_1 \times (0, T)$, defined by*

$$u^{out}(x, t) + u^{1,in}_{ad}(x, t) = \sum_{j\geq 0} \nu^j u_j(x, t) + \sum_{j\geq 2} \nu^j U^{1,*}_{ad,j}(\frac{-x}{\nu}, t). \tag{3.15}$$

The first non vanishing term in the inner expansion $u^{1,in}_{ad}$ is

$$U^{1,*}_{ad,2}(y, t) = -\frac{1}{a^4}\mathcal{R}(u_0 - u^1_a)(0, t)e^{-ay}. \tag{3.16}$$

Proof. The multiscale analysis is similar to the one given in Section 3.1.1, except that the domain is now $(-L_1, 0)$, and that the vanishing right hand side in the boundary condition is replaced by $u^1_{ma}(0, \cdot)$, given by Lemma 3.2. Only the outer expansion in $u^1_{ma}(0, \cdot)$ is taken into account, since the boundary layer is at $x = L_2$. The outer expansion is the same as the outer expansion of u, and we write

$$u^1_{ad}(x, t) \approx u^{1,out}_{ad}(x, t) + u^{1,in}_{ad}(x, t) = \sum_{j\geq 0} \nu^j U^1_{ad,j}(x, -\frac{x}{\nu}, t),$$

with $U^1_{ad,j}(x, -\frac{x}{\nu}, t) = u^1_{ad,j}(x, t) + U^{1,*}_{ad,j}(-\frac{x}{\nu}, t)$. Therefore Eqs. (3.5), (3.6) remain valid, only the boundary condition on the right has to be changed into

$$\begin{aligned} G_{-1} &:= -a\partial_y U^1_{ad,0}(0, 0, \cdot) = 0, \\ G_j &:= -a\partial_y U^1_{ad,j+1}(0, 0, \cdot) + \mathcal{L}_a U^j_{ad,j}(0, 0, \cdot) - u^1_{ma,j}(0, \cdot) = 0, \ j \geq 0. \end{aligned} \tag{3.17}$$

The *zeroth order term* is $U^1_{ad,0}(x, y, t) = u_0(x, t)$ because of the homogeneous initial condition. The following terms in the outer expansion can be computed recursively with the same data as for u, therefore $u^{1,out}_{ad} = u^{out}$. We now turn to the inner expansion. The *first order term* is given by $\partial_y U^{1,*}_{ad,1} = \alpha^{1,*}_1(t)e^{-ay}$. Using the boundary condition $G_0 = 0$ leads to

$$a\,\alpha^{1,*}_1(t) = \mathcal{L}_a u_0(0, t) - u^1_{ma,0}(0, t) = (\mathcal{L}_a u_0 - f)(0, t) = 0.$$

This shows that, with the homogeneous initial conditions, $U^{1,*}_{ad,1}$ is zero. For the *second order term*, we obtain $\partial_y U^{1,*}_{ad,2}(y, t) = \alpha^{1,*}_2(t)e^{-ay}$, and with the boundary condition $G_1 = 0$, we get

$$a\,\alpha^{1,*}_2(t) = \mathcal{L}_a u_1(0, t) - u^1_{ma,1}(0, t) = \partial_x^2 u_0(0, t) - \partial_x^2 u^1_a(0, t).$$

Using furthermore (3.13), and that $\mathcal{L}_a u_0 = \mathcal{L}_a u^1_a = f$, we deduce that

$$\mathcal{R}\,u_0 - (a\partial_x)^2 u_0 = \mathcal{L}^0_{ma} f = \mathcal{R}\,u^1_a - (a\partial_x)^2 u^1_a,$$

which implies

$$a\,\alpha^{1,*}_2(t) = \frac{1}{a^2}\mathcal{R}(u_0 - u^1_a)(0, t).$$

We thus obtain for the second order term of the inner expansion, since the initial condition is zero,

$$U^{1,*}_{ad,2}(y,t) = -\frac{1}{a^4}\mathcal{R}(u_0 - u^1_a)(0,t)e^{-ay}. \quad \square$$

This finishes the multiscale analysis of the first iteration of the factorization algorithm.

We now start the second iteration, with the first step (2.11), which requires the expansion of u^2_a in Ω_2. There is again only an *outer expansion*, given by $u^{2,out}_a = \sum_j \nu^j u^2_{a,j}$, where the coefficients are solutions of transport problems. The first few are given by

$$\text{Order 0} \qquad \begin{cases} \mathcal{L}_a u^2_{a,0} = f \text{ in } \Omega_2 \times (0,T), \\ u^2_{a,0}(0,\cdot) = u_0(0,\cdot), \\ u^2_{a,0}(\cdot,0) = h, \end{cases} \tag{3.18}$$

$$\text{Order 1} \qquad \begin{cases} \mathcal{L}_a u^2_{a,1} = 0 \text{ in } \Omega_2 \times (0,T), \\ u^2_{a,1}(0,\cdot) = u_1(0,\cdot), \\ u^2_{a,1}(\cdot,0) = 0, \end{cases} \tag{3.19}$$

$$\text{Order 2} \qquad \begin{cases} \mathcal{L}_a u^2_{a,2} = 0 \text{ in } \Omega_2 \times (0,T), \\ u^2_{a,2}(0,\cdot) = u_2(0,\cdot) - \dfrac{1}{a^4}\mathcal{R}(u_0(0,\cdot) - g^0_{ad}), \\ u^2_{a,2}(\cdot,0) = 0, \end{cases} \tag{3.20}$$

From (3.18), we see that

$$u^2_{a,0} = u_0 \text{ in } \Omega_2 \times (0,T). \tag{3.21}$$

The second step (2.12) of the second iteration gives u^2_{ma} in Ω_2, from which we will only need the outer expansion, as for the first iteration.

Lemma 3.4. *There is a unique formal multiscale approximation to u^2_{ma} in $\Omega_2 \times (0,T)$, defined by*

$$u^{2,out}_{ma}(x,t) + u^{2,in}_{ma}(x,t) = \sum_{j\geq 0} \nu^j u^2_{ma,j}(x,t) + \sum_{j\geq 1} \nu^j U^{2,*}_{ma,j}\left(\frac{L_2 - x}{\nu}, t\right). \tag{3.22}$$

The first terms in the outer expansion $u^{2,out}_{ma}$ *are given at $x = 0$ by*

$$\begin{aligned} u^2_{ma,0} = f, \quad u^2_{ma,1} &= \mathcal{L}_a u_1, \quad u^2_{ma,2} = \mathcal{L}_a u_2, \\ u^2_{ma,3} &= \mathcal{L}_a u_3 - \frac{1}{a^6}\mathcal{R}^2(u_0 - g^0_{ad}) - \frac{1}{a^4}\mathcal{R}\partial_x^2 u_0. \end{aligned} \tag{3.23}$$

Proof. We determine the outer expansion for u^2_{ma} in the form $\sum_{j\geq 0}\nu^j u^2_{ma,j}(x,t)$, by inserting it into the differential equation, and obtain

$$u^2_{ma,0} = f, \quad \mathcal{L}^0_{ma} u^2_{ma,j} + a^2 u^2_{ma,j+1} = \mathcal{R}\, u^2_{a,j}, \quad j \geq 0. \tag{3.24}$$

This determines by induction the value of $u^2_{ma,j}$ in $\Omega_2 \times (0,T)$. First $u^2_{ma,0} = u^1_{ma,0} = f$, and by (3.19) and (3.21) we find

$$a^2 u^2_{ma,1} = \mathcal{R}\, u^2_{a,0} - \mathcal{L}^0_{ma} f \overset{\substack{(3.2)\\(3.21)}}{=} \mathcal{R}\, u_0 - \mathcal{L}^0_{ma}\mathcal{L}_a u_0 \overset{(3.13)}{=} a^2\partial_x^2 u_0 \overset{(3.3)}{=} a^2\mathcal{L}_a u_1,$$

and

$$a^2 u^2_{ma,2} = \mathcal{R}\, u^2_{a,1} - \mathcal{L}^0_{ma}\mathcal{L}_a u_1 \overset{(3.13)}{=} \mathcal{R}(u^2_{a,1} - u_1) + a^2\partial_x^2 u_1 \overset{(3.3)}{=} \mathcal{R}(u^2_{a,1} - u_1) + a^2\mathcal{L}_a u_2.$$

Due to the boundary condition at $x = 0$ for $u^2_{a,1}$ in (3.19), the first term on the right hand side vanishes at $x = 0$, and we obtain

$$u^2_{ma,2}(0,\cdot) = \mathcal{L}_a u_2(0,\cdot).$$

For the next term, we find

$$\begin{aligned} a^2 u^2_{ma,3} &= \mathcal{R}\, u^2_{a,2} - \mathcal{L}^0_{ma} u^2_{ma,2} = \mathcal{R}\, u^2_{a,2} - \mathcal{L}^0_{ma}\left(\frac{1}{a^2}\mathcal{R}(u^2_{a,1} - u_1) + \mathcal{L}_a u_2\right) \\ &= \mathcal{R}\, u^2_{a,2} - \mathcal{L}^0_{ma}\mathcal{L}_a u_2 - \frac{1}{a^2}\mathcal{L}^0_{ma}\mathcal{R}(u^2_{a,1} - u_1) \\ &\overset{(3.13)}{=} \mathcal{R}\left(u^2_{a,2} - u_2 - \frac{1}{a^2}\mathcal{L}^0_{ma}(u^2_{a,1} - u_1)\right) + (a\partial_x)^2 u_2, \end{aligned}$$

which gives

$$a^2(u^2_{ma,3} - \mathcal{L}_a u_3) = \mathcal{R}(u^2_{a,2} - u_2 - \frac{1}{a^2}\mathcal{L}^0_{ma}(u^2_{a,1} - u_1)). \tag{3.25}$$

We can now evaluate the expression above at $x = 0$. Since $\mathcal{L}^0_{ma} + \mathcal{L}_a = 2(\partial_t + c)$, and since $u^2_{a,1}$ and u_1 coincide at $x = 0$, we obtain

$$\mathcal{L}^0_{ma}(u^2_{a,1} - u_1)(0,\cdot) = -\mathcal{L}_a(u^2_{a,1} - u_1)(0,\cdot) \overset{(3.3)}{\underset{(3.19)}{=}} \partial_x^2 u_0(0,\cdot).$$

Inserting this last equality together with the boundary condition for $u^2_{a,2}$ in (3.20) into (3.25), we obtain

$$a^2(u^2_{ma,3} - \mathcal{L}_a u_3)(0,\cdot) = -\frac{1}{a^4}\mathcal{R}^2(u_0(0,\cdot) - g^0_{ad}) - \frac{1}{a^2}\mathcal{R}\partial_x^2 u_0(0,\cdot). \quad \square$$

We finally arrive at the expansion of the third step (2.13) of the second iteration, i.e., u^2_{ad} in Ω_2.

Lemma 3.5. *There is a unique formal multiscale approximation to u^2_{ad} in $\Omega_1 \times (0,T)$, defined by*

$$u^{out}(x,t) + u^{2,in}_{ad}(x,t) = \sum_{j\geq 0}\nu^j u_j(x,t) + \sum_{j\geq 4}\nu^j U^{2,*}_{ad,j}(\frac{-x}{\nu}, t). \tag{3.26}$$

The first non vanishing term in the inner expansion $u^{2,in}_{ad}$ is

$$U^{2,*}_{ad,4}(y,\cdot) = -\frac{1}{a^8}\mathcal{R}(\mathcal{R}(u_0(0,\cdot) - g^0_{ad}) + a^2\partial_x^2 u_0(0,\cdot))e^{-ay}. \tag{3.27}$$

Proof. The proof is similar to the proof for the first iteration; we search an expansion of the form

$$u^2_{ad}(x,t) \approx \sum_{j\geq 0}\nu^j U^2_{ad,j}(x, -\frac{x}{\nu}, t) = u^{out} + \sum_{j\geq 0}\nu^j U^{2,*}_{ad,j}(-\frac{x}{\nu}, t).$$

The boundary condition for the second iteration gives the expansion

$$\begin{aligned} G_{-1} &:= -a\partial_y U^{2,*}_{ad,0}(0,\cdot) = 0, \\ G_j &:= -a\partial_y U^{2,*}_{ad,j+1}(0,\cdot) + \mathcal{L}_a u_j(0,\cdot) + (\partial_t + c)U^{2,*}_{ad,j}(0,\cdot) - u^2_{ma,j}(0,\cdot) = 0, \; j \geq 0. \end{aligned} \tag{3.28}$$

As before, the *zeroth order term* $U^{2,*}_{ad,0}$ and the *first order term* $U^{2,*}_{ad,1}$ vanish. For the *second order term*, we obtain by integration $\partial_y U^{2,*}_{ad,2}(y,t) = \alpha_2^{2,*}(t)e^{-ay}$, and using the boundary condition

$$a\partial_y U^{2,*}_{ad,2}(0,\cdot) = (\mathcal{L}_a u_1 - u^2_{ma,1})(0,\cdot) = 0 \text{ by } (3.23)$$

gives $U^{2,*}_{ad,2} = 0$. For the *third order term*, we get $\partial_y U^{2,*}_{ad,3} = \alpha_3^{2,*}(t)e^{-ay}$ and using the boundary condition gives

$$a\partial_y U^{2,*}_{ad,3}(0,\cdot) = (\mathcal{L}_a u_2 - u^2_{ma,2})(0,\cdot) = 0 \text{ by } (3.23).$$

Finally, for the *fourth order term*, we get $\partial_y U^{2,*}_{ad,4} = \alpha_4^{2,*}(t)e^{-ay}$, and with the boundary condition

$$a\partial_y U^{2,*}_{ad,4}(0,\cdot) = (\mathcal{L}_a u_3 - u^2_{ma,3})(0,\cdot) = \frac{1}{a^6}\mathcal{R}^2(u_0(0,\cdot) - g^0_{ad}) + \frac{1}{a^4}\mathcal{R}\partial_x^2 u_0(0,\cdot) \text{ by } (3.23).$$

We thus obtain

$$U^{2,*}_{ad,4}(y,\cdot) = -\frac{1}{a^8}\mathcal{R}(\mathcal{R}(u_0(0,\cdot) - g^0_{ad}) + a^2\partial_x^2 u_0(0,\cdot))e^{-ay},$$

which completes the proof. $\square$

We can now compare the expansions for u and the iterates to obtain a precise error estimate for the factorization algorithm when ν becomes small:

Theorem 3.6. *In the case of positive advection, the multiscale approximations of the solution u, and the approximations u^1_a, u^1_{ad}, u^2_a and u^2_{ad} obtained by the Factorization algorithm (2.11), (2.12), (2.13) are given by*

$$u(x,t) \approx u^{out}(x,t) + u^{in}(x,t) = \sum_{j\geq 0}\nu^j u_j(x,t) + \sum_{j\geq 2}\nu^j U^*_j(\frac{L_2 - x}{\nu}, t), \; x \in \Omega,$$

$$u^1_a(x,t) = u_0(x,t), \; x \in \Omega_2,$$

$$u_a^2(x,t) \approx u_a^{out}(x,t) = u_0(x,t) + \sum_{j\ge 1} \nu^j u_{a,j}^2(x,t),\ x \in \Omega_2,$$
$$u_{ad}^1(x,t) \approx u^{out}(x,t) + u_{ad}^{1,in}(x,t) = \sum_{j\ge 0} \nu^j u_j(x,t) + \sum_{j\ge 2} \nu^j U_{ad,j}^{1,*}(\frac{-x}{\nu},t),\ x \in \Omega_1,$$
$$u_{ad}^2(x,t) \approx u^{out}(x,t) + u_{ad}^{2,in}(x,t) = \sum_{j\ge 0} \nu^j u_j(x,t) + \sum_{j\ge 4} \nu^j U_{ad,j}^{2,*}(\frac{-x}{\nu},t),\ x \in \Omega_1,$$

with

$$\begin{aligned} U_2^*(y,t) &= -\frac{1}{a^2}\partial_x^2 u_0(L_2,t)e^{-ay},\\ U_{ad,2}^{1,*}(y,t) &= -\frac{1}{a^4}\mathcal{R}(u_0 - u_a^1)(0,t)e^{-ay},\\ U_{ad,4}^{2,*}(y,t) &= -\frac{1}{a^8}\mathcal{R}(\mathcal{R}(u_0(0,t) - g_{ad}^0(t)) + a^2\partial_x^2 u_0(0,t))e^{-ay}. \end{aligned} \tag{3.29}$$

We therefore have the error estimates

$$\|u - u_a^2\|_{L^2(\Omega_2\times(0,T))} \sim \nu\|e_{a,1}^2\|_{L^2(\Omega_2\times(0,T))}, \tag{3.30}$$

$$\|u - u_{ad}^1\|_{L^2(\Omega_1\times(0,T))} \sim \frac{\nu^{\frac{5}{2}}}{\sqrt{2a^9}}\|\mathcal{R}(u_0(0,\cdot) - g_{ad}^0)\|_{L^2(0,T)}, \tag{3.31}$$

$$\|u - u_{ad}^2\|_{L^2(\Omega_1\times(0,T))} \sim \frac{\nu^{\frac{9}{2}}}{\sqrt{2a^{17}}}\|\mathcal{R}(\mathcal{R}(u_0(0,\cdot) - g_{ad}^0) + a^2\partial_x^2 u_0(0,\cdot))\|_{L^2(0,T)} \tag{3.32}$$

with $e_{a,1}^2$ defined by

$$\mathcal{L}_a e_{a,1}^2 = \partial_x^2 u_0, \quad e_{a,1}^2(\cdot,0) = 0, \quad e_{a,1}^2(0,\cdot) = 0. \tag{3.33}$$

Proof. The multiscale expansions we have proved already in the corresponding lemmas. Using them to estimate $u - u_a^2$, we obtain

$$(u - u_a^2)(x,t) \approx \sum_{j\ge 1} \nu^j (u_j - u_{a,j}^2)(x,t) - \frac{\nu^2}{a}\partial_x^2 u_0(L_2,t)e^{-a\frac{L_2-x}{\nu}} + \sum_{j\ge 3} \nu^j U_j^*(\frac{L_2-x}{\nu},t),$$

where we replaced already the term in U_2^* using (3.29). For any $j > 2$, U_j^* is obtained from U_{j-1}^* by integration of the differential equation in the y variable $\underline{F}_j^* = 0$, i.e. we have to solve the differential equation (see last equation in (3.8))

$$(a\partial_y + \partial_y^2)U_j^* = (\partial_t + c)U_{j-1}^*.$$

Since U_2^* equals a constant times e^{-ay}, see (3.29), each $U_j^*, j > 2$, is a polynomial of degree at most $j-2$ in y with time dependent coefficients, multiplied by e^{-ay}. The L^2 norm of $U_j^*(\frac{L_2-x}{\nu},t)$ is therefore a constant times $\sqrt{\nu}$. Thus the inner expansion is negligible and the leading term comes from the outer expansion,

$$(u - u_a^2)(x,t) \sim \nu(u_1(x,t) - u_{a,1}^2(x,t)),$$

and we therefore obtain

$$\|u - u_a^2\|_{L^2(\Omega_2\times(0,T))} \sim \nu\|u_1 - u_{a,1}^2\|_{L^2(\Omega_2\times(0,T))}.$$

For the advection–reaction–diffusion expansion in Ω_1, we get

$$(u - u_{ad}^1)(x,t) \sim -\frac{\nu^2}{a^2}\partial_x^2 u_0(L_2,t)e^{-a\frac{L_2-x}{\nu}} + \frac{\nu^2}{a^4}\mathcal{R}(u_0 - u_a^1)(0,t)e^{a\frac{x}{\nu}}.$$

In Ω_1, the L^2 norm of $e^{-a\frac{L_2-x}{\nu}}$ decays exponentially in ν, while the norm of $e^{a\frac{x}{\nu}}$ is equivalent to $\sqrt{\frac{\nu}{2a}}$. Therefore

$$\|u - u_{ad}^1\|_{L^2(\Omega_1\times(0,T))} \sim \sqrt{\frac{\nu}{2a}}\frac{\nu^2}{a^4}\|\mathcal{R}(u_0(0,\cdot) - u_a^1(0,\cdot))\|_{L^2(0,T)},$$

and similarly for the second step. □

Remark 3.2. Continuing this process, it is easy to see that $u_a^3 \sim u_0$, and hence no further improvement of the approximation can be obtained.

3.2. The case of negative advection

We now consider $a < 0$, and first perform a multiscale analysis of the advection–reaction–diffusion equation (2.2), before studying the factorization algorithm in detail.

3.2.1. Multiscale analysis of the advection–reaction–diffusion equation

The details of this analysis can be found in [22], we just give an outline for completeness.

Lemma 3.7. *There is a unique formal multiscale solution of the mixed Cauchy problem (2.2) in $\Omega \times (0, T)$, defined by*

$$u^{out}(x,t) + u^{in}(x,t) = \sum_{j\geq 0} \nu^j u_j(x,t) + \sum_{j\geq 0} \nu^j U_j^*(\frac{x+L_1}{\nu}, t), \tag{3.34}$$

where each term in the outer expansion *is solution of a transport equation*

$$\mathcal{L}_a u_0 = f, \qquad u_0(x,0) = h(x), \qquad u_0(L_2,t) = g_2, \tag{3.35}$$

$$\mathcal{L}_a u_j = \partial_x^2 u_{j-1}, \qquad u_j(x,0) = 0, \qquad u_j(L_2,t) = 0. \tag{3.36}$$

Each term $U_j^*(y,t)$ of the inner expansion *is a polynomial of degree j in y multiplied by e^{ay}, with coefficients depending on $g_1(t)$ and of $u_i(-L_1,t)$ for $0 \leq i \leq j$. The first non vanishing term in the inner expansion is*

$$U_0^*(y,t) = (g_1(t) - u_0(-L_1,t))e^{ay}. \tag{3.37}$$

Proof. We seek a multiscale expansion of the solution u of (2.2) in $(-L_1, L_2)$, in the form

$$u(x,t) \approx \sum_{j\geq 0} \nu^j U_j(x, \frac{x+L_1}{\nu}, t), \quad U_j(x,y,t) = u_j(x,t) + U_j^*(y,t).$$

In the sense of formal series, we thus obtain

$$\mathcal{L}_a u \approx \frac{a}{\nu}\partial_y U_0 + \sum_{j\geq 0} \nu^j(\mathcal{L}_a U_j + a\,\partial_y U_{j+1}),$$

$$\mathcal{L}_{ad} u \approx \frac{1}{\nu}(a\partial_y - \partial_y^2)U_0 + (a\partial_y - \partial_y^2)U_1 + \mathcal{L}_a U_0 + \sum_{j\geq 1} \nu^j\left((a\partial_y - \partial_y^2)U_{j+1} + \mathcal{L}_a U_j - \partial_x^2 U_{j-1}\right).$$

Defining the operator in the y variable $\mathcal{L}_- := a\partial_y - \partial_y^2$, and collecting terms in ν, we have a formal solution of the advection–reaction–diffusion equation if and only if

$$\begin{aligned} F_{-1} &:= \mathcal{L}_- U_0 = 0, \\ F_0 &:= \mathcal{L}_- U_1 + \mathcal{L}_a U_0 - f = 0, \\ F_j &:= \mathcal{L}_- U_{j+1} + \mathcal{L}_a U_j - \partial_x^2 U_{j-1} = 0,\ j \geq 1. \end{aligned} \tag{3.38}$$

Similarly to (3.6), the expansion of the initial condition is

$$U_0(x,y,0) = h(x), \quad U_j(x,y,0) = 0, \quad \forall x \in (-L_1, L_2),\ \forall y \in (0, +\infty), \tag{3.39}$$

and for the boundary condition on the left, we get

$$U_0(-L_1, 0, t) = g_1(t),\ U_j(-L_1, 0, t) = 0,\ j \geq 1. \tag{3.40}$$

From $F_{-1} = 0$ in (3.38), we see that $\partial_y U_0^*(y,t) = \alpha_0(t)e^{ay}$, and with the same splitting of $F_j = 0$ we used in (3.8), we obtain

$$\begin{aligned} \underline{F}_0 &:= \mathcal{L}_a u_0 - f = 0, \quad \underline{F}_j := \mathcal{L}_a u_j - \partial_x^2 u_{j-1} = 0,\ j \geq 1, \\ \underline{F}_j^* &:= \mathcal{L}_- U_{j+1}^* + (\partial_t + c)U_j^* = 0,\ j \geq 0. \end{aligned} \tag{3.41}$$

The outer expansion is determined by transport equations into the negative x direction,

$$\begin{aligned} &\mathcal{L}_a u_0 = f, &&u_0(x,0) = h(x), &&u_0(L_2,t) = g_2(t), \\ &\mathcal{L}_a u_j = \partial_x^2 u_{j-1}, &&u_j(x,0) = 0, &&u_j(L_2,t) = 0, \quad j \geq 1. \end{aligned}$$

As for the inner expansion, the functions U_j^* are computed recursively using $\underline{F}_{j-1}^*$, with zero initial data, and boundary condition at $y = 0$ given by

$$U_0^*(0,\cdot) = g_1 - u_0(-L_1,\cdot), \quad U_j^*(0,\cdot) = -u_j(-L_1,\cdot) \text{ for } j \geq 1.$$

To start the recursion, U_0^* is given by

$$U_0^*(y,t) = (g_1(t) - u_0(-L_1,t))e^{ay}.$$

At step j, U_j^* is a polynomial of degree j in the y variable with coefficients depending on $g_1(t)$ and the boundary values $u_j(-L_1,t)$.

$$U_j^*(y,t) = e^{ay}\sum_{i=1}^{j}\alpha_i(t)y^i. \quad \Box$$

According to the lemma, close to $x = L_2$, there is no boundary layer, and the outer expansion is valid. On the other hand, the inner expansion U_j^* decays faster than any polynomial in ν at $x = L_2$.

3.2.2. Analysis of the factorization algorithm

The first transport equation in (2.14) yields a solution u_a^1 in $\Omega_2 \times (0,T)$ which is infinitely smooth, and not depending on ν. We thus have

$$u_a^1 = u_0 \text{ in } \Omega_2 \times (0,T). \tag{3.42}$$

We next compute a multiscale expansion of u_a^2, defined by (2.15).

Lemma 3.8. *The solution of the second advection equation (2.15) in $\Omega_2 \times (0,T)$ in the factorization algorithm is*

$$u_a^2 = \frac{a^2}{\nu}u_0 + u_{a,1}^2, \tag{3.43}$$

where

$$u_{a,1}^2 = a^2u_1 + \mathcal{L}_{ma}^0 u_0. \tag{3.44}$$

Proof. Replacing u_a^1 by u_0 in (2.15), and inserting the regular expansion

$$\frac{1}{\nu}\sum_{j\geq 0}\nu^j u_{a,j}^2(x,t)$$

into the differential equation (2.15) with $\mathcal{L}_{ma} = \mathcal{L}_{ma}^0 + \frac{a^2}{\nu}$ yields

$$\begin{aligned}
&\mathcal{L}_a u_{a,0}^2 = a^2 f, && u_{a,0}^2(\cdot,0) = a^2 h, && u_{a,0}^2(L_2,\cdot) = a^2u_0(L_2,\cdot),\\
&\mathcal{L}_a u_{a,1}^2 = \mathcal{R}\,u_0, && u_{a,1}^2(\cdot,0) = \mathcal{L}_{ma}^0 u_0(\cdot,0), && u_{a,1}^2(L_2,\cdot) = \mathcal{L}_{ma}^0 u_0(L_2,\cdot),\\
&\mathcal{L}_a u_{a,j}^2 = 0, && u_{a,j}^2(\cdot,0) = 0, && u_{a,j}^2(L_2,\cdot) = 0, \quad j \geq 2.
\end{aligned} \tag{3.45}$$

This determines $u_{a,0}^2 = a^2u_0$ and $u_{a,j}^2 = 0$ for $j \geq 2$ in Ω_2; for $u_{a,1}^2$, we define $v := u_{a,1}^2 - \mathcal{L}_{ma}^0 u_0$, and compute

$$\mathcal{L}_a v = \mathcal{R}\,u_0 - \mathcal{L}_a\mathcal{L}_{ma}^0 u_0 = a^2\partial_x^2 u_0.$$

Since v vanishes at $t = 0$ and at $L_2 = 0$, it is equal to a^2u_1 in $\Omega_2 \times (0,T)$. $\Box$

We finally give a multiscale expansion of u_{ad}, solution of the advection–reaction–diffusion equation (2.16) in Ω_1.

Lemma 3.9. *There is a unique formal multiscale solution of the mixed Cauchy problem (2.16) in $\Omega_1 \times (0,T)$, of the form*

$$u_{ad}^{out}(x,t) + u_{ad}^{in}(x,t) = \sum_{j\geq 0}\nu^j u_{ad,j}(x,t) + \sum_{j\geq 0}\nu^j U_{ad,j}^*(\frac{x+L_1}{\nu},t), \tag{3.46}$$

with $u_{ad,j} = u_j$ for $j \leq 1$, and for $j \geq 2$,

$$\mathcal{L}_a u_{ad,j} = \partial_x^2 u_{ad,j-1}, \quad u_{ad,j}(x,0) = 0, \quad u_{ad,j}(0,t) = -\frac{1}{a^2}\mathcal{L}_{ma}^0 u_{ad,j-1}(0,t).$$

Each term $U_{ad,j}^*(y,t)$ of the inner expansion *is a polynomial of degree j in y multiplied by e^{ay}, with coefficients depending on $g_1(t)$ and of $u_{ad,i}(-L_1,t)$ for $0 \leq i \leq j$. The first terms in the* inner expansion *are*

$$U_{ad,j}^* = U_j^* \text{ for } j \leq 1.$$

Proof. Recall that $\mathcal{L}_{ma} = \mathcal{L}_{ma}^0 + \frac{a^2}{\nu}$, with $\mathcal{L}_{ma}^0 = \partial_t + c - a\partial_x$. We have the same boundary layer at $x = -L_1$ as in Section 3.2.1. Therefore we will focus on what happens at $x = 0$. The boundary condition at $x = 0$ can be written as

$$\mathcal{L}_{ma}u_{ad}(0,\cdot) = u_a^2(0,\cdot) = (\frac{a^2}{\nu}u_0 + u_{a,1}^2)(0,\cdot). \tag{3.47}$$

We seek an expansion of u_{ad} of the form

$$u_{ad}^{out}(x,t)=\sum_{j\geq 0}\nu^j u_{ad,j}(x,t).$$

Inserting this expansion into the differential equation as before and collecting terms gives

$$\mathcal{L}_a u_{ad,0}=f,\quad \mathcal{L}_a u_{ad,j}=\partial_x^2 u_{ad,j-1},\quad j\geq 1. \tag{3.48}$$

The initial condition has the expansion

$$u_{ad,0}(x,0)=h(x),\quad u_{ad,j}(x,0)=0,\quad j\geq 1, \tag{3.49}$$

and from the transmission condition at $x=0$, $\mathcal{L}_{ma}u_{ad}(0,\cdot)=u_a^2(0,\cdot)$, we obtain

$$\begin{aligned}
&a^2u_{ad,0}=a^2u_0,\\
&a^2u_{ad,1}+\mathcal{L}_{ma}^0u_{ad,0}=u_{a,1}^2,\\
&a^2u_{ad,j+1}+\mathcal{L}_{ma}^0u_{ad,j}=0,\quad j\geq 1.
\end{aligned} \tag{3.50}$$

From the zeroth order term equations, we obtain $u_{ad,0}$ by solving

$$\mathcal{L}_a u_{ad,0}=f,\quad u_{ad,0}(x,0)=h(x),\quad u_{ad,0}(0,\cdot)=u_0(0,\cdot).$$

This shows that $u_{ad,0}=u_0$ in $\Omega_1\times(0,T)$. Moreover, for any (j,k), $\partial_t^j\partial_x^k u_{ad,0}(0,\cdot)=\partial_t^j\partial_x^k u_0(0,\cdot)$. For the first order term, we get

$$\mathcal{L}_a u_{ad,1}=\partial_x^2u_0,\quad u_{ad,1}(x,0)=0,\quad u_{ad,1}(0,\cdot)=\frac{1}{a^2}(u_{a,1}^2-\mathcal{L}_{ma}^0u_{ad,0})(0,\cdot)=u_1(0,\cdot).$$

Therefore,

$$u_{ad,j}\equiv u_j \text{ for } j\leq 1 \text{ in } \Omega_1.$$

At order 2, $\mathcal{L}_a u_{ad,2}=\partial_x^2u_1$ in Ω_1. We verify that $u_{ad,2}\neq u_2$ by considering the boundary condition at $x=0$, $-\mathcal{L}_{ma}^0u_{ad,1}/a^2$, and showing that $\mathcal{L}_a(-\mathcal{L}_{ma}^0u_{ad,1}/a^2)=\partial_x^2u_1-Ru_1\neq\partial_x^2u_1$.

The inner expansion is obtained as in the proof of Lemma 3.7 □

Using these lemmas we can now obtain the following error estimates:

Theorem 3.10. *In the case of negative advection, we obtain for the factorization algorithm the error estimates*

$$\|u-u_a^1\|_{L^2_{x,t}}\sim\nu\|u_1\|_{L^2_{x,t}},\quad \|u-u_{ad}\|_{L^2_{x,t}}\sim\nu^2\|u_2-u_{ad,2}\|_{L^2_{x,t}},$$

where $\|\cdot\|_{L^2_{x,t}}$ *stands for the* L^2 *norm in the considered spatial domain and on the time interval* $(0,T)$.

Proof. Since $u_a^1=u_0$, we have

$$u-u_a^1\sim\nu u_1.$$

By the lemmas above, we obtain

$$u-u_{ad}\sim\nu^2(u_2-u_{ad,2})+\nu^2(U_2^*-U_{ad,2}^*).$$

From the form of the coefficients of the outer and inner expansions, we deduce

$$\|u_2-u_{ad,2}\|_{L^2_{x,t}}=\mathcal{O}(1),\quad \|U_2^*-U_{ad,2}^*\|_{L^2_{x,t}}=\mathcal{O}(\nu).$$

Therefore

$$\|u-u_{ad}\|_{L^2_{x,t}}\sim\nu^2\|u_2-u_{ad,2}\|_{L^2_{x,t}}.\quad\square$$

Remark 3.3. If the second advection equation (2.15) defining u_a^2 was replaced by

$$\begin{cases}
\mathcal{L}_a u_a^2=\dfrac{a^2}{\nu}f+\mathcal{R}(u_0+\nu u_1) \text{ in } \Omega_2\times(0,T),\\
u_a^2(L_2,\cdot)=0,\\
u_a^2(\cdot,0)=0,
\end{cases} \tag{3.51}$$

then the error would be $\mathcal{O}(\nu^3)$. This would add the solution of a third transport equation, that defining u_1.

4. Multiscale analysis of the variational algorithm

We will not give in this case the complete analysis, since it is very similar to the one above; we will only present the dominant terms in the multiscale expansions. We have to study again the two cases for the advection direction separately.

4.1. Positive advection

We expect the advection–reaction–diffusion solution u_{ad}^V of (2.3) to have the same outer expansion as u, and u_a^V to have only an outer expansion,

$$u_{ad}^V(x,t) = \sum_{j\geq 0} \nu^j u_j(x,t) + \sum_{j\geq 0} \nu^j U_{ad,j}^{V,*}(-\frac{x}{\nu},t), \quad u_a^V(x,t) = \sum_{j\geq 0} \nu^j u_{a,j}^V(x,t).$$

The computations in the subdomains are decoupled in the variational algorithm. We start with u_{ad}^V in Ω_1. The boundary condition $\nu\partial_x u_{ad}^V = 0$ at $x = 0$ yields boundary conditions for the inner expansion,

$$\partial_y U_{ad,0}^{V,*}(0,t) = 0, \quad \partial_y U_{ad,j+1}^{V,*}(0,t) = \partial_x u_j(0,t), \quad j \geq 0,$$

which allow us to compute

$$U_{ad,0}^{V,*} = 0, \quad U_{ad,1}^{V,*}(y,t) = -\frac{1}{a}\partial_x u_0(0,t)e^{-ay}.$$

The sequence of transport problems for Ω_2 is

$$\mathcal{L}_a u_{a,0}^V = f, \quad \mathcal{L}_a u_{a,j}^V = 0,\ j \geq 1,$$

with zero initial values, since h vanishes in Ω_2. The transmission condition at $x = 0$, $u_a^V(0,\cdot) = u_{ad}^V(0,\cdot)$ translates into a sequence of conditions,

$$u_{a,j}^V(0,\cdot) = u_j(0,\cdot) + U_{ad,j}^{V,*}(0,\cdot).$$

Therefore, $u_{a,0}^V = u_0$, and from $u_{a,1}^V(0,\cdot) = u_1(0,\cdot) - \frac{1}{a}\partial_x u_0(0,\cdot)$, we conclude that

$$(u_{ad}^V - u)(x,t) \sim \frac{\nu}{a}\partial_x u_0(0,t)e^{a\frac{x}{\nu}}, \quad u_a^V - u \sim \nu e_{a,1}^V,$$

with

$$\mathcal{L}_a e_{a,1}^V = -\partial_x^2 u_0, \quad e_{a,1}^V(0,\cdot) = -\frac{1}{a}\partial_x u_0(0,\cdot), \quad e_{a,1}^V(\cdot,0) = 0. \tag{4.1}$$

Hence

$$\|u_{ad}^V - u\|_{L^2_{x,t}} \sim \sqrt{\frac{\nu^3}{2a^3}}\|\partial_x u_0(0,\cdot)\|_{L^2_t}, \quad \|u_a^V - u\|_{L^2_{x,t}} \sim \nu\|e_{a,1}^V\|_{L^2_{x,t}}. \tag{4.2}$$

Remark 4.1. Comparing with the result of the factorization algorithm in Theorem 3.6, we can see that the error in Ω_1 is $\mathcal{O}(\nu^{\frac{3}{2}})$ instead of $\mathcal{O}(\nu^{\frac{9}{2}})$, and the error in Ω_2 is of the same order, the problems (3.47) and (4.1) are slightly different, due to the boundary layer in Ω_1 at $x = 0$.

4.2. Negative advection

We first note that u_a does not depend on ν and coincides with u_0 in Ω_2. For u_{ad}, we expect as in Section 3.2.2 a boundary layer at $x = -L_1$, and an outer expansion

$$u_{ad}^V(x,t) \approx \sum_{j\geq 0} \nu^j u_{ad,j}^V(x,t).$$

Inserting this expansion into the differential equation as before gives

$$\mathcal{L}_a u_{ad,0}^V = f, \quad \mathcal{L}_a u_{ad,j}^V = \partial_x^2 u_{ad,j-1}^V, \quad j \geq 1. \tag{4.3}$$

The expansion of the initial condition is

$$u_{ad,0}^V(x,0) = h(x), \quad u_{ad,j}^V(x,0) = 0, \quad j \geq 1. \tag{4.4}$$

The only difference with the analysis in Section 3.2.2 comes from the boundary data, which becomes

$$-\nu\partial_x u_{ad}^V(0,\cdot) + a u_{ad}^V(0,\cdot) = a u_0(0,\cdot),$$

and is expanded as

$$u^V_{ad,0}(0,\cdot) = u_0(0,\cdot), \quad -\partial_x u^V_{ad,j-1}(0,\cdot) + a u^V_{ad,j}(0,\cdot) = 0, \quad j \geq 1. \tag{4.5}$$

Therefore the zeroth order term $u^V_{ad,0}$ coincides with u_0. For the first order term, we get the equation

$$\mathcal{L}_a u^V_{ad,1} = \partial_x^2 u_0, \quad u^V_{ad,1}(\cdot,0) = 0, \quad u^V_{ad,1}(0,\cdot) = \frac{1}{a}\partial_x u_0(0,\cdot),$$

which shows that $e^V_{ad,1} = u^V_{ad,1} - u_1$ is solution of

$$\mathcal{L}_a e^V_{ad,1} = 0, \quad e^V_{ad,1}(0,\cdot) = (\frac{1}{a}\partial_x u_0 - u_1)(0,\cdot), \quad (e^V_{ad,1})(\cdot,0) = 0, \tag{4.6}$$

and

$$\|u - u^V_{ad}\|_{L^2_{x,t}} \sim \nu \|e^V_{ad,1}\|_{L^2_{x,t}}, \quad \|u - u^V_a\|_{L^2_{x,t}} \sim \nu \|u_1\|_{L^2_{x,t}}. \tag{4.7}$$

5. Algorithm with non variational conditions

As before, we proceed in two steps, depending on the advection direction.

5.1. The case of positive advection

We seek again a multiscale expansion of the form

$$u^{NV}_{ad}(x,t) = \sum_{j\geq 0} \nu^j u_j(x,t) + \sum_{j\geq 0} \nu^j U^{NV,*}_{ad,j}(-\frac{x}{\nu}, t), \quad u_a(x,t) = \sum_{j\geq 0} \nu^j u^{NV}_{a,j}(x,t).$$

Using similar arguments as before, we obtain the sequence of transport problems in $\Omega_2 \times (0,T)$,

$$\mathcal{L}_a u^{NV}_{a,0} = f, \quad \mathcal{L}_a u^{NV}_{a,j} = 0, \ j \geq 1,$$

with $u^{NV}_{a,0} = h$ at time 0 and vanishing initial conditions for $j \geq 1$. The transmission conditions translate into

$$\begin{aligned}
&\partial_y U^{NV,*}_{ad,0}(0,t) = 0,\\
&u_j(0,t) + U^{NV,*}_{ad,j}(0,t) = u^{NV}_{a,j}(0,t),\\
&\partial_x u_j(0,t) - \partial_y U^{NV,*}_{ad,j+1}(0,t) = \partial_x u^{NV}_{a,j}(0,t).
\end{aligned}$$

From this we see that $U^{NV,*}_{ad,0}(0,t) = 0$, therefore $u_0(0,t) = u^{NV}_{a,0}(0,t)$ and $u^{NV}_{a,0} = u_0$. Using the transport equation for $u^{NV}_{a,0}$ and u_0, we deduce that $\partial_x u_0(0,t) = \partial_x u^{NV}_{a,0}(0,t)$. Inserting this into the transmission condition yields $\partial_y U^{NV,*}_{ad,1}(0,t) = 0$, which implies that $U^{NV,*}_{ad,1} = 0$. Then using the transport equations again, we get

$$\partial_x u_1(0,t) - \partial_x u^{NV}_{a,1}(0,t) = \frac{1}{a}\partial_x^2 u_0(0,t).$$

Inserting this into the transmission condition yields

$$\partial_y U^{NV,*}_{ad,2}(0,t) = \frac{1}{a}\partial_x^2 u_0(0,t),$$

and $U^{NV,*}_{ad,2}(y,t) = -\frac{1}{a^2}\partial_x^2 u_0(0,t)e^{-ay}$. Therefore we obtain that

$$(u^{NV}_{ad} - u)(x,t) \sim -\frac{\nu^2}{a^2}\partial_x^2 u_0(0,t)e^{a\frac{x}{\nu}}, \quad u^{NV}_a - u \sim \nu e^{NV}_{a,1},$$

where $e^{NV}_{a,1} = u^{NV}_{a,1} - u_1$ is solution in $\Omega_2 \times (0,T)$ of

$$\mathcal{L}_a e^{NV}_{a,1} = -\partial_x^2 u_0, \quad \partial_x e^{NV}_{a,1}(0,\cdot) = -\frac{1}{a}\partial_x^2 u_0(0,\cdot), \quad (e^{NV}_{ad,1})(\cdot,0) = 0, \tag{5.1}$$

which gives the estimates

$$\|u^{NV}_{ad} - u\|_{L^2_{x,t}} \sim \sqrt{\frac{\nu^5}{2a^5}}\|\partial_x^2 u_0(0,\cdot)\|_{L^2_t}, \quad \|u^{NV}_a - u\|_{L^2_{x,t}} \sim C\nu\|e^{NV}_{a,1}\|_{L^2_{x,t}}. \tag{5.2}$$

5.2. Negative advection

In this case the advective solution is the same as in the variational case,

$$u_a^{NV} = u_a^V = u_0,$$

and the outer expansion of the advection–reaction–diffusion equation is defined by

$$u_{ad}^{NV}(x,t) \approx \sum_{j\geq 0} \nu^j u_{ad,j}^{NV}(x,t).$$

Inserting this expansion into the differential equation as before gives

$$\mathcal{L}_a u_{ad,0}^{NV} = f, \quad \mathcal{L}_a u_{ad,j}^{NV} = \partial_x^2 u_{ad,j-1}^{NV}, \quad j \geq 1. \tag{5.3}$$

The expansion of the initial condition is

$$u_{ad,0}^{NV}(x,0) = h(x), \quad u_{ad,j}^{NV}(x,0) = 0, \quad j \geq 1. \tag{5.4}$$

The only difference with the analysis in the variational case comes from the transmission condition, which yields

$$u_{ad}^{NV}(0,\cdot) = u_0(0,\cdot),$$

and is expanded as

$$u_{ad,0}^{NV}(0,\cdot) = u_0(0,\cdot), \quad u_{ad,j}^{NV}(0,\cdot) = 0, \quad j \geq 1. \tag{5.5}$$

Therefore the zeroth order term $u_{ad,0}^{NV}$ coincides with u_0. For the first order term, we get the equation

$$\mathcal{L}_a u_{ad,1}^{NV} = \partial_x^2 u_0, \quad u_{ad,1}^{NV}(0,\cdot) = 0, \quad u_{ad,1}^{NV}(\cdot,0) = 0,$$

which shows that $e_{ad,1}^{NV} = u_{ad,1}^{NV} - u_1$ is solution of

$$\mathcal{L}_a e_{ad,1}^{NV} = 0, \quad e_{ad,1}^{NV}(0,\cdot) = -u_1(0,\cdot), \quad (e_{ad,1}^{NV})(\cdot,0) = 0, \tag{5.6}$$

and

$$\|u - u_{ad}^{NV}\|_{L^2_{x,t}} \sim \nu \|e_{ad,1}^{NV}\|_{L^2_{x,t}}, \quad \|u - u_a^{NV}\|_{L^2_{x,t}} \sim \nu \|u_1\|_{L^2_{x,t}}. \tag{5.7}$$

6. Numerical experiments

We start with a numerical experiment in 1D on the domain $\Omega := (-1, 1)$, to illustrate the asymptotic performance of the various coupling methods predicted by our analysis. We use as our model problem

$$\begin{cases} u_t + au_x - \nu u_{xx} + cu = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \{-1\} \times (0,T), \\ u_t + au_x + cu = 0 & \text{on } \{1\} \times (0,T), \\ u(\cdot,0) = h & \text{in } \Omega, \end{cases}$$

with $a = 1$, $c = 1$, $T = 0.5$ and varying ν. We use as initial condition $h(x) = e^{-100(x+0.5)^2}$, and as right hand side the function

$$f(x,t) = f_1(t) f_2(x),$$
$$f_1(t) = 10 \sin^4(4\pi(t-0.05))\chi_{t>0.05},$$
$$f_2(x) = -e^{-30(x-0.5)^2} + e^{-30(x+0.5)^2}.$$

For the discretization, we use a Crank–Nicolson scheme for the advection–reaction–diffusion equation and an implicit upwind scheme for the advection equation, with $\Delta t = \Delta x = 1.5625\,10^{-5}$. The viscous domain is $\Omega_1 = (-1, 0)$ and the inviscid one is $\Omega_2 = (0, 1)$. For the non variational algorithm (2.7)–(2.8), we introduced a relaxation in the iteration to obtain convergence, i.e.

$$(u_a^{NV})^k(0,\cdot,\cdot) = \theta(u_a^{NV})^{k-1}(0,\cdot,\cdot) + (1-\theta)(u_{ad}^{NV})^k(0,\cdot,\cdot),$$

where we used $\theta = 1/(450\sqrt{\nu})$ based on a heuristic to ensure good convergence.

We show in Fig. 6.1 the asymptotic performance of the various coupling methods when ν becomes small.

We can clearly see the asymptotic behavior predicted by our analysis, and also the predicted hierarchy of quality of the coupled solution. This hierarchy remains even when ν is not small, which our asymptotic analysis cannot predict, and thus the new coupling method based on factorization is really giving a better coupled solution, also when ν is not small.

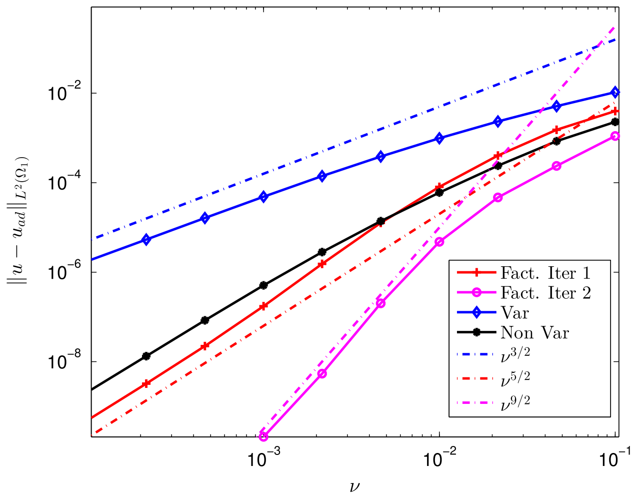

Fig. 6.1. Asymptotic performance of the various coupling methods in 1D compared to our theoretical estimates.

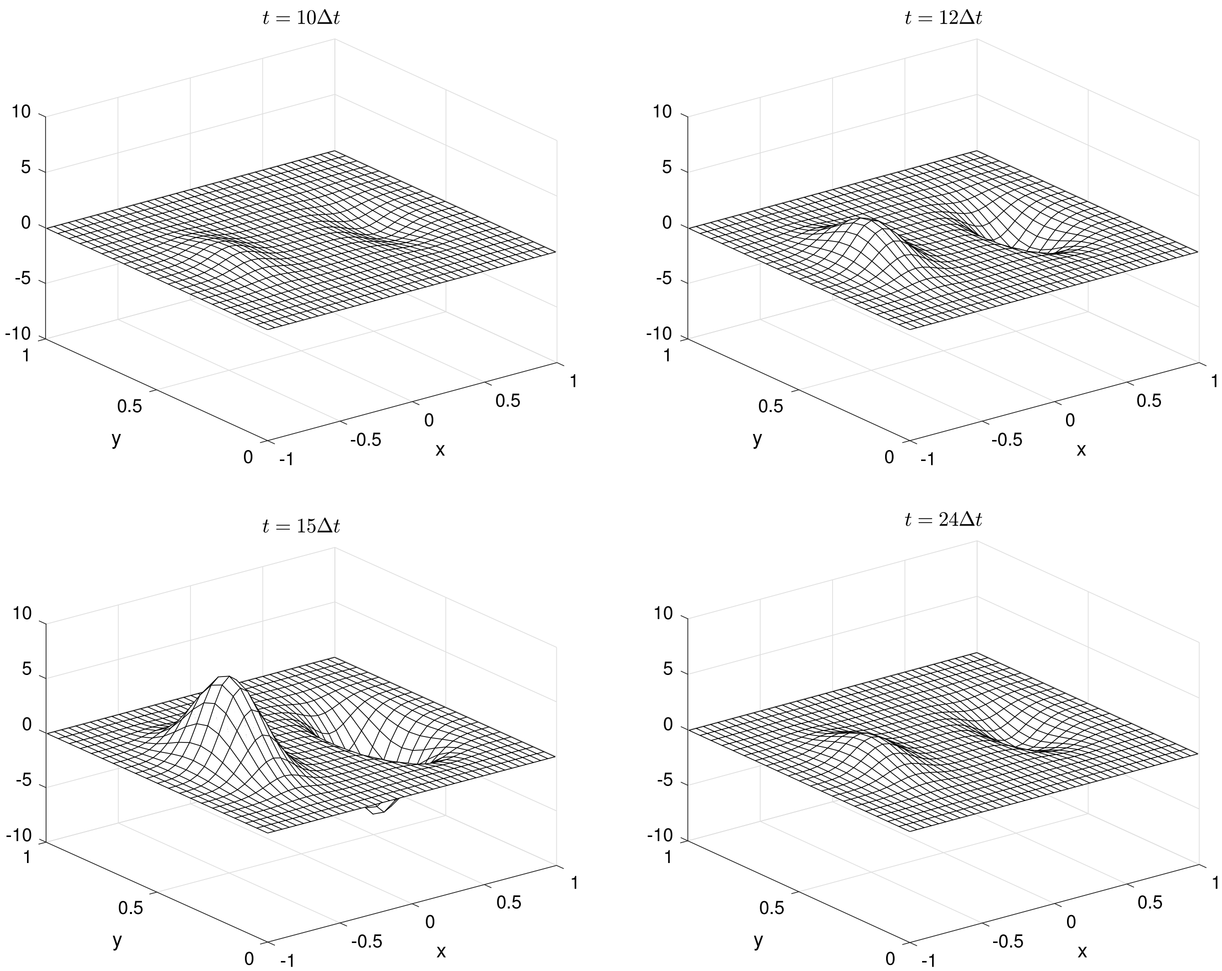

Fig. 6.2. Snapshots of the right hand side function at times $t = 10\Delta t$, $12\Delta t$, $15\Delta t$ and $24\Delta t$.

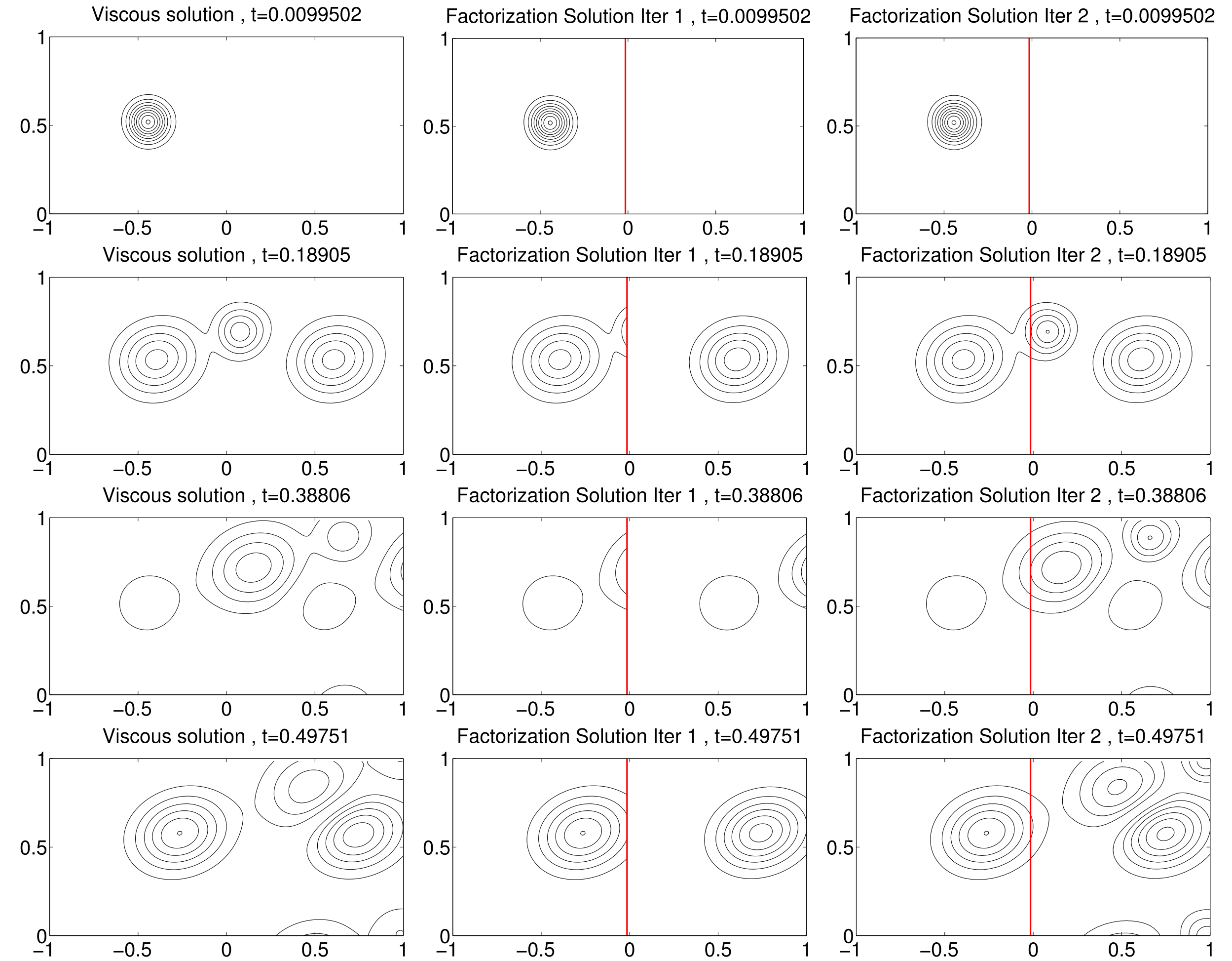

Fig. 6.3. From top to bottom: several snapshots at several time steps in the x–y plane. From left to right: viscous solution, solution of Algorithm (2.11)–(2.12)-(2.13) at iteration $k = 1$, and at iteration $k = 2$.

We now want to compare the quality of the coupling methods numerically for a given viscosity and a two dimensional problem, which also goes beyond our analysis, posed in the domain $\Omega = (-1, 1) \times (0, 1)$,

$$\begin{cases} \partial_t u + \mathbf{a} \cdot \nabla u - \nu \Delta u + cu = f & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \{-1\} \times (0, 1) \times (0, T), \\ \partial_t u + \mathbf{a} \cdot \nabla u + cu = 0 & \text{on } \{1\} \times (0, 1) \times (0, T), \\ u(\cdot, \cdot, 0) = h & \text{in } \Omega, \end{cases}$$

and we impose periodic boundary conditions in the y-direction. The physical parameters are $\mathbf{a} = (3, 1)$, $c = 1$, $\nu = 0.01$ and $T = 0.5$. We use as initial condition $h(x, y) = e^{-100((x+0.5)^2+(y-0.5)^2)}$, and as right hand side the function

$$\begin{aligned} f(x, y, t) &= f_1(t) f_2(x, y), \\ f_1(t) &= 10 \sin^4(4\pi(t - 0.05))\chi_{t>0.05}, \\ f_2(x, y) &= -e^{-30((x-0.5)^2+(y-0.5)^2)} + e^{-30((x+0.5)^2+(y-0.5)^2)}, \end{aligned}$$

an illustration of which is shown in Fig. 6.2.

For the discretization, we use again a Crank–Nicolson scheme for the advection–reaction–diffusion equation and an implicit upwind scheme for the advection equation, with spatial steps $\Delta x = \Delta y = 10^{-2}$ and time step $\Delta t = \Delta x$. The viscous domain is $\Omega_1 = (-1, 0) \times (0, 1)$ and the inviscid one is $\Omega_2 = (0, 1) \times (0, 1)$.

In the left column of Fig. 6.3, we show snapshots of the viscous solution at several instances in time. In the second and third columns, we show the solution obtained with the factorization algorithm (2.11)–(2.12)–(2.13) at iteration one and two. We see that with only one iteration, the coupled solution in the viscous region is already very good, and the second

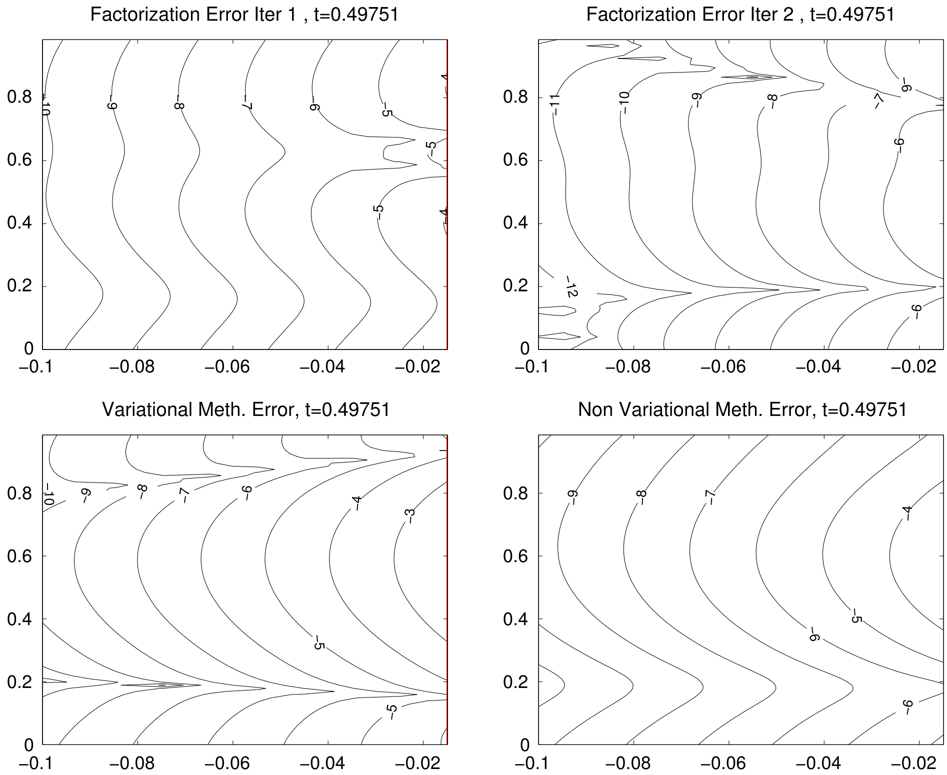

Fig. 6.4. Level sets of the error in Ω_1 in the x-y plane for the different coupling methods at the final time.

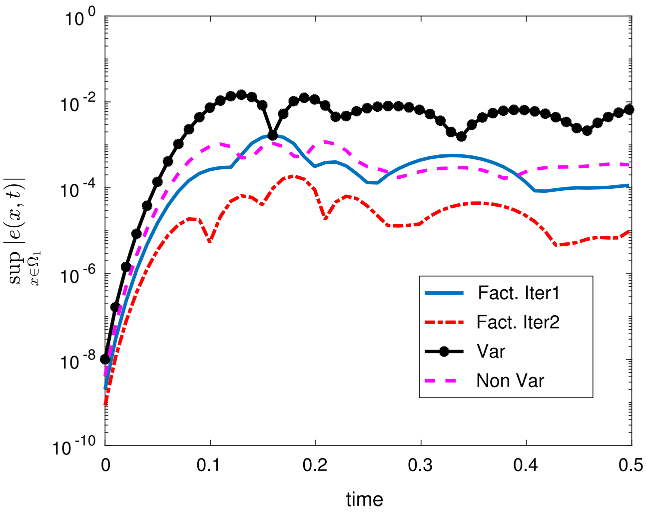

Fig. 6.5. Error $\sup_{x\in\Omega_1} |e(x,t)|$ as a function of time t, where e stands for the error between the viscous solution and the coupled one.

iteration advects this good solution through (2.11) into the inviscid region, and from this produces an even better solution in the viscous region.

To get a quantitative comparison with the variational algorithm (2.3)–(2.4) and the non variational algorithm (2.7)–(2.8), we computed the errors in the viscous region. We show in Fig. 6.4 the level sets of the errors at the end of the time interval, and in Fig. 6.5 how the L^∞ norm of the error in space evolves as a function of time.

We clearly see that also in two spatial dimensions the best results are obtained with the factorization method after two iterations, as predicted by our multiscale analysis for the one dimensional problem. Even after one iteration, the factorization method gives smaller errors than the non variational method, and the variational method gives by far the largest error, two orders of magnitude larger close to the interface than the factorization method. Our multiscale analysis in one dimension thus reliably predicts the quality of the different coupling methods, also in higher space dimensions.

7. Conclusion

Using formal multiscale expansions, we have obtained error estimates for three heterogeneous domain decomposition algorithms for the coupling of time dependent advection–reaction–diffusion equations with advection reaction equations. Our error estimates show that in the case of positive advection, one can obtain with the factorization algorithm L^2 errors in the diffusive region which are $\mathcal{O}(\nu^{9/2})$ after two iterations. The first iteration gives already $\mathcal{O}(\nu^{5/2})$, a result which can only be achieved with a fully converged non-variational heterogeneous domain decomposition method after many iterations. The variational heterogeneous domain decomposition method only performs one iteration, but also gives a much larger error of $\mathcal{O}(\nu^{3/2})$ which cannot be improved any more. In the case of negative advection, the factorization method gives an error of $\mathcal{O}(\nu^2)$ in the diffusive region, whereas the other algorithms only give errors $\mathcal{O}(\nu)$ for comparable computational cost, since each algorithm only solves one expensive diffusive problem in the same region. In the regions where the diffusion is neglected, all the algorithms have the same error term $\mathcal{O}(\nu)$. We showed with numerical experiments that our one dimensional asymptotic results also predict very well the behavior of the three coupling algorithms in two spatial dimensions; the factorization algorithm gave in our experiments for a fixed viscosity a one to two orders of magnitude more accurate solution in the important viscous region. In the active research area of heterogeneous domain decomposition methods, new coupling techniques continue to be developed, for example the recently proposed one based on interface control [24], and it will be interesting to compare the quality of coupled solutions obtained with this new technique using the multiscale approach we presented here.

Acknowledgments

We would like to thank Guy Metivier for the fruitful discussions on multiscale analysis, and two anonymous referees for their valuable suggestion to do a careful numerical comparison of the methods in two space dimensions.

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