# Stability of Perfectly Matched Layers for Maxwell's Equations in Rectangular Solids

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**Abstract**

Perfectly matched layers are extensively used to compute approximate solutions for Maxwell's equations in $\mathbb{R}^{1+3}$ using a bounded computational domain, usually a rectangular solid. A smaller rectangular domain of interest is surrounded by layers designed to absorb outgoing waves in perfectly reflectionless manner. On the boundary of the computational domain, an absorbing boundary condition is imposed that is necessarily imperfect. The method replaces the Maxwell equations by a larger system, and introduces absorption coefficients positive in the layers. Well posedness of the resulting initial boundary value problem is proved here for the first time. The Laplace transform of a resulting Helmholtz system is studied. For positive real values of the transform variable $\tau$, the Helmholtz system has a unique solution from a variational form that yields limited regularity for rectangular domains. When $\tau$ is not real the complex variational form loses positivity. We smooth the domain and, in spite of this loss, construct $H^2$ solutions with uniform $L^2$ estimates. Using the $H^2$ regularity, we deduce Maxwell from Helmholtz, then remove the smoothing. The boundary condition at the smoothed boundary must be carefully chosen. A method of Jerison-Kenig-Mitrea is extended to compensate the nonpositivity of the flux.

**Keywords.** Perfectly matched layers, Maxwell's equations, absorbing boundary conditions, trihedral corner, stability.

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# 1 Introduction

This paper proves well posedness of algorithms that compute approximate solutions of Maxwell's equations

$$\partial_t E - \operatorname{curl} B = -j\,, \qquad \partial_t B + \operatorname{curl} E = 0, \tag{1.1}$$

on the unbounded domain $\mathbb{R}_t \times \mathbb{R}^3$. The algorithms use a bounded computational domain $\mathcal{Q} \subset \mathbb{R}^3$. The values of the fields are sought on a smaller domain of interest denoted $\mathcal{Q}_I$. Perfectly matched layers are employed in $\mathcal{Q} \setminus \mathcal{Q}_I$.

The current density $j$ and charge density $\rho$ supported in $\{t \geq 0\} \times \mathcal{Q}_I$ are given and satisfy the conservation identity

$$\partial_t \rho \;=\; -\mathrm{div}\; j. \tag{1.2}$$

Taking the divergence of (1.1) implies that,

$$\partial_t(\operatorname{div} E) + \operatorname{div} j \;=\; \partial_t(\operatorname{div} B) \;=\; 0.$$

Since $E, B, j$ are supported in $t \geq 0$, it follows that $E$ and $B$ satisfy

$$\operatorname{div} E \;=\; \rho, \qquad \text{and}, \qquad \operatorname{div} B = 0\,. \tag{1.3}$$

A fundamental difficulty posed by the bounded computational domain is that the boundary $\partial\mathcal{Q}$ is not physical. Waves in $\mathbb{R}^3$ simply pass through it undisturbed. A numerical algorithm needs to mimic that invisibility. This is done by introducing a boundary condition chosen to approximately reproduce this transparency. Such conditions let waves leave $\mathcal{Q}$ so that solutions viewed in $\mathcal{Q}$ seem to lose energy. The conditions are called *absorbing boundary conditions* [19]. In dimension $d > 1$ there is invariably some reflection that pollutes the approximation. A remedy with extremely high computational cost is to enlarge $\mathcal{Q}$ so that the reflections do not have time to come back. A second remedy is to modify the equations in a layer about $\mathcal{Q}_I$ to absorb waves. The aim is that waves that reach $\partial\mathcal{Q}$ are weak so that their reflections are small. However even very cleverly constructed layers themselves reflect and can create errors analogous to those from the boundary conditions. In the early nineties perfectly reflectionless layers were constructed.

### 1.1 The PML strategy

**Assumption 1.1** *$\mathcal{Q}$ is a rectangular solid, and there is a rectangular solid $\mathcal{Q}_I$ with $\overline{\mathcal{Q}_I} \subset \mathcal{Q}$ called the* **domain of interest***. Assume that the charge density $\rho$ and current density $j$ are supported in $[0,\infty[\times\overline{\mathcal{Q}_I}$ and satisfy* (1.2).

The PML strategy introduces a new larger system of equations in $\mathcal{Q}$. A linear combination of the new unknowns yields the approximate solution in $\mathcal{Q}_I$. There are three goals.

$*$ In $\mathcal{Q} \setminus \overline{\mathcal{Q}}_I$ solutions of the augmented system are damped.

$*$ There is *no reflection at all* from the layers in $\mathcal{Q} \setminus \overline{\mathcal{Q}}_I$.

$*$ The strategy is easy to implement.

Such perfectly matched layers for Maxwell's equations were constructed by Bérenger [8, 9]. The existence of such layers is truly remarkable. After Bérenger, related PML were introduced [35, 13, 42, 28, 39, 43]. They all have at their core the stretched equations introduced below. Our analysis of the stretched equations suffices to analyse them all (see Section 9).

The combination of perfectly matched layers for Maxwell's equations with standard absorbing boundary conditions on $\partial\mathcal{Q}$ is amazingly efficient. It is deservedly one of the most widely used algorithms in computational physics. As a result of this massive experience, the behavior of the algorithm in practice is well understood. Four desirable properties are observed.

• **Stability**. The computed solutions are bounded on bounded time intervals.

• **Corners are easy**. Absorbing conditions at the smooth parts of the boundary suffice.

• **Absorbing layers.** The computed waves decrease as they pass through the regions with nonvanishing absorptions (introduced in Section 1.3).

• **Uniform stability.** The computed solutions are bounded in $\{t \geq 0\}$.

There is an enormous gap between what is understood in practice and what is proved. In this paper we prove that the algorithm is well posed. Solutions with perfectly matched layers and absorbing boundary conditions at $\partial\mathcal{Q}$ exist, are uniquely determined, and remain bounded on bounded intervals of time. In addition, the boundary conditions that determine the solutions are imposed only at the smooth points of $\partial\mathcal{Q}$. This proves the first two bullets. The last two bulleted properties are outstanding open problems.

The well posedness implies an error estimate. The error in the domain of interest is bounded by a constant times the size of the solution of the PML boundary value problem near the boundary of the computational domain. The proof given in Section 9.2 depends only on perfection and well posedness of the PML boundary value problem.

The well posedness has remained unsolved for more than thirty years for several reasons.

• The computational domain has trihedral corners. The boundary conditions change when one passes from one face of the rectangular solid to another. Such hyperbolic problems are little developed.

• The PML involve absorption coefficients $\sigma_j(x_j)$ that vanish identically on $\mathcal{Q}_I$. These variable coefficients defeat exact solution strategies for the absorbing boundary conditions.

• All PML introduce auxiliary variables. The augmented systems are degenerate in one way or another. They are plagued by characteristic boundaries, loss of ellipticity, weak hyperbolicity, *etc.*

To formulate precise results begin with some definitions.

**Definition 1.1** *Define a* $\mathbb{C}^6$ *valued function* $u$ *and a* $6\times 6$ *sytem of differential operators* $\mathcal{A}$ *by,*

$$u \ := \ (E,B), \qquad \mathcal{A}(\partial) \ := \ \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}, \qquad \partial = (\partial_1, \partial_2, \partial_3)\,.$$

*Equation* (1.1) *is equivalent to*

$$\partial_t u \ + \ \mathcal{A}(\partial) u \ = \ f, \qquad f \ := \ (-j, 0)\,. \tag{1.4}$$

*Define real antisymmetric* $3\times 3$ *matrices* $A_k$ *and* $6\times 6$ *real symmetric matrices* $\mathcal{A}_k$*,*

$$A_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$A_3 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{A}_k := \begin{pmatrix} 0 & A_k \\ -A_k & 0 \end{pmatrix}.$$

*Then* $\mathcal{A}(\partial) = \sum \mathcal{A}_k \partial_k$*. Define*

$$\mathcal{A}(\xi) \ := \ \sum_k \xi_k \mathcal{A}_k \ = \ \begin{pmatrix} 0 & -\xi\wedge \\ \xi\wedge & 0 \end{pmatrix}. \tag{1.5}$$

**Definition 1.2** *Denote by* $\mathcal{Q} = \mathcal{Q}(L_1, L_2, L_3)$ *the rectangular solid*

$$\mathcal{Q} \ := \ \big\{x \in \mathbb{R}^3 \ : \ |x_j| < L_j/2, \quad j = 1, 2, 3\big\}.$$

*The boundary of* $\mathcal{Q}$ *has six open faces* $\Gamma_k$ *with* $1 \le k \le 6$*. For* $i = 1, 2, 3$*,*

$$\Gamma_i := \big\{x_i = -L_i/2, \text{ and, } |x_j| < L_j/2 \text{ for } j \ne i\big\},$$
$$\Gamma_{i+3} := \big\{x_i = L_i/2, \text{ and, } |x_j| < L_j/2 \text{ for } j \ne i\big\}.$$

*For a point* $x \in \Gamma_k$*,* $\nu(x)$ *denotes the outward unit normal to* $\mathcal{Q}$ *at* $x$*. The singular subset* $\mathcal{S} \subset \partial\mathcal{Q}$ *consists of the points of the boundary so that* $|x_j| = L_j/2$ *for more than one value of* $j$*.*

## 1.2 Energy flux and absorbing boundary condition

The energy law for real solutions of Maxwell's equations is recalled and used to choose the absorbing boundary condition.

**Definition 1.3** *For vectors* $v$ *and* $w$ *in* $\mathbb{C}^d$ *the dot product is defined as* $v \cdot w := \sum_k v_k w_k$ *the sum of the products of the components. The* $\mathbb{C}^d$ *scalar product equal to* $v \cdot \overline{w}$ *is denoted* $(v, w)$*.*

**Remark 1.1** *If vectors* $v(\tau)$ *and* $w(\tau)$ *depend holomorphically on* $\tau$ *then* $v(\tau) \cdot w(\tau)$ *depends holomorphically on* $\tau$*. The* $\mathbb{C}^d$ *scalar product need not be holomorphic.*

Suppose that $\Omega \subset \mathbb{R}^{1+3}$ is a smoothly bounded open set. Suppose that $(E, B)$ is a real $C^1$ solution of Maxwell's equations with $\rho = 0$ and $j = 0$ on $\Omega$. The energy density is defined as $(\|E\|^2 + \|B\|^2)/2$. Compute three equivalent expressions for the flux of energy through an element of surface $d\Sigma$ with unit outward normal $\nu$. For real solutions,

$$\partial_t\big(\|E\|^2 + \|B\|^2\big)/2 = \big(E \cdot E_t + B \cdot B_t\big) = (E, B) \cdot (E, B)_t \,.$$

Maxwell's equations yield

$$E \cdot E_t + B \cdot B_t \ = \ E \cdot \operatorname{curl} B - B \cdot \operatorname{curl} E \,.$$

The vector identity $\operatorname{div}(u \wedge v) = v \cdot \operatorname{curl} u - u \cdot \operatorname{curl} v$ yields Poynting's identity,

$$\partial_t\big(\|E\|^2 + \|B\|^2\big)/2 \,+\, \operatorname{div}\big(E \wedge B\big) \ = \ 0.$$

Integrating and using the Divergence Theorem yields

$$\partial_t \int_\Omega (\|E\|^2 + \|B\|^2)/2 \; dx \;+\; \int_{\partial\Omega} (E \wedge B) \cdot \nu \; d\Sigma \;=\; 0.$$

The outward energy flux through $d\Sigma$ is equal to

$$(E \wedge B) \cdot \nu \; d\Sigma \,. \tag{1.6}$$

Compute the flux using the matricial form of the Maxwell operator. With $u = (E, B)$,

$$u \cdot \big(u_t \;+\; \sum \mathcal{A}_k \partial_k u\big) \;=\; 0.$$

Since the $\mathcal{A}_k$ are real and symmetric this yields the conservation law

$$\partial_t \|u\|^2 \;+\; \sum_k \partial_k \big(u \cdot \mathcal{A}_k u\big) \;=\; 0.$$

Integrating over $\Omega$ yields the second formula for the flux,

$$\partial_t \int_\Omega \|u\|^2 \; dx \;+\; \int_{\partial\Omega} u \cdot (\mathcal{A}(\nu) u) \; d\Sigma \;=\; 0.$$

**Definition 1.4** *For a hermitian matrix $A$, $\mathcal{E}^\pm(A)$ denote the spectral subspaces corresponding to strictly positive and strictly negative eigenvalues. Denote by $\pi^\pm(A)$ the orthogonal projections on those spaces. Denote the nullspace as $\mathcal{E}^0(A)$ with projector $\pi^0(A)$.*

Lemma 2.1 shows that the eigenvalues of $\mathcal{A}(\nu)$ are $\pm 1$ and 0. Therefore

$$u \cdot (\mathcal{A}(\nu) u) \;=\; \big\|\pi^+(\mathcal{A}(\nu)) u\big\|^2 \;-\; \big\|\pi^-(\mathcal{A}(\nu)) u\big\|^2.$$

Thus the outward flux of energy is,

$$\frac{1}{2}\, u \cdot (\mathcal{A}(\nu) u) \; d\Sigma \;=\; \frac{1}{2} \,\Big(\big\|\pi^+(\mathcal{A}(\nu)) u\big\|^2 \;-\; \big\|\pi^-(\mathcal{A}(\nu)) u\big\|^2\Big) d\Sigma. \tag{1.7}$$

The boundary of the computational domain $\mathcal{Q}$ is an artificial boundary. A natural choice of boundary condition is to require that $\pi^-(A(\nu))(E, B) = 0$. This says that the incoming part of the energy flux from (1.7) vanishes at $\partial\mathcal{Q}$. The outward flux is maximized.

**Definition 1.5** *If $\mathcal{O} \subset \mathbb{R}^3$ is an embedded 2-manifold, $\omega \in \mathcal{O}$, $\nu$ is a unit normal vector to $\mathcal{O}$ at $\omega$, and $V \in T_\omega(\mathcal{O})$, the tangential component of $V$, denoted $V_{tan}$, is defined as $V - (V \cdot \nu)\nu$.*

Example 2.4 shows that the absorbing boundary condition $\pi^-(A(\nu))(E,B) = 0$ holds if and only if $E_{tan} = B_{tan} \wedge \nu$. With this boundary condition on $\cup\Gamma_k$, the outward energy flux density is non negative. For *sufficiently regular solutions* this implies that the energy in $\mathcal{Q}$ is non increasing. The theory of symmetric positive boundary value problems of Friedrichs [20, 27, 36] shows that the initial boundary value problem for Maxwell's equations has weak solutions and that strong solutions are unique. The fact that the boundary is irregular defeats Friedrichs' mollifier strategy to prove the uniqueness of weak solutions in $L^2([0,T] \times \mathcal{Q})$.

Part 1 of [22] proves that this boundary value problem for the Maxwell equations in rectangular geometry is well posed. The proof relies on the fact that if $u$ is a harmonic function on a Lipschitz domain $\mathcal{Q}$ with $u|_{\partial\mathcal{Q}} \in H^1(\partial\mathcal{Q})$ then $u \in H^{3/2}(\mathcal{Q})$ (see [25, 14, 32]). In Section 8.2, uniqueness for this Maxwell boundary value problem on $\mathcal{Q}$ is used in our proof of uniqueness for the stretched boundary value problem treated in Theorem 1.15.

### 1.3 The split and stretched equations

This section introduces the absorption coefficients and associated stretched equations and coordinate stretchings.

**Assumption 1.2** *The split equations involve non negative* **absorption coefficients** $\sigma_k \in C^\infty(\mathbb{R})$. *Assume that* $\sigma_k(x_k)$ *vanishes for all* $x = (x_1, x_2, x_3) \in \overline{\mathcal{Q}_I}$ *and that for all* $\ell \in \mathbb{N}$, *the derivative* $\partial^\ell \sigma_k \in L^\infty(\mathbb{R})$.

**Remark 1.2** *The absorption* $\sigma_k(x_k)$ *is typically strictly positive on a neighborhood of* $\Gamma_k \cup \Gamma_{k+3}$. *As a function of* $x_k$, $\sigma_k$ *usually increases from value zero in* $\mathcal{Q}_I$ *to a maximum at* $\partial\mathcal{Q}$ *(see [8, 9]). The strips in* $\mathcal{Q}$ *where* $\sigma_k > 0$ *are called* **absorbing layers**.

Bérenger's splitting introduces $\widetilde{U} := (U^1, U^2, U^3)$ with $U^k$ taking values in $\mathbb{R}^6$. The split equations imposed on $\widetilde{U}$ are

$$\big(\partial_t + \sigma_k(x_k)\big)U^k \ + \ \mathcal{A}_k \partial_k (U^1 + U^2 + U^3) \ = \ f_k\,, \qquad k = 1,2,3\,. \tag{1.8}$$

The $k^{\text{th}}$ equation has the $\partial_k$ derivatives from Maxwell's system. The source terms $f_k$ are chosen so that $f_1 + f_2 + f_3 = (-j, 0) := f$. The most symmetrical choice is $f_k = f/3$. Another common choice is $f_1 = f$ and $f_2 = f_3 = 0$. To cover both treat

$$f_k := c_k f, \quad f = (-j, 0), \quad c_k \geq 0, \quad c_1 + c_2 + c_3 = 1. \tag{1.9}$$

Our method yields the same conclusion for each of these choices with estimates uniform for $c_k$ as in (1.9).

**Definition 1.6** *The equations* (1.8) *with* (1.9) *are called the* **split equations**. *For a solution of the split equations, the* **associated electric and magnetic fields** *are defined by* $(E, B) := U^1 + U^2 + U^3$. *The restriction of* $E, B$ *to* $\mathcal{Q}_I$ *is the output of Bérenger's method.*

The analysis of the pure initial value problem on $\mathbb{R}^{1+d}$ for the split equations corresponds to a computational domain $\mathcal{Q} = \mathbb{R}^3$. It is *much* simpler than the mixed initial boundary value problem with boundary conditions on $\partial\mathcal{Q}$ with its edges and corners.

The first proofs of well posedness for Bérenger's system on $\mathbb{R}^3$ with non-constant $\sigma_k$ differentiated the equations to construct a large system for a highly heterogeneous mix of derivatives. The large system has constant coefficient principal part that is symmetrized by a Fourier multiplier $S(D)$, see [29, 28, 34, 30, 21].

The estimates in all of the above papers involve subtle constructions. Though the layers are called absorbing, no energy decay law for Bérenger's split Maxwell system is known. If the splitting is applied to systems very close to the isotropic Maxwell system the layers are amplifying. Examples are wave equations with symbol $\tau^2 - q(\partial_x, \partial_x)$ with real positive definite anisotropic quadratic form $q$ with axes of inertia not aligned with the coordinate axes [21, 4]. Another example is the anisotropic Maxwell system even with axes aligned with the coordinate axes [5].

There are other PML's for which the analysis of the pure initial value problem is straightforward. Some, for example those in Section 9 are symmetric hyperbolic systems. For isotropic Maxwell in rectangular geometry with absorbing boundary conditions at $\partial\mathcal{Q}$ our analysis treats them all. As in [22], the analysis works for Maxwell's equations with variable $\epsilon(x)$ provided that $\epsilon(x)$ is scalar and constant outside $\mathcal{Q}_I$.

The solutions of (1.8) are supported in $t \geq 0$. If the solutions grow no faster than $e^{Mt}$, then their Laplace transforms are holomorphic in $\mathrm{Re}\,\tau > M$. The Laplace transform, indicated with a hat, of (1.8) is

$$\big(\tau + \sigma_k(x_k)\big)\widehat{U}^k \; + \; \mathcal{A}_k \partial_k(\widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3) \; = \; \widehat{f}_k\,, \qquad k = 1, 2, 3\,. \tag{1.10}$$

**Definition 1.7** *For* $\mathrm{Re}\,\tau > 0$, *define the stretched derivatives*

$$\widetilde{\partial}_k \;:=\; \frac{\tau}{\tau + \sigma_k(x_k)}\;\partial_k, \qquad k = 1, 2, 3, \qquad \widetilde{\partial} \;:=\; \big(\widetilde{\partial}_1\,,\, \widetilde{\partial}_2\,,\, \widetilde{\partial}_3\big).$$

*Define stretched vector operators as,*

$$\widetilde{\text{grad}} := \big(\widetilde{\partial}_1, \widetilde{\partial}_2, \widetilde{\partial}_3\big), \qquad \widetilde{\text{div}} := \big(\widetilde{\partial}_1, \widetilde{\partial}_2, \widetilde{\partial}_3\big)\cdot\,, \qquad \widetilde{\text{curl}} := \big(\widetilde{\partial}_1, \widetilde{\partial}_2, \widetilde{\partial}_3\big)\wedge\,.$$

Multiply the $k^{\text{th}}$ equation of (1.10) by $\tau/(\tau+\sigma_k(x_k))$. The sum of the three resulting equations shows that $(\widehat{E},\widehat{B})$ satisfies

$$\big(\tau + \sum_k \mathcal{A}_k \widetilde{\partial}_k\big)\big(\widehat{E},\widehat{B}\big) \;=\; \sum_k \frac{\tau \widehat{f}_k}{\tau+\sigma_k(x_k)} \;=\; \widehat{f}. \tag{1.11}$$

where Assumption 1.1 and Assumption 1.2 are used in the last equality. Equivalently,

$$\tau\widehat{E} \,-\, \widetilde{\text{curl}}\,\widehat{B} \;=\; -\widehat{j}\sum_k \frac{\tau c_k}{\tau+\sigma_k(x_k)} \;=\; -\widehat{j}\,, \qquad \tau\widehat{B} \,+\, \widetilde{\text{curl}}\,\widehat{E} \;=\; 0. \tag{1.12}$$

**Definition 1.8** *The equivalent systems* (1.11) *and* (1.12) *satisfied by* $\widehat{E},\widehat{B}$ *are called the* **stretched Maxwell equations** *or simply the* **stretched equations***.*

The stretched equations are at the heart of all PML that we know and have at least two interpretations in terms of changes of variables.

**Definition 1.9** *For* $\text{Re}\,\tau > 0$*, define the coordinate stretchings* $X_k(\tau, x_k)$ *as the solutions of ordinary differential equation initial value problems*

$$\frac{\partial X_k}{\partial x_k} \;=\; \frac{\tau+\sigma_k(x_k)}{\tau}\,, \qquad X_k(\tau,0) \;=\; 0\,. \tag{1.13}$$

For $\tau \in ]0,\infty[$, $\partial X_k/\partial x_k \ge 1$ so $x \mapsto X(\tau,x) := (X_1,X_2,X_3)$ is a diffeomorphism from $\mathbb{R}^3$ onto itself. The inverse is denoted $X \mapsto x = x(\tau,X)$. Thus $x$ and $X$ are global coordinates on $\mathbb{R}^3$. A function $f(x)$ corresponds the the function $\underline{f}(X)$ when

$$f(x) \;=\; \underline{f}(X(\tau,x)), \qquad \text{equivalently}, \qquad \underline{f}(X) \;=\; f(x(\tau,X))\,.$$

For $\tau > 0$ fixed, if $\underline{P}(\tau,X,\partial_X)$ is a system of partial differential operators in the $X$ variables it induces an operator on functions $f(x)$ as follows. To a function $f(x)$ compute the corresponding function $\underline{f}(X)$, apply $\underline{P}$ yielding $\underline{Pf}$, then compute the function of $x$ that corresponds to $\underline{Pf}$. Write $P \leftrightarrow \underline{P}$ to indicated corresponding pairs.

**Lemma 1.10** *For $\tau \in ]0,\infty[$ one has the pair of corresponding operators*

$$\frac{\tau}{\tau+\sigma_j(x_j)}\,\frac{\partial}{\partial x_j} \;\leftrightarrow\; \frac{\partial}{\partial X_j}\,, \qquad \frac{\partial}{\partial x_j} \;\leftrightarrow\; \frac{\tau+\sigma_j(x_j(\tau,X_j))}{\tau}\,\frac{\partial}{\partial X_j}\,. \tag{1.14}$$

**Proof.** To prove the first identity, compute

$$\frac{\partial}{\partial x_j} = \sum_k \frac{\partial X_k}{\partial x_j}\,\frac{\partial}{\partial X_k} = \frac{\tau+\sigma_j(x_j)}{\tau}\,\frac{\partial}{\partial X_j}, \qquad \frac{\tau}{\tau+\sigma_j(x_j)}\,\frac{\partial}{\partial x_j} = \frac{\partial}{\partial X_j}.$$

For the second, remark that if $P_1(\tau,x,\partial_x) \leftrightarrow \underline{P}_1(\tau,X,\partial_X)$ and $P_2(\tau,x,\partial_x) \leftrightarrow \underline{P}_2(\tau,X,\partial_X)$, then $\underline{P}_1\,\underline{P}_2 \leftrightarrow P_1\,P_2$. This product rule implies the second identity upon writing

$$\frac{\partial}{\partial x_j} \;=\; \frac{\tau+\sigma_j(x_j)}{\tau}\Big(\frac{\tau}{\tau+\sigma_j(x_j)}\,\frac{\partial}{\partial x_j}\Big)\,. \qquad \Box$$

The following lemma yields a definition of stretched partial differential operators.

**Lemma 1.11** *Suppose that for $\tau \in ]0,\infty[$, $P \leftrightarrow \underline{P}$ is a pair of corresponding operators. Suppose that $\underline{P}(\tau,X,\partial_X) = \sum a_\alpha(\tau,X)\partial_X^\alpha$ with coefficients $a_\alpha(\tau,x) \in C^\infty(\{\mathrm{Re}\,\tau>0\}\times\mathbb{R}^3)$ holomorphic in $\tau$. Then, $P$ has a unique extension to* $\mathrm{Re}\,\tau > 0$ *with smooth coefficients that are holomorphic in $\tau$.*

**Proof.** Uniqueness follows by uniqueness of analytic continuation. The product rule shows that for $\tau$ real

$$P(\tau,x,\partial_x) \;=\; \sum_\alpha a_\alpha(\tau,X(\tau,x))\;\Pi_{k=1}^3\Big(\frac{\tau+\sigma_k(X_k(\tau,x_k))}{\tau}\,\frac{\partial}{\partial x_k}\Big)^{\alpha_k}\,.$$

The coefficients of the operator on the right are smooth in $\tau, x$ and holomorphic in $\tau$ by inspection. $\Box$

**Definition 1.12** *The extension $P$ from Lemma 1.11 is called the* **stretched operator** *corresponding to $\underline{P}$.*

Summarizing, the stretched family is defined for real $\tau$ by the change of coordinates. Then for complex $\tau$ by holomorphic continuation. Lemma 1.11 sets the stage for proofs by analytic continuation as in the next Lemma. The correspondence is used in this paper for $\underline{P}$ with constant coefficients in which case the stretched operator is the original operator with $\partial_X$ replaced by $\widetilde{\partial}_x$.

A second version of the stretching correspondence is *complex scaling.* Compute $\partial(\mathrm{Re}\ X_k)/\partial x_k = 1 + \sigma_k\ \mathrm{Re}\ \overline{\tau}/|\tau|^2 \geq 1$. Therefore, for complex $\tau$ fixed with $\mathrm{Re}\ \tau > 0$, the map $x_k \mapsto X_k(\tau, x_k)$ is a diffeomorphism of $\mathbb{R}$ to a submanifold $\mathcal{G}_k(\tau) \subset \mathbb{C}$ with $\dim_{\mathbb{R}} \mathcal{G}_k(\tau) = 1$. The map $x \mapsto X$ is a diffeomorphism of $\mathbb{R}^3$ to the submanifold $\mathcal{G}(\tau) := \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \subset \mathbb{C}^3$ with $\dim_{\mathbb{R}} \mathcal{G}(\tau) = 3$. The functions $x_1, x_2, x_3$ are global coordinates on $\mathcal{G}(\tau)$. The submanifolds $\mathcal{G}(\tau)$ depend analytically on $\tau$. In Lemma 1.11, $P(\tau, x, \partial_x)$ is viewed as an operator on $\mathbb{R}^3$ depending parametrically on $\tau$. An alternative version views $x$ as coordinates on $\mathcal{G}(\tau)$. Then if $P(\tau, x, \partial_x)$ is a $\kappa \times \kappa$ system of differential operators from the trivial bundle over $\mathcal{G}(\tau)$ with fiber $\mathbb{R}^\kappa$ to itself. Lemma 1.11 characterizes that family of operators by the following properties.

- For $\tau \in ]0, \infty[$, the family is equal to the stretched operators on $\mathbb{R}^3$.
- The family of operators depends holomorphically on $\tau$.

**Lemma 1.13** *Denote by $\widetilde{\Delta}$ the stretched operator from Definition 1.12 corresponding to $\Delta_X$. Then, the stretched vector operators satisfy,*

$$\begin{aligned} &\widetilde{\mathrm{div}}(\Phi \wedge \Psi) = (\widetilde{\mathrm{curl}}\,\Phi) \cdot \Psi - (\widetilde{\mathrm{curl}}\,\Psi) \cdot \Phi, \qquad \widetilde{\Delta} = \widetilde{\mathrm{div}}\,\widetilde{\mathrm{grad}}, \\ &\widetilde{\Delta}\, I_{3\times 3} = \widetilde{\mathrm{grad}}\,\widetilde{\mathrm{div}} - \widetilde{\mathrm{curl}}\,\widetilde{\mathrm{curl}}, \qquad \widetilde{\mathrm{div}}\,\widetilde{\mathrm{curl}} = 0, \qquad \widetilde{\mathrm{curl}}\,\widetilde{\mathrm{grad}} = 0. \end{aligned} \tag{1.15}$$

**Proof.** For $\tau$ real, the stretched version follows from the unstretched version by coordinate stretching. The terms in the stretched identities are holomorphic in $\{\mathrm{Re}\ \tau > 0\}$. Since the identities hold on the real axes, they follow by analytic continuation for all $\mathrm{Re}\ \tau > 0$. $\square$

Applying $\widetilde{\mathrm{div}}$ to the stretched Maxwell equations (1.12) and using the continuity equation in Assumption 1.1 yields the stretched divergence identities

$$\tau\, \widetilde{\mathrm{div}}\, \widehat{E} \;=\; -\,\widetilde{\mathrm{div}}\, \widehat{j} \;=\; \tau\, \widehat{\rho}, \qquad \tau\, \widetilde{\mathrm{div}}\, \widehat{B} = 0. \tag{1.16}$$

When $\tau \in ]0, \infty[$, the stretched Maxwell equations are real and symmetric in the sense of Friedrichs [20]. That is, the matrices multiplying $\partial_k$ are real and symmetric. For $\tau \in ]0, \infty[$ and large, the stretched system is positive in his sense. However, when $\tau$ is not real the coefficients are not even hermitian symmetric. The split system (1.10) is never symmetric and never hermitian.

## 1.4 Main Theorem for the stretched equations

Consider (1.11). At a boundary point in $\Gamma_j$ the normal matrix is equal to

$$\sum \nu_k \, \frac{\tau}{\tau + \sigma_k(x_k)} \, \mathcal{A}_k \;=\; \frac{\tau}{\tau + \sigma_j(x_j)} \, \nu_j \, \mathcal{A}_j \,, \qquad \nu_j = \pm 1.$$

When $\tau > 0$ this is a strictly positive scalar multiple of the normal matrix for the Maxwell system. The energy flux argument in Section 1.2 shows that the boundary condition $\pi^-(\nu)(\widehat{E}, \widehat{B}) = 0$ is a natural choice for the stretched equations with $\tau > 0$. The $\widehat{E}, \widehat{B}$ are holomorphic on $\mathrm{Re}\, \tau > M$. Therefore, if this choice is made for $\tau > 0$, then by analytic continuation $\pi^-(\nu)(\widehat{E}, \widehat{B}) = 0$ is satisfied on $\mathrm{Re}\, \tau > M$. The conclusion is that for the rectangular domain $\mathcal{Q}$, a natural boundary condition for the stretched equations is $\pi^-(\nu)(\widehat{E}, \widehat{B}) = 0$ on $\partial\mathcal{Q} \setminus \mathcal{S}$. Theorem 1.15 solves the resulting mixed initial boundary value problem.

The well posedness result for the stretched equations involves boundary traces. If $W = (F, G) \in L^2(\mathcal{Q})$ satisfies the stretched Maxwell equations with source $j, \rho$ vanishing on a neighborhood of $\partial\mathcal{Q}$ then on a neighborhood of the boundary, $W, \widetilde{\mathrm{div}}\, W, \widetilde{\mathrm{curl}}\, W$ are square integrable. Proposition 5.4 shows that $W$ has a well defined trace in $H^{-1/2}(\partial Q)$. Therefore, the boundary traces in **ii** and **iv** of Theorem 1.15 make sense.

**Definition 1.14** *If $\mathcal{O} \subset \mathbb{R}^3$ is open and $K \subset \mathcal{O}$ is compact. define $L^2_K(\mathcal{O}) := \big\{ f \in L^2(\mathcal{O}) \; ; \; \mathrm{supp}\, f \subset K \big\}$. The spaces $C^\infty_K(\mathcal{O})$ and $H^1_K(\mathcal{O})$ are defined similarly.*

**Theorem 1.15** *Assume that Assumption 1.2 is satisfied. There is an $M_0 > 0$ so that if $M > M_0$, $J$ and $R$ satisfying $-\widetilde{\mathrm{div}}\, J = \tau R$ are holomorphic on $\mathrm{Re}\, \tau > M$ with values in $L^2_{\overline{\mathcal{Q}_I}}(\mathcal{Q})$, then there is a unique $W(\tau) = (F(\tau), G(\tau))$ that satisfies* **i, ii, iii, iv**.

**i.** *$W$ is holomorphic on $\{\mathrm{Re}\, \tau > M\}$ with values in $L^2(\mathcal{Q})$.*

**ii.** *$W_{tan}$ is holomorphic on $\{\mathrm{Re}\, \tau > \mathrm{M}\}$ with values in $L^2(\cup \Gamma_k)$.*

**iii.** *For all $\mathrm{Re}\, \tau > M$, the following equations are satisfied on $\mathcal{Q}$,*

$$\tau F - \widetilde{\mathrm{curl}}\, G = -J, \quad \tau G + \widetilde{\mathrm{curl}}\, F = 0, \quad \widetilde{\mathrm{div}}\, G = 0, \quad \widetilde{\mathrm{div}} F = R. \quad (1.17)$$

**iv.** *$F_{tan} = G_{tan} \wedge \nu$ (equivalently $\pi^-(\nu)(F, G) = 0$) on $\cup \Gamma_k$.*

*In addition, with constant independent of $R, J, \tau$,*

$$(\mathrm{Re}\, \tau)^2 \| W(\tau) \|^2_{L^2(\mathcal{Q})} + (\mathrm{Re}\; \tau) \| W_{tan}(\tau) \|^2_{L^2(\cup \Gamma_k)} \lesssim \| R(\tau), J(\tau) \|^2_{L^2(\mathcal{Q})}. \quad (1.18)$$

**Remark 1.3** *In addition, $W|_{\partial\mathcal{Q}}$ is holomorphic with values in $L^2(\partial\mathcal{Q})$ with $\|W|_{\partial\mathcal{Q}}\|_{L^2(\partial\mathcal{Q})} \lesssim |\tau|^{1/2}(\mathrm{Re}\ \tau)^{-1}\|R, J\|_{L^2(\mathcal{Q})}$ (see (8.6)). For any $\eta < 1$, $\nabla W \in L^2(\eta\mathcal{Q})$ with $\|\nabla W\|_{L^2(\eta\mathcal{Q})} \leq C_\eta|\tau|(\mathrm{Re}\ \tau)^{-1}\|R, J\|_{L^2(\mathcal{Q})}$ (see (8.7)).*

**Example 1.1 Perfect matching.** *Theorem 1.15 implies perfect matching when $\mathcal{Q} = \mathbb{R}^3$. Compare $W$ with a second family of solutions, $W_0$, corresponding to the unstretched case $\sigma = 0$. Perfect matching asserts that $W|_{\mathcal{Q}_I} = W_0|_{\mathcal{Q}_I}$.*

**Proof of perfect matching from [21].** For positive real $\tau$, Equation (1.14) asserts that $\widetilde{\partial}_j$ in the $x$ coordinates is equal to $\partial/\partial X_j$ in the $X$ coordinates. Therefore, the change of variable conjugates the stretched operator in $x$ to the unstretched operator (equivalently the operator with $\sigma = 0$) in $X$.

In addition for $\tau$ real both operators have easy estimates. Both of these operators are invertible for $\tau \in ]M, \infty[$. The change of variable is equal to the identity on $\mathcal{Q}_I$. Therefore, when one solves these different equations with the same source supported in $\mathcal{Q}_I$ the two solutions agree on $\mathcal{Q}_I$.

As both $W$ and $W_0$ are holomorphic on $\mathrm{Re}\ \tau > M$, it follows by analytic continuation that $W|_{\mathcal{Q}_I} = W_0|_{\mathcal{Q}_I}$ for $\mathrm{Re}\ \tau > M$. $\square$

**Remark 1.4** *For a bounded computational domain $\mathcal{Q}$ the perfect matching is destroyed. Waves reaching the external boundary are partly reflected and pollute the computation in $\mathcal{Q}_I$. This effect is mitigated by the decay of wave in the layers. The imperfections decrease with increasing thickness of the layers. Proving that the PML for Maxwell's equations have this observed dissipative behavior is an outstanding open problem.*

## 1.5 Main Theorem for the split equations

**Theorem 1.16** *Assume that assumptions 1.1 and 1.2 are satisfied. There is an $M_0 > 0$ so that if $\mu > M_0$ and $\rho, j \in e^{\mu t}L^2(\mathbb{R} \times \mathcal{Q}_I)$, then there is a unique solution $\widetilde{U} = \big(U^1, U^2, U^3\big) \in L^2(\mathbb{R}; H^{-1}(\mathcal{Q}))$, satisfying $\widetilde{U} = 0$ for $t \leq 0$ and the split boundary value problem (1.8,1.9) on $\mathbb{R} \times \mathcal{Q}$ with,*

$$V := U^1 + U^2 + U^3 := (F, G) \in e^{\mu t}L^2(\mathbb{R} \times \mathcal{Q}),$$
$$V_{tan} \in e^{\mu t}L^2(\mathbb{R} \times \Gamma_k), \quad 1 \leq k \leq 6,$$
$$F_{tan} = G_{tan} \wedge \nu \quad \text{(equivalently } \pi^-(\nu)(F, G) = 0) \quad \text{on} \quad \mathbb{R} \times \cup\Gamma_k .$$

*In addition, $V$ and the components $\widetilde{U}^k$ satisfy*

$$
\begin{aligned}
\mu\big\|e^{-\mu t}\,V\big\|_{L^2(\mathbb{R}\times\mathcal{Q})} \ +\ \mu^{1/2}\big\|e^{-\mu t}\,V_{tan}\big\|_{L^2(\mathbb{R}\times\cup\Gamma_k)}\ + \qquad\qquad \\
\mu\big\|e^{-\mu t}(U^k,\partial_t U^k)\big\|_{L^2(\mathbb{R}:H^{-1}(\mathcal{Q}))}\ \lesssim\ \big\|e^{-\mu t}(\rho,j)\big\|_{L^2(\mathbb{R}\times\mathcal{Q})}.
\end{aligned}
\tag{1.19}
$$

**Remark 1.5 i.** *The output of Bérenger's algorithm is the restriction of $V$ to $\mathcal{Q}_I$. With this in mind, estimate* (1.19) *resembles the estimate for the Maxwell's equations themselves.*

**ii.** *The restriction of $U^k$ to $\eta\mathcal{Q}$ with $\eta<1$ satisfies*

$$
\mu\,\|e^{-\mu t}\,U^k\|_{L^2(\mathbb{R}\times\eta\mathcal{Q}))}\ \le\ C_\eta\,\|e^{-\mu t}(\partial_t\rho,\partial_t j)\|_{L^2(\mathbb{R}\times\mathcal{Q})}.
\tag{1.20}
$$

**iii. Uniform stability**. *Estimate* (1.19) *allows the possibility of exponential growth in time. Proving uniform boundedness, suggested by computational experience, is an oustanding open problem.*

*For the special case of $\mathcal{Q}=\mathbb{R}^3$ with constant $\sigma_k$, uniform stability can be attacked by exact solution in Fourier, or by constructing decreasing nonnegative functionals see for instance [6, 3].*

A number of works began the assault on the theorems of this paper. Begin with important steps for the analogous problems without PML. Variational forms for the Maxwell equations with the absorbing boundary condition use div-curl formulations. Excellent expositions in the time harmonic setting are Nicaise-Tomecz [32] and Costabel-Dauge-Nicaise [16, Section 4.5]. Their formulation is strictly positive when applied with $\operatorname{Re}\tau>0$. Our variational formulation is a stretched cousin of theirs. We analysed the unstretched Maxwell equations in rectangular domains with the absorbing boundary condition in [22]. We analysed the *unstretched* Maxwell equations in rectangular domains with the absorbing boundary condition in [22]. It opened the door for the present problem. The uniqueness assertion from that paper is used in the last paragraph of the uniqueness proof in Section 8.2. Costabel [14] showed the pertinence of the Jerison Kenig Theorems for Maxwell regularity in presence of corners like ours.

For problems with PML we started in [21] with a variational treatment by Laplace transformation of Bérenger's layers with $\mathcal{Q}=\mathbb{R}^3$. In particular we proved perfection.

Diaz-Joly in [17, 18] treat the stretched wave operator $\partial_t^2-\widetilde{\Delta}$ with Dirichlet or Neumann boundary conditions at $\partial\mathcal{Q}$. They prove uniform stability and an amazing explicit solution formula in spite of the variable absorptions.

The Diaz-Joly boundary conditions are perfectly reflecting, the opposite of absorbing. The elegant analysis uses reflection arguments that fail for the ABC for their problem. We know of no wellposed boundary conditions for time dependent Maxwell that have a reflection principle.

A problem with difficulty greater than that of the stretched wave equation with absorbing boundary condition is the PML method applied to the Pauli system in a rectangle with ABC. It is analysed in [23]. That paper introduced the strategy of solving a Helmholtz system on a smoothed domain then using smoothness to derive the original Pauli equations. To our surprise, the analysis on the smoothed domain for the Maxwell PML is much more difficult.

Bramble-Pasciak [10] consider the PML applied to the time harmonic Helmholtz systems coming from Maxwell's equations with perfect conductor boundary conditions on the external boundary. The conservative boundary is a great technical simplification because flux terms vanish (see Remark 7.3). Bramble-Pasciak consider only the stretched time harmonic Helmholtz problems.

Bécache-Kachanovska-Wess [7] apply the PML method to the Maxwell equations in two dimensions with perfect conductor boundary conditions in planar wave guides. They analyse the resulting Helmholtz equation for the magnetic field. The two dimensionality and the conservative boundary condition are significant aids. They prove exponential decay in the one dimensional layer. This gives hope for decay in layers in $\mathbb{R}^{1+3}$.

### 1.6 Outline of the proof

The key result is Theorem 1.15 asserting existence uniqueness and holomorphy in $\tau$ of solutions of the stretched Maxwell system. Just as solutions of Maxwell's system are solutions of D'Alembert's wave equation, the solution of the stretched Maxwell system satisfies a Helmholtz system, introduced in Section 3.

On $\partial\mathcal{Q}$ this Helmholtz system requires six boundary conditions. Two come from the boundary condition $\pi^-(\nu)(E,B)=0$. An additional pair are the divergence relations $\operatorname{div} E = \operatorname{div} B = 0$. The final pair comes from the stretched Maxwell equations themselves. The additional condition, introduced in Section 3.2, is $\pi^+(\mathcal{M}\nu)(\tau - \widetilde{\operatorname{curl}}\, E\,,\, \tau + \widetilde{\operatorname{curl}}\, B) = 0$ with $\mathcal{M}$ from Definition 3.3. The resulting boundary value problem is solved on a domain $\mathcal{Q}^\delta$ that is a smoothing of $\mathcal{Q}$ on a $\delta$-neighborhood of the singular set $\mathcal{S}$.

Though solutions of the stretched Maxwell boundary value problem automatically satisfy the resulting Helmholtz boundary value problem, the converse is proved only for real $\tau$ and only for smooth solutions. That smoothness requires the smoothed domain. That the stretched boundary value problem is satisfied for general $\tau$ follows by analytic continuation.

Section 4 shows that the Helmholtz boundary value problem satisfies Lopatinski's coercivity condition, and depends holomorphically on $\tau$. The boundary data for this problem are sections of vector bundles that depend holomorphically on $\tau$.

The hardest part of the analysis is the derivation of estimates for the solutions of the stretched Maxwell boundary value problem on $\mathcal{Q}^\delta$. One set of estimates comes from the Helmholtz boundary value problem. It is classical [14, 15, 32, 16] that the absorbing conditions are associated to the div-rot quadratic form of the Laplace operator. That quadratic form has a natural but not obvious extension to the stretched problem. It relies on Green's identities from Section 5, and, the spectral projections of Section 2.

In Section 5.3 the quadratic form is used to prove the invertibility of stretched Helmholtz boundary value problem for $\tau \in ]0, \infty[$.

For complex $\tau$ there are two difficulties in applying the bilinear form from Section 5. The first is that neither the volume integrals nor the boundary terms have a sign. They both are almost positive. The second is that the boundary term yields at best an estimate for the $\pi^+(\mathcal{M}\nu)$ projection of the boundary data. The $\pi^-(\mathcal{M}\nu)$ projections enter in error terms. This does not happen for the perfect conductor boundary conditions.

Section 6 presents the technique for overcoming the second obstacle. We derive estimates for the stretched Maxwell system that are analogues of those of Jerison-Kenig [25] for harmonic functions and M. Mitrea [31] for Maxwell's equations. For stretched Maxwell, the estimates assert that the $\pi^-(\mathcal{M}\nu)$ projection of the trace is estimated in terms of $\pi^+(\mathcal{M}\nu)$ projection and the stretched divergence and curl.

The natural multiplier in Green's identity is not the complex conjugate of $(E, B)$. The natural space is the set of fields with square integrable $\widetilde{\mathrm{div}}$ and $\widetilde{\mathrm{curl}}$. The complex conjugate of such a field need not satisfy this constraint. In Section 7, appropriate multipliers are found. The $\delta$-independent estimates for the stretched Maxwell boundary value problem on $\mathcal{Q}^\delta$ are proved in Section 7.

Theorem 8.1 of Section 8.1 solves the stretched Maxwell boundary value problem on $\mathcal{Q}^\delta$. The Laplace transform of the solution is constructed as a solution of the stretched Helmholtz boundary value problem. Section 8.2 solves the stretched Maxwell equations on $\mathcal{Q}$ with its trihedral corners by passage to the limit $\delta \to 0$. The holomorphy in $\tau$ makes it easier to prove convergence. The uniqueness of the solution is proved by showing that the Laplace transform vanishes for $\tau$ real. That follows from the uniqueness theorem in Part 1 of [22]. Section 8.3 solves the split equations on $\mathcal{Q}$ using the solvability of stretched Maxwell. Section 9 shows that the methods used to solve the Bérenger's split equations also treat other PMLs. It also proves a new error estimate. The proof works for any perfectly matched and well posed PML.

## 2 Symbol spectral decomposition

The formulas of this section are used throughout. Formula (1.7) for the energy flux leads to the absorbing boundary condition for $\tau$ real. In Section 3.2, the holomorphy of Lemma 2.7 is used to find the boundary conditions for complex $\tau$. The projectors appear in the bilinear form $a((E,B),(E',B'))$ from Definition 5.8 that is used to analyse the stretched and unstretched Helmholtz boundary value problems. Proposition 5.10 proves, using the projectors, that the boundary term after integration by parts in $a$ vanishes if $(E,B)$ satisfies the stretched Maxwell equations at $\partial\mathcal{Q}^\delta$.

**Lemma 2.1** *With notation from Definition 1.4, for $\xi \in \mathbb{R}^3 \setminus 0$, the real $6\times 6$ symmetric matrix $\mathcal{A}(\xi)$ from* (1.5) *has eigenvalues $\pm|\xi|$ and $0$. Each eigenvalue has multiplicity two. The eigenspaces, positive homogeneous of degree zero in $\xi$, are given by*

$$\begin{aligned}
\mathcal{E}^0(\mathcal{A}(\xi)) &= \big\{(\mathbf{e},\mathbf{b}) \ :\ \mathbf{e} \parallel \xi \ \text{ and } \ \mathbf{b} \parallel \xi\big\}, \\
\mathcal{E}^+(\mathcal{A}(\xi)) &= \big\{(\mathbf{e},\mathbf{b}) \ :\ \mathbf{b} \perp \xi \ \text{ and } \ \mathbf{e} = \mathbf{b}\wedge\xi/|\xi|\big\}, \\
\mathcal{E}^-(\mathcal{A}(\xi)) &= \big\{(\mathbf{e},\mathbf{b}) \ :\ \mathbf{b} \perp \xi \ \text{ and } \ \mathbf{e} = -\mathbf{b}\wedge\xi/|\xi|\big\}.
\end{aligned} \tag{2.1}$$

**Proof.** By homogeneity, it suffices to consider $|\xi| = 1$. Each of the subspaces on the right is two dimensional. For the eigenvalues $\pm 1$, compute

$$\mathcal{A}(\xi)\begin{pmatrix}\mathbf{e}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}0 & -\xi\wedge\\ \xi\wedge & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}-\xi\wedge\mathbf{b}\\ \xi\wedge\mathbf{e}\end{pmatrix}.$$

It follows that $(\mathbf{e},\mathbf{b})$ in the formulas are eigenvectors with eigenvalue $\pm 1$. The eigenvalue 0 is easier. Since the spaces on the right of (2.1) span $\mathbb{C}^6$ it follows that they contain all the eigenvectors. $\square$

**Example 2.1** *It follows that for $\tau \in \mathbb{C}$ and $\xi \in \mathbb{R}^3 \setminus 0$*

$$\det \big(\tau I - \mathcal{A}(\xi)\big) \;=\; \tau^2(\tau^2 - \xi \cdot \xi)^2 \,. \tag{2.2}$$

*Since the left hand side is a homogeneous degree six polynomial in $(\tau, \xi)$, it follows (for example by Taylor series) that this identity holds for all $(\tau, \xi) \in \mathbb{C}^{1+3}$.*

**Definition 2.2** *When dealing with the Maxwell system we use the shorthand $\mathcal{E}^{\pm}(\xi)$ and $\mathcal{E}^0(\xi)$ for $\mathcal{E}^{\pm}(\mathcal{A}(\xi))$ and $\mathcal{E}^0(\mathcal{A}(\xi))$. Similarly for the spectral projectors.*

**Example 2.2** *For a real unit vector $\xi$ the map $\mathbf{b} \mapsto (\pm\mathbf{b} \wedge \xi\,,\, \mathbf{b})/\sqrt{2}$ is an isometry from the set of vectors orthogonal to $\xi$ to $\mathcal{E}^{\pm}(\xi)$.*

**Definition 2.3** *Define*

$$\mathcal{Z} \;:=\; \big\{\xi \in \mathbb{C}^3 \setminus 0 \,:\, \xi_1^2 + \xi_2^2 + \xi_3^2 = 0\big\}\,.$$

*$\mathcal{Z}$ is a closed conic subset of $\mathbb{C}^3 \setminus 0$. $(\mathbb{C}^3 \setminus 0) \setminus \mathcal{Z}$ is open, conic, connected, and contains $\mathbb{R}^3 \setminus 0$.*

**Example 2.3** *For $\xi \in \mathbb{C}^3 \setminus 0$, define subspaces $W_j(\xi) \subset \mathbb{C}^3$ by*

$$W_1 \;:=\; \mathbb{C}\,\xi\,, \qquad \text{and,} \qquad W_2 \;:=\; \big\{v \in \mathbb{C}^3 \,:\, v \cdot \xi = 0\big\}.$$

*Then $\dim W_1 = 1$ and $\dim W_2 = 2$. In addition, $W_1 \cap W_2 = \{0\}$ if and only if $\xi \notin \mathcal{Z}$. In that case, $\mathbb{C}^3 = W_1 \oplus W_2$.*

**Definition 2.4** *For $0 \neq \xi \in \mathbb{C}^3 \setminus \mathcal{Z}$, denote $\pi_\xi$ and $\pi_{\xi^\perp}$ the projections in $\mathrm{Hom}(\mathbb{C}^3)$ given by*

$$\pi_\xi v \;:=\; \frac{v \cdot \xi}{\xi \cdot \xi}\,\xi, \qquad \text{and,} \qquad \pi_{\xi^\perp} \;:=\; I \;-\; \pi_\xi\,.$$

*The decomposition $I = \pi_\xi + \pi_{\xi^\perp}$ corresponds to $W_1 \oplus W_2 = \mathbb{C}^3$.*

**Remark 2.1** **i.** *For $\xi$ in $\mathbb{R}^3 \setminus 0$, $\pi_{\xi^\perp}$ is equal to the orthogonal projection onto the orthogonal complement of $\xi$. When $\xi$ is not real, $\pi_{\xi^\perp}$ is not equal to the orthogonal projection onto the orthogonal complement of $\xi$.*

**ii.** *The map* $\{(\mathbb{C}^3\setminus 0)\setminus\mathcal{Z}\} \ni \xi \mapsto \pi_{\xi^\perp}$ *is holomorphic with values in* $\mathrm{Hom}(\mathbb{C}^3)$. *For* $\xi$ *in* $\mathbb{R}^3\setminus 0$, $\pi_{\xi^\perp}$ *is equal to the orthogonal projection onto the orthogonal complement of* $\xi$. *These properties characterize* $\pi_{\xi^\perp}$.

**iii.** *For* $c \in \mathbb{C}\setminus 0$ *and* $0 \neq \xi \in \mathbb{C}^3\setminus\mathcal{Z}$, $\pi_{c\xi} = \pi_\xi$ *and* $\pi_{(c\xi)^\perp} = \pi_{\xi^\perp}$.

**iv.** *For* $\xi \in \mathbb{R}^3\setminus 0$,

$$\xi \wedge (\xi \wedge w) \;=\; -(\xi\cdot\xi)\Big(w - \frac{(\xi\cdot w)\xi}{\xi\cdot\xi}\Big) \;=\; -(\xi\cdot\xi)\,\pi_{\xi^\perp} w\,. \tag{2.3}$$

*The terms of the identity are holomorphic functions of* $\xi$ *on the connected set* $\mathbb{C}^3\setminus\mathcal{Z}$. *Identity* (2.3) *extends by analytic continuation to* $\xi \in \mathbb{C}^3\setminus\mathcal{Z}$.

**Lemma 2.5** *Suppose that* $\xi \in \mathbb{R}^3$ *is a unit vector.*

**i.** *The span of the non zero eigenspaces of* $\mathcal{A}(\xi)$ *is the set of* $\mathbf{e},\mathbf{b}$ *with* $\mathbf{e}\cdot\xi = 0$ *and* $\mathbf{b}\cdot\xi = 0$. *In addition, for arbitrary* $(\mathbf{e},\mathbf{b})$,

$$\pi^0(\xi)\big(\mathbf{e},\mathbf{b}\big) \;=\; \Big((\mathbf{e}\cdot\xi)\xi\,,\,(\mathbf{b}\cdot\xi)\xi\Big).$$

**ii.** *For* $(\mathbf{e},\mathbf{b})$ *satisfying* $\mathbf{e}\cdot\xi = 0$, *and* $\mathbf{b}\cdot\xi = 0$,

$$\pi^+(\xi)\big(\mathbf{e},\mathbf{b}\big) \;=\; \frac{1}{2}\,\Big(\mathbf{b}\wedge\xi + \mathbf{e}\;,\;\mathbf{b} - \mathbf{e}\wedge\xi\Big)\,.$$

**iii.** *For arbitrary* $(\mathbf{e},\mathbf{b})$,

$$\begin{aligned}\pi^+(\xi)\big(\mathbf{e},\mathbf{b}\big) \;&=\; \pi^+(\xi)\big(\pi_{\xi^\perp}\mathbf{e}\,,\,\pi_{\xi^\perp}\mathbf{b}\big)\\ &=\; \frac{1}{2}\,\Big((\pi_{\xi^\perp}\mathbf{b})\wedge\xi + \pi_{\xi^\perp}\mathbf{e}\;,\;\pi_{\xi^\perp}\mathbf{b} - (\pi_{\xi^\perp}\mathbf{e})\wedge\xi\Big).\end{aligned}$$

**iv.** *The projection on the negative eigenspace is given by* $\pi^-(\xi) = \pi^+(-\xi)$.

**Proof. i.** The span of the eigenspaces with non zero eigenvalues is the orthogonal complement of the kernel. Part **i** follows from the description of the kernel in Lemma 2.1.

**ii.** Using Example 2.2, there are unique vectors $\mathbf{b}_\pm$ perpendicular to $\xi$ so that

$$(\mathbf{e},\mathbf{b}) \;=\; (\mathbf{b}_+\wedge\xi\,,\,\mathbf{b}_+)/\sqrt{2} \;+\; (-\mathbf{b}_-\wedge\xi\,,\,\mathbf{b}_-)/\sqrt{2}, \tag{2.4}$$

with the $\mathbf{b}_\pm$ term being the projection on $\mathcal{E}^\pm(\xi)$. Equation (2.4) holds if and only if

$$\mathbf{b}_+ + \mathbf{b}_- \;=\; \sqrt{2}\,\mathbf{b}\,, \qquad \mathbf{b}_+\wedge\xi - \mathbf{b}_-\wedge\xi \;=\; \sqrt{2}\,\mathbf{e}\,.$$

The wedge product of the second with $\xi$ shows that it is equivalent to

$$-\mathbf{b}_+ \;+\; \mathbf{b}_- \;=\; \sqrt{2}\,\mathbf{e}\wedge\xi\,.$$

Adding (resp. subtracting) from the first equation implies that $\mathbf{b}_- = (\mathbf{b} + \mathbf{e}\wedge\xi)/\sqrt{2}$ (resp. $\mathbf{b}_+ = (\mathbf{b} - \mathbf{e}\wedge\xi)/\sqrt{2}$). Therefore

$$\begin{aligned}\pi^+(\xi)(\mathbf{e},\mathbf{b}) \;&=\; (\mathbf{b}_+\wedge\xi\,,\,\mathbf{b}_+)/\sqrt{2} \;=\; \big((\mathbf{b}-\mathbf{e}\wedge\xi)\wedge\xi\;,\;\mathbf{b}-\mathbf{e}\wedge\xi\big)/2\\ &=\; \big(\mathbf{b}\wedge\xi+\mathbf{e}\,,\,\mathbf{b}-\mathbf{e}\wedge\xi\big)/2,\end{aligned}$$

proving **ii**.

**iii.** For general $(\mathbf{e},\mathbf{b})$, part **i** implies that the projection on the span of the nonzero eigenspaces is equal to $(\pi_{\xi^\perp}\mathbf{e}\,,\,\pi_{\xi^\perp}\mathbf{b})$. Then **ii** implies **iii**.

**iv.** Follows from $\mathcal{A}(-\xi) = -\mathcal{A}(\xi)$. $\square$

**Example 2.4** *For real unit vectors $\xi$, the span of the nonpositive eigenspaces $\mathcal{E}^-(\xi)\oplus\mathcal{E}^0(\xi) = \mathrm{Ker}\,\pi^+(\xi)$ is equal to the set of vectors $\mathbf{e},\mathbf{b}$ satisfying $\pi_{\xi^\perp}\mathbf{b} - (\pi_{\xi^\perp}\mathbf{e})\wedge\xi = 0$. Taking the wedge product with $\xi$ shows that this is equivalent to $(\pi_{\xi^\perp}\mathbf{b})\wedge\xi + \pi_{\xi^\perp}\mathbf{e} = 0$. In addition,*

$$\frac{\|\pi^+(\xi)(\mathbf{e},\mathbf{b})\|}{\sqrt{2}} \le \|\pi_{\xi^\perp}\mathbf{b}-(\pi_{\xi^\perp}\mathbf{e})\wedge\xi\| = \|(\pi_{\xi^\perp}\mathbf{b})\wedge\xi+\pi_{\xi^\perp}\mathbf{e}\| \le \sqrt{2}\,\|\pi^+(\xi)(\mathbf{e},\mathbf{b})\|.$$

**Example 2.5** *Equation* (2.2) *implies that for $\xi\in\mathbb{C}^3\setminus 0$ the eigenvalues of $\mathcal{A}(\xi)$ are equal to $0$ and the two square roots of $\xi\cdot\xi$. Analytic continuation on paths that wind about $\mathcal{Z}$ can pass from one root to the other. There is no holomorphic function on $\mathbb{C}^3\setminus\mathcal{Z}$ whose square is equal to $\xi\cdot\xi$.*

**Definition 2.6** *Let*

$$\Omega := \{\xi\in\mathbb{C}^3 : \mathrm{Re}\,(\xi\cdot\xi) > 0\} \;=\; \{\xi\in\mathbb{C}^3 : |\,\mathrm{Im}\;\xi| < |\,\mathrm{Re}\;\xi|\}\,.$$

*For $\xi\in\Omega$, define $(\xi\cdot\xi)^{1/2}$ to be the unique square root of $\xi\cdot\xi$ that has strictly positive real part. The function $\Omega\ni\xi\mapsto(\xi\cdot\xi)^{1/2}$ is holomorphic.*

The eigenvalues are holomorphic on larger sets, for example $\{\xi\in\mathbb{C}^3 : \xi\cdot\xi\in\mathbb{C}\setminus]-\infty,0]\}$. The extension to $\Omega$ suffices for our needs.

**Lemma 2.7** *For $\Omega$ from Definition 2.6 and $\xi \in \Omega$, $\mathcal{A}(\xi)$ has eigenvalues* $0$ *and $\pm(\xi \cdot \xi)^{1/2}$. Each eigenvalue has an eigenspace of dimension two. The spectral projections are holomorphic on $\Omega$ and are given by,*

$$\begin{aligned}
\pi^+(\xi)\big(\mathbf{e},\mathbf{b}\big) &= 2^{-1}\big((\xi\cdot\xi)^{-1/2}\mathbf{b}\wedge\xi + \pi_{\xi^\perp}\mathbf{e}\ ,\ \pi_{\xi^\perp}\mathbf{b} - (\xi\cdot\xi)^{-1/2}\mathbf{e}\wedge\xi\big),\\
\pi^0(\xi)(\mathbf{e},\mathbf{b}) &= \big(\pi_\xi\mathbf{e}, \pi_\xi\mathbf{b}\big),\\
\pi^-(\xi) &= -\pi^+(-\xi).
\end{aligned}$$

**Proof.** The eigenvalues and their algebraic multiplicities follow from (2.2). The contour integral representations show that the spectral projections are holomorphic in $\xi$. For $\xi \in \mathbb{R}^3 \setminus 0$ one has the formulas for the projectors and,

$$\mathcal{A}(\xi)\pi^\pm(\xi) = \pm(\xi\cdot\xi)^{1/2}\pi^\pm(\xi), \qquad \mathcal{A}(\xi)\pi^0(\xi) = 0. \tag{2.5}$$

By analytic continuation, the identities (2.5) and the formulas for the spectral projections extend to $\Omega$. $\square$

**Example 2.6** *For $\xi \in \Omega$, $\pi^\pm(\xi)$, $\pi_\xi$ and $\pi_{\xi^\perp}$ are related by*

$$\begin{pmatrix}\pi_{\xi^\perp} & 0\\ 0 & \pi_{\xi^\perp}\end{pmatrix} = \pi^+(\xi) + \pi^-(\xi), \qquad \text{and,} \qquad \begin{pmatrix}\pi_\xi & 0\\ 0 & \pi_\xi\end{pmatrix} = \pi^0(\xi). \tag{2.6}$$

# 3 Stretched Helmholtz boundary value problem

The stretched Maxwell equations are solved on smoothed domains $\mathcal{Q}^\delta$. Section 3.2 shows that on the smoothed parts there is a boundary condition forced by holomorphy in $\tau$. Proposition 3.4 shows that solutions of the resulting stretched Maxwell boundary value problem satisfy a Helmholtz boundary value problem on $\mathcal{Q}^\delta$. It also proves a partial converse: when $\tau$ is real, solutions of the Helmholtz boundary value problem satisfy the stretched Maxwell system. The proof requires the smoothness of $\mathcal{Q}^\delta$. Section 8.1 uses the partial converse for real $\tau$ to show that for the holomorphic family the stretched Maxwell system is satisfied for all $\tau$.

## 3.1 Smoothed domains $\mathcal{Q}^\delta$

The construction of a holomorphic family of solutions $(E(\tau), B(\tau))$ to the stretched Maxwell equations on $\mathcal{Q}$ is carried out by solving the same equations on a sequence of domains $\mathcal{Q}^\delta$ for $0 < \delta \leq \delta_0$ increasing to $\mathcal{Q}$ as $\delta$ decreases. The $\mathcal{Q}^\delta$ are obtained by smoothing the corners and edges of $\mathcal{Q}$.

$\mathcal{Q}$ is convex with $0 \in \mathcal{Q}$. Express $\partial\mathcal{Q}$ in spherical coordinates $(r,\omega) \in ]0,\infty[\times S^2$ as the graph $r = \phi(\omega)$. Then $\phi$ is lipschitzian on $S^2$. Denote by $\underline{\mathcal{S}} := \phi^{-1}(\mathcal{S}) \subset S^2$ the preimage of the singular set $\mathcal{S}$ of $\partial\mathcal{Q}$.

**Definition 3.1** *For $0 < \delta < 1$ choose a sequence of smooth convex domains $\mathcal{Q}^\delta := \{(r,\omega) : r < \phi^\delta(\omega)\}$ with $\phi^\delta \in C^\infty(S^2)$ increasing to $\phi$ as $\delta \searrow 0$. The functions $\phi^\delta$ are required to satisfy, $\phi^\delta = \phi$ on $\{\omega \in S^2 : \mathrm{dist}(\omega, \underline{\mathcal{S}}) > \delta\}$, and, $\sup_\delta \|\nabla_\omega \phi^\delta\|_{L^\infty(S^2)} < \infty$.*

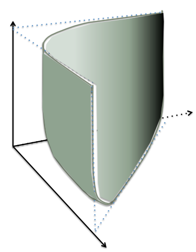

Figure 1: A smoothed corner of $\mathcal{Q}^\delta$

**Remark 3.1** *Except for a $\delta$ neighhorhood of $\mathcal{S}$, the normal vectors $\nu$ to $\partial\mathcal{Q}^\delta$ are parallel to the coordinate axes, see Definition 3.1.*

**Definition 3.2** *With $\mathcal{Q}_I$ from Assumption 1.1, choose $0 < \delta_0 < 1$ so that $\mathcal{Q}^{\delta_0}$ contains $\overline{\mathcal{Q}_I}$ in its interior.*

### 3.2 Absorbing boundary conditions on $\partial\mathcal{Q}^\delta$

This subsection chooses a boundary condition to impose on the curved parts of $\partial\mathcal{Q}^\delta$. On the flat parts of the boundary impose the condition $\pi^-(\nu)(E,B) = 0$. This convention implies that taking the limit as $\delta \to 0$ recovers the boundary conditions on $\partial\mathcal{Q}$ in Theorem 1.16. The choice of boundary conditions on the curved parts requires thought.

**Definition 3.3** *Define*

$$\Pi(\tau, x) \;:=\; \prod_{i=1}^{3} \frac{\tau + \sigma_i(x_i)}{\tau}, \tag{3.1}$$

*and,*

$$\mathcal{M}(\tau,x) := \begin{pmatrix} \mathcal{M}_{11} & 0 & 0 \\ 0 & \mathcal{M}_{22} & 0 \\ 0 & 0 & \mathcal{M}_{33} \end{pmatrix}, \qquad \text{with} \quad \mathcal{M}_{ii}(\tau,x) \ := \ \prod_{j\neq i} \frac{\tau+\sigma_j(x_j)}{\tau}\,.$$

Then

$$\mathcal{M}_{11}\ \mathcal{M}_{22}\ \mathcal{M}_{33} \ = \ \Pi^2, \qquad \frac{\tau}{\tau+\sigma_i(x_i)} \ = \ \Pi^{-1}\mathcal{M}_{ii}, \quad \widetilde{\partial} \ = \ \Pi^{-1}\,\mathcal{M}\,\partial\,. \tag{3.2}$$

The operator $\mathcal{A}(\widetilde{\partial})$ is given by

$$\sum_i \mathcal{A}_i\ \widetilde{\partial}_i = \sum_i \frac{\tau}{\tau+\sigma_i(x_i)}\ \mathcal{A}_i\ \partial_i = \mathcal{A}(\Pi^{-1}\mathcal{M}\partial) = \Pi^{-1}\,\mathcal{A}(\mathcal{M}\partial). \tag{3.3}$$

The normal matrix associated to $\mathcal{A}(\widetilde{\partial})$ is equal to

$$\sum_i \frac{\tau}{\tau+\sigma_i(x_i)}\ \mathcal{A}_i\ \nu_i \ = \ \Pi^{-1}\mathcal{A}(\mathcal{M}\nu)\,. \tag{3.4}$$

For $\tau$ real, it is natural to impose at all $x\in\partial\mathcal{Q}^\delta$ the boundary condition

$$\pi^-\big(\mathcal{M}(\tau,x)\nu\big)\big(E,B\big) \ = \ 0. \tag{3.5}$$

This choice is characterized by the following three desireable features. First, it gives the desired answer when $\nu$ is parallel to one of the coordinate axes. Therefore it gives the right limit when $\delta\to 0$ since in the limit the normals are parallel to coordinate axes. Second, it depends holomorphically on $\tau$. Third it maximizes the outward energy flux when $\tau$ is real. Indeed, $A(\mathcal{M}\nu)$ has eigenvalues $\pm\|\mathcal{M}\nu\|$ and 0 each with multiplicity two. Therefore the outward flux is equal to one half of

$$(A(\mathcal{M}\nu)u)\cdot u \ = \ \|\mathcal{M}\nu\|\,\|\pi^+(\mathcal{M}\nu)u\|^2 \ - \ \|\mathcal{M}\nu\|\,\|\pi^-(\mathcal{M}\nu)u\|^2\,. \tag{3.6}$$

The boundary condition (3.5) enforces the vanishing of the nonpositive or incoming part. The resulting outward flux is as positive as possible.

Seek $(E,B)$ holomorphic in $\tau$. In that case the left hand side of (3.5) is holomorphic in $\{\mathrm{Re}\ \tau > \max_{k,x_k}\sigma_k(x_k)\}$. Indeed, suppressing dependence on $k, x_k$ one has for $\mathrm{Re}\ \tau>0$,

$$\frac{\tau+\sigma}{\tau} = 1+\frac{\sigma\overline{\tau}}{|\tau|^2} \qquad \text{so,} \qquad \mathrm{Re}\ \frac{\tau+\sigma}{\tau} \geq 1, \quad \text{and,} \quad \Big|\mathrm{Im}\ \frac{\tau+\sigma}{\tau}\Big| \leq \frac{\sigma}{|\tau|}.$$

Therefore for $\mathrm{Re}\,\tau > \max_{k,x_k} \sigma_k(x_k)$, $|\mathrm{Re}\,\mathcal{M}\nu| > |\mathrm{Im}\,\mathcal{M}\nu|$. The holomorphy of the left hand side of (3.6) follows from Lemma 2.7.

It is natural to impose that $\pi^-(\mathcal{M}\nu)(E,B)|_{\partial\mathcal{Q}^\delta} = 0$ when $\tau$ is real. By analytic continuation $\pi^-(\mathcal{M}\nu)(E,B)|_{\partial\mathcal{Q}^\delta} = 0$ on $\{\mathrm{Re}\,\tau > \max_{k,x_k} \sigma_k(x_k)\}$. This is our choice of absorbing boundary condition for the stretched problem.

### 3.3 Near equivalence of Maxwell and Helmholtz

**Proposition 3.4** *Suppose that $K$ is a compact subset of $\mathcal{Q}^\delta$, $\mathrm{Re}\,\tau > 0$ and $\rho(x), j(x) \in L^2_K(\mathcal{Q}^\delta)$ satisfy the continuity equation $\tau\rho = -\widetilde{\mathrm{div}}\, j$. Consider two boundary value problems for $(E,B) \in H^2(\mathcal{Q}^\delta)$ with first boundary condition $\pi^-(\mathcal{M}\nu)(E,B) = 0$ on $\partial\mathcal{Q}^\delta$.*

**i.** *$E, B$ satisfies the stretched Maxwell's equations,*

$$\tau E - \widetilde{\mathrm{curl}}\, B = -j, \quad \tau B + \widetilde{\mathrm{curl}}\, E = 0, \quad \widetilde{\mathrm{div}}\, E = \rho, \quad \widetilde{\mathrm{div}}\, B = 0. \tag{3.7}$$

**ii.** *$E, B$ satisfies the boundary value problem for the stretched Helmholtz equation with additional boundary conditions,*

$$\begin{aligned} \tau^2 E - \widetilde{\Delta} E = -\tau j - \widetilde{\mathrm{grad}}\, \rho &\qquad \text{on} \quad \mathcal{Q}^\delta, \\ \tau^2 B - \widetilde{\Delta} B = \widetilde{\mathrm{curl}}\, j &\qquad \text{on} \quad \mathcal{Q}^\delta, \\ \pi^+(\mathcal{M}\nu)\big(\tau E - \widetilde{\mathrm{curl}}\, B\,,\, \tau B + \widetilde{\mathrm{curl}}\, E\big) \;=\; 0 &\qquad \text{on} \quad \partial\mathcal{Q}^\delta, \\ \widetilde{\mathrm{div}}\, E \;=\; \widetilde{\mathrm{div}}\, B \;=\; 0 &\qquad \text{on} \quad \partial\mathcal{Q}^\delta. \end{aligned} \tag{3.8}$$

*They are related as follows.*

**I. i $\Longrightarrow$ ii**.

**II.** *There is an $M > 0$ so that for real $\tau \in ]M, \infty[$,* **ii $\Longrightarrow$ i**.

**Proof of Proposition 3.4. I.** Use (1.15). The identity

$$\mathcal{A}(\widetilde{\partial})^2 = -\begin{pmatrix} \widetilde{\mathrm{curl}}\,\widetilde{\mathrm{curl}} & 0 \\ 0 & \widetilde{\mathrm{curl}}\,\widetilde{\mathrm{curl}} \end{pmatrix} = -\begin{pmatrix} -\widetilde{\Delta}\, I_3 + \widetilde{\mathrm{grad}}\,\widetilde{\mathrm{div}} & 0 \\ 0 & -\widetilde{\Delta}\, I_3 + \widetilde{\mathrm{grad}}\,\widetilde{\mathrm{div}} \end{pmatrix}$$

implies that

$$\Big(\tau\, I_6 - \mathcal{A}(\widetilde{\partial})\Big)\Big(\tau\, I_6 + \mathcal{A}(\widetilde{\partial})\Big) = \begin{pmatrix} (\tau^2 - \widetilde{\Delta}) I_3 + \widetilde{\mathrm{grad}}\,\widetilde{\mathrm{div}} & 0 \\ 0 & (\tau^2 - \widetilde{\Delta}) I_3 + \widetilde{\mathrm{grad}}\,\widetilde{\mathrm{div}} \end{pmatrix}.$$

Apply to $(E,B)$ using (3.7) to find

$$\Big(\tau^2 I_6 + \widetilde{\rm curl}\,\widetilde{\rm curl}\Big)(E,B) \;=\; \Big(\tau\, I_6 - \mathcal{A}(\widetilde\partial)\Big)(-j,0) - \big(\widetilde{\rm grad}\,\widetilde{\rm div}\,E,0\big).$$

This is equivalent to the first two lines in (3.8).

In $\overline{\mathcal{Q}}^\delta$, the functions $\tau E - \widetilde{\rm curl}B, \tau B + \widetilde{\rm curl}E, \widetilde{\rm div}E, \widetilde{\rm div}B$ vanish on a neighborhood of $\partial\mathcal{Q}^\delta$. This implies the third and fourth lines of (3.8). This completes the proof of **I.**

**II.** Suppose $\tau \in ]0,\infty[$. Define

$$(F,G) \;:=\; \big(\tau I \;+\; \mathcal{A}(\widetilde\partial)\big)(E,B) + (j,0).$$

The first two equations from (3.7) are satisfied if and only if $(F,G)=0$.

The second line of (3.8) implies that $\widetilde{\rm curl}\, j \in L^2(\mathcal{Q}^\delta)$. The continuity equation implies that $\widetilde{\rm div}\, j \in L^2(\mathcal{Q}^\delta)$. Since $j$ is supported in $K$ the fact that $(\widetilde{\rm curl},\widetilde{\rm div})$ is an (overdetermined) elliptic system on $\mathbb{R}^3$ implies that $j\in\ H^1_K(\mathcal{Q}^\delta)$. Since $E\in H^2(\mathcal{Q}^\delta)$ it follows that $(F,G)\in H^1(\mathcal{Q}^\delta)$.

The third line of equation (3.8) together with the fact that $\rho$ and $j$ vanish at the boundary imply that on $\partial\mathcal{Q}^\delta$,

$$\pi^+(\mathcal{M}\nu)\big(F,G\big) \;=\; \pi^+(\mathcal{M}\nu)\Big(\tau E - \widetilde{\rm curl}\,B\,,\, \tau B + \widetilde{\rm curl}\,E\Big) \;=\; 0\,. \tag{3.9}$$

Computing $(\tau^2-\widetilde\Delta)\widetilde{\rm div}\,E = \widetilde{\rm div}(\tau^2-\widetilde\Delta)E$ using the Helmholtz equation for $E$ together with $\tau\rho = -\widetilde{\rm div}\,j$ yields $(\tau^2-\widetilde\Delta)(\widetilde{\rm div}\,E-\rho)=0$. By hypothesis $\widetilde{\rm div}E-\rho\in H^1(\mathcal{Q}^\delta)$ and vanishes at $\partial\mathcal{Q}^\delta$. For $\tau\in]0,\infty[$, this implies that $\widetilde{\rm div}\,E=\rho$ on $\overline{\mathcal{Q}}^\delta$. Similarly $\widetilde{\rm div}\,B=0$ on $\overline{\mathcal{Q}}^\delta$.

Define $W:=(F,G)$. The stretched Helmholtz equations, $\widetilde{\rm div}\,E=\rho$, and $\widetilde{\rm div}\,B=0$, imply that $(\tau I - \mathcal{A}(\widetilde\partial))W \;=\; 0$. Note the minus sign. Compute using the symmetry of $\mathcal{A}_k$ in the second step to show that

$$\begin{aligned}\int_{\mathcal{Q}^\delta} W\cdot\mathcal{A}(\widetilde\partial)W\,dx \;&=\; \sum\int_{\mathcal{Q}^\delta} W\cdot\frac{\tau}{\tau+\sigma_k}\mathcal{A}_k\partial_k W\,dx\\ &=\; \sum\int_{\mathcal{Q}^\delta}\frac{\tau}{\tau+\sigma_k}\mathcal{A}_k W\cdot\partial_k W\,dx\end{aligned}$$

An integration by parts, with $\nu$ denoting the unit outward normal to $\mathcal{Q}^\delta$, yields

$$\begin{aligned}&\int_{\mathcal{Q}^\delta} W\cdot\mathcal{A}(\widetilde\partial)W\,dx \;=\\ -&\int_{\mathcal{Q}^\delta}\sum\mathcal{A}_k\partial_k\big(\frac{\tau}{\tau+\sigma_k}W\big)\cdot W dx + \int_{\partial\mathcal{Q}^\delta}\sum\frac{\tau\nu_k}{\tau+\sigma_k}\mathcal{A}_k W\cdot W d\Sigma.\end{aligned} \tag{3.10}$$

Definition 3.3 shows that the first term on the right of (3.10) is equal to

$$-\int_{\mathcal{Q}^\delta} \mathcal{A}(\widetilde{\partial})W \cdot W dx - \int_{\mathcal{Q}^\delta} \Big( \sum \mathcal{A}_k \partial_k \frac{\tau}{\tau + \sigma_k} \Big) W \cdot W dx. \tag{3.11}$$

The second term on the right of (3.10) is equal to

$$\int_{\partial \mathcal{Q}^\delta} \sum \frac{\tau \nu_k}{\tau + \sigma_k} \mathcal{A}_k W \cdot W \, d\Sigma \;=\; \int_{\partial \mathcal{Q}^\delta} \Pi^{-1} \mathcal{A}(\mathcal{M}\nu) W \cdot W \, d\Sigma. \tag{3.12}$$

In the last four identities use $\mathcal{A}(\widetilde{\partial})W = \tau W$ to find

$$\int_{\mathcal{Q}^\delta} \Big[ 2\tau - \sum \mathcal{A}_k \partial_k \Big( \frac{\tau}{\tau + \sigma_k} \Big) \Big] W \cdot W dx = \int_{\partial \mathcal{Q}^\delta} \Pi^{-1} \mathcal{A}(\mathcal{M}\nu) W \cdot W d\Sigma. \tag{3.13}$$

Equation (3.9) yields $\pi^+(\mathcal{M}\nu)W = 0$ on $\partial \mathcal{Q}^\delta$. Therefore, the integrand in the boundary integral on the right of (3.13) is nonpositive thanks to (3.6). Thus, the integral is nonpositive.

Choose $M > 0$ so that for $\tau > M$ the real symmetric matrix in square brackets on the left is everywhere greater than the identity matrix. For such $\tau$, the left hand side is $\geq 0$. The only way that (3.13) can hold is if both sides vanish. The vanishing of the left hand side yields $W = 0$. This completes the proof. $\square$

## 4 Lopatinski, Fredholm, and analyticity

This section shows that the second boundary value problem in Proposition 3.4 satisfies Lopatinski's ellipticity condition. The boundary value problem has unknowns $(E, B)$ of dimension six. The equation $\pi^-(\mathcal{M}\nu)(E, B) = 0$ is two boundary conditions of Dirichlet type since $\pi^-$ has rank equal to two. The last three equations in (4.1) are four boundary conditions expressing the normal derivative in terms of tangential derivatives,

$$\begin{aligned}
\tau^2 E - \widetilde{\Delta} E \;&=\; f_1 && \text{on} \quad \mathcal{Q}^\delta, \\
\tau^2 B - \widetilde{\Delta} B \;&=\; f_2 && \text{on} \quad \mathcal{Q}^\delta, \\
\pi^-(\mathcal{M}\nu)\big(E, B\big) \;&=\; g_1 && \text{on} \quad \partial\mathcal{Q}^\delta, \\
\pi^+(\mathcal{M}\nu)\big(\tau E - \widetilde{\mathrm{curl}} B \,,\, \tau B + \widetilde{\mathrm{curl}} E\big) \;&=\; g_2 && \text{on} \quad \partial\mathcal{Q}^\delta, \\
\widetilde{\mathrm{div}}\, E \;&=\; g_3 && \text{on} \quad \partial\mathcal{Q}^\delta, \\
\widetilde{\mathrm{div}}\, B \;&=\; g_4 && \text{on} \quad \partial\mathcal{Q}^\delta.
\end{aligned} \tag{4.1}$$

Five properties are proved with constant $M$ independent of $\delta$.

• For Re $\tau > M$, problem (4.1) satisfies Lopatinski's criterion describing coercive elliptic boundary value problems (Proposition 4.1 and Corollary 4.2).

• The operator $\mathbf{L}$ mapping $(E, B)$ to $(f_1, f_2, g_1, g_2, g_3, g_4)$ is Fredholm (Corollary 4.6) with index independent of $\tau$ and the absorptions $\sigma_j$.

• $\mathbf{L}$ depends holomorphically on $\tau \in \mathbb{C}$ for Re $\tau > M$ (Proposition 4.7).

• $\mathbf{L}$ is invertible for $\tau \in ]0, \infty[$ (Proposition 5.12).

• Analytic Fredholm Theory implies that for each $\delta$ the operator is invertible except at a discrete subset $\mathbb{D}(\delta) \subset \{\text{Re } \tau > M\}$ (Proposition 5.13).

The hard uniform estimates for solutions of (4.1) are in Sections 6 and 7.

### 4.1 Lopatinski condition and Fredholm property

The verification of Lopatinski's condition (see [2, 40, 24]), begins with the unstretched boundary value problem.

**Proposition 4.1** *For each $\tau \in \mathbb{C}$ and $\delta \in ]0, \delta_0]$, the unstretched boundary value problem*

$$\begin{aligned} \tau^2 B - \Delta B &= f_1 \qquad \text{on} \quad \mathcal{Q}^\delta, \\ \tau^2 E - \Delta B &= f_2 \qquad \text{on} \quad \mathcal{Q}^\delta, \\ \pi^-(\nu)\big(E, B\big) &= g_1 \qquad \text{on} \quad \partial\mathcal{Q}^\delta, \\ \pi^+(\nu)\big(\tau E - \text{curl } B\,,\, \tau B + \text{curl } E\big) &= g_2 \qquad \text{on} \quad \partial\mathcal{Q}^\delta, \\ \text{div } E &= g_3 \qquad \text{on} \quad \partial\mathcal{Q}^\delta, \\ \text{div } B &= g_4 \qquad \text{on} \quad \partial\mathcal{Q}^\delta, \end{aligned} \tag{4.2}$$

*satisfies Lopatinski's condition characterizing coercive elliptic problems.*

**Proof.** The verification at a point of $\partial\mathcal{Q}^\delta$ demands that one replaces $\mathcal{Q}^\delta$ by the half-space with the same tangent plane and outward unit normal $\nu$. Next one drops lower order terms in the equations and the boundary conditions. The resulting problem is rotation invariant. Thus it suffices to verify for the half space $\{x_1 > 0\}$ with $\nu = (-1, 0, 0)$.

The Lopatinski condition concerns solutions of the resulting problem with all source terms equal to zero. It considers solutions of the form

$$(E, B) = e^{i(\xi_2 x_2 + \xi_3 x_3)}\, w(x_1), \quad (\xi_2, \xi_3) \in \mathbb{R}^2 \backslash 0, \quad w(x_1) \to 0 \text{ when } x_1 \to +\infty\,.$$

The condition is satisfied when zero is the only such solution.

The boundary value problem with source terms equal to zero is

$$\begin{aligned} \Delta E = \Delta B &= 0 \qquad \text{on} \quad \{x_1 > 0\}, \\ \pi^-(\nu)\big(E, B\big) &= 0 \qquad \text{on} \quad \{x_1 = 0\}, \\ \pi^+(\nu)(-\operatorname{curl} B\,,\, \operatorname{curl} E) &= 0 \qquad \text{on} \quad \{x_1 = 0\}, \\ \operatorname{div} E \;=\; \operatorname{div} B &= 0 \qquad \text{on} \quad \{x_1 = 0\}. \end{aligned}$$

The Laplace equations and decay at infinity hold if and only if there are constant vectors $\mathbf{e} \in \mathbb{C}^3$ and $\mathbf{b} \in \mathbb{C}^3$ so that

$$w = e^{i\xi_1 x_1}(\mathbf{e}, \mathbf{b}), \qquad \xi_1^2 + \xi_2^2 + \xi_3^2 = 0, \qquad \operatorname{Im} \xi_1 > 0\,. \tag{4.3}$$

Lopatinski's condition requires that the only such solution that satisfies the boundary conditions is the trivial solution. For each $\xi_2, \xi_3$ there is a six dimensional space of solutions $w$ to (4.3). This shows that six boundary conditions are needed.

By rotational invariance it suffices to verify Lopatinski's condition for $(\xi_2, \xi_3) = (0, 1)$ in which case, $\xi = (\xi_1, \xi_2, \xi_3) = (i, 0, 1)$. The boundary conditions hold if and only if

$$\pi^-(\nu)\big(\mathbf{e}, \mathbf{b}\big) = \mathbf{0}, \quad \pi^+(\nu)\big(-\xi \wedge \mathbf{b}\,,\, \xi \wedge \mathbf{e}\big) = \mathbf{0}, \quad \text{and} \quad \xi \cdot \mathbf{e} = \xi \cdot \mathbf{b} = \mathbf{0}\,.$$

First analyse the boundary condition $0 = \pi^-(\nu)(\mathbf{e}\,,\, \mathbf{b}) = \pi^-(\nu)(\mathbf{e}_{tan}, \mathbf{b}_{tan})$. Lemma 2.1 shows that this is equivalent to

$$(\mathbf{e}_{tan}, \mathbf{b}_{tan}) \in \mathcal{E}^+(\nu) \quad \Leftrightarrow \quad \mathbf{e}_{tan} = \mathbf{b}_{tan} \wedge \nu \quad \Leftrightarrow \quad (0, e_2, e_3) \;=\; (0, b_3, -b_2)\,.$$

The divergence conditions $i\mathbf{e}_1 + \mathbf{e}_3 = 0$ and $i\mathbf{b}_1 + \mathbf{b}_3 = 0$ yield,

$$\mathbf{e} = (-ib_2, b_3, -b_2), \qquad \text{and} \qquad \mathbf{b} = (ib_3, b_2, b_3).$$

The curl boundary condition is

$$0 \;=\; \pi^+(\nu)\big(-\xi \wedge \mathbf{b}\,,\, \xi \wedge \mathbf{e}\big) \;=\; \pi^+(\nu)\big(-(\xi \wedge \mathbf{b})_{tan}\,,\, (\xi \wedge \mathbf{e})_{tan}\big).$$

Equivalently, $\big(-(\xi \wedge \mathbf{b})_{tan}\,,\, (\xi \wedge \mathbf{e})_{tan}\big) \in \mathcal{E}^-(\nu)$. Lemma 2.1 implies that this holds if and only if

$$-(\xi \wedge \mathbf{b})_{tan} \;=\; -\big(\xi \wedge \mathbf{e})_{tan} \wedge \nu\,. \tag{4.4}$$

For any vector $\mathbf{v}$, $(\xi \wedge \mathbf{v})_{tan} = (0, v_1 - iv_3, iv_2)$. Therefore

$$(\xi \wedge \mathbf{b})_{tan} = (0, 0, ib_2), \quad (\xi \wedge \mathbf{e})_{tan} = (0, 0, ib_3), \quad (\xi \wedge \mathbf{e})_{tan} \wedge \nu = (0, ib_3, 0).$$

Comparing the first and last shows that if (4.4) holds then $b_2 = b_3 = 0$. Therefore $\mathbf{e} = \mathbf{b} = 0$ verifying Lopatinski's condition. $\square$

**Corollary 4.2** *There is an $M > 0$ so that for $|\tau| > M$ and $0 < \delta < \delta_0$, the stretched boundary value problem* (4.1)*, satisfies Lopatinski's condition characterizing coercive elliptic problems.*

**Proof.** Since $\mathcal{M}(\tau, x) \to I$ and $\nu(\tau, x) \to \nu$ as $\tau \to \infty$, the problems obtained by dropping lower order terms converge to those of Proposition 4.1. Since the Lopatinski condition is inherited by nearby operators, the result follows. $\square$

The source term $g_1$ takes values in the negative spectral subspace of $\mathcal{A}(\mathcal{M}\nu)$. Consider the action of the operators $\tau^2 - \widetilde{\Delta}$ on $H^2(\mathcal{Q}^\delta)$. Since $g_1$ is a trace at the boundary, the natural space for $g_1$ is

$$\Big\{ g \in H^{3/2}(\partial \mathcal{Q}^\delta) \,:\, \pi^-(\mathcal{M}\nu) g = g \Big\} . \tag{4.5}$$

Denote by $\mathcal{E}^-(\tau, x)$ the negative spectral subspace of $\mathcal{A}(\mathcal{M}(\tau, x)\nu(\tau, x))$. Then an equivalent definition is that $g_1$ is an $H^{3/2}$ section of the vector bundle over $\partial\mathcal{Q}^\delta$ whose fiber at $x$ is $\mathcal{E}^-(\tau, x)$. Similarly, the natural space for $g_2$ is

$$\Big\{ g \in H^{1/2}(\partial \mathcal{Q}^\delta) \,:\, \pi^+(\mathcal{M}\nu) g = g \Big\} . \tag{4.6}$$

**Definition 4.3** *For* $\mathrm{Re}\ \tau > 0$ *the closed subspace of $H^{3/2}(\partial\mathcal{Q}^\delta)$ defined in* (4.5) *is denoted $H^{3/2}_{\mathcal{E}^-(\tau,x)}(\partial\mathcal{Q}^\delta)$. The space* (4.6) *is denoted $H^{1/2}_{\mathcal{E}^+(\tau,x)}(\partial\mathcal{Q}^\delta)$.*

Analysis of this space of sections of a $\tau$ dependent vector bundle is reduced to the case of a $\tau$ independent Sobolev space as follows. Treat $H^{3/2}_{\mathcal{E}^-(\tau,x)}(\partial\mathcal{Q}^\delta)$. The space $H^{1/2}_{\mathcal{E}^+(\tau,x)}(\partial\mathcal{Q}^\delta)$ is analogous. For $g \in H^{3/2}(\partial\mathcal{Q}^\delta)$,

$$g = \pi^-(\mathcal{M}\nu) g \quad \Longleftrightarrow \quad \pi^+(\mathcal{M}\nu) g = 0 \quad \text{and} \quad \pi^0(\mathcal{M}\nu) g = 0 \,.$$

Lemma 2.7 shows that with $g = (\mathbf{e}, \mathbf{b})$ this is equivalent to

$$\mathbf{e} \cdot (\mathcal{M}\nu) = \mathbf{b} \cdot (\mathcal{M}\nu) = 0 \quad \text{and} \quad (\pi_{(\mathcal{M}\nu)^\perp} \mathbf{b}) \wedge (\mathcal{M}\nu) + \pi_{(\mathcal{M}\nu)^\perp} \mathbf{e} = 0.$$

For $|\tau|$ large, $\mathcal{M}\nu = \nu + O(1/|\tau|)$. It follows that the two dimensional space $\mathcal{E}^-(\tau, x)$ is very close to $\mathcal{E}^-(\nu)$. The latter is given by the pair of equations independent of $\tau$,

$$\mathbf{e} \cdot \nu = \mathbf{b} \cdot \nu = 0 \quad \text{and} \quad (\pi_{\nu^\perp} \mathbf{b}) \wedge \nu + \pi_{\nu^\perp} \mathbf{e} = 0.$$

A vector $\mathbf{e}, \mathbf{b}$ satisfies these equations if and only if $(\mathbf{e}, \mathbf{b}) = (\mathbf{e}_{tan}, \mathbf{b}_{tan})$ and $\mathbf{b}_{tan} \wedge \nu + \mathbf{e}_{tan} = 0$. Therefore, for each $x$ the linear map

$$\mathcal{E}^-(\nu(x)) \ni (\mathbf{e}(x), \mathbf{b}(x)) \quad \mapsto \quad \mathbf{e}(x)_{tan} \in T_x(\partial \mathcal{Q}^\delta) \tag{4.7}$$

is an invertible map from a two dimensional space to another. It depends smoothly on $x$.
It follows that there is an $M > 0$ so that the nearby map

$$\mathcal{E}^-(\mathcal{M}(\tau, x)\nu(x)) \ni (\mathbf{e}(x), \mathbf{b}(x)) \quad \mapsto \quad \mathbf{e}(x)_{tan} \ \in T_x(\partial \mathcal{Q}^\delta) \tag{4.8}$$

is also invertible. It depends smoothly on $\tau, x$ and analytically on $\tau$.

**Definition 4.4** *Denote by $R^-(\tau, x)$ the inverse from the last sentence. Define $R^+(\tau, x)$ similarly.*

The equation $\pi^-(\mathcal{M}\nu)g = g$ holds if and only if $(\mathbf{e}, \mathbf{b}) = R^-(\tau, x)\mathbf{e}_{tan}$. The space $H^{3/2}_{\mathcal{E}^-(\tau,x)}(\partial \mathcal{Q}^\delta)$ is exactly the set of functions of the form $R^-(\tau, x)\mathbf{e}_{tan}$ where $\mathbf{e}_{tan}$ is a tangential vector field in $H^{3/2}(\partial \mathcal{Q}^\delta)$. The set of such fields is independent of $\tau$. Analogously, $\mathcal{E}^+(\mathcal{M}\nu)$ has equation $(\mathbf{e}, \mathbf{b}) = R^+(\tau, x)\mathbf{e}_{tan}$.

**Definition 4.5** *A continuous linear operator from one Banach space to another is Fredholm when its kernel is finite dimensional and its range is closed with finite dimensional cokernel. The index of a Fredholm operator $S$ is defined as* $\dim \ker S \, - \, \dim \operatorname{coker} S$.

Recall that a continuous curve of Fredholm operators has constant index.

**Corollary 4.6** *For* $\operatorname{Re} \tau > M$*, the operator* $\mathbf{L}$ *that sends* $E, B \in H^2(\mathcal{Q}^\delta)$ *to*

$$\Big(\tau^2 E - \widetilde{\Delta} E \,,\, \tau^2 B - \widetilde{\Delta} B \,,\, \pi^-(\mathcal{M}\nu)\big(E, B)|_{\partial \mathcal{Q}^\delta} \,,$$
$$\pi^+(\mathcal{M}\nu)\big(\tau E - \widetilde{\operatorname{curl}}\, B \,,\, \tau B + \widetilde{\operatorname{curl}}\, E\big)|_{\partial \mathcal{Q}^\delta} \,,\, (\widetilde{\operatorname{div}}\, E)\big|_{\partial \mathcal{Q}^\delta} \,,\, (\widetilde{\operatorname{div}}\, B)\big|_{\partial \mathcal{Q}^\delta}\Big)$$

*is a continuous and Fredholm operator with values in*

$$L^2(\mathcal{Q}^\delta) \times L^2(\mathcal{Q}^\delta) \times H^{3/2}_{\mathcal{E}^-(\tau,x)}(\partial \mathcal{Q}^\delta) \times H^{1/2}_{\mathcal{E}^+(\tau,x)}(\partial \mathcal{Q}^\delta) \times H^{1/2}(\partial \mathcal{Q}^\delta) \times H^{1/2}(\partial \mathcal{Q}^\delta) \,. \tag{4.9}$$

*Both the kernel and the annihilator of the range contain only smooth functions. The index of the operator is independent of $\tau$ with* $\operatorname{Re} \tau > M$ *and the nonnegative absorptions.*

**Proof.** The Fredholm and smoothness assertions are the basic results of the theory of elliptic boundary value problems satisfying Lopatinski's condition (see Agmon Douglis Nirenberg [2], Taylor [40, Vol II], Hörmander [24, Vol III, Theorem 20.1.8]).

To prove that the index is constant, consider the continuous family of Fredholm maps that for $\mu \in [0,1]$ takes $\tau(\mu) = (1-\mu)\tau + \mu$ and $\sigma_j(\mu) = (1-\mu)\sigma_j$. This connects the operator at $\tau, \sigma$ to that at $1, 0$. Multiplication by $R^-(\tau, x)$ is a linear isomorphism from $H^{3/2}_{\mathcal{E}^-(\tau,x)}(\partial\mathcal{Q}^\delta)$ to the fixed Hilbert space of $H^{3/2}(\partial\mathcal{Q}^\delta)$ tangential fields. Similarly for $H^{1/2}_{\mathcal{E}^+(\tau,x)}(\partial\mathcal{Q}^\delta)$. In this way the Fredholm maps are conjugated to maps with values in a fixed space. The index of the resulting family is does not depend on $\tau$. Therefore the index of the original family does not depend on $\tau$. $\square$

## 4.2 Analyticity

The source terms in the $\pi^\pm$ boundary conditions in (4.1) take values in spaces that depend on $\tau$. They are first converted to source terms in $\tau$-independent spaces. With $R^\pm$ from Definition 4.4, the boundary value problem (4.1) takes the form,

$$\begin{aligned}
\big(\tau^2 - \widetilde{\Delta}\big)(E,B) &= (f_1, f_2) && \text{on} \quad \mathcal{Q}^\delta, \\
\pi^-(\mathcal{M}\nu)\big(E,B\big) &= R^-(\tau,x)\underline{g}_1 && \text{on} \quad \partial\mathcal{Q}^\delta, \\
\pi^+(\mathcal{M}\nu)\big(\tau E - \widetilde{\operatorname{curl}}\, B\,,\, \tau B + \widetilde{\operatorname{curl}}\, E\big) &= R^+(\tau,x)\underline{g}_2 && \text{on} \quad \partial\mathcal{Q}^\delta, \\
(\widetilde{\operatorname{div}}\, E, \widetilde{\operatorname{div}}\, B) &= (g_3, g_4) && \text{on} \quad \partial\mathcal{Q}^\delta.
\end{aligned} \tag{4.10}$$

The source term $\underline{g}_1$ takes values in the space of $H^{3/2}(\partial\mathcal{Q}^\delta)$ tangential fields. and $\underline{g}_2$ in the $H^{1/2}(\partial\mathcal{Q}^\delta)$ tangential fields. The coefficients of the operators depend differentiably on $\tau, x$ and analytically on $\tau$.

**Proposition 4.7** *Suppose that at $\underline{\tau} \in \mathbb{C}$, the Fredholm operator* $\mathbf{L}$ *from Corollary 4.6 is invertible. Then* $\mathbf{L}$ *is invertible on a neighborhood of $\underline{\tau}$. If the source terms $f, \underline{g}_j$ depend analytically on $\tau$, then the corresponding solution of* (4.10) *is an analytic function on a neighborhood of $\underline{\tau}$ with values in $H^2(\mathcal{Q}^\delta)$.*

**Proof.** The invertibility of $\mathbf{L}$ for $\tau = a + ib$ near $\underline{\tau}$ follows by Neumann's series. Standard elliptic theory shows that the map $a, b \mapsto (E, B)$ is infinitely differentiable with values in $H^2(\mathcal{Q}^\delta)$. The derivatives satisfy the system obtained by differentiating the system and boundary conditions with respect to $a, b$.

To prove analyticity it suffices to show that $\partial(E,B)/\partial\overline{\tau} = 0$. All the coefficients and the $f, g$ are analytic in $\tau$. Therefore, differentiating the boundary value problem shows that $\partial(E,B)/\partial\overline{\tau}$ satisfies system (4.10) with all sources equal to zero. That $\partial(E,B)/\partial\overline{\tau} = 0$ follows from the invertibility. $\square$

# 5 Green's identities

*A priori* estimates require integrations by parts using Green's identities for $\widetilde{\text{div}}$ and $\widetilde{\text{curl}}$. They and associated trace theorems for the spaces $H_{\widetilde{\text{curl}}}$ and $H_{\widetilde{\text{div}}}$ are derived in Section 5.1. The unstretched versions are standard [11, 12]. Definition 5.8 introduces a bilinear form $a((E,B),(E',B'))$ associated to $\tau^2 - \widetilde{\Delta}$. Proposition 5.10 and Remark 5.2 relate $a$ to the spectral projections from Section 2 and to the stretched Maxwell and Helmholtz boundary value problems in Proposition 3.4. For $\tau \in ]0,\infty[$ the quadratic form of $a$ is strictly positive. Using this, Section 5.3 proves that $\mathbf{L}$ from Corollary 4.6 is invertible when $\tau \in ]0,\infty[$.

## 5.1 Stretched Green's identities

### 5.1.1 Stretched div, curl and grad identities

The identities involve $\Pi, \mathcal{M}, \mathcal{M}_{ii}$ from Definition 3.3.

**Lemma 5.1** *The following differential identities hold for $v \in C^1$,*

$$\Pi(\tau,x)\,\widetilde{\text{div}}\,v \;=\; \text{div}(\mathcal{M}v)\,, \tag{5.1}$$

$$\widetilde{\text{div}}(\Phi\wedge\Psi) \;=\; (\widetilde{\text{curl}}\,\Phi)\cdot\Psi \;-\; (\widetilde{\text{curl}}\,\Psi)\cdot\Phi, \tag{5.2}$$

$$\Pi(\tau,x)\,\widetilde{\text{grad}}\,\phi = \big(\partial_1(\mathcal{M}_{11}\phi)\,,\,\partial_2(\mathcal{M}_{22}\phi)\,,\,\partial_3(\mathcal{M}_{33}\phi)\big) = \mathcal{M}\ \text{grad}\ \phi. \tag{5.3}$$

**Proof.** For (5.1) compute using (3.2),

$$\begin{aligned}
\Pi(\tau,x)\,\widetilde{\operatorname{div}}\, v &= \Big(\prod_{j=1}^{3}\frac{\tau+\sigma_j(x_j)}{\tau}\Big)\Big(\sum_{i=1}^{3}\frac{\tau}{\tau+\sigma_i(x_i)}\partial_i v_i\Big)\\
&= \sum_{i=1}^{3}\Big(\prod_{j\neq i}\frac{\tau+\sigma_j(x_j)}{\tau}\Big)\partial_i v_i = \sum_{i=1}^{3}\mathcal{M}_{ii}\partial_i v_i\\
&= \sum_{i=1}^{3}\partial_i\big(\mathcal{M}_{ii}v_i\big) \qquad \text{since}\ \ \partial_i\mathcal{M}_{ii}=0\\
&= \operatorname{div}(\mathcal{M}v)\,.
\end{aligned} \tag{5.4}$$

For (5.3) use $\partial_i\mathcal{M}_{ii}=0$ again to compute,

$$\begin{aligned}
\Pi(\tau,x)\ \widetilde{\operatorname{grad}}\,\phi &= \Pi(\tau,x)\Big(\frac{\tau}{\tau+\sigma_1}\partial_1\phi\,,\ \frac{\tau}{\tau+\sigma_2}\partial_2\phi\,,\ \frac{\tau}{\tau+\sigma_3}\partial_3\phi\Big)\\
&= \Big(\mathcal{M}_{11}\partial_1\phi\,,\ \mathcal{M}_{22}\partial_2\phi\,,\ \mathcal{M}_{33}\partial_3\phi\Big) = \mathcal{M}\ \operatorname{grad}\phi\\
&= \Big(\partial_1(\mathcal{M}_{11}\phi)\,,\ \partial_2(\mathcal{M}_{22}\phi)\,,\ \partial_3(\mathcal{M}_{33}\phi)\Big).
\end{aligned}$$

Identity (5.2) is similar and is left to the reader. $\square$

**Lemma 5.2** *For $\phi,\Phi,\Psi$ in $H^1(\mathcal{Q}^\delta)$ the folllowing integral identities hold,*

$$\int_{\mathcal{Q}^\delta}\Pi(\tau,x)\,\widetilde{\operatorname{curl}}\,\Phi\cdot\Psi\,dx-\int_{\mathcal{Q}^\delta}\Pi(\tau,x)\,\widetilde{\operatorname{curl}}\,\Psi\cdot\Phi\,dx = \int_{\partial\mathcal{Q}^\delta}(\Psi\wedge(\mathcal{M}\,\nu))\cdot\Phi\,d\Sigma, \tag{5.5}$$

$$\int_{\mathcal{Q}^\delta}\Pi(\tau,x)\ \Phi\cdot\widetilde{\operatorname{grad}}\,\phi\,dx+\int_{\mathcal{Q}^\delta}\Pi(\tau,x)\,\phi\ \widetilde{\operatorname{div}}\,\Phi\ dx = \int_{\partial\mathcal{Q}^\delta}\phi\,\Phi\cdot(\mathcal{M}\nu)\ d\Sigma. \tag{5.6}$$

**Proof.** For (5.5), using (5.1) and the divergence theorem shows that for $\Phi,\Psi$ in $H^1(\mathcal{Q}^\delta)$,

$$\int_{\mathcal{Q}^\delta}\Pi(\tau,x)\,\widetilde{\operatorname{div}}(\Phi\wedge\Psi)\,dx=\int_{\mathcal{Q}^\delta}\operatorname{div}(\mathcal{M}(\Phi\wedge\Psi))\,dx=\int_{\partial\mathcal{Q}^\delta}(\mathcal{M}(\Phi\wedge\Psi))\cdot\nu\,dx.$$

Since $\mathcal{M}$ is diagonal,

$$(\mathcal{M}(\Phi\wedge\Psi))\cdot\nu=(\Phi\wedge\Psi)\cdot\mathcal{M}\,\nu=-(\Psi\wedge\Phi)\cdot\mathcal{M}\,\nu=(\Psi\wedge\mathcal{M}\,\nu)\cdot\Phi\,.$$

This proves (5.5) using (5.2).

Formula (5.6) comes from (5.3). Indeed,

$$\begin{aligned}
\int_{\mathcal{Q}^\delta} \Pi(\tau,x)\, \widetilde{\mathrm{grad}}\,\phi\cdot\Phi\,dx &= \int_{\mathcal{Q}^\delta}\sum \Phi_j\partial_j(\mathcal{M}_{jj}\phi)\,dx \\
&= -\int_{\mathcal{Q}^\delta}\sum \mathcal{M}_{jj}\phi\,\partial_j\Phi_j\,dx + \int_{\partial\mathcal{Q}^\delta}\sum \phi\,\mathcal{M}_{jj}\Phi_j\nu_j\,d\Sigma \\
&= -\int_{\mathcal{Q}^\delta}\phi\sum \mathcal{M}_{jj}\partial_j\Phi_j\,dx + \int_{\partial\mathcal{Q}^\delta}\phi\sum \Phi_j\mathcal{M}_{jj}\nu_j\,d\Sigma \\
&= -\int_{\mathcal{Q}^\delta}\Pi\,\phi\sum \frac{\mathcal{M}_{jj}}{\Pi}\,\partial_j\Phi_j\,dx + \int_{\partial\mathcal{Q}^\delta}\phi\,\Phi\cdot(\mathcal{M}\nu)\,d\Sigma \\
&= -\int_{\mathcal{Q}^\delta}\Pi\,\phi\,\widetilde{\mathrm{div}}\,\Phi\,dx + \int_{\partial\mathcal{Q}^\delta}\phi\,\Phi\cdot(\mathcal{M}\nu)\,d\Sigma. \qquad \square
\end{aligned}$$

**Definition 5.3** *Define Hilbert spaces and their analogues for $\mathcal{Q}$,*

$$\begin{aligned}
H_{\widetilde{\mathrm{curl}}}(\mathcal{Q}^\delta) &:= \Big\{\Phi\in L^2(\mathcal{Q}^\delta) : \widetilde{\mathrm{curl}}\,\Phi\in L^2(\mathcal{Q}^\delta)\Big\}, \\
H_{\widetilde{\mathrm{div}}}(\mathcal{Q}^\delta) &:= \Big\{\Phi\in L^2(\mathcal{Q}^\delta) : \widetilde{\mathrm{div}}\,\Phi\in L^2(\mathcal{Q}^\delta)\Big\}.
\end{aligned}$$

The set of restrictions of $C_0^\infty(\mathbb{R}^3)$ to $\mathcal{Q}^\delta$, denoted $C^\infty_{(0)}\big(\overline{\mathcal{Q}^\delta}\big)$, is dense in each of the three spaces $H_{\widetilde{\mathrm{curl}}}(\mathcal{Q}^\delta)$, $H_{\widetilde{\mathrm{div}}}(\mathcal{Q}^\delta)$, and $H_{\widetilde{\mathrm{div}}}(\mathcal{Q}^\delta)\cap H_{\widetilde{\mathrm{curl}}}(\mathcal{Q}^\delta)$. The density is proved by convolution with a kernel supported in cones towards the interior. Similarly for $\mathcal{Q}$.

**Proposition 5.4** *The map*

$$C^\infty_{(0)}\big(\overline{\mathcal{Q}^\delta}\big)\times C^\infty(\partial\mathcal{Q}^\delta) \ni (\Phi,\Psi) \quad\mapsto\quad \int_{\partial\mathcal{Q}^\delta}(\Phi\wedge\Psi)\cdot\mathcal{M}\nu\;d\Sigma \quad\in\quad \mathbb{C}$$

*extends uniquely to a continuous bilinear form on $H_{\widetilde{\mathrm{curl}}}(\mathcal{Q}^\delta)\times H^{1/2}(\partial\mathcal{Q}^\delta)$. Therefore, for $\Phi\in H_{\widetilde{\mathrm{curl}}}(\mathcal{Q}^\delta)$, $\Phi|_{\partial\mathcal{Q}^\delta}\wedge\mathcal{M}\nu$ is a well defined element of $H^{-1/2}(\partial\mathcal{Q}^\delta)$. Similarly, the map*

$$\big(\phi,\Psi\big)\mapsto\int_{\mathcal{Q}^\delta}\phi\Psi\cdot\mathcal{M}\nu\,d\Sigma$$

*is a continuous bilinear form on $H^{1/2}(\partial\mathcal{Q}^\delta)\times H_{\widetilde{\mathrm{div}}}(\mathcal{Q}^\delta)$. For $\Psi\in H_{\widetilde{\mathrm{div}}(\mathcal{Q}^\delta)}$, $\Psi|_{\partial\mathcal{Q}^\delta}\cdot\mathcal{M}\nu$ is a well defined element of $H^{-1/2}(\partial\mathcal{Q}^\delta)$. The same assertions are true for $\mathcal{Q}$.*

**Proof.** For the trace, prove the $\widetilde{\mathrm{curl}}$ assertions. Given $\Psi \in H^{1/2}(\partial\mathcal{Q}^\delta)$ choose $\underline{\Psi} \in H^1(\mathbb{R}^3)$ whose trace at $\partial\mathcal{Q}^\delta$ is equal to $\Psi$ and $\|\underline{\Psi}\|_{H^1(\mathbb{R}^3)} \lesssim \|\Psi\|_{H^{1/2}}(\partial\mathcal{Q}^\delta)$. Equation (5.5) shows that with constants independent of $\delta$,

$$\begin{aligned}\Big|\int_{\partial\mathcal{Q}^\delta} (\Phi\wedge\mathcal{M}\nu)\cdot\Psi\, d\Sigma\Big| \ &\lesssim\ \|\Phi, \widetilde{\mathrm{curl}}\Phi\|_{L^2(\mathcal{Q}^\delta)} \|\underline{\Psi}, \widetilde{\mathrm{curl}}\underline{\Psi}\|_{L^2(\mathcal{Q}^\delta)} \\ &\lesssim\ \|\Phi, \widetilde{\mathrm{curl}}\Phi\|_{L^2(\mathcal{Q}^\delta)} \|\Psi\|_{H^{1/2}(\partial\mathcal{Q}^\delta)}.\end{aligned}$$

The second assertion follows from (5.6) in the same way. $\square$

**Corollary 5.5** *The map $C^\infty_{(0)}(\overline{\mathcal{Q}}) \ni \Phi \mapsto \Phi|_{\partial\mathcal{Q}}$ extends uniquely to a continous map from $H_{\widetilde{\mathrm{curl}}}(\mathcal{Q}) \cap H_{\widetilde{\mathrm{div}}}(\mathcal{Q})$ to $H^{-1/2}(\partial\mathcal{Q})$.*

**Proof.** That $C^\infty_{(0)}(\overline{\mathcal{Q}})$ is dense in $H_{\widetilde{\mathrm{curl}}}(\mathcal{Q}) \cap H_{\widetilde{\mathrm{div}}}(\mathcal{Q})$ is proved by convolution. To prove continuity, test $\Phi$ by dense elements of $H^{1/2}(\partial\mathcal{Q})$. The set of restrictions of element of $C^\infty_0(\mathbb{R}^3)$ so that $\Psi$ vanishes in a neighborhood of $\mathcal{S}$ is dense in $H^{1/2}(\partial\mathcal{Q})$ (see [22, Lemma 2.7.**i**] where the assertion has a typo. It should be $H^{-1/2}(\mathbb{R}^3)$). For such test functions

$$\int_{\partial\mathcal{Q}} \Phi\cdot\Psi\ d\Sigma\ =\ \sum_k \int_{\Gamma_k} \Phi\cdot\Psi\ d\Sigma\,.$$

Definition 2.4 together with formula (2.3) imply that for all $w \in \mathbb{C}^3$ and $\xi \in \mathbb{C}^3 \setminus \mathcal{Z}$,

$$\mathbf{w}\ =\ \frac{(\xi\cdot\mathbf{w})\xi}{\xi\cdot\xi}\ -\ \frac{\xi\wedge(\xi\wedge\mathbf{w})}{\xi\cdot\xi}\,.$$

Use this on $\Gamma_k$ with $\xi = \mathcal{M}\nu$, Re $\tau > 0$ and $\mathbf{w} = \Phi$. On each face $\Gamma_k$, $\mathcal{M}\nu$ is a constant vector that is not in $\mathcal{Z}$. Find that

$$\int_{\Gamma_k} \Phi\cdot\Psi\ d\Sigma\ =\ \int_{\Gamma_k}\Big(\frac{((\mathcal{M}\nu)\cdot\Phi)(\mathcal{M}\cdot\nu)}{(\mathcal{M}\nu)\cdot(\mathcal{M}\nu)}\ -\ \frac{(\mathcal{M}\nu)\wedge((\mathcal{M}\nu)\wedge\boldsymbol{\Phi})}{(\mathcal{M}\nu)\cdot(\mathcal{M}\nu)}\Big)\cdot\Psi\ d\Sigma\,.$$

Since $\Psi$ is compactly supported in $\Gamma_k$ one has the upper bound

$$\lesssim\ \Big(\|(\mathcal{M}\nu)\cdot\Phi\|_{H^{-1/2}(\partial\mathcal{Q})}\ +\ \|(\mathcal{M}\nu)\wedge\Phi\|_{H^{-1/2}(\partial\mathcal{Q})}\Big)\|\Psi\|_{H^{1/2}(\partial\mathcal{Q})}\,.$$

Corollary 5.5 follows from Proposition 5.4. $\square$

### 5.1.2 Stretched $\operatorname{curl}\operatorname{curl}$ and $\operatorname{grad}\operatorname{div}$ identities

**Lemma 5.6** *For $w, \Phi \in H^2(\mathcal{Q}^\delta) \times H^1(\mathcal{Q}^\delta)$ one has the two identities,*

$$\begin{aligned}\int_{\mathcal{Q}^\delta} \Pi(\tau,x)\ \widetilde{\operatorname{curl}}\,\widetilde{\operatorname{curl}}\, w \cdot \Phi\ dx\ -\ \int_{\mathcal{Q}^\delta} \Pi(\tau,x)\ \widetilde{\operatorname{curl}}\, w \cdot \widetilde{\operatorname{curl}}\,\Phi\ dx \\ = -\int_{\partial\mathcal{Q}^\delta} (\widetilde{\operatorname{curl}}\, w \wedge \mathcal{M}\,\nu)\ \cdot \Phi\ d\Sigma\,,\end{aligned} \tag{5.7}$$

$$\begin{aligned}\int_{\mathcal{Q}^\delta} \Pi\,(\tau,x)\ \widetilde{\operatorname{grad}}\,\widetilde{\operatorname{div}}\, w \cdot \Phi\ dx + \int_{\mathcal{Q}^\delta} \Pi(\tau,x)\,\widetilde{\operatorname{div}}\, w\ \widetilde{\operatorname{div}}\,\Phi\ dx \\ = \int_{\partial\mathcal{Q}^\delta} \widetilde{\operatorname{div}}\, w\ \Phi \cdot (\mathcal{M}\nu)\ d\Sigma.\end{aligned} \tag{5.8}$$

**Remark 5.1** *When $\tau$ is not real, $\mathcal{M}\nu$ is a complex vector. It is not a normal.*

**Proof of Lemma 5.6.** Applying (5.5) to $\Psi := \widetilde{\operatorname{curl}}\, w$ proves identity (5.7). Choosing $\phi = \widetilde{\operatorname{div}}\, w$ in (5.6) yields (5.8). $\square$

**Lemma 5.7** *For $\Phi \in H^2(\mathcal{Q}^\delta)$ and $\Psi \in H^1(\mathcal{Q}^\delta)$,*

$$\begin{aligned}&\int_{\mathcal{Q}^\delta} \Pi(\tau,x)\,\Psi \cdot (\tau^2\Phi - \widetilde{\Delta}\Phi)\,dx = \\ &\qquad \int_{\mathcal{Q}^\delta} \Pi(\tau,x)\Big(\tau^2\,\Phi\cdot\Psi + \widetilde{\operatorname{curl}}\,\Phi\cdot\widetilde{\operatorname{curl}}\,\Psi + \widetilde{\operatorname{div}}\,\Phi\,\widetilde{\operatorname{div}}\,\Psi\Big)\,dx \\ &\qquad\quad - \int_{\partial\mathcal{Q}^\delta} \big(\widetilde{\operatorname{curl}}\,\Phi \wedge (\mathcal{M}\nu) + \widetilde{\operatorname{div}}\,\Phi\ (\mathcal{M}\nu)\ \big)\cdot\Psi\ d\Sigma.\end{aligned} \tag{5.9}$$

**Proof.** Equation (1.15) shows that for any $\Phi \in H^1(\mathcal{Q}^\delta)$,

$$-\int_{\mathcal{Q}^\delta} \Pi(\tau,x)\,\Phi\cdot\widetilde{\Delta}\,\Psi\,dx = -\int_{\mathcal{Q}^\delta} \Pi(\tau,x)\,\Phi\cdot(\widetilde{\operatorname{grad}}\,\widetilde{\operatorname{div}}\,\Psi - \widetilde{\operatorname{curl}}\,\widetilde{\operatorname{curl}}\Psi)\,dx\,.$$

Using the two second order formulas from Lemma 5.6 yields

$$\begin{aligned}-\int_{\mathcal{Q}^\delta} \Pi(\tau,x)\Psi\cdot\widetilde{\Delta}\Phi\,dx = \int_{\mathcal{Q}^\delta} \Pi(\tau,x)\,\widetilde{\operatorname{curl}}\Phi\cdot\widetilde{\operatorname{curl}}\Psi\,dx + \int_{\mathcal{Q}^\delta} \Pi(\tau,x)\,\widetilde{\operatorname{div}}\Phi\,\widetilde{\operatorname{div}}\Psi\,dx \\ - \int_{\partial\mathcal{Q}^\delta} (\widetilde{\operatorname{curl}}\Phi \wedge (\mathcal{M}\nu))\cdot\Psi\ d\Sigma - \int_{\partial\mathcal{Q}^\delta} \Psi\cdot(\mathcal{M}\nu)\ \widetilde{\operatorname{div}}\,\Phi\ d\Sigma.\end{aligned}$$

Adding the $\tau^2$ term to each side yields (5.9). $\square$

## 5.2 The bilinear form $a((E,B),(E',B'))$

When the absorptions are all equal to zero, the form $a$ from the next definition is related to formula (1.8) in [14]. For the definition, Proposition 5.4 proves that $\pi_{(\mathcal{M}\nu)^\perp}\Phi$ is a well defined element of $H^{-1/2}(\partial\mathcal{Q}^\delta)$.

**Definition 5.8** *With the notations from Definition 2.4, define*

$$\mathbb{V} \ = \ \Big\{ \Phi \in H_{\widetilde{\mathrm{curl}}}(\mathcal{Q}^\delta) \cap H_{\widetilde{\mathrm{div}}}(\mathcal{Q}^\delta) \ \ : \pi_{(\mathcal{M}\nu)^\perp}\Phi \in L^2(\partial\mathcal{Q}^\delta) \Big\}\,.$$

*Define a symmetric continuous bilinear form* $b : \mathbb{V}\times\mathbb{V} \to \mathbb{C}$ *by*

$$\begin{aligned} b(\Phi,\Psi) \ &:= \ \tau \int_{\partial\mathcal{Q}^\delta} \pi_{(\mathcal{M}\nu)^\perp}\Phi \cdot \pi_{(\mathcal{M}\nu)^\perp}\Psi \ \ d\Sigma \ + \\ &\int_{\mathcal{Q}^\delta} \Pi(\tau,x)\Big(\tau^2\,\Phi\cdot\Psi + \widetilde{\mathrm{div}}\,\Phi\cdot\widetilde{\mathrm{div}}\,\Psi + \widetilde{\mathrm{curl}}\Phi\cdot\widetilde{\mathrm{curl}}\Psi\Big)\,dx. \end{aligned} \tag{5.10}$$

*Define a symmetric continuous bilinear form* $a : (\mathbb{V}\times\mathbb{V})\times(\mathbb{V}\times\mathbb{V}) \to \mathbb{C}$ *by*

$$a\big((E,B)\,,\,(E',B')\big) \ := \ b(E,E') \ + \ b(B,B')\,. \tag{5.11}$$

*For* $(E,B) \in H_{\widetilde{\mathrm{curl}}}(\mathcal{Q}^\delta)\cap H_{\widetilde{\mathrm{div}}}(\mathcal{Q}^\delta)$ *define* $\mathcal{D}(E,B)$ *as*

$$\begin{aligned} \mathcal{D}(E,B) := \Big(&\tau\,\pi_{(\mathcal{M}\nu)^\perp}E + \widetilde{\mathrm{curl}}E\wedge(\mathcal{M}\nu) + \widetilde{\mathrm{div}}E\ (\mathcal{M}\nu), \\ &\tau\,\pi_{(\mathcal{M}\nu)^\perp}B + \widetilde{\mathrm{curl}}B\wedge(\mathcal{M}\nu) + \widetilde{\mathrm{div}}B\ (\mathcal{M}\nu)\Big). \end{aligned} \tag{5.12}$$

**Lemma 5.9** *For* $(E,B)\in H^2(\mathcal{Q}^\delta)$*, and* $(E',B')\in H^1(\mathcal{Q}^\delta)$*,*

$$\begin{aligned} a\big((E,B)\,,\,(E'\,,\,B')\big) = &\int_{\mathcal{Q}^\delta} \Pi(\tau,x)\,\big(\big(\tau^2 - \widetilde{\Delta}\big)(E,B)\big)\cdot(E',B')\,dx \\ &+ \int_{\partial\mathcal{Q}^\delta} \mathcal{D}(E,B)\cdot(E',B')\,d\Sigma. \end{aligned} \tag{5.13}$$

**Proof of Lemma 5.9.** Rewriting identity (5.9) yields

$$\begin{aligned} &\int_{\mathcal{Q}^\delta} \Pi(\tau,x)\Big(\tau^2\Phi\cdot\Psi + \widetilde{\mathrm{curl}}\,\Phi\cdot\widetilde{\mathrm{curl}}\,\Psi + \widetilde{\mathrm{div}}\,\Phi\ \widetilde{\mathrm{div}}\,\Psi\Big)\,dx = \\ &\int_{\mathcal{Q}^\delta} \Pi(\tau,x)\Psi\cdot(\tau^2\Phi - \widetilde{\Delta}\Phi)\,dx + \int_{\partial\mathcal{Q}^\delta} \big(\widetilde{\mathrm{curl}}\,\Phi\wedge(\mathcal{M}\nu) + \widetilde{\mathrm{div}}\Phi\ \mathcal{M}\nu\ \big)\cdot\Psi\,d\Sigma. \end{aligned}$$

This yields

$$
\begin{aligned}
b(\Phi,\Psi) \;&=\; \int_{\mathcal{Q}^\delta} \Pi(\tau,x)(\tau^2\Phi - \widetilde{\Delta}\Phi)\cdot\Psi\,dx \;+ \\
&\int_{\partial\mathcal{Q}^\delta} \tau(\pi_{(\mathcal{M}\nu)^\perp}\Phi)\cdot(\pi_{(\mathcal{M}\nu)^\perp}\Psi) + \big(\widetilde{\mathrm{curl}}\Phi\wedge(\mathcal{M}\nu) + \widetilde{\mathrm{div}}\Phi\ \mathcal{M}\nu\big)\cdot\Psi\,d\Sigma.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
b(\Phi,\Psi) \;&=\; \int_{\mathcal{Q}^\delta} \Pi(\tau,x)\Psi\cdot(\tau^2\Phi - \widetilde{\Delta}\Phi)\,dx \\
&\quad + \int_{\partial\mathcal{Q}^\delta} \Big(\tau\pi_{(\mathcal{M}\nu)^\perp}\Phi + \widetilde{\mathrm{curl}}\,\Phi\wedge(\mathcal{M}\nu) + \widetilde{\mathrm{div}}\,\Phi\ \mathcal{M}\nu\ \Big)\cdot\Psi\ d\Sigma.
\end{aligned}
\tag{5.14}
$$

Combining (5.14) for $(E,E')$ and $(B,B')$ completes the proof of Lemma 5.9. □

The next proposition together with (5.13) show how intimately linked the form $a$, the boundary term $\mathcal{D}$, and, the stretched Maxwell boundary value problem are.

**Proposition 5.10** *For $\tau\in\mathbb{C}$ with $|\tau|>M$ and $(E,B)\in H^2(\mathcal{Q}^\delta)$,*

$$
\begin{aligned}
\pi^0(\mathcal{M}\nu)\,\mathcal{D}(E,B) \;&=\; \Big((\widetilde{\mathrm{div}}\,E)\mathcal{M}\nu\,,\,(\widetilde{\mathrm{div}}\,B)\mathcal{M}\nu\Big),\\
\pi^\pm(\mathcal{M}\nu)\mathcal{D}(E,B) \;&=\; \pi^\pm(\mathcal{M}\nu)\Big(\tau E - \widetilde{\mathrm{curl}}\,B\,,\,\tau B + \widetilde{\mathrm{curl}}\,E\Big)\big(\mathcal{M}\nu\cdot\mathcal{M}\nu\big)^{1/2}.
\end{aligned}
$$

**Remark 5.2** *If $(E,B)$ is a solution of the stretched Maxwell boundary value problem or the Helmholtz boundary value problem from Proposition 3.4, then $\mathcal{D}(E,B)=0$ since $\pi^\mu(\mathcal{M}\nu)\mathcal{D}(E,B)=0$ for $\mu\in\{0,+,-\}$.*

**Proof.** Use Lemma 2.7. The $\pi^0$ identity follows from the definition (5.12) of $\mathcal{D}$. Indeed, when $\pi^0(\mathcal{M}\nu)$ from (2.6) is applied to the right hand side of (5.12), it annihilates all the terms except $(\widetilde{\mathrm{div}}E)M\nu$ and $(\widetilde{\mathrm{div}}B)M\nu$ yielding the desired conclusion.

For the $\pi^+(\mathcal{M}\nu)$ identity, use $\pi^+(\mathcal{M}\nu)\pi_{\mathcal{M}\nu}=0$ to compute

$$
\begin{aligned}
\pi^+(\mathcal{M}\nu)\big(E,B\big) &= \pi^+(\mathcal{M}\nu)\big((\pi_{\mathcal{M}\nu}+\pi_{(\mathcal{M}\nu)^\perp})E\,,\,(\pi_{\mathcal{M}\nu}+\pi_{(\mathcal{M}\nu)^\perp})B\big)\\
&= \pi^+\big(\mathcal{M}\nu)(\pi_{(\mathcal{M}\nu)^\perp}E\,,\,\pi_{(\mathcal{M}\nu)^\perp}B\big).
\end{aligned}
\tag{5.15}
$$

Use the formula of part **ii** of Lemma 2.5. Compute for arbitrary $\mathbf{v}$ and $\mathbf{w}$ and $\xi \in \mathbb{R}^3 \setminus 0$. Since $\mathbf{v} \wedge \xi$ and $\mathbf{w} \wedge \xi$ are orthogonal to $\xi$,

$$\begin{aligned}\pi^+(\xi)(\mathbf{v} \wedge \xi, \mathbf{w} \wedge \xi) \;&=\; \frac{1}{2}\Big(\mathbf{v} \wedge \xi + (\mathbf{w} \wedge \xi) \wedge \xi/|\xi|\,,\, \mathbf{w} \wedge \xi + (-\mathbf{v} \wedge \xi) \wedge \xi/|\xi|\Big)\\ &=\; \frac{1}{2}\Big(-|\xi|\,\mathbf{w} + \mathbf{v} \wedge \xi\,,\, |\xi|\,\mathbf{v} + \mathbf{w} \wedge \xi\Big) \;=\; |\xi|\,\pi^+(\xi)(-\mathbf{w}, \mathbf{v})\,.\end{aligned}$$

This implies that for $\tau \in ]0, \infty[$

$$\pi^+(\mathcal{M}\nu)(\mathbf{v} \wedge \mathcal{M}\nu, \mathbf{w} \wedge \mathcal{M}\nu) \;=\; \pi^+(\mathcal{M}\nu)(-\mathbf{w}, \mathbf{v})(\mathcal{M}\nu \cdot \mathcal{M}\nu)^{1/2}. \tag{5.16}$$

The terms of the identity are holomorphic in $|\tau| > M$. By analytic continuation it follows that the identity holds for all such $\tau$. Therefore,

$$\pi^+(\mathcal{M}\nu)\big(\widetilde{\mathrm{curl}}E \wedge \mathcal{M}\nu\,,\, \widetilde{\mathrm{curl}}B \wedge \mathcal{M}\nu\big) \;=\; \pi^+(\mathcal{M}\nu)\big(-\widetilde{\mathrm{curl}}B\,,\, \widetilde{\mathrm{curl}}E\big)(\mathcal{M}\nu \cdot \mathcal{M}\nu)^{1/2}. \tag{5.17}$$

Equations (5.15) and (5.17) yield the $\pi^+(\mathcal{M}\nu)$ identity of the proposition for $|\tau| > M$. For the $\pi^-(\mathcal{M}\nu)$ identity compute, using (5.16),

$$\begin{aligned}\pi^-(\mathcal{M}\nu)\mathcal{D}(E, B) \;&=\; \pi^+(-\mathcal{M}\nu)\mathcal{D}(E, B)\\ &=\; \pi^+(-\mathcal{M}\nu)\Big(\tau E - \widetilde{\mathrm{curl}}B\,,\, \tau B + \widetilde{\mathrm{curl}}E\Big)(\mathcal{M}\nu \cdot \mathcal{M}\nu)^{1/2}\\ &=\; \pi^-(\mathcal{M}\nu)\Big(\tau E - \widetilde{\mathrm{curl}}B\,,\, \tau B + \widetilde{\mathrm{curl}}E\Big)(\mathcal{M}\nu \cdot \mathcal{M}\nu)^{1/2}.\end{aligned}$$

This completes the proof of the Proposition 5.10. $\square$

## 5.3 Invertibility for $\tau \in ]0, \infty[$

For $\tau \in ]0, \infty[$ and real test functions, $a((E, B), (E, B))$ and $b(E, E)$ from Definition 5.8 are sums of nonnegative terms. This together with Proposition 5.10 yield a proof of the invertibility of the operator $\mathbf{L}$. This positivity fails for non real $\tau$. That is the heart of the difficulty of this paper. The proof of surjectivity relies on the following lemma.

**Lemma 5.11** *Extend $\nu$ to a smooth unit vector field on a neighborhood of $\partial\mathcal{Q}^\delta$ by defining $\nu$ to be constant on the normals to $\partial\mathcal{Q}^\delta$. Suppose that $\sigma = 0$ and $\tau \in \mathbb{C}$. Then for $\mu \in \{+, -, 0\}$ and $x \in \partial\mathcal{Q}^\delta$,*

$$\pi^\mu(\nu(x))\mathcal{D}(E, B) = (\nu(x) \cdot \partial_x)\big(\pi^\mu(\nu(x))(E, B)\big) + P_\mu(E, B), \tag{5.18}$$

*where $P_\mu : C^\infty_{\mathcal{E}^\mu}(\partial\mathcal{Q}^\delta) \to C^\infty(\partial\mathcal{Q}^\delta; \mathbb{R}^6)$ is a tangential differential operator of order one with smooth coefficients.*

**Remark 5.3 i.** *With this choice of* $\nu$, $\nu\cdot\partial$ *commutes with* $\pi^\mu(\nu(x))$.
**ii.** *If a family of fields* $(E,B)$ *has* $(E,B)|_{\partial\mathcal{Q}^\delta}$ *fixed while the traces of* $(\nu\cdot\partial)(\pi^\mu(\nu)(E,B))$ *exhaust* $C^\infty_{\mathcal{E}^\mu}(\partial\mathcal{Q}^\delta)$, *then the values of* $\pi^\mu(\nu)\mathcal{D}(E,B)$ *exhaust* $C^\infty_{\mathcal{E}^\mu}(\partial\mathcal{Q}^\delta)$.

**Proof of Lemma 5.11.** Since $\sigma=0$ the problem is invariant by rotation. Therefore it suffices to treat $x\in\partial\mathcal{Q}^\delta$ with $\nu(x)=(1,0,0)$. At $x$ separating the $\partial_1$ terms in Proposition 5.10 from the others yields

$$\begin{aligned}\pi^\pm(\nu)\mathcal{D}(E,B) &= \pi^\pm(\nu)(\tau E-\operatorname{curl}B,\tau B+\operatorname{curl}E)\\ &= \pi^\pm(\nu)\big((0,\partial_1B_3,-\partial_1B_2),(0,-\partial_1E_3,\partial_1E_2)\big)+P_\pm(E,B)\end{aligned}$$

with tangential operators $P_\pm$. The proof of the $\pm$ case is completed by

$$\begin{aligned}\pi^\pm(\nu)\big((0,&\partial_1B_3,-\partial_1B_2),(0,-\partial_1E_3,\partial_1E_2)\big)=\pi^\pm(\nu)\partial_1\big((0,B_3,-B_2),(0,-E_3,E_2)\big)\\ &=\partial_1\big(\pi^\pm(\nu)(0,B_3,-B_2),(0,-E_3,E_2)\big)=\partial_1\big(\pi^\pm(\nu)(E,B)\big).\end{aligned}$$

For $\mu=0$ compute

$$\begin{aligned}\pi^0(\nu)\mathcal{D}(E,B) &= \pi^0(\nu)((\operatorname{div}E)\nu,(\operatorname{div}B)\nu)\\ &= \pi^0(\nu)(\partial_1E_1\nu,\partial_1B_1\nu)+\pi^0(\nu)((\partial_2E_2+\partial_3E_3)\nu,(\partial_2B_2+\partial_3B_2)\nu)\\ &= \partial_1\pi^0(E_1\nu,B_1\nu)+((\partial_2E_2+\partial_3E_3)\nu,(\partial_2B_2+\partial_3B_3)\nu).\end{aligned}$$

$E_1\nu$ and $E$ differ by a tangential vector and $B$ similarly. Thus, $\pi^0(\nu)(E_1\nu,B_1\nu)=\pi^0(\nu)(E,B)$ completing the proof. $\square$

**Proposition 5.12** *For* $\delta\in]0,\delta_0]$ *and* $\tau\in]0,\infty[$, *the Fredholm map* $\mathbf{L}$ *from Corollary 4.6 is invertible.*

**Proof. Injectivity.** Suppose that $E,B\in C^\infty(\mathcal{Q}^\delta)$ belongs to the kernel. Since the operator is real it is sufficient to prove that there are no real elements of the kernel. For such elements take $E',B'=E,B$. The equation $\mathbf{L}=0$, equation (5.13), and Proposition 5.10 imply that $a((E,B),(E,B))=b(E,E)+b(B,B)=0$. Equation (5.10) shows that each of $b(E,E)$ and $b(B,B)$ is a sum of three nonnegative terms. Therefore all six terms vanish, so $\int_{\mathcal{Q}^\delta}\Pi(\tau,x)\,(E\cdot E+B\cdot B)\,dx=0$. Thus $E=B=0$ proving injectivity.

**Surjectivity**. It suffices to prove that the index is equal to zero. Since the index is independent of $\tau,\sigma$ it suffices to treat $\tau>0$ and $\sigma=0$. In that case, $\widetilde{\Delta}=\Delta$, $\Pi(\tau,\sigma)=1$, and $\mathcal{M}(\tau,\nu)=I$. Suppose that $\big(\underline{E},\underline{B},\underline{h}^-,\underline{h}^+,\underline{h}_3,\underline{h}_4\big)$

belongs to the annihilator of the range of $\mathbf{L}$. Elliptic regularity implies that they are smooth,

$$\underline{E}, \underline{B} \in C^\infty(\overline{\mathcal{Q}}^\delta; \mathbb{R}^3), \quad \underline{h}^\pm \in C^\infty_{\mathcal{E}^\pm(\tau,x)}(\partial\mathcal{Q}^\delta), \quad \underline{h}_3, \underline{h}_4 \in C^\infty(\partial\mathcal{Q}^\delta; \mathbb{R}).$$

They annihilate the range when for all $E, B \in C^\infty(\overline{\mathcal{Q}^\delta})$,

$$\begin{aligned} 0 = \int_{\mathcal{Q}^\delta} (\tau^2 - \Delta)E \cdot \underline{E} + (\tau^2-\Delta)B \cdot \underline{B}\, dx + \\ \int_{\partial\mathcal{Q}^\delta} \Big(\pi^-(\nu)(E,B)\cdot \underline{h}^- + \pi^+(\nu)(\tau E - \operatorname{curl} B, \tau B + \operatorname{curl} E)\cdot \underline{h}^+ \\ + (\operatorname{div} E)\, \underline{h}_3 + (\operatorname{div} B)\, \underline{h}_4 \Big)\, d\Sigma. \end{aligned} \tag{5.19}$$

Taking $(E, B) \in C_0^\infty(\mathcal{Q}^\delta)$, the boundary terms vanish. This yields

$$(\tau^2 - \Delta)\underline{E} = (\tau^2 - \Delta)\underline{B} = 0. \tag{5.20}$$

In (5.19), the last two terms in the boundary integral satisfy

$$\int_{\partial\mathcal{Q}^\delta} \pi^+(\nu)(\tau E - \operatorname{curl} B, \tau B + \operatorname{curl} E)\cdot \underline{h}^+\, d\Sigma = \int_{\partial\mathcal{Q}^\delta} \pi^+(\nu)\mathcal{D}(E,B)\cdot \underline{h}^+\, d\Sigma,$$
$$\int_{\partial\mathcal{Q}^\delta} (\operatorname{div} E)\, \underline{h}_3 + (\operatorname{div} B)\, \underline{h}_4\, d\Sigma = \int_{\partial\mathcal{Q}^\delta} \pi^0(\nu)\mathcal{D}(E,B)\cdot(\underline{h}_3\nu, \underline{h}_4\nu)\, d\Sigma.$$

Write $u = (E, B)$, $(\tau^2 - \Delta)u = ((\tau^2-\Delta)E, (\tau^2-\Delta)B)$, *etc.* Then (3.1) is equivalent to

$$\begin{aligned} 0 = \int_{\mathcal{Q}^\delta} (\tau^2 - \Delta)u \cdot \underline{u}\, dx + \\ \int_{\partial\mathcal{Q}^\delta} \big(\pi^-(\nu)u\cdot \underline{h}^- + \pi^+(\nu)\mathcal{D}(u)\cdot \underline{h}^+ + \pi^0(\nu)\mathcal{D}(u)\cdot(\underline{h}_3\nu, \underline{h}_4\nu)\big)\, d\Sigma. \end{aligned} \tag{5.21}$$

For $f \in C^\infty_{\mathcal{E}^-(\nu)}(\partial\mathcal{Q}^\delta)$, choose $E, B \in C^\infty(\overline{\mathcal{Q}^\delta})$ to be the unique solution of the strictly dissipative symmetric hyperbolic boundary value problem with boundary of constant multiplicity [27, 36]

$$\tau E - \operatorname{curl} B = \tau B - \operatorname{curl} E = 0 \ \text{ on } \ \mathcal{Q}^\delta, \quad \pi^-(\nu)(E,B) = f \ \text{ on } \ \partial\mathcal{Q}^\delta. \tag{5.22}$$

For this choice of $E, B$, all terms of (5.21) vanish except the $\underline{h}^-$ term. Thus $\int_{\partial\mathcal{Q}^\delta} f\cdot \underline{h}^-\, d\Sigma = 0$ for arbitrary $f \in C^\infty_{\mathcal{E}^-}$, so $\underline{h}^- = 0$.

Lemma 5.9 gives formulas for $a(u,\underline{u})$ and $a(\underline{u},u)$. Subtracting using (5.20) and the symmetry of $a$ yields

$$\int_{\mathcal{Q}^\delta} (\tau^2-\Delta)u\cdot\underline{u}\,dx + \int_{\mathcal{Q}^\delta} \mathcal{D}(u)\cdot\underline{u}\,d\Sigma = \int_{\partial\mathcal{Q}^\delta} \mathcal{D}(\underline{u})\cdot u\,d\Sigma.$$

Using this in (5.21) yields the identity involving only traces at the boundary,

$$\begin{aligned}\int_{\partial\mathcal{Q}^\delta} \Big(&\mathcal{D}(u)\cdot\underline{u} - \mathcal{D}(\underline{u})\cdot u \\ &- \pi^+(\nu)\mathcal{D}(u)\cdot\underline{h}^+ - \pi^0(\nu)\mathcal{D}(u)\cdot(\underline{h}_3\nu,\underline{h}_4\nu)\Big)\,d\Sigma \;=\; 0.\end{aligned} \tag{5.23}$$

Next show that (5.23) implies that $\underline{E},\underline{B}\in\operatorname{Ker}\mathbf{L}$. The Helmholtz equation (5.20) has already been proved.

Derive the boundary condition $\pi^-(\nu)\underline{u}=0$. Choose $u$ with $\pi^+(\nu)\mathcal{D}(u)=0$, $\pi^0(\nu)\mathcal{D}(u)=0$, and, $u|_{\partial\mathcal{Q}^\delta}=0$. Then only the $\mathcal{D}(u)\cdot\underline{u}$ term remains. Thus

$$0=\int_{\partial\mathcal{Q}^\delta}\mathcal{D}(u)\cdot\underline{u}\,d\sigma = \int_{\partial\mathcal{Q}^\delta}\pi^-(\nu)\mathcal{D}(u)\cdot\underline{u}\,d\sigma = \int_{\partial\mathcal{Q}^\delta}\pi^-(\nu)\mathcal{D}(u)\cdot\pi^-(\nu)\underline{u}\,d\sigma.$$

Remark 5.3 shows that choosing arbitrary values for $(\nu\cdot\partial)(\pi^-(\nu)u)$, the values of $\pi^-(\nu)\mathcal{D}(u)$ are arbitrary. It follows that $\pi^-(\nu)\underline{u}|_{\partial\mathcal{Q}^\delta}=0$.

Since $\pi^-(\nu)\underline{u}=0$, $\int_{\partial\mathcal{Q}^\delta}\mathcal{D}(u)\cdot\underline{u}\,d\sigma=0$ for $u$ so that $\pi^+(\nu)\mathcal{D}(u)=0$ and $\pi^0(\nu)\mathcal{D}(u)=0$. For such $u$, (5.23) implies that $\int_{\partial\mathcal{Q}^\delta}\mathcal{D}(\underline{u})\cdot u\,d\sigma=0$. Lemma 5.11 implies that the values $\pi^-(\nu)u|_{\partial\mathcal{Q}^\delta}$ for such $u$ are arbitrary. Conclude that $0=\pi^-(\nu)\mathcal{D}(\underline{u})=\pi^-(\nu)(\tau\underline{E}-\operatorname{curl}\underline{B},\tau\underline{B}+\operatorname{curl}\underline{E})$ verifying a second boundary condition.

Finally, choose test functions $u$ with $u|_{\partial\mathcal{Q}^\delta}=0$ and $\pi^\pm(\nu)\mathcal{D}(u)=0$. Varying the normal derivative of $\pi^0(\nu)\mathcal{D}(u)$ shows that $\pi^0(\nu)\mathcal{D}(u)$ takes arbitrary values. Using this in (5.23) implies that $0=\pi^0(\nu)\mathcal{D}(\underline{u})|_{\partial\mathcal{Q}^\delta}$. Equivalently $\operatorname{div}\underline{E}=\operatorname{div}\underline{B}=0$ verifying the final boundary condition. Therefore $\underline{E},\underline{B}\in\operatorname{Ker}\mathbf{L}$, so $\underline{E}=\underline{B}=0$.

What remains of (5.23) is

$$\int_{\partial\mathcal{Q}^\delta}\Big(\pi^+(\nu)\mathcal{D}(E,B)\cdot\underline{h}^+ + \pi^0(\nu)\mathcal{D}(E,B)\cdot(\underline{h}_3\nu,\underline{h}_4\nu)\Big)\,d\Sigma=0,$$

with $E,B$ arbitrary smooth fields. On $\partial\mathcal{Q}^\delta$, $(\pi^+(\nu)\mathcal{D}(u),\pi^0(\nu)\mathcal{D}(u))$ is an arbitrary section of $\mathcal{E}^+(\tau,x)\times\mathcal{E}^0(\tau,x)$. Therefore, $\underline{h}^+=0$ and $h_3=h_4=0$. This completes the proof that the annihilator of the range is trivial. $\square$

**Proposition 5.13** *For $\delta \in ]0, \delta_0]$ there is a discrete set $\mathbb{D}(\delta) \subset \{\mathrm{Re}\ \tau > M\}$ so that for $\tau$ in the complement of $\mathbb{D}(\delta)$, the Fredholm map from Corollary 4.6 is invertible. The inverse is meromorphic with values in the set of bounded operators from the space* (4.9) *to $H^2(\mathcal{Q}^\delta)$.*

**Proof.** Once invertibility is established for $\tau \in ]0, \infty[$, Proposition 5.13 follows from the Analytic Fredholm Theorem [37, Theorem VI.14]. To apply this theorem, rewrite the boundary value problem in the form (4.10) with source terms $\underline{g}_1, \underline{g}_2$ in $\tau$-independent Sobolev spaces. The map $E, B \mapsto (f, \underline{g}_1, \underline{g}_2, g_3, g_4)$ is then Fredholm. It is defined for $|\tau| > M$ and is analytic on that set. It is invertible for $\tau$ large and real. The Analytic Fredholm Theorem implies that it is invertible on the complement of a discrete subset of $|\tau| > M$ and the inverse is meromorphic. $\square$

For $\tau \in \{\mathrm{Re}\ \tau > M\} \setminus \mathbb{D}(\delta)$, denote by $(\tau^2 - \widetilde{\Delta})^{-1} \in \mathrm{Hom}(L^2(\mathcal{Q}^\delta); H^2(\mathcal{Q}^\delta))$ the inverse operator of the problem *with homogeneous boundary data*. In Section 7.3 we prove that for $M$ sufficiently large, $(\tau^2 - \widetilde{\Delta})^{-1}(\widetilde{\mathrm{curl}}\, j, -\tau j - \widetilde{\mathrm{grad}}\, \rho)$ has no poles in $\mathrm{Re}\ \tau > M$. This follows from uniform estimates on $\{\mathrm{Re}\ \tau > M\} \setminus \mathbb{D}(\delta)$. Their proofs occupy Sections 6 and 7.

# 6 Boundary estimate à la Jerison-Kenig-Mitrea

This section and the next yield the fundamental estimate of Theorem 7.6. The estimate of Section 6 is invoked in the proof of that Theorem after equation (7.8). We advise skipping this section till it is needed at that point. The estimate of Section 6 has a long history for harmonic functions dating to Rellich [38] and Payne-Weinberger [33]. It was promoted to a result of central importance in studying boundary value problems in Lipschitz domains by Jerison and Kenig [26, 25]. Mitrea [31] introduced a version appropriate to Maxwell's equations. The present section extends Mitrea's work to the stretched Maxwell equations. The extended estimate is slightly weaker than in the unstretched case.

Recall the case of harmonic functions on $\mathcal{Q}^\delta$. The Dirichlet condition $u = f$ on $\partial\mathcal{Q}^\delta$ immediately controls $\|\nabla_{tan} u\|_{L^2(\mathcal{Q}^\delta)}$. The Neumann condition $\nu \cdot \nabla u = g$ controls $\|\nabla_n u\|_{L^2(\partial\mathcal{Q}^\delta)}$. The Jerison-Kenig estimate shows that for harmonic functions $\|\nabla_{tan} u\|_{L^2(\partial\mathcal{Q}^\delta)}$ and $\|\nabla_n u\|_{L^2(\partial\mathcal{Q}^\delta)}$ are of the same magnitude so the full tangential trace is bounded in terms of either Dirichlet or Neumann data. In (6.1), $\|E|_{\partial\mathcal{Q}^\delta}\|_{L^2(\partial\mathcal{Q}^\delta}$, (resp. $\|\pi_{(\mathcal{M}\nu)^\perp} E|_{\partial\mathcal{Q}^\delta}\|_{L^2(\partial\mathcal{Q}^\delta)}$) appears on the left (resp. right). The second is a part of the trace estimated using the lower bound (7.2) for the variational form $a$.

In (6.1), the full trace appears on the left and the boundary data $\pi_{(\mathcal{M}\nu)^\perp} E$ appears on the right.

**Proposition 6.1** *There are constants $C, M$ so that for all $\delta \in ]0, \delta_0[$, $\tau \in \{\mathrm{Re}\,\tau > M\}$, and, $E \in H^1(\mathcal{Q}^\delta)$,*

$$\begin{aligned} \|E\|^2_{L^2(\partial\mathcal{Q}^\delta)} \ \le\ C\Big( & \|\pi_{(\mathcal{M}\nu)^\perp} E\|^2_{L^2(\partial\mathcal{Q}^\delta)} + \\ & + \frac{1}{|\tau|}\,\|E\|^2_{L^2(\mathcal{Q}^\delta)} \ +\ \|E\|_{L^2(\mathcal{Q}^\delta)}\,\|\widetilde{\mathrm{div}}\,E\,,\ \widetilde{\mathrm{curl}}\,E\|_{L^2(\mathcal{Q}^\delta)}\Big). \end{aligned} \tag{6.1}$$

## 6.1 Unstretched estimate of Mitrea

**Definition 6.2** *Denote by $\Theta$ the radial vector field $(x_1, x_2, x_3)$.*

Mitrea [31] proves the following identity for complex $E$,

$$\begin{aligned} \mathrm{div}\,\Big\{\frac{1}{2}\,|E|^2\,\Theta\ -\ \mathrm{Re}\big((\overline{E}\cdot\Theta)E\big)\Big\}\ &= \\ \mathrm{Re}\,\Big\{\frac{1}{2}\,|E|^2\ &-\overline{E}\cdot\Theta\ \mathrm{div}\,E + (\overline{E}\wedge\Theta)\cdot\mathrm{curl}\,E\Big\}. \end{aligned} \tag{6.2}$$

Integrating this identity, the flux through the boundary has density

$$\frac{1}{2}\,|E|^2\ \Theta\cdot\nu\ -\ \mathrm{Re}\big((\overline{E}\cdot\Theta)\ (E\cdot\nu)\big)\,. \tag{6.3}$$

Decompose $\Theta = \Theta_{tan} + \Theta_n$ with $\Theta_n = (\Theta\cdot\nu)\nu$. Similarly $E_n = (E\cdot\nu)\nu$. There is a $c > 0$ so that for all $\delta$, $\Theta\cdot\nu \ge c$ on $\partial\mathcal{Q}^\delta$.

The contribution of $\Theta_n$ to the second summand in (6.3) is equal to

$$\mathrm{Re}\big((\overline{E}\cdot\nu)\ (\Theta\cdot\nu)\ (E\cdot\nu)\big)\ =\ (\Theta\cdot\nu)|E_n|^2.$$

Since $|E|^2 = |E_{tan}|^2 + |E_n|^2$, the flux density is equal to

$$\frac{1}{2}\,\Big(|E_{tan}|^2\ -\ |E_n|^2\Big)\Theta\cdot\nu\ -\ \mathrm{Re}\big((\overline{E}_{tan}\cdot\Theta_{tan})\ (E\cdot\nu)\big). \tag{6.4}$$

The second term is $\lesssim |E_{tan}||E_n|$. Integrating (6.2) over $\mathcal{Q}^\delta$ and using the strict positivity of $\Theta\cdot\nu$ yields Mitrea's estimate,

$$\begin{aligned} \Big|\|E_{tan}\|^2_{L^2(\partial\mathcal{Q}^\delta)} - \|E_n\|^2_{L^2(\partial\mathcal{Q}^\delta)}\Big|\ &\lesssim\ \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}\|E_n\|_{L^2(\partial\mathcal{Q}^\delta)}\ + \\ &+\ \|E\|_{L^2(\mathcal{Q}^\delta)}\Big(\|E\|_{L^2(\mathcal{Q}^\delta)}\ +\ \|\mathrm{div}\,E, \mathrm{curl}\,E\|_{L^2(\mathcal{Q}^\delta)}\Big)\,. \end{aligned}$$

## 6.2 Stretched Mitrea estimate

This section proves the stretched analogue (6.5) that is weaker than the unstretched version because of the $1/|\tau|$ term on the right.

**Proposition 6.3** *With constant independent of* $\delta \in [0,\delta_0]$, $E \in H^1(\mathcal{Q}^\delta)$, *and,* $\tau$ *with* $\mathrm{Re}\,\tau > 1$,

$$\begin{aligned} \big|\|E_{tan}\|^2_{L^2(\partial\mathcal{Q}^\delta)} - \|E_n\|^2_{L^2(\partial\mathcal{Q}^\delta)}\big| \ &\lesssim \ \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}\|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} \\ +\frac{1}{|\tau|}\|E\|^2_{L^2(\partial\mathcal{Q}^\delta)} + \|E\|_{L^2(\mathcal{Q}^\delta)}&\Big(\|E\|_{L^2(\mathcal{Q}^\delta)} + \|\widetilde{\mathrm{div}}\,E, \widetilde{\mathrm{curl}}\,E\|_{L^2(\mathcal{Q}^\delta)}\Big). \end{aligned} \tag{6.5}$$

Begin with a stretched version of (6.2).

**Lemma 6.4** *If* $E \in C^1(\mathcal{Q}^\delta)$ *and the coordinate stretchings* $X(\tau,x)$ *are from Definition 1.9, then for all* $\tau \in \{\mathrm{Re}\,\tau > 0\}$,

$$\begin{aligned} &\widetilde{\mathrm{div}}\Big\{\frac{1}{2}\,|E|^2\,X(\tau,x)\ -\ \mathrm{Re}\big((\overline{E}\cdot X(\tau,x))E\big)\Big\}\ = \\ &\qquad \mathrm{Re}\,\Big\{\frac{1}{2}\,|E|^2\ -\ (\overline{E}\cdot X(\tau,x))\,\widetilde{\mathrm{div}}E + (\overline{E}\wedge X(\tau,x))\cdot\widetilde{\mathrm{curl}}E\Big\}. \end{aligned} \tag{6.6}$$

**Proof of Lemma 6.4.** For $\tau \in ]0,\infty[$, $X(\tau,x)$ maps $\mathcal{Q}^\delta$ to its image $\underline{\mathcal{Q}}^\delta(\tau)$. For a vector field $F_{\mathcal{Q}^\delta}$ defined on $\mathcal{Q}^\delta$, associate the field $F_{\underline{\mathcal{Q}}^\delta}$ on $\underline{\mathcal{Q}}^\delta$ defined by $F_{\underline{\mathcal{Q}}^\delta}\big(X(\tau,x)\big) = F_{\mathcal{Q}^\delta}(x)$. The value of $F_{\underline{\mathcal{Q}}^\delta}$ at $X(\tau,x)$ is equal to the value of $F_{\mathcal{Q}^\delta}$ at $x$. Formula (1.14) yields

$$\mathrm{div}_X F_{\underline{\mathcal{Q}}^\delta}\big|_{X=X(\tau,x)} = \widetilde{\mathrm{div}}\,F_{\mathcal{Q}^\delta}(x), \quad \text{and} \quad \mathrm{curl}_X\,F_{\underline{\mathcal{Q}}^\delta}\big|_{X=X(\tau,x)} = \widetilde{\mathrm{curl}}\,F_{\mathcal{Q}^\delta}(x).$$

For the radial field $\Psi(X) = (X_1, X_2, X_3)$, the corresponding field in $x$ is $X(\tau,x)$. The identity (6.2) on $\underline{\mathcal{Q}}^\delta(\tau)$ is

$$\begin{aligned} &\mathrm{div}_X\Big\{\frac{1}{2}\,|\underline{E}|^2\,\Psi\ -\ \mathrm{Re}\big((\overline{\underline{E}}\cdot\Psi)\underline{E}\big)\Big\}\ = \\ &\mathrm{Re}\,\Big\{\frac{1}{2}\,|\underline{E}|^2\ - (\overline{\underline{E}}\cdot\Psi)\mathrm{div}_X\underline{E} + (\overline{\underline{E}}\wedge\Psi)\cdot\mathrm{curl}_X\underline{E}\Big\}. \end{aligned} \tag{6.7}$$

This proves (6.6) for $\tau \in ]0,\infty[$.

For all $x \in \mathcal{Q}^\delta$, each of the terms of (6.6) is holomorphic in $\{\mathrm{Re}\,\tau > 0\}$. Analytic continuation from $]0,\infty[$ implies that (6.6) holds on $\mathrm{Re}\,\tau > 0$. $\square$

**Proof of Proposition 6.3.** Define

$$\begin{aligned}\mathcal{N} \;:=\; &\|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}\|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} + \\ &\qquad \|E\|_{L^2(\mathcal{Q}^\delta)}\Big(\|E\|_{L^2(\mathcal{Q}^\delta)} \;+\; \|\widetilde{\operatorname{div}}\, E\,,\; \widetilde{\operatorname{curl}}\, E\|_{L^2(\mathcal{Q}^\delta)}\Big)\,.\end{aligned} \tag{6.8}$$

Multiply (6.6) by $\Pi(\tau,x)$ from (3.1). Using (5.1) and integrating over $\mathcal{Q}^\delta$ yields the bound,

$$\Big|\int_{\partial\mathcal{Q}^\delta} \mathcal{M}\nu\cdot\Big(\frac{1}{2}\,|E|^2\,X(\tau,x)\;-\;\operatorname{Re}\big((\overline{E}\cdot X(\tau,x))E\big)\Big)d\Sigma\Big| \;\lesssim\; \mathcal{N}. \tag{6.9}$$

From Definition 3.3, $\mathcal{M} = I + O(1/|\tau|)$. In addition $X(\tau,x) - \Theta(x) = O(1/|\tau|)$ uniformly on compact sets in $x$. Therefore

$$\begin{aligned}\Big|&\int_{\partial\mathcal{Q}^\delta} \mathcal{M}\nu\,\cdot\Big(\frac{1}{2}\,|E|^2\,X(\tau,x)\;-\;\operatorname{Re}\big((\overline{E}\cdot X(\tau,x))E\big)\Big)d\Sigma\;- \\ &\int_{\partial\mathcal{Q}^\delta}\nu\cdot\Big(\frac{1}{2}\,|E|^2\,\Theta\;-\;\operatorname{Re}\big((\overline{E}\cdot\Theta)E\big)\Big)d\Sigma\Big| \;\lesssim\; \frac{1}{|\tau|}\,\|E\|^2_{L^2(\partial\mathcal{Q}^\delta)}.\end{aligned} \tag{6.10}$$

Equation (6.4) implies that

$$\begin{aligned}\Big|\nu\cdot\Big(\frac{1}{2}\,|E|^2\,\Theta\;-\;\operatorname{Re}\big((\overline{E}\cdot\Theta)E\big)\Big)\;-\frac{1}{2}\,\Big(|E_{tan}|^2\;-\;|E_n|^2\Big)(\Theta,\nu)\Big| \\ \lesssim\;\big\|E_{tan}\big\|_{L^2(\partial\mathcal{Q}^\delta)}\big\|E_n\big\|_{L^2(\partial\mathcal{Q}^\delta)}.\end{aligned} \tag{6.11}$$

Therefore,

$$\begin{aligned}\Big|\|E_{tan}\|^2_{L^2(\partial\mathcal{Q}^\delta)} - \|E_n\|^2_{L^2(\partial\mathcal{Q}^\delta)}\Big| \lesssim \big\|E_{tan}\big\|_{L^2(\partial\mathcal{Q}^\delta)}\big\|E_n\big\|_{L^2(\partial\mathcal{Q}^\delta)} \\ +\Big|\int_{\partial\mathcal{Q}^\delta}\nu\cdot\Big(\frac{1}{2}\,|E|^2\,\Theta\;-\;\operatorname{Re}\big((\overline{E}\cdot\Theta)E\big)\Big)d\Sigma\Big|.\end{aligned} \tag{6.12}$$

Combining (6.8), (6.9), (6.10), and, (6.12) completes the proof of (6.5). $\square$

**Proof of Proposition 6.1.** Define

$$\mathcal{R} := \frac{1}{|\tau|}\|E\|^2_{L^2(\partial\mathcal{Q}^\delta)} + \;\|E\|_{L^2(\mathcal{Q}^\delta)}\Big(\|E\|_{L^2(\mathcal{Q}^\delta)}\;+\;\|\widetilde{\operatorname{div}}\,E, \widetilde{\operatorname{curl}}\,E\|_{L^2(\mathcal{Q}^\delta)}\Big).$$

Proposition 6.3 shows that

$$\Big|\|E_{tan}\|^2_{L^2(\partial\mathcal{Q}^\delta)} - \|E_n\|^2_{L^2(\partial\mathcal{Q}^\delta)}\Big| \;\lesssim\; \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}\|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} \;+\; \mathcal{R},$$

so

$$\|E_n\|^2_{L^2(\partial\mathcal{Q}^\delta)} \ \lesssim\ \|E_{tan}\|^2_{L^2(\partial\mathcal{Q}^\delta)} \ +\ \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}\|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} \ +\ \mathcal{R}.$$

Bound $\|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}\|E_n\|_{L^2(\partial\mathcal{Q}^\delta)}$ by a small constant times $\|E_n\|^2_{L^2(\partial\mathcal{Q}^\delta)}$ plus a large constant times $\|E_{tan}\|^2_{L^2(\partial\mathcal{Q}^\delta)}$ and absorb the $E_n$ term on the left to find

$$\|E_n\|^2_{L^2(\partial\mathcal{Q}^\delta)} \ \lesssim\ \|E_{tan}\|^2_{L^2(\partial\mathcal{Q}^\delta)} \ +\ \mathcal{R}.$$

Adding $\|E_{tan}\|^2_{L^2(\partial\mathcal{Q}^\delta)}$ to both sides yields

$$\|E\|^2_{L^2(\partial\mathcal{Q}^\delta)} \ \lesssim\ \|E_{tan}\|^2_{L^2(\partial\mathcal{Q}^\delta)} \ +\ \mathcal{R}. \tag{6.13}$$

Use $\mathcal{M} = I + O(1/|\tau|)$ to show that

$$\mathcal{M}\nu - \nu = O(1/|\tau|), \qquad \text{so}, \qquad \pi_{(\mathcal{M}\nu)^\perp} - \pi_{\nu^\perp} \ =\ O(1/|\tau|). \tag{6.14}$$

Since $\pi_{\nu^\perp}E = E_{tan}$, it follows that

$$\|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)} \lesssim \|\pi_{(\mathcal{M}\nu)^\perp}E\|_{L^2(\partial\mathcal{Q}^\delta)} + \frac{1}{|\tau|}\,\|E\|_{L^2(\partial\mathcal{Q}^\delta)},$$

so,

$$\|E_{tan}\|^2_{L^2(\partial\mathcal{Q}^\delta)} \lesssim \|\pi_{(\mathcal{M}\nu)^\perp}E\|^2_{L^2(\partial\mathcal{Q}^\delta)} + \frac{1}{|\tau|^2}\,\|E\|^2_{L^2(\partial\mathcal{Q}^\delta)}\,.$$

Insert this estimate in (6.13) to find

$$\|E\|^2_{L^2(\partial\mathcal{Q}^\delta)} \lesssim \|\pi_{(\mathcal{M}\nu)^\perp}E\|^2_{L^2(\partial\mathcal{Q}^\delta)} + \mathcal{R} + \frac{1}{|\tau|^2}\,\|E\|^2_{L^2(\partial\mathcal{Q}^\delta)}\,.$$

Choose $M$ so that for $|\tau| > M$ the last term can be absorbed on the left. This completes the proof of Proposition 6.1. □

# 7 Stretched Maxwell estimate on $\mathcal{Q}^\delta$

This section proves the *a priori* estimate of Theorem 7.6. Lemma 5.9, Proposition 5.10, and Remark 5.2, imply that for a solution $(E,B)$ of the stretched Maxwell boundary value problem,

$$a((E,B),(E',B')) = \int_{\mathcal{Q}^\delta} \Pi(\tau,x)\ (\tau^2 - \widetilde{\Delta})(E,B)\cdot(E',B')\ dx.$$

The $\tau^2 - \widetilde{\Delta}$ term is estimated in terms of the sources $\rho, j$. The strategy is to pick test function $E', B'$ so that $|a((E,B),(E',B'))|$ is large. The subtle choice, in Section 7.2, relies on the approximation identity in Section 7.1, and yields the lower bound of Proposition 7.3. That in turn yields Theorem 7.6. In Section 8 the main theorems are proved by combining Theorem 7.6 with the Fredholm theory of Sections 4 and 5.3.

## 7.1 $\widetilde{\operatorname{curl}}\,\Phi - \operatorname{curl}(\mathcal{M}^{-1}\Phi) \;=\; O(1/|\tau|)$

**Definition 7.1** *With notation from Definition 3.3,* **i.** $Q := \mathcal{M}\overline{\mathcal{M}}^{\,-1}$ *is the diagonal matrix with entries* $\mathcal{M}_{ii}\,(\overline{\mathcal{M}_{ii}})^{-1}$ *of modulus equal to one.*

**ii.**

$$\mathcal{N} \;:=\; \begin{pmatrix} 0 & -\mathcal{M}_{33}\,\partial_3\mathcal{M}_{22} & \mathcal{M}_{22}\,\partial_2\mathcal{M}_{33} \\ \mathcal{M}_{33}\,\partial_3\mathcal{M}_{11} & 0 & -\mathcal{M}_{11}\,\partial_1\mathcal{M}_{33} \\ -\mathcal{M}_{22}\,\partial_2\mathcal{M}_{11} & \mathcal{M}_{11}\,\partial_1\mathcal{M}_{22} & 0 \end{pmatrix}.$$

**Remark 7.1** **i.** *The coordinate directions are eigenvectors of* $\mathcal{M}$ *and* $Q$. **ii.** $Q$ *is unitary.* **iii.** *As* $|\tau| \to \infty$, $\Pi - I$, $\mathcal{M} - I$, $Q - I$, *and* $\mathcal{N}$ *are all* $O(1/|\tau|)$.

Part **iii** of Remark 7.1 shows that equation (7.1) implies the estimate of the section title.

**Lemma 7.2** *For all* $\Phi \in H^1$,

$$\widetilde{\operatorname{curl}}\,\Phi \;=\; \Pi\,\mathcal{M}^{-1}\operatorname{curl}\,\left(\mathcal{M}^{-1}\Phi\right) + \frac{1}{\Pi}\mathcal{N}\mathcal{M}^{-1}\Phi\,. \tag{7.1}$$

**Proof of Lemma 7.2.** Compute

$$\Pi(\tau,x)\,\widetilde{\operatorname{curl}}\,\Phi = \Pi(\tau,x)\begin{pmatrix}\widetilde{\partial}_1\\ \widetilde{\partial}_2\\ \widetilde{\partial}_3\end{pmatrix}\wedge\begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix} = \begin{pmatrix}\mathcal{M}_{11}\partial_1\\ \mathcal{M}_{22}\partial_2\\ \mathcal{M}_{33}\partial_3\end{pmatrix}\wedge\begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix}$$

$$= \begin{pmatrix}\mathcal{M}_{11}\partial_1\\ \mathcal{M}_{22}\partial_2\\ \mathcal{M}_{33}\partial_3\end{pmatrix}\wedge\begin{pmatrix}\mathcal{M}_{11}\frac{\Phi_1}{\mathcal{M}_{11}}\\ \mathcal{M}_{22}\frac{\Phi_2}{\mathcal{M}_{22}}\\ \mathcal{M}_{33}\frac{\Phi_3}{\mathcal{M}_{33}}\end{pmatrix}$$

$$= \begin{pmatrix}\mathcal{M}_{22}\partial_2\left(\mathcal{M}_{33}\frac{\Phi_3}{\mathcal{M}_{33}}\right) - \mathcal{M}_{33}\partial_3\left(\mathcal{M}_{22}\frac{\Phi_2}{\mathcal{M}_{22}}\right)\\ \mathcal{M}_{33}\partial_3\left(\mathcal{M}_{11}\frac{\Phi_1}{\mathcal{M}_{11}}\right) - \mathcal{M}_{11}\partial_1\left(\mathcal{M}_{33}\frac{\Phi_3}{\mathcal{M}_{33}}\right)\\ \mathcal{M}_{11}\partial_1\left(\mathcal{M}_{22}\frac{\Phi_2}{\mathcal{M}_{22}}\right) - \mathcal{M}_{22}\partial_2\left(\mathcal{M}_{11}\frac{\Phi_1}{\mathcal{M}_{11}}\right)\end{pmatrix}$$

$$= \begin{pmatrix}\mathcal{M}_{22}\mathcal{M}_{33}\left(\partial_2\frac{\Phi_3}{\mathcal{M}_{33}} - \partial_3\frac{\Phi_2}{\mathcal{M}_{22}}\right) + (\mathcal{M}_{22}\partial_2\mathcal{M}_{33})\frac{\Phi_3}{\mathcal{M}_{33}} - (\mathcal{M}_{33}\partial_3\mathcal{M}_{22})\frac{\Phi_2}{\mathcal{M}_{22}}\\ \mathcal{M}_{33}\mathcal{M}_{11}\left(\partial_3\frac{\Phi_1}{\mathcal{M}_{11}} - \partial_1\frac{\Phi_3}{\mathcal{M}_{33}}\right) + (\mathcal{M}_{33}\partial_3\mathcal{M}_{11})\frac{\Phi_1}{\mathcal{M}_{11}} - (\mathcal{M}_{11}\partial_1\mathcal{M}_{33})\frac{\Phi_3}{\mathcal{M}_{33}}\\ \mathcal{M}_{11}\mathcal{M}_{22}\left(\partial_1\frac{\Phi_2}{\mathcal{M}_{22}} - \partial_2\frac{\Phi_1}{\mathcal{M}_{11}}\right) + (\mathcal{M}_{11}\partial_1\mathcal{M}_{22})\frac{\Phi_2}{\mathcal{M}_{22}} - (\mathcal{M}_{22}\partial_2\mathcal{M}_{11})\frac{\Phi_1}{\mathcal{M}_{11}}\end{pmatrix}.$$

Using the definition of $\mathcal{N}$, this identity is equivalent to

$$\Pi(\tau,x)\,\widetilde{\operatorname{curl}}\,\Phi = \begin{pmatrix}\mathcal{M}_{22}\mathcal{M}_{33} & 0 & 0\\ 0 & \mathcal{M}_{11}\mathcal{M}_{33} & 0\\ 0 & 0 & \mathcal{M}_{11}\mathcal{M}_{22}\end{pmatrix}\operatorname{curl}(\mathcal{M}^{-1}\Phi) + \mathcal{N}\mathcal{M}^{-1}\Phi.$$

Dividing by $\Pi$ and using Definition 3.3 yields (7.1). $\square$

## 7.2 Lower bound for $a((E,B)\,,\,(E',B'))$ for well chosen $E', B'$

Given $(E,B)$, seek $E'$ so that $\widetilde{\operatorname{curl}}\,E' \approx \overline{\widetilde{\operatorname{curl}}\,E}$ so $\widetilde{\operatorname{curl}}\,E\cdot\widetilde{\operatorname{curl}}\,E' \approx |\widetilde{\operatorname{curl}}\,E|^2$. Since $\widetilde{\operatorname{curl}}$ is not real, it is not sufficient to take $E' = \overline{E}$. The result of the preceding section yields

$$\overline{\widetilde{\operatorname{curl}}\,E'} \;\approx\; \overline{\operatorname{curl}\left(\mathcal{M}^{-1}E'\right)} \;=\; \operatorname{curl}\left(\overline{\mathcal{M}^{-1}E'}\right), \qquad \widetilde{\operatorname{curl}}\,E \approx \operatorname{curl}(\mathcal{M}^{-1}E).$$

Comparing suggests the important idea of choosing $E', B'$ so that

$$\mathcal{M}^{-1}E' \;=\; \overline{\mathcal{M}^{-1}E}\,, \quad \text{equivalently,} \quad E' \;=\; \mathcal{M}\,\overline{\mathcal{M}^{-1}E} \;=\; Q\,\overline{E}\,,$$
$$\mathcal{M}^{-1}B' \;=\; \overline{\mathcal{M}^{-1}B}\,, \quad \text{equivalently,} \quad B' \;=\; Q\,\overline{B}\,.$$

When $\tau$ is real the positivity of $a$ suffices to prove well posedness. When $\tau$ is complex this is not the case. The critical juncture is at (7.8) where the full trace appears as an error term that cannot be absorbed by positive terms from $a$. Proposition 6.1 is used to estimate this term.

**Proposition 7.3** *There are positive constants $C, M$ so that for all $\tau$ with* $\mathrm{Re}\,\tau > M$, $\delta \in ]0, \delta_0[$, *and all* $W = (E, B) \subset H^1(\mathcal{Q}^\delta)$ *with* $\mathrm{supp}(\widetilde{\mathrm{div}}\, E\,,\, \widetilde{\mathrm{div}}\, B) \subset \mathcal{Q}_I$, *and* $W' = (E', B') = (Q\overline{E}, Q\overline{B})$,

$$
\begin{aligned}
(\mathrm{Re}\,\tau)\,|\tau|\,\big\|W\big\|^2_{L^2(\mathcal{Q}^\delta)} \;+\; |\tau|\,\big\|\pi_{(\mathcal{M}\nu)^\perp} W\big\|^2_{L^2(\partial\mathcal{Q}^\delta)} \\
+\; \frac{\mathrm{Re}\,\tau}{|\tau|}\,\big\|\widetilde{\mathrm{curl}}\, W\,,\, \widetilde{\mathrm{div}}\, W\big\|^2_{L^2(\mathcal{Q}^\delta)} \;\le\; C\,\big|a(W, W')\big|\,.
\end{aligned}
\tag{7.2}
$$

The proof begins with a sequence of lemmas analysing the individual terms in $a((E, B), (E', B'))$.

**Lemma 7.4** *With $E' := Q\overline{E}$, one has*

$$
\widetilde{\mathrm{curl}}\, E' \;=\; \big(I + O(1/|\tau|)\big)\,\overline{\widetilde{\mathrm{curl}}\, E} \;+\; O(1/|\tau|)\overline{E}. \tag{7.3}
$$

*where $O(1/|\tau|)$ denotes a matrix valued function with $L^\infty(\mathbb{R}^3)$-norm $\lesssim 1/|\tau|$ as $|\tau| \to \infty$. For the divergence one has*

$$
\widetilde{\mathrm{div}}\, E' \;=\; \overline{\widetilde{\mathrm{div}}\, E} \;+\; O(1/|\tau|)\overline{E} \;+\; O(1/|\tau|)\overline{\nabla E}\,. \tag{7.4}
$$

**Remark 7.2** *The term $O(1/|\tau|)\overline{\nabla E}$ involves all derivatives of $E$ and not just $\widetilde{\mathrm{div}}\, E$ and $\widetilde{\mathrm{curl}}\, E$.*

**Proof of Lemma 7.4.** Apply (7.1) to $E'$. Since $Q = \mathcal{M}\overline{\mathcal{M}}^{-1}$,

$$
\begin{aligned}
\widetilde{\mathrm{curl}}\, E' &= \Pi \mathcal{M}^{-1}\,\mathrm{curl}\,\left(\mathcal{M}^{-1}E'\right) + \frac{1}{\Pi}\mathcal{N}\mathcal{M}^{-1}E' \\
&= \Pi \mathcal{M}^{-1}\,\mathrm{curl}\,\left(\overline{\mathcal{M}^{-1}E}\right) + \frac{1}{\Pi}\mathcal{N}\overline{\mathcal{M}^{-1}E}.
\end{aligned}
\tag{7.5}
$$

Apply (7.1) to $E$ and conjugate to find

$$
\overline{\widetilde{\mathrm{curl}}\, E} = \overline{\Pi \mathcal{M}^{-1}}\;\mathrm{curl}\,\left(\overline{\mathcal{M}^{-1}E}\right) + \left(\overline{\Pi}\right)^{-1}\overline{\mathcal{N}}\,\overline{\mathcal{M}^{-1}E}\,.
$$

Solve this equation for curl $\left(\overline{\mathcal{M}^{-1}E}\right)$ and insert in (7.5) to find,

$$
\widetilde{\mathrm{curl}}\, E' = \left(\Pi\,/\,\overline{\Pi}\right)\mathcal{M}^{-1}\overline{\mathcal{M}}\left(\overline{\widetilde{\mathrm{curl}}\, E} - \left(\overline{\Pi}\right)^{-1}\overline{\mathcal{N}}\,\overline{\mathcal{M}^{-1}E}\right) + \frac{1}{\Pi}\,\mathcal{N}\,\overline{\mathcal{M}^{-1}E}.
$$

Using Remark 7.1, this gives (7.3).

For the divergence identity, compute using (5.1)

$$\overline{\Pi}\ \overline{\widetilde{\operatorname{div}} E} \;=\; \overline{\operatorname{div} \mathcal{M} E} \;=\; \operatorname{div} \overline{\mathcal{M} E}\,. \tag{7.6}$$

Since $Q$ and $\mathcal{M}$ are diagonal so commute,

$$\Pi \,\widetilde{\operatorname{div}}\, E' \;=\; \Pi \,\widetilde{\operatorname{div}}\, Q\overline{E} \;=\; \operatorname{div} \mathcal{M} Q \overline{E} \;=\; \operatorname{div} Q \mathcal{M} \overline{E}\,.$$

The definition of $Q$ yields

$$Q\mathcal{M}\overline{E} \;=\; Q\mathcal{M}\overline{\mathcal{M}}^{-1}\overline{\mathcal{M}}\,\overline{E} \;=\; Q^2\,\overline{\mathcal{M}E}\,.$$

Therefore

$$\begin{aligned}\Pi \,\widetilde{\operatorname{div}}\, E' \;&=\; \operatorname{div} Q^2 \overline{\mathcal{M}E} \;=\; \operatorname{div} \overline{\mathcal{M}E} \;+\; \operatorname{div}\big((I-Q^2)\overline{\mathcal{M}E}\big)\\ &=\; \overline{\Pi}\ \overline{\widetilde{\operatorname{div}} E} \;+\; \operatorname{div}\big((I-Q^2)\overline{\mathcal{M}E}\big).\end{aligned}$$

Therefore

$$\widetilde{\operatorname{div}}\, E' \;=\; \overline{\widetilde{\operatorname{div}} E} \;+\; \big(I - \Pi^{-1}\,\overline{\Pi}\big)\overline{\widetilde{\operatorname{div}} E} \;+\; \Pi^{-1}\,\operatorname{div}\big((I-Q^2)\overline{\mathcal{M}E}\big).$$

When the derivatives in the divergence hit $(I-Q^2)\overline{\mathcal{M}}$ they generate terms $O(1/|\tau|)\overline{E}$. When the derivatives in the last divergence hit the field $\overline{E}$, they generate $O(1/|\tau|)\nabla\overline{E}$. This completes the proof of the Lemma. $\Box$

The next lemma shows that the terms in $a((E,B),(E',B'))$ differ from positive quantities by terms $O(1/|\tau|)$.

**Lemma 7.5** *Define $E' := Q\overline{E}$. There is a constant so that in $\overline{\mathcal{Q}}$ for all unit vectors $\omega \in \mathbb{R}^3$*

$$\Big|\Pi\ \widetilde{\operatorname{curl}}\, E \cdot \widetilde{\operatorname{curl}}\, E' \;-\; \big|\widetilde{\operatorname{curl}}\, E\big|^2\Big| \;\lesssim\; \frac{1}{|\tau|}\ \big|\widetilde{\operatorname{curl}}\, E\big|\ \big(\big|\widetilde{\operatorname{curl}}\, E\big| \;+\; |E|\big)\,,$$

$$\Big|\Pi\ \widetilde{\operatorname{div}}\, E \;\cdot\; \widetilde{\operatorname{div}}\, E' \;-\; |\widetilde{\operatorname{div}} E|^2\Big| \;\lesssim\; \frac{1}{|\tau|}\ |\widetilde{\operatorname{div}}\, E|\ \Big(|\nabla E| \;+\; |E|\Big),$$

$$\Big|\pi_{(\mathcal{M}\omega)^\perp} E \cdot \pi_{(\mathcal{M}\omega)^\perp} E' \;-\; \big|\pi_{(\mathcal{M}\omega)^\perp} E\big|^2\Big| \;\lesssim\; \frac{1}{|\tau|}\big|E\big|^2.$$

**Proof of Lemma 7.5.** The first two estimates follow from (7.3) and (7.4) together with Remark 7.1. For the third, compute

$$\pi_{(\mathcal{M}\omega)^\perp} E' \;=\; \pi_{(\mathcal{M}\omega)^\perp} Q\overline{E} \;=\; \pi_{\omega^\perp}\overline{E} \;+\; (\pi_{(\mathcal{M}\omega)^\perp} Q - \pi_{\omega^\perp})\,\overline{E}.$$

To estimate the second summand on the right use,

$$\|\pi_{(\mathcal{M}\omega)^\perp} - \pi_{\omega^\perp}\|_{L^\infty(\partial\mathcal{Q}^\delta)} \;=\; O(1/\tau), \quad \text{and,} \quad \|Q - I\|_{L^\infty(\partial\mathcal{Q}^\delta)} \;=\; O(1/\tau).$$

Thanks to the reality of $\pi_{\omega^\perp}$,

$$\pi_{(\mathcal{M}\omega)^\perp} E' \;=\; \pi_{\omega^\perp}\overline{E} \;+\; O(1/|\tau|)\;\overline{E} \;=\; \overline{\pi_{\omega^\perp} E} \;+\; O(1/|\tau|)\;\overline{E}.$$

Continuing,

$$\pi_{\omega^\perp} E \;=\; \pi_{(\mathcal{M}\omega)^\perp} E \;+\; (\pi_{\omega^\perp} \;-\; \pi_{(\mathcal{M}\omega)^\perp}) E \;=\; \pi_{(\mathcal{M}\omega)^\perp} E \;+\; O(1/|\tau|)\; E\,.$$

Therefore

$$\Big|\pi_{(\mathcal{M}\omega)^\perp} E' \;-\; \overline{\pi_{(\mathcal{M}\omega)^\perp} E}\Big| \;\lesssim\; \frac{1}{|\tau|}\;|E|\,.$$

The desired estimate follows. $\square$

**Proof of Proposition 7.3.** Define

$$\mathcal{G} \;:=\; \tau^2\|W\|^2_{L^2(\mathcal{Q}^\delta)} \;+\; \tau\|\pi_{(\mathcal{M}\nu)^\perp} W\|^2_{L^2(\partial\mathcal{Q}^\delta)} \;+\; \|\widetilde{\operatorname{div}}\, W\,, \widetilde{\operatorname{curl}}\, W\|^2_{L^2(\mathcal{Q}^\delta)}.$$

The summands agree with those of $a(W, W')$ to leading order in $\tau$.

• Lemmas 7.4 and 7.5 imply that

$$\Big|\mathcal{G} \;-\; a(W,W')\Big| \;\lesssim\; |\tau|\|W\|^2_{L^2(\partial\mathcal{Q}^\delta)} \;+\; \frac{1}{|\tau|}\int_{\mathcal{Q}^\delta} |\widetilde{\operatorname{div}}\, W|\;|\nabla W|\;dx$$
$$+\frac{1}{|\tau|}\,\big\|\widetilde{\operatorname{div}}\, W\,, \widetilde{\operatorname{curl}}\, W\big\|_{L^2(\mathcal{Q}^\delta)}\Big(\big\|\widetilde{\operatorname{div}}\, W\,, \widetilde{\operatorname{curl}}\, W\big\|_{L^2(\mathcal{Q}^\delta)} + \|W\|_{L^2(\mathcal{Q}^\delta)}\Big).$$

To estimate the integral on the right it is important that $\widetilde{\operatorname{div}}\, E$ has support in $\mathcal{Q}_I$ so the integral is $\le \|\widetilde{\operatorname{div}}\, W\|_{L^2(\mathcal{Q}_I)}\|\nabla W\|_{L^2(\mathcal{Q}_I)}$. The overdetermined system $\widetilde{\operatorname{div}}, \widetilde{\operatorname{curl}}$ is elliptic uniformly in $\operatorname{Re}\tau \ge 1$. It follows that with a constant independent of such $\tau$ and $\delta$,

$$\|\nabla W\|_{L^2(\mathcal{Q}_I)} \;\lesssim\; \|\widetilde{\operatorname{div}}\, W\,,\; \widetilde{\operatorname{curl}}\, W\|_{L^2(\mathcal{Q}^\delta)} \;+\; \|W\|_{L^2(\mathcal{Q}^\delta)}\,. \tag{7.7}$$

Define

$$N_1 := \|W\|^2_{L^2(\mathcal{Q}^\delta)},\;\; N_2 := \|\pi_{(\mathcal{M}\nu)^\perp} W\|^2_{L^2(\partial\mathcal{Q}^\delta)},\;\; N_3 := \|\widetilde{\operatorname{div}}\, W\,, \widetilde{\operatorname{curl}}\, W\|^2_{L^2(\mathcal{Q}^\delta)}.$$

Therefore,

$$\Big|\mathcal{G} \;-\; a(W,W')\Big| \;\lesssim\; \|W\|^2_{L^2(\partial\mathcal{Q}^\delta)} \;+\; |\tau| N_1 \;+\; \frac{1}{|\tau|}\;\Big(N_3 + (N_3 N_1)^{1/2}\Big). \tag{7.8}$$

Proposition 6.1 implies that

$$\|W\|^2_{L^2(\partial\mathcal{Q}^\delta)} \ \lesssim\ |\tau|^{-1}N_1 + N_2 + (N_1N_3)^{1/2}\,. \tag{7.9}$$

Combining (7.8) and (7.9) yields

$$\Big|\mathcal{G} \ -\ a(W,W')\Big| \ \lesssim\ |\tau|\,N_1 + N_2 + |\tau|^{-1}N_3 + (N_1N_3)^{1/2}\,. \tag{7.10}$$

• Derive a lower bound for $\mathcal{G}$ by considering its real and imaginary parts,

$$\begin{aligned} \mathrm{Re}\ \mathcal{G} \ &=\ \big((\mathrm{Re}\ \tau)^2 - (\mathrm{Im}\,\tau)^2\big)N_1 + \mathrm{Re}\ \tau\ N_2 \ +\ N_3, \\ \mathrm{Im}\,\mathcal{G} \ &=\ \big(2\,\mathrm{Im}\tau\ \mathrm{Re}\ \tau\big)N_1 \ +\ \mathrm{Im}\,\tau\ N_2\,. \end{aligned} \tag{7.11}$$

Where $|\,\mathrm{Im}\ \tau| \le \mathrm{Re}\ \tau/2$, the equation for the real part of $\mathcal{G}$ implies that for $\mathrm{Re}\ \tau > M_1$,

$$|\tau|^2N_1 + |\tau|N_2 + N_3 \lesssim |\,\mathrm{Re}\ \mathcal{G}|. \tag{7.12}$$

For the more difficult region $\{|\,\mathrm{Im}\ \tau| > \mathrm{Re}\ \tau/2\}$, the lower bound for $|\mathcal{G}|$ is weaker. When $\mathrm{Im}\ \tau \ne 0$, the second equation in (7.11) is equivalent to

$$\frac{\mathrm{Im}\,\mathcal{G}}{\mathrm{Im}\,\tau} \ =\ \big(2\,\mathrm{Re}\,\tau\big)N_1 \ +\ N_2\,. \tag{7.13}$$

Where $\{|\,\mathrm{Im}\ \tau| > \mathrm{Re}\ \tau/2\}$, this implies

$$(\mathrm{Re}\ \tau)N_1 + N_2 \ \lesssim\ \frac{|\mathrm{Im}\,\mathcal{G}|}{|\tau|}\ . \tag{7.14}$$

Multiplying by $|\tau|^2/(\mathrm{Re}\ \tau)$ yields for $\mathrm{Re}\ \tau > M_2$,

$$|\tau|^2\,N_1 + \frac{|\tau|^2}{\mathrm{Re}\ \tau}\,N_2 \ \lesssim\ \frac{|\tau|}{\mathrm{Re}\ \tau}|\,\mathrm{Im}\ \mathcal{G}|. \tag{7.15}$$

Estimate $N_3$ using the first equation in (7.11) together with (7.15) to find,

$$N_3 \ \lesssim\ |\tau|^2N_1 + |\mathrm{Re}\,\mathcal{G}| \ \lesssim\ \frac{|\tau|}{\mathrm{Re}\ \tau}\ |\,\mathrm{Im}\ \mathcal{G}| \ +\ |\mathrm{Re}\,\mathcal{G}|.$$

This together with (7.15) yields

$$|\tau|^2\,N_1 + \frac{|\tau|^2}{\mathrm{Re}\ \tau}\,N_2 \ +\ N_3 \ \lesssim\ \frac{|\tau|}{\mathrm{Re}\,\tau}\ |\mathcal{G}|\,.$$

Multiplying by $(\mathrm{Re}\ \tau)/|\tau|$ yields

$$(\mathrm{Re}\ \tau)|\tau|N_1 + |\tau|N_2 + \frac{\mathrm{Re}\ \tau}{|\tau|}N_3 \ \lesssim\ \ |\mathcal{G}|\,. \tag{7.16}$$

One has the larger lower bound (7.12) on the complementary region. Therefore (7.16) holds for all $\text{Re }\tau > M_2$. The left hand side of (7.16) and the left hand side of (7.2) are the same. Denote these left hand sides as

$$\mathcal{H} \;:=\; (\text{Re }\tau)|\tau|N_1 + |\tau|N_2 + \frac{\text{Re }\tau}{|\tau|}N_3 \;\lesssim\; |\mathcal{G}|\,. \tag{7.17}$$

• Next show that

$$|\tau|\,N_1 + N_2 + |\tau|^{-1}N_3 + (N_1N_3)^{1/2} \;\lesssim\; \frac{1}{\text{Re }\tau}\,\mathcal{H}\,. \tag{7.18}$$

The equality in (7.17) estimates the first three terms. For the last, estimate

$$\big(N_1N_3\big)^{1/2} = \big(|\tau|N_1\big)^{1/2}\big(|\tau|^{-1}N_3\big)^{1/2} \le \frac{1}{2}\Big(|\tau|N_1 + |\tau|^{-1}N_3\Big) \le \frac{1}{\text{Re }\tau}\,\mathcal{H}.$$

With the shorthand $a = a(W, W')$ and $\text{Re }\tau > M_3$ estimate (7.10) yields

$$\mathcal{H} \;\lesssim\; |\mathcal{G}| \;\lesssim\; \big|a\big| \;+\; \big|\mathcal{G} - a\big| \;\lesssim\; \big|a\big| \;+\; \frac{1}{\text{Re }\tau}\,\mathcal{H}\,.$$

For $\text{Re }\tau > M_3$, $\mathcal{H} \lesssim \big|a\big|$. This completes the proof of Proposition 7.3. □

**Remark 7.3** *The analysis of conservative boundary conditions is easier than absorbing conditions. Conservative conditions have flux equal to zero for $\tau \in ]0,\infty[$. By analytic continuation the flux vanishes for all $\tau$. For the absorbing conditions, the flux is positive for $\tau \in ]0,\infty[$. That does not continue. The flux for complex $\tau$ in that case is the sum of positive term plus a term that is smaller in powers of $\tau$ but is not absorbable. This occurs at equation* (7.8)*. The remedy is Proposition 6.1 .*

### 7.3 Stretched Maxwell equation estimate on $\mathcal{Q}^\delta$

The estimate of the next Theorem is a key step toward Theorem 1.15. It concernes sources $\rho(x), j(x)$ independent of $\tau$ and does not assert existence.

**Theorem 7.6** *There are positive constants $C, M$ depending on $\mathcal{Q}_I$ and $\sigma_k$, and independent of $\delta \in ]0, \delta_0[$, $\tau \in \{\text{Re }\tau > M\}$, so that for $(j,\rho) \in L^2_{\mathcal{Q}_I}(\mathcal{Q}^\delta)$, and $W = (E, B) \in H^2(\mathcal{Q}^\delta)$ that satisfies the stretched Maxwell boundary* *value problem,*

$$\begin{aligned}
\tau E \ - \ \widetilde{\operatorname{curl}}\, B \ &= \ -j \quad \text{ on } \ \mathcal{Q}^\delta, \\
\tau B \ + \ \widetilde{\operatorname{curl}}\, E \ &= \ 0 \qquad \text{ on } \ \mathcal{Q}^\delta, \\
\widetilde{\operatorname{div}}\, B \ &= \ 0 \qquad \text{ on } \ \mathcal{Q}^\delta, \\
\widetilde{\operatorname{div}}\, E \ &= \ \rho \qquad \text{ on } \ \mathcal{Q}^\delta, \\
\pi^-(\mathcal{M}\nu) W \ &= \ 0 \qquad \text{ on } \ \partial\mathcal{Q}^\delta,
\end{aligned} \tag{7.19}$$

*one has*

$$(\mathrm{Re}\ \tau)^2 \|W\|^2_{L^2(\mathcal{Q}^\delta)} + (\mathrm{Re}\ \tau) \|\pi_{(\mathcal{M}\nu)^\perp} W\|^2_{L^2(\partial\mathcal{Q}^\delta)} \ \le \ C\, \|\rho, j\|^2_{L^2(\mathbb{R}^3)}. \tag{7.20}$$

**Proof.** The stretched Maxwell equations imply the stretched Helmholtz equation $(\tau^2 - \widetilde{\Delta}) W \ = (-\tau j - \widetilde{\operatorname{grad}}\, \rho\,,\, \widetilde{\operatorname{curl}}\, j)$ on $\mathcal{Q}^\delta$. Use the notation $\mathcal{H}$ from (7.17). The source terms in the Helmholtz equations are

$$(f_1, f_2) \ := \ (-\tau j - \widetilde{\operatorname{grad}}\, \rho\,,\, \widetilde{\operatorname{curl}}\, j). \tag{7.21}$$

Inequality (7.2) shows that with $(E', B')$ from Proposition 7.3, $\mathcal{H} \lesssim \big| a((E,B),(E',B')) \big|$. Lemma 5.9 implies that

$$a\big((E,B),(E',B')\big) = \int_{\mathcal{Q}^\delta} \Pi(f_1 \cdot E' + f_2 \cdot B')\, dx + \int_{\partial\mathcal{Q}^\delta} \mathcal{D}(E,B) \cdot (E',B')\, d\Sigma.$$

The formulas for the spectral projections of $\mathcal{D}(E,B)$ in Proposition 5.10 show that $\mathcal{D}(E,B) = 0$ for the solutions of (7.19), so

$$a((E,B),(E',B')) \ = \ \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \Big( (-\tau j - \widetilde{\operatorname{grad}}\, \rho) \cdot E' \ + \ \widetilde{\operatorname{curl}}\, j \cdot B' \Big)\, dx\,.$$

Since $j$ and $\rho$ vanish at the boundary, integration by parts using Lemma 5.2 yields

$$a((E,B),(E',B')) \ = \ \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \Big( (-\tau j \cdot E' + \rho \cdot \widetilde{\operatorname{div}}\, E' \ + \ j \cdot \widetilde{\operatorname{curl}}\, B' \Big)\, dx\,.$$

The Cauchy-Schwarz inequality together with the inequalities of Lemma 7.4 yield,

$$\big| a((E,B),(E',B')) \big| \lesssim \big\| \rho, j \big\|_{L^2(\mathcal{Q}^\delta)} \Big( \| \tau W \|_{L^2(\mathcal{Q}^\delta)} + \| \widetilde{\operatorname{div}}\, W, \widetilde{\operatorname{curl}}\, W \|_{L^2(\mathcal{Q}^\delta)} \Big).$$

The Maxwell equations imply that

$$\| \tau W \|_{L^2(\mathcal{Q}^\delta)} \ \lesssim \ \| \rho, j \|_{L^2(\mathcal{Q}^\delta)} \ + \ \| \widetilde{\operatorname{div}}\, W, \widetilde{\operatorname{curl}}\, W \|_{L^2(\mathcal{Q}^\delta)}.$$

Therefore,

$$\big|a((E,B),(E',B'))\big| \lesssim \|\rho, j\big\|^2_{L^2(\mathcal{Q}^\delta)} + \|\rho, j\big\|_{L^2(\mathcal{Q}^\delta)} \|\widetilde{\operatorname{div}}\, W, \widetilde{\operatorname{curl}}\, W\|_{L^2(\mathcal{Q}^\delta)}.$$

Estimate the second summand on the right as

$$\begin{aligned}\Big(\frac{|\tau|}{\mathrm{Re}\ \tau}\|\rho, j\big\|^2_{L^2(\mathcal{Q}^\delta)}\Big)^{1/2}&\Big(\frac{\mathrm{Re}\ \tau}{|\tau|}\|\widetilde{\operatorname{div}}\, W, \widetilde{\operatorname{curl}}\, W\|^2_{L^2(\mathcal{Q}^\delta)}\Big)^{1/2} \\ &\le\ \frac{1}{2}\Big(\frac{1}{\epsilon}\frac{|\tau|}{\mathrm{Re}\ \tau}\|\rho, j\big\|^2_{L^2(\mathcal{Q}^\delta)}\ +\ \epsilon\frac{\mathrm{Re}\ \tau}{|\tau|}\|\widetilde{\operatorname{div}}\, W, \widetilde{\operatorname{curl}}\, W\|^2_{L^2(\mathcal{Q}^\delta)}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{H}\ \le\ C\big|a\big|\ &\le\ \frac{C}{2\epsilon}\,\frac{|\tau|}{\mathrm{Re}\ \tau}\|\rho, j\big\|^2_{L^2(\mathcal{Q}^\delta)}\ +\ \frac{C\epsilon}{2}\,\frac{\mathrm{Re}\ \tau}{|\tau|}\|\widetilde{\operatorname{div}}\, W, \widetilde{\operatorname{curl}}\, W\|^2_{L^2(\mathcal{Q}^\delta)} \\ &\le\ \frac{C}{2\epsilon}\,\frac{|\tau|}{\mathrm{Re}\ \tau}\|\rho, j\big\|^2_{L^2(\mathcal{Q}^\delta)}\ +\ \frac{C\epsilon}{2}\,\mathcal{H}\,.\end{aligned}$$

Choose $\epsilon = 1/C$. Absorb the second summand on the right in the left hand side to find $\mathcal{H} \lesssim (|\tau|/\mathrm{Re}\ \tau)\|\rho, j\big\|^2_{L^2(\mathcal{Q}^\delta)}$. Multiplying by $(\mathrm{Re}\ \tau)/|\tau|$ yields

$$\begin{aligned}(\mathrm{Re}\ \tau)^2\|W\|^2_{L^2(\mathcal{Q}^\delta)}\ +\ \mathrm{Re}\ \tau\,\|\pi_{(\mathcal{M}\nu)^\perp} W\|^2_{L^2(\partial\mathcal{Q}^\delta)}\ +& \\ +\ |\tau|^{-1}\big\|\widetilde{\operatorname{curl}}\, W\,, \widetilde{\operatorname{div}}\, W\big\|^2_{L^2(\mathcal{Q}^\delta)}\ \lesssim\ \|\rho, j\|_{L^2(\mathbb{R}^3)}\,.&\end{aligned}\tag{7.22}$$

This completes the proof of Theorem 7.6. □

**Remark 7.4** *In our earlier work on the internal corners for Bérenger's algorithm in Part II of [22], and also for the external corners for the Pauli system [23], the estimates analogous to* (7.22) *were stronger. The variational forms used Dirichlet's integral rather than a div-curl form.*

# 8 Proofs of the main Theorems 8.1, 1.16, 1.15

The order is crucial. First solve Helmholtz on $\mathcal{Q}^\delta$. Derive the stretched Maxwell equations on $\mathcal{Q}^\delta$ in Theorem 8.1. This requires the smooth boundary. Solve the stretched equations on $\mathcal{Q}$ by passing to the limit $\delta = 0$. Use [22] for uniqueness of the stretched equations on $\mathcal{Q}$ when $\tau$ is real. Uniqueness for all $\tau$ follows by analytic continuation, proving Theorem 1.15. The solution of the stretched equations yields a solution of the split equations via the Laplace transform. Uniqueness for the split equations follows from uniqueness for the stretched equations, proving Theorem 1.16.

## 8.1 Solving the stretched Maxwell equations on $\mathcal{Q}^\delta$

Proposition 3.4, Proposition 5.13, and Theorem 7.6 are used to solve the stretched Maxwell system on $\mathcal{Q}^\delta$.

**Theorem 8.1** *Assume that Assumption 1.2 is satisfied. There is an $M_0 > 0$ so that if $M > M_0$, $J$ and $R$ satisfying $-\widetilde{\operatorname{div}}\, J = \tau R$ are holomorphic on $\operatorname{Re}\tau > M$ with values in $L^2_{\overline{\mathcal{Q}_I}}(\mathcal{Q})$, then there is a unique $W(\tau) = (F(\tau), G(\tau))$ holomorphic from $\{\operatorname{Re}\tau > M\}$ to $H^2(\mathcal{Q}^\delta)$ that satisfies the stretched Maxwell boundary value problem with absorbing boundary condition*

$$
\begin{aligned}
\tau F \;-\; \widetilde{\operatorname{curl}}\, G \;&=\; -J \qquad && \text{on } \ \mathcal{Q}^\delta, \\
\tau G \;+\; \widetilde{\operatorname{curl}}\, F \;&=\; 0 \qquad && \text{on } \ \mathcal{Q}^\delta, \\
\big(\widetilde{\operatorname{div}}\, F, \widetilde{\operatorname{div}}\, G\big) \;&=\; (R, 0) \qquad && \text{on } \ \mathcal{Q}^\delta, \\
\pi^-(\mathcal{M}\nu)(F,G) \;&=\; 0 \qquad && \text{on } \ \partial\mathcal{Q}^\delta.
\end{aligned}
\tag{8.1}
$$

*The solution satisfies with constant independent of $\delta$ and $\tau$,*

$$
\begin{aligned}
(\operatorname{Re}\tau)^2 \|W(\tau)\|^2_{L^2(\mathcal{Q}^\delta)} + (\operatorname{Re}\tau)\, \|\pi_{(\mathcal{M}\nu)^\perp} W(\tau)\|^2_{L^2(\partial\mathcal{Q}^\delta)} \\
\lesssim\; \|R(\tau), J(\tau)\|^2_{L^2(\mathcal{Q}^\delta)}.
\end{aligned}
\tag{8.2}
$$

**Proof. Existence.** For $\tau \in \{\operatorname{Re}\tau > M\} \setminus \mathbb{D}(\delta)$ denote by $(\tau^2 - \widetilde{\Delta})^{-1}$ the inverse of the Helmholtz boundary value problem with homogeneous boundary conditions from Proposition 5.13. Define

$$
(F, G) \;:=\; W \;=\; (\tau^2 - \widetilde{\Delta})^{-1}\big( -\tau J - \widetilde{\operatorname{grad}}\, R\,,\, \widetilde{\operatorname{curl}}\, J\big)\,.
$$

Then $W$ is holomorphic on $\{\operatorname{Re}\tau > M\} \setminus \mathbb{D}(\delta)$ and satisfies the Helmholtz boundary value problem with homogeneous boundary conditions.

For $\tau$ real and large, Part II of Proposition 3.4 implies that $W$ satisfies the stretched Maxwell boundary value problem (8.1).

The equations in each line of the stretched Maxwell boundary value problem are analytic on $\{\operatorname{Re}\tau > M\}\setminus\mathbb{D}(\delta)$ and are satisfied for $\tau$ large and real. Since $\{\operatorname{Re}\tau > M\} \setminus \mathbb{D}(\delta)$ is connected, they vanish for all $\tau \in \{\operatorname{Re}\tau > M\} \setminus \mathbb{D}(\delta)$ by analytic continuation. Therefore, (8.1) is satisfied for $\tau \in \{\operatorname{Re}\tau > M\} \setminus \mathbb{D}(\delta)$.

Estimate (7.20) applies for all $\tau \in \{\operatorname{Re}\tau > M\}\setminus\mathbb{D}(\delta)$. Therefore (8.2) holds on this punctured neighborhood.

Since $\mathbb{D}(\delta)$ is discrete, the singularities of the function $W(\tau)$ at the points of $\mathbb{D}(\delta)$ are isolated. The estimate shows that $W$ is bounded on a punctured neighborhood of each singularity so the singlularities are all removable. Therefore $W$ is holomorphic on $\mathrm{Re}\,\tau > M$. Equation (8.1) and estimate (8.2) extend by continuity from the complement of $\mathbb{D}(\delta)$ to $\mathrm{Re}\,\tau > M$. This completes the proof.

**Uniqueness.** It suffices to show that the only solution with sources $R = 0$ and $J = 0$ is identically zero. Since it is holomorphic on $\mathrm{Re}\ \tau > M$ it suffices to show that it vanishes on $]c, \infty[$ for $c$ sufficiently large. For those $c$ the operator $\tau I + \sum_k \mathcal{A}_k \widetilde{\partial}_k$ is symmetric and strictly positive in the sense of Friedrichs. In addition the boundary condition is dissipative and the solution belongs to $H^2(\mathcal{Q}^\delta)$. Uniqueness follows immediately from the identity

$$\mathrm{Re}\Big( \big(\tau I + \sum_k \mathcal{A}_k \widetilde{\partial}_k\big) W \,, W \Big)_{L^2(\mathcal{Q}^\delta)} \;=\; 0\,,$$

after an integration by parts as in the proof of Proposition 3.4. $\square$

## 8.2 Solving the stretched Maxwell equations on $\mathcal{Q}$

Passing to the limit $\delta \searrow 0$ solves the stretched Maxwell boundary value problem on $\mathcal{Q}$.

**Proof of Theorem 1.15. Existence.** Define $W^\delta = (F^\delta, G^\delta)$ on $\mathcal{Q}^\delta$ to be the solution from Theorem 8.1.

With constant independent of $\delta, \tau$,

$$(\mathrm{Re}\ \tau)^2 \big\|W^\delta\big\|^2_{L^2(\mathcal{Q}^\delta)} + (\mathrm{Re}\ \tau)\big\|\pi_{(\mathcal{M}\nu)^\perp} W^\delta\big\|^2_{L^2(\partial\mathcal{Q}^\delta)} \lesssim \big\|R, J\big\|^2_{L^2(\mathcal{Q}^\delta)} = \big\|R, J\big\|^2_{L^2(\mathcal{Q})}. \tag{8.3}$$

Proposition 6.1 yields the following estimate for $(\mathrm{Re}\ \tau)\|W^\delta\|^2_{L^2(\partial\mathcal{Q}^\delta)}$,

$$\begin{aligned}&(\mathrm{Re}\ \tau)\|W^\delta\|^2_{L^2(\partial\mathcal{Q}^\delta)} \lesssim (\mathrm{Re}\ \tau)\|\pi_{(\mathcal{M}\nu)^\perp} W^\delta\|^2_{L^2(\partial\mathcal{Q}^\delta)} + \\ &\frac{\mathrm{Re}\ \tau}{|\tau|}\,\|W^\delta\|^2_{L^2(\mathcal{Q}^\delta)} + (\mathrm{Re}\ \tau)\|W^\delta\|_{L^2(\mathcal{Q}^\delta)}\,\|\widetilde{\mathrm{div}}\, W^\delta, \widetilde{\mathrm{curl}}\, W^\delta\|_{L^2(\mathcal{Q}^\delta)}.\end{aligned} \tag{8.4}$$

The first two terms on the right are estimated by (8.3). The first two lines of (8.1) together with (8.3) imply

$$\|\widetilde{\mathrm{div}}\, W^\delta\,,\, \widetilde{\mathrm{curl}}\, W^\delta\|_{L^2(\mathcal{Q}^\delta)} \;\lesssim\; \|\tau W^\delta\|_{L^2(\mathcal{Q}^\delta)} + \|J\|_{L^2(\mathcal{Q}^\delta)} \;\lesssim\; \frac{|\tau|}{\mathrm{Re}\ \tau}\,\big\|R, J\big\|_{L^2(\mathcal{Q})}\,.$$

Therefore,

$$(\text{Re } \tau)\|W^\delta\|_{L^2(\mathcal{Q}^\delta)}\|\widetilde{\text{div}} W^\delta \,, \, \widetilde{\text{curl}}\, W^\delta\|_{L^2(\mathcal{Q}^\delta)} \lesssim \frac{|\tau|}{(\text{Re } \tau)}\, \|R, J\|^2_{L^2(\mathcal{Q})}. \tag{8.5}$$

Using (8.5) in (8.4) yields

$$\big\|W^\delta\big\|_{L^2(\partial\mathcal{Q}^\delta)} \lesssim \frac{|\tau|^{1/2}}{\text{Re } \tau}\, \big\|R, J\big\|_{L^2(\mathcal{Q})}\,. \tag{8.6}$$

As in the derivation of (7.7), $\widetilde{\text{div}}, \widetilde{\text{curl}}$ is an overdetermined elliptic system. It follows that $W^\delta$ satisfies for any $0 < \eta < 1$,

$$\|\nabla W^\delta\|_{L^2(\eta\mathcal{Q})} \ \le\ C_\eta \|W^\delta, \widetilde{\text{div}}\, W^\delta, \widetilde{\text{curl}}\, W^\delta\|_{L^2(\mathcal{Q}^\delta)} \ \le\ \frac{C_\eta\, |\tau|}{\text{Re } \tau} \|R, J\|_{L^2(\mathcal{Q})}\,. \tag{8.7}$$

Choose a sequence $\delta(n)$ decreasing to zero. Since the $\mathcal{Q}^\delta$ increase as $\delta$ decreases it follows that for $m \le n$, $\mathcal{Q}^{\delta(n)} \supset \mathcal{Q}^{\delta(m)}$ and $(\partial\mathcal{Q}^{\delta(n)}) \cap \Gamma_k \supset (\partial\mathcal{Q}^{\delta(m)}) \cap \Gamma_k$. The Cantor diagonal method yields a subsequence, still denoted $\delta(n)$, so that for each $m$,

- $W^{\delta(n)}$ converges weakly in $L^2(\mathcal{Q}^{\delta(m)})$ to a limit $W_m$,
- $W^{\delta(n)}|_{\Gamma_k \cap \partial\mathcal{Q}^{\delta(m)}}$ converges weakly in $L^2(\Gamma_k \cap \partial\mathcal{Q}^{\delta(m)})$ to a limit $\Lambda_{m,k}$.

Passing to the limit in estimate (8.3) for $W^{\delta(n)}$ yields with constants independent of $m$,

$$(\text{Re } \tau)^2\big\|W_m\big\|^2_{L^2(\mathcal{Q}^{\delta(m)})} + (\text{Re } \tau)\sum_k \big\|\pi_{(\mathcal{M}\nu)^\perp}\Lambda_{m,k}\big\|^2_{L^2(\Gamma_k \cap \mathcal{Q}^{\delta(m)})} \lesssim \big\|R, J\big\|^2_{L^2(\mathcal{Q})}. \tag{8.8}$$

Since the $\mathcal{Q}^{\delta(n)}$ are increasing, for each $m > 1$, one has for $n \ge m$, $W_n = W_m$ on $\mathcal{Q}^{\delta(m)}$. Define $(F, G) = W \in L^2(\mathcal{Q})$ by $W = W_m$ on $\mathcal{Q}^{\delta(m)}$.

Similarly for $n \ge m$, $\Lambda_{n,k} = \Lambda_{m,k}$ on $\Gamma_k \cap \partial\mathcal{Q}^{\delta(m)}$. Define $\Lambda_k \in L^2(\Gamma_k \cap \partial\mathcal{Q})$ by $\Lambda_k = \Lambda_{m,k}$ on $\Gamma_k \cap \partial\mathcal{Q}^{\delta(m)}$. Estimate (8.8) implies,

$$(\text{Re } \tau)^2\|W\|^2_{L^2(\mathcal{Q})} \ +\ (\text{Re } \tau)\sum_k \|\pi_{(\mathcal{M}\nu)^\perp}\Lambda_k\|^2_{L^2(\Gamma_k)} \ \lesssim\ \|R, J\|^2_{L^2(\mathcal{Q})}.$$

The stretched Maxwell equations on $\mathcal{Q}$ follow from the equations on $\mathcal{Q}^{\delta(n)}$ on passing to the limit $n \to \infty$. In particular, $W \in H_{\widetilde{div}}(\mathcal{Q}) \cap H_{\widetilde{curl}}(\mathcal{Q})$.

Define $\Lambda \in L^2(\partial\mathcal{Q})$ to be the unique element so that $\Lambda|_{\Gamma_k} = \Lambda_k$ for all $k$. Corollary 5.5 shows that $W$ has a well defined trace in $H^{-1/2}(\partial\mathcal{Q})$. By continuity of the trace, the restriction of the trace of $W$ to $\Gamma_k$ is equal to $\Lambda_k$. Therefore

$$W|_{\partial\mathcal{Q}} \;-\; \Lambda \in H^{-1/2}(\partial\mathcal{Q}), \qquad \text{and} \qquad \text{supp}\,\big(W|_{\partial\mathcal{Q}} \;-\; \Lambda\big) \;\subset\; \mathcal{S}.$$

The only element of $H^{-1/2}(\partial\mathcal{Q})$ with such small support is 0 (see [22, Lemma 2.7.**i**]). It follows that $W|_{\partial\mathcal{Q}} = \Lambda \in L^2(\partial\mathcal{Q})$. In addition,

$$(\text{Re }\tau)^2 \big\|W\big\|^2_{L^2(\mathcal{Q})} + (\text{Re }\tau)\,\big\|\pi_{(M\nu)^\perp} W\big\|^2_{L^2(\partial\mathcal{Q})} \;\lesssim\; \big\|R, J\big\|^2_{L^2(\mathcal{Q})}\,. \tag{8.9}$$

Since $\nu$ is a unit vector parallel to the $k^{\text{th}}$ axis one has

$$\|\pi_{(\mathcal{M}\nu)^\perp} W\|^2_{L^2(\Gamma_k)} \;=\; \|(\mathcal{M}_{kk})^2 W_{tan}\|^2_{L^2(\Gamma_k)} \;\gtrsim\; \|W_{tan}\|^2_{L^2(\Gamma_k)},$$

yielding

$$(\text{Re }\tau)^2 \big\|W\big\|^2_{L^2(\mathcal{Q})} + (\text{Re }\tau)\,\sum_k \big\|W_{tan}\big\|^2_{L^2(\Gamma_k)} \;\lesssim\; \big\|R, J\big\|^2_{L^2(\mathcal{Q})}\,. \tag{8.10}$$

For any $\underline{\delta} > 0$ the holomorphy of $W : \{\text{Re }\tau > M\} \to L^2(\mathcal{Q}^{\underline{\delta}})$ follows from the fact that it is the weak limit of a bounded family of holomorphic functions $W^{\delta(n)}(\tau)$. It follows that for any $\underline{\delta}$, $W : \{\text{Re }\tau > M\} \to L^2(\mathcal{Q}^{\underline{\delta}})$ is holomorphic.

Indeed, to show that $W$ is holomorphic with values in $L^2(\mathcal{Q})$ it is sufficient to show that $\tau \mapsto \ell(W(\tau))$ is holomorphic for each $\ell$ in the dual of $L^2(\mathcal{Q})$. Since $W \in L^\infty\big(\{\text{Re }\tau > M\}\,;\, L^2(\mathcal{Q})\big)$, it suffices to show that $\ell(v(\tau))$ is holomorphic for $\ell$ in a dense subset. Indeed if $\ell$ is the limit of $\ell_j$ for which the result is true, estimate

$$\big|\ell(W(\tau)) \;-\; \ell_j(W(\tau))\big| \;\le\; \|\ell - \ell_j\| \sup_{\text{Re }\tau > M} \|W(\tau)\|_{L^2(\mathcal{Q})}, \quad \text{on} \quad \text{Re }\tau > M.$$

This proves that $\ell(W(\tau))$ is the uniform limit of the holomorphic functions $\ell_j(W(\tau))$ so is holomorphic.

Take the dense set to be the functionals $W \mapsto \int W \cdot \Phi\, dx$ with $\Phi \in C_0^\infty(\mathcal{Q})$. For each such $\Phi$, $\Phi \in C_0^\infty(\mathcal{Q}^\delta(n))$ for $n \ge m$. Then for $n \ge m$, $\ell(W(\tau)) = \lim \ell(W^{\delta(n)}(\tau))$ with $\ell(W^{\delta(n)}(\tau))$ holomorphic from Theorem 8.1.

An entirely analogous argument proves the analyticity of $W(\tau)|_{\partial\mathcal{Q}}$ with values in $L^2(\partial\mathcal{Q})$. In this case the functionals $\int_{\partial\mathcal{Q}} W \cdot \Psi\, d\Sigma$ with $\Psi$ smooth on the boundary and vanishing on a neighborhood of $\mathcal{S}$ are dense. This completes the proof of existence.

**Uniqueness.** It suffices to show that if the source terms $R, J$ are identically equal to zero, then the solution must vanish. By analytic continuation, it suffices to show that the solution vanishes for $\tau$ large and real.

For $\tau > 0$ real, the change of variable $X(\tau, x)$ from Definition 1.9 maps $\mathcal{Q}$ to the rectangle $\underline{\mathcal{Q}}$. The stretched Maxwell equations are conjugated to the unstretched Maxwell equations in the variable $X$. For $\tau$ real and $\nu$ parallel to one of the coordinate axes $A(\mathcal{M}\nu)$ is a positive multiple of $A(\nu)$ so $\pi^-(\mathcal{M}\nu) = \pi^-(\nu)$.

Therefore $V(X)$ on $\underline{Q}$ defined by $V\big(X(\tau,x)\big) \;=\; W(x)$ belongs to $L^2(\underline{\mathcal{Q}})$ and satisfies

$$\Big(\tau I + \sum_k \mathcal{A}_k\, \frac{\partial}{\partial X_k}\Big) V = 0 \quad \text{on} \;\; \underline{\mathcal{Q}}, \qquad \pi^-(\nu) V \;=\; 0, \quad \text{on} \;\; \partial\underline{\mathcal{Q}} \setminus \underline{\mathcal{S}}\,.$$

The trace $V|_{\partial\underline{\mathcal{Q}}\setminus\underline{\mathcal{S}}} \in L^2(\partial\underline{\mathcal{Q}}\setminus\underline{\mathcal{S}})$. Denote by $\Lambda$ the unique element of $L^2(\partial\underline{\mathcal{Q}})$ whose restriction to $\partial\underline{\mathcal{Q}}\setminus\underline{\mathcal{S}}$ is equal to $V|_{\partial\underline{\mathcal{Q}}\setminus\mathcal{S}}$. Then $V|_{\partial\mathcal{Q}} - \Lambda$ is an element of $H^{-1/2}(\partial\mathcal{Q})$ supported in $\mathcal{S}$. The difference therefore vanishes so $V|_{\partial\underline{\mathcal{Q}}} = \underline{\Lambda} \in L^2(\partial\underline{\mathcal{Q}})$.

This shows that $V$ satisfies the hypotheses of the uniqueness Theorem 2.13 of [22] that implies $V = 0$. Pulling back to the $x$ variables shows that $W = (F, G) = 0$ for $\tau \in ]M, \infty[$. By analytic continuation they vanish on $\{\mathrm{Re}\,\tau > M\}$. This completes the proof of uniqueness and therefore Theorem 1.15. $\square$

## 8.3 Solving Bérenger's split equations on $\mathcal{Q}$

The solution is constructed from its Laplace transform. The Paley-Wiener Theorem 8.2 yields properties of the solution from those of its Laplace transform. The Laplace transform of a distribution $F$ supported in $t \geq 0$ and so that $e^{-Mt}F \in L^2(\mathbb{R})$, is defined for all $\mathrm{Re}\,\tau > M$ by

$$\widehat{F}(\tau) \;:=\; \int e^{-\tau t}\; F(t)\; dt\,.$$

Our functions $F$ take values in a Hilbert space $H$. The Laplace transform takes values in $H$. It is defined and holomorphic in a half space $\mathrm{Re}\,\tau > M$.

**Theorem 8.2** *The Laplace transforms of functions $F$ supported in $t \geq 0$ and so that $e^{-Mt}F \in L^2(\mathbb{R}\,;\, H)$ are exactly the functions $G(\tau)$ holomorphic in* $\mathrm{Re}\,\tau > M$ *with values in $H$ and so that*

$$\sup_{\lambda > M} \int_{\mathrm{Re}\,\tau = \lambda} \big\|\widehat{F}(\tau)\big\|_H^2\; |d\tau| \;<\; \infty\,.$$

*In this case* $\widehat{F}(\tau)$ *has trace at* $\mathrm{Re}\ \tau = M$ *that is square integrable and*

$$\int e^{-2Mt}\ \|F(t)\|_H^2\, dt = \sup_{\lambda > M} \int_{\mathrm{Re}\,\tau=\lambda} \big\|\widehat{F}(\tau)\big\|_H^2\, |d\tau| = \int_{\mathrm{Re}\,\tau=M} \big\|\widehat{F}(\tau)\big\|_H^2\, |d\tau|.$$

**Proof of Theorem 1.16.** Define $M$ from Theorem 1.15.

**Uniqueness.** Suppose that $\widetilde{U} = (U^1, U^2, U^3)$ is a solution.

The Laplace transform of the condition $V := \sum U^j \in e^{\mu t} L^2(\mathbb{R} \times \mathcal{Q})$ is that $W := \widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3$ satisfies **i** of Theorem 1.15.

The Laplace transform of the condition $V_{tan} \in e^{\mu t} L^2(\Gamma_k)$ implies that $W_{tan} : \{\mathrm{Re}\ \tau > \mu\} \to L^2(\Gamma_k)$ is holomorphic.

The Laplace transform of the split equations (1.8) imply the split equations

$$\big(\tau + \sigma_k(x_k)\big)\widehat{U}^k \ + \ \mathcal{A}_k \partial_k W \ = \ \widehat{f}_k\,, \quad k = 1,2,3. \tag{8.11}$$

Summing implies that $W$ satisfies **iii** of Theorem 1.15.

The Laplace transform of the boundary condition $\pi^-(A(\nu))V = 0$ shows that $W$ satisfies **iv** of Theorem 1.15.

Therefore $W$ must be equal to the unique solution of the stretched Maxwell system from Theorem 1.15. This uniquely determines $W$.

The split equations (8.11) then determine $\widehat{U}^k$.

This completes the proof that $\widehat{U}^k$ is uniquely determined. Since the Laplace transform is injective, this proves the uniqueness of $\widetilde{U}$.

**Existence. • Construction of a candidate solution.** Define $W(\tau)$ to be the solution from Theorem 1.15, associated to $R = \widehat{\rho}$ and $J = \widehat{j}$. Consider five holomorphic functions on $\mathrm{Re}\ \tau > M$,

$$\begin{aligned} W(\tau) &\quad \text{with values in} \quad L^2(\mathcal{Q}), \\ (\tau + \sigma_k(x_k))^{-1} A_k \partial_k W(\tau) &\quad \text{with values in} \quad H^{-1}(\mathcal{Q}), \quad k = 1,2,3, \\ W_{tan}(\tau) &\quad \text{with values in} \quad L^2(\cup \Gamma_k). \end{aligned}$$

The estimate of Theorem 1.15 implies that

$$(\mathrm{Re}\ \tau)^2 \big\|W(\tau)\big\|^2_{L^2(\mathcal{Q})} \ + \ (\mathrm{Re}\ \tau) \sum_k \big\|W_{tan}(\tau)\big\|^2_{L^2(\Gamma_k)} \lesssim \big\|\widehat{\rho}(\tau), \widehat{j}(\tau)\big\|_{L^2(\mathcal{Q})}.$$

It follows that for all $k$,

$$(\mathrm{Re}\ \tau)^2\, |\tau|\, \big\|(\tau + \sigma_k(x_k))^{-1}\, A_k \partial_k W(\tau)\big\|^2_{H^{-1}(\mathcal{Q})} \lesssim \big\|\widehat{\rho}(\tau), \widehat{j}(\tau)\big\|_{L^2(\mathcal{Q})}.$$

Theorem 8.2 implies that there are uniquely determined functions

$$\begin{aligned} V(t) &\in e^{\mu t} L^2(\mathbb{R}\,;\, L^2(Q)), \\ U^k(t) &\in e^{\mu t} L^2(\mathbb{R}\,;\, H^{-1}(\mathcal{Q})), \quad k = 1, 2, 3, \\ \Psi(t) &\in e^{\mu t} L^2(\mathbb{R}\,;\, L^2(\partial \mathcal{Q})), \end{aligned}$$

supported in $t \geq 0$ so that,

$$\widehat{V} = W, \quad \widehat{U}^k = -(\tau + \sigma_k(x_k))^{-1} A_k \partial_k W, \quad \widehat{\Psi}|_{\cup \Gamma_k} = W_{tan}(\tau)\,. \tag{8.12}$$

Not only is $\widehat{U}^k$ estimated in terms of $W$ but so is $|\tau|\widehat{U}^k$. This yields,

$$\begin{aligned} &\mu^2 \big\| e^{-\mu t} V \big\|^2_{L^2(\mathbb{R}; L^2(\mathcal{Q}))} + \mu \sum_k \big\| e^{-\mu t} \Psi \big\|^2_{L^2(\mathbb{R}; L^2(\Gamma_k))} + \\ &\quad \mu^2 \big\| e^{-\mu t} \{U^i, \partial_t U^i\} \big\|^2_{L^2(\mathbb{R}; H^{-1}(\mathcal{Q}))} \lesssim \big\| e^{-\mu t} \{\rho, j\} \big\|^2_{L^2(\mathbb{R}; L^2(\mathcal{Q}))}. \end{aligned} \tag{8.13}$$

• **Verification of Bérenger's split equations for $V$.**
The stretched system satisfied by $W$ from (1.11) is

$$\tau W \;+\; \sum_k \frac{\tau}{\tau + \sigma_k(x_k)}\, \mathcal{A}_k\, \partial_k W \;=\; (0, -\widehat{j}(\tau))\,.$$

Since $W = \widehat{V}$,

$$\tau \widehat{V} \;+\; \sum_k \frac{\tau}{\tau + \sigma_k(x_k)}\, \mathcal{A}_k\, \partial_k \widehat{V} \;=\; (0, -\widehat{j}(\tau))\,. \tag{8.14}$$

The middle identity of (8.12) implies that

$$\tau\, \widehat{U}^k \;+\; \frac{\tau}{\tau + \sigma_k(x_k)}\, \mathcal{A}_k\, \partial_k \widehat{V} \;=\; (0, -\widehat{j}_k(\tau))\,. \tag{8.15}$$

Summing on $k$ yields

$$\tau \sum_k \widehat{U}^k \;+\; \sum_k \frac{\tau}{\tau + \sigma_k(x_k)}\, \mathcal{A}_k\, \partial_k \widehat{V} \;=\; (0, -\widehat{j}(\tau))\,. \tag{8.16}$$

Subtracting (8.14) from (8.16) show that $\widehat{V} = \sum_k \widehat{U}^k$. Therefore $V = \sum_i U^i$. Inserting this identity in (8.15) yields

$$\tau\, \widehat{U}^k \;+\; \frac{\tau}{\tau + \sigma_k(x_k)}\, \mathcal{A}_k\, \partial_k \sum_i \widehat{U}^i \;=\; (0, -\widehat{j}_k(\tau))\,. \tag{8.17}$$

Multiplying (8.17) by $(\tau + \sigma_k(x_k))/\tau$ and using the fact that $\sigma_k = 0$ on the support of $j_k$ yields (1.8) with $f_k = (-j_k, 0)$. Since that is the Laplace transform of the $k^{\rm th}$ split Bérenger equation, this completes the verification of the Bérenger split system on $\mathbb{R} \times \mathcal{Q}$.

• **Verification of the boundary condition** $\pi^-(\nu)V = 0$ **on** $\Gamma_k$**.**

Write $V = (E, B)$. Part **iii** of Lemma 2.5 shows that on $\cup_k \Gamma_k$,

$$\pi^-(\nu)V \;=\; \Big(E_{tan} - B_{tan} \wedge \nu\,,\, B_{tan} + E_{tan} \wedge \nu\Big)\,.$$

By construction the right hand side belongs to $e^{\mu t} L^2(\mathbb{R} \times \cup \Gamma_k)$ and is supported in $t \geq 0$. To prove that the right hand side vanishes it suffices to show that its Laplace transform vanishes. The transform is holomorphic on $\mathrm{Re}\ \tau > M$ with values in $L^2(\cup \Gamma_k)$. To show that it vanishes it suffices to show that it vanishes for $\tau \in ]M, \infty[$.

By construction, $\pi^-(\mathcal{M}\nu)\widehat{V}(\tau) = 0$ on $\cup \Gamma_k$ for $\mathrm{Re}\ \tau > M$. When $\tau$ is real $\mathcal{M}(\tau, x)$ is a positive diagonal matrix. When $x \in \Gamma_k$, the normal is parallel to one of the coordinate axes. Therefore $\mathcal{M}\nu$ is a positive multiple of $\nu$. Thus for those $\tau, x$, $\pi^-(\mathcal{M}\nu) = \pi^-(\nu)$. Therefore, for $\tau \in ]M, \infty[$, $\pi^-(\nu)\widehat{V}(\tau) = 0$ on $\Gamma_k$. By analytic continuation in $\tau$ it follows that $\pi^-(\nu)\widehat{V}(\tau) = 0$ on $\Gamma_k$ holds on $\mathrm{Re}\ \tau > M$. This is the Laplace transform of $\pi^-(\nu)V(t)|_{\Gamma_k}$. Therefore $\pi^-(\nu)V(t)|_{\Gamma_k} = 0$. This completes the proof of Theorem 1.16. $\square$

**Proof of part ii of Remark 1.5.** Equation (8.11) implies that

$$(\mathrm{Re}\ \tau)\|\widehat{U}^k\|_{L^2(\eta\mathcal{Q})} \;\leq\; \frac{\mathrm{Re}\ \tau}{|\tau|}\big(\|\nabla W\|_{L^2(\eta\mathcal{Q})} \;+\; \|\widehat{\jmath}\|_{L^2(\mathcal{Q})}\big) \;\leq\; C_\eta\,\|\widehat{\rho}, \widehat{\jmath}\|_{L^2(\mathcal{Q})}.$$

The Paley-Wiener Theorem 8.2 then implies (1.20). $\square$

# 9 Other PML and an error estimate

In our experience, all perfectly matched layers for Maxwell's equations are built around the stretched Maxwell system, so can be analysed using Theorem 1.15. This section analyses a handful of other methods to illustrate this point.

Bérenger's splitting has several drawbacks. First, the Cauchy problem for the split equations even with constant coefficients is only weakly well-posed. Second, the split equations do not resemble problem from physics. The anisotropic medium of Sacks *et al* [39] addresses both. It leads to the stretched Maxwell system of Chew and Weedon see [13]. It was used by Ziolkowski [44] to produce potentially realizable absorbers. It was modified by Abarbanel and Gottlieb [1] and extended by Turkel-Yefet to $\mathbb{R}^3$, [41].

We analyse the Turkel-Yefet system, equation (5.4) in [41]. It includes as special cases the others listed above. We add a non zero current $j$. The unknowns are modified electric and magnetic fields $(\widetilde{E}, \widetilde{B}) \in \mathbb{R}^3 \times \mathbb{R}^3$, augmented by 6 auxiliary functions $(P, Q) \in \mathbb{R}^3 \times \mathbb{R}^3$ to yield a twelve dimensional system. Define diagonal constant coefficient matrices,

$$
\begin{aligned}
S &:= \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3), \\
M &:= \operatorname{diag}(\sigma_2 + \sigma_3 - \sigma_1\,,\ \sigma_3 + \sigma_1 - \sigma_2\,,\ \sigma_1 + \sigma_2 - \sigma_3), \\
N &:= \operatorname{diag}\big((\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)\,,\ (\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)\,,\ (\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)\big).
\end{aligned}
$$

Assumptions 1.1 and 1.2 imply that $S, M, N$ vanish on $\mathcal{Q}_I$.

The unknowns $\widetilde{E}, \widetilde{B}, P, Q$ are required to satisfy the symmetric hyperbolic system

$$
\begin{aligned}
\partial_t \widetilde{E} - \operatorname{curl} \widetilde{B} + M\widetilde{E} + NP &= -j, \\
\partial_t \widetilde{B} + \operatorname{curl} \widetilde{E} + M\widetilde{B} + NQ &= 0, \\
\partial_t P + SP - \widetilde{E} &= 0, \\
\partial_t Q + SQ - \widetilde{B} &= 0.
\end{aligned}
\tag{9.1}
$$

The auxiliary variable $P$ resembles the polarization in electromagnetism and the variable $Q$ is a magnetic analogue. In contrast to Bérenger's splitting, the pure initial value problem is well posed thanks to the symmetric hyperbolicity.

The electric and magnetic fields of interest are defined by

$$
E := \widetilde{E} - SP, \qquad \text{and,} \qquad B := \widetilde{B} - SQ. \tag{9.2}
$$

Therefore $E = \widetilde{E}$ and $B = \widetilde{B}$ on $\mathcal{Q}_I$. The absorbing boundary condition $\pi^-(\nu)(\widetilde{E}, \widetilde{B}) = 0$ is imposed on $\cup\Gamma_k$. The boundary value problem for (9.1) is symmetric hyperbolic with maximal dissipative boundary condition. This yields $e^{\mu t} L^\infty(\mathbb{R}; L^2(\mathcal{Q}))$ estimates for sufficiently smooth solutions $\widetilde{E}, \widetilde{B}, P, Q$. One concludes existence of weak solutions and uniqueness of sufficiently smooth solutions. Uniqueness of weak solutions is not at all easy. Section 9.1 shows that the results of this paper suffice to fill that gap. Interestingly, the uniqueness proof for the unstretched Maxwell equations on $\mathcal{Q}$ does not apply. That problem is analysed in [22] by a hidden ellipticity argument using the divergence identities. In contrast, (9.1) does not have good divergence identities. Indeed, taking the divergence of the equations yields terms $\mathrm{div}(M\widetilde{E}, M\widetilde{B}, NP, NQ, QP, SQ)$ that involve all derivatives of $\widetilde{E}, \widetilde{B}, P, Q$, not just their divergence and curl.

## 9.1 Equivalence of Turkel-Yefet and stretched Maxwell

*For solutions of the Turkel-Yefet system, the fields $(E, B)$ satisfy the stretched Maxwell system.*

**Proof.** The Laplace transformed system, with hats omitted for easy of reading, implies

$$(\tau I + S)P \;=\; \widetilde{E}, \qquad \text{and}, \qquad (\tau I + S)Q \;=\; \widetilde{B}.$$

Inserting this into the first equations yields

$$\begin{aligned} \big(\tau I + M + N(\tau I + S)^{-1}\big)\widetilde{E} \;\; - \mathrm{curl}\, \widetilde{B} \;&=\; -j, \\ \big(\tau I + M + N(\tau I + S)^{-1}\big)\widetilde{B} \;+\; \mathrm{curl}\, \widetilde{E} \;&=\; 0. \end{aligned}$$

Factor $(\tau I + S)^{-1}$ to find,

$$\begin{aligned} \big((\tau I + M)(\tau I + S) + N\big)(\tau I + S)^{-1}\widetilde{E} \;-\; \mathrm{curl}\, \widetilde{B} \;&=\; -j, \\ \big((\tau I + M)(\tau I + S) + N\big)(\tau I + S)^{-1}\widetilde{B} \;+\; \mathrm{curl}\, E \;&=\; 0. \end{aligned}$$

Multiply by $\tau$ and use $(\tau I + S)E = \tau\widetilde{E}$ and $(\tau I + S)B = \tau\widetilde{B}$, to find

$$\begin{aligned} \big((\tau I + M)(\tau I + S) + N\big)E \;-\; \mathrm{curl}(\tau I + S)B \;&=\; -\tau j, \\ \big((\tau I + M)(\tau I + S) + N\big)B \;+\; \mathrm{curl}(\tau I + S)E \;&=\; 0. \end{aligned} \tag{9.3}$$

All the matrices involved are diagonal, so commute. The first coefficient of $(\tau I + M)(\tau I + S) + N$ is equal to

$$\begin{aligned} &(\tau + \sigma_2 + \sigma_3 - \sigma_1)(\tau + \sigma_1) \;+\; (\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) \\ &\qquad\qquad = \; \tau^2 \;+\; \tau(\sigma_2 + \sigma_3) \;+\; \sigma_2\sigma_3 \;=\; (\tau + \sigma_2)(\tau + \sigma_3). \end{aligned}$$

The first component of $\mathrm{curl}(\tau I + S)B$ is equal to $\partial_2((\tau + \sigma_3)B_3) \;-\; \partial_3((\tau + \sigma_2)B_2)$. Since $\sigma_j$ depends only on $x_j$, this is equal to

$$(\tau + \sigma_3)\,\partial_2 B_3 \;-\; (\tau + \sigma_2)\,\partial_3 B_2.$$

The first line of (9.3) is equal to

$$(\tau + \sigma_2)(\tau + \sigma_3)E_1 - ((\tau + \sigma_3)\partial_2 B_3 - (\tau + \sigma_2)\partial_3 B_2) = -\tau j_1.$$

Multiply by $\tau$ and divide by $(\tau+\sigma_2)(\tau+\sigma_3)$ to obtain

$$\tau E_1 - \Big(\frac{\tau}{\tau+\sigma_2}\partial_2 B_3 - \frac{\tau}{\tau+\sigma_3}\partial_3 B_2\Big) = -\frac{\tau^2}{(\tau+\sigma_2)(\tau+\sigma_3)} j_1 = -j_1.$$

A similar computation for $\tau\, E_j, \tau B_j$, shows that $E, B$ satisfy the stretched Maxwell system

$$\tau E \; - \; \widetilde{\mathrm{curl}}\, B \; = \; -j, \qquad \tau B \; + \; \widetilde{\mathrm{curl}}\, E \; = \; 0\,. \qquad \square$$

Conversely, if $E, B$ is a solution of the stretched equations,

$$\partial_t P + SP \; = \; \widetilde{E} \; = \; E + SP, \qquad \text{so,} \qquad \partial_t P \; = \; E.$$

Similarly $\partial_t Q = B$, determing $P, Q$. With $(\widetilde{E}, \widetilde{B}) = (E + SP, B + SQ)$ they generate a solution of the Turkel-Yefet system.

*Perfect matching when the computational domain is* $\mathbb{R}^3$ *follows.*

The Turkel-Yefet system (9.1) on $\mathcal{Q}$ with boundary condition $\pi^-(\nu)(E, B) = 0$ on $\cup\Gamma_k$, implies that their $E, B$ satisfy the stretched Maxwell boundary value. Theorem 1.15 implies that *the Turkel-Yefet system in rectangular geometry with absorbing boundary condition is well posed.*

## 9.2 An error estimate

Perfect matching plus well posedness of the boundary value problem on $\mathcal{Q}$ implies an error estimate in terms of the size of the computed quantities near the boundary. We present the case of Turkel-Yefet.

**Proposition 9.1** *Denote by* $\mathbf{K}$ *the system of partial differential operators on the left of* (9.1). *Suppose that* $0 < \eta/\mathrm{dist}(\mathcal{Q}_I, \partial\mathcal{Q}) < 1/2$. *Denote by* $\widetilde{u}$ *the solution of the boundary value problem defined by* (9.1) *with the absorbing boundary condition.* $E$ *and* $B$ *from* (9.2) *are the fields associated to* $\widetilde{u}$. *Then,*

$$\|\mathrm{Error\,in}\, E, B\|_{L^\infty([0,T];L^2(\mathcal{Q}_I))} \le C_{T,\eta}\, \|\widetilde{E}, \widetilde{B}\|_{L^1([0,T];L^2(\{\mathrm{dist}(x,\partial\mathcal{Q})\le\eta\}))}. \tag{9.4}$$

**Proof.** Denote by $\widetilde{u}_{\mathbb{R}^3}$ the solution of (9.1) on $\mathbb{R}^{1+3}$ that vanishes for $t \le 0$. Denote by $\widetilde{E}_{\mathbb{R}^3}, \widetilde{B}_{\mathbb{R}^3}$ the corresponding fields. Perfect matching shows that $(\widetilde{E}_{\mathbb{R}^3}, \widetilde{B}_{\mathbb{R}^3})|_{\mathcal{Q}_I}$ is equal to the Maxwell field $\widetilde{u}_{\mathbb{R}^3}$ on $\mathcal{Q}_I$.

Choose $\psi \in C_0^\infty(\mathcal{Q})$ with $\psi = 1$ on $\{x \in \mathcal{Q} : \mathrm{dist}(x, \partial\mathcal{Q}) \ge \eta\}$. Compute

$$\mathbf{K}(\psi\widetilde{u}) = \psi\mathbf{K}\widetilde{u} + [\mathbf{K}, \psi]\widetilde{u} = (0, -j) + [\mathbf{K}, \psi]\widetilde{u}. \tag{9.5}$$

The lower order terms in $\mathbf{K}$ and the last lines in (9.1) commute with $\psi$. Therefore the commutator term in (9.5) is order zero and is supported in $\{\mathrm{dist}(x, \partial \mathcal{Q}) \le \eta\}$. Therefore

$$\|[\mathbf{K}, \psi]\widetilde{u}(t)\|_{L^2(\mathcal{Q})} \ \le\ C_\eta\, \|\widetilde{E}(t), \widetilde{B}(t)\|_{L^2(\mathrm{dist}(x,\partial\mathcal{Q})\le\eta)}. \tag{9.6}$$

Combining estimates (9.5) and (9.6) yields

$$\|\mathbf{K}\big(\psi\widetilde{u} - \widetilde{u}_{\mathbb{R}^3}\big)(t)\|_{L^2(\mathcal{Q})} \ \le\ C_\eta\, \|\widetilde{E}(t), \widetilde{B}(t)\|_{L^2(\mathrm{dist}(x,\partial\mathcal{Q})\le\eta)}.$$

Well posedness of the symmetric hyperbolic system on $\mathbb{R}^3$ yields

$$\|\psi\widetilde{u} - \widetilde{u}_{\mathbb{R}^3}\|_{L^\infty([0,T];L^2(\mathcal{Q}))} \ \le\ C_{T,\eta}\, \|\widetilde{E}, \widetilde{B}\|_{L^1([0,T];L^2(\mathrm{dist}(x,\partial\mathcal{Q})\le\eta))}. \tag{9.7}$$

Since $\psi\widetilde{u}|_{\mathcal{Q}_I} = \widetilde{u}|_{\mathcal{Q}_I}$, $(\widetilde{E}, \widetilde{B})|_{\mathcal{Q}_I} = (E, B)_{\mathcal{Q}_I}$, and, $(\widetilde{E}_{\mathbb{R}^3}, \widetilde{B}_{\mathbb{R}^3})|_{\mathcal{Q}_I}$ is exact,

$$\|\text{Error in } E, B\|_{L^\infty([0,T];L^2(\mathcal{Q}_I))} \ \le\ \|\psi\widetilde{u} - \widetilde{u}_{\mathbb{R}^3}\|_{L^\infty([0,T];L^2(\mathcal{Q}_I))}. \tag{9.8}$$

Combining (9.7) and (9.8) yields (9.4). $\square$

**Remark 9.1** *Estimate* (9.4) *is not sharp. Errors come from waves that cross the layer, are reflected at $\partial\mathcal{Q}$, then cross the layer. If waves decrease by a factor $f$ on crossing the layer and by $r$ upon reflection at the absorbing boundary, the error is expected to be $\sim f^2 r$. The right hand side of* (9.4) *is the size of waves reaching the boundary that is expected to be $\sim f$.*

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