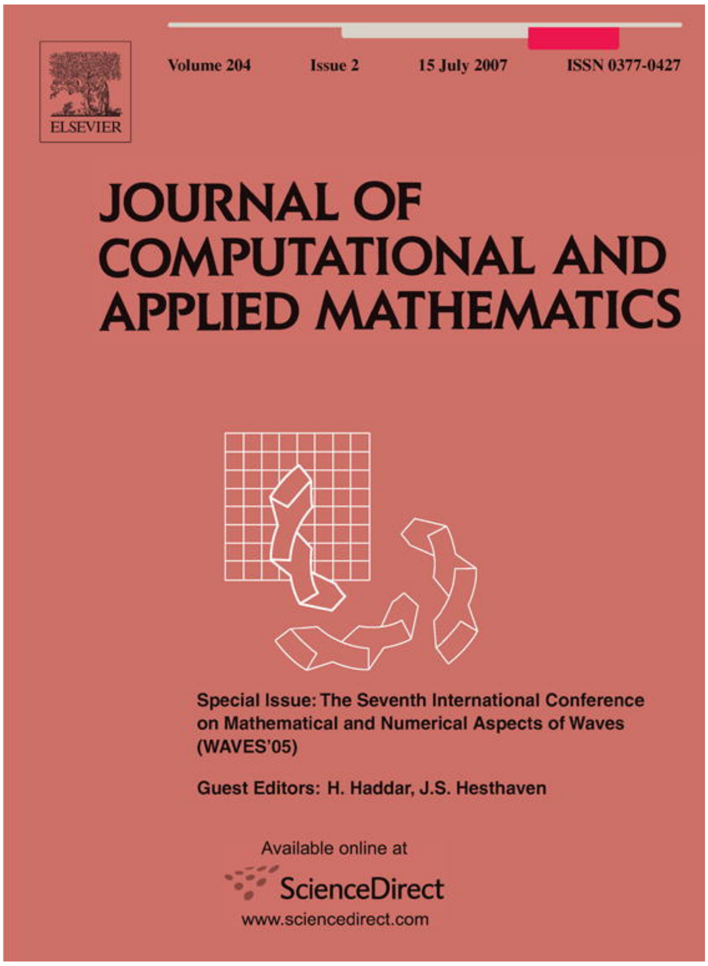
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Dirichlet to Neumann map for domains with corners and approximate boundary conditions

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Abstract

We present in this paper the Dirichlet to Neumann operator for the wave equation on a straight wedge in $\mathbb{R}^2$, using Fourier integral operators. As a consequence, we recover the classical approximate boundary conditions of orders 1 and 2.
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1. Statement of the problem

We consider the wave equation in $\mathbb{R}^2$:

$$\begin{cases} \partial_{tt}U - \Delta U = 0 & \text{in } (0,T)\times\mathbb{R}^2, \\ U(0,\cdot) = u_0, \quad \partial_t U(0,\cdot) = u_1, \end{cases} \tag{1}$$

with initial data u_0 and u_1 compactly supported in a bounded domain B (we could as well consider a diffraction problem). We want to compute the unique solution of (1) in some domain $\Omega_I \supset B$. Therefore, we introduce a computational domain $\Omega \supset \Omega_I$, and search for absorbing boundary conditions on the boundary $\Gamma = \partial\Omega$, i.e., operators $\mathscr{B}$ such that a function u satisfying

$$\begin{cases} \partial_{tt}u - \Delta u = 0 & \text{in } (0,T)\times\Omega, \\ u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = u_1 & \text{in } \Omega, \\ \mathscr{B}u = 0 & \text{on } (0,T)\times\Gamma, \end{cases} \tag{2}$$

be as close as possible to U in Ω_I. To this end, the theory of absorbing boundary conditions has been developed by Engquist and Majda in [5], where the strategy, extended in [9] to parabolic problems, consists of writing the Dirichlet

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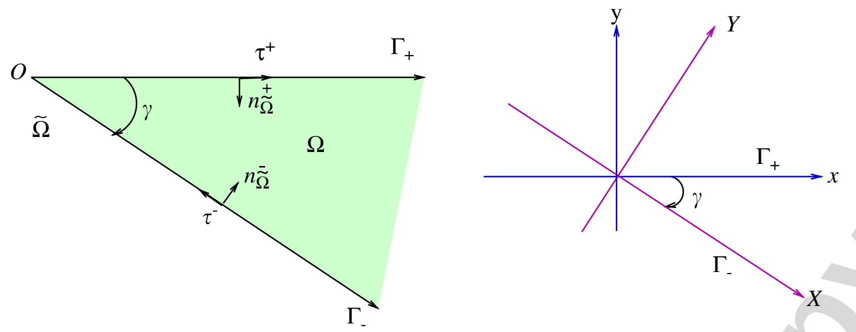

Fig. 1. Geometric notations for the corner.

to Neumann map for $\widetilde{\Omega}=\mathbb{R}^2\backslash\Omega$, then expanding the symbol of this pseudo-differential operator, truncating the series, and making approximations for incidence on the boundary close to normal. Other strategies have been developed for constant coefficient problems, in order to compute U exactly in Ω, such as PML [3] or fast solvers [1]. Higher order absorbing boundary conditions can be obtained as well, and Kreiss theory [12] or energy estimates [8] show that the boundary value problem in Ω is well-posed. For example, the second order boundary condition for the half-space $\{x<0\}$ is

$$\partial_{tt}u+\partial_{tx}u-\tfrac{1}{2}\,\partial_{yy}u=0. \tag{3}$$

An open and puzzling question remains the case of a corner. Obviously, the pseudo-differential approach cannot be used, since it relies on smooth functions.

However, it is tempting to try and use the absorbing boundary conditions (3) on each side of the wedge. With the notation in Fig. 1, this gives

$$\begin{cases}\partial_{tt}u-\partial_{tY}u-\tfrac{1}{2}\,\partial_{XX}u=0 & \text{on } (0,T)\times\Gamma_-,\\ \partial_{tt}u+\partial_{ty}u-\tfrac{1}{2}\,\partial_{xx}u=0 & \text{on } (0,T)\times\Gamma_+.\end{cases} \tag{4}$$

In [5], it was pointed out that in a right wedge (i.e., with $\gamma=\pi/2$), a "corner condition" was missing, in order to determine the discrete solution. The authors of [5] suggested to use a discretization of the first order absorbing boundary condition, transparent for the angle $\pi/4$. Later on, in [10], it was proved that the smoothest solution (i.e., in $H^{3-\varepsilon}((0,T)\times\Omega)$) to the wave equation in the right wedge with boundary conditions (4) satisfies the following equation at the corner:

$$(\tfrac{3}{2}\,\partial_t u-\partial_x u+\partial_y u)(0,0,t)=0. \tag{5}$$

The proof relies on the isotropy of the Laplace operator in the usual coordinates (the tangential derivative on one face is the normal derivative on the other), supplemented with energy estimates and regularity considerations (see [2,4]). Therefore, it does not extend to angles different from $\pi/2$. This corner condition coupled with (4) corresponds to imposing $u|_\Gamma$ in $H^2(\Gamma)$ where Γ is the full boundary.

This remark, together with results proved in [6], Sections 2 and 5, is the starting point of our strategy for the corner. We use the Helmholtz equation, and construct the explicit form of the Dirichlet to Neumann map. It will be treated as an operator on the Dirichlet datum, which is the trace of u on the whole boundary. We adopt the formalism introduced in [6] for the diffraction problem with a vanishing Dirichlet boundary condition, and generalized in [11] for the diffraction by an impedance boundary condition.

In Section 2, we first summarize some well-known results for the regular case. In Section 3, we compute analytically the solution in the exterior wedge, from which we deduce the Dirichlet to Neumann operator in Section 4. The main result is Theorem 6. Finally we establish in Section 5 a more tractable form for the Dirichlet to Neumann map (Theorem 9), from which we deduce the classical approximate boundary conditions of orders 1 and 2.

2. The Dirichlet to Neumann map: The regular case

The Dirichlet to Neumann map is defined as follows. Let Γ be a regular closed curve in $\mathbb{R}^2$ enclosing a domain Ω. We denote by $\widetilde{\Omega}$ the complement set of $\bar{\Omega}$ in $\mathbb{R}^2$. Let h in $\mathscr{D}(\mathbb{R}\times\Gamma)$ be such that $h\equiv 0$ for $t\leqslant 0$, and let w be the solution of the problem

$$\begin{cases}\partial_{tt}w-\Delta w=0 \quad \text{in } (0,T)\times\widetilde{\Omega},\\ w(0,\cdot)=0, \quad \partial_t w(0,\cdot)=0 \ \text{ in } \widetilde{\Omega},\\ w=h \quad \text{on } (0,T)\times\Gamma.\end{cases} \tag{6}$$

We call the Dirichlet to Neumann map for $\widetilde{\Omega}$ the map

$$N_{\widetilde{\Omega}}: h\mapsto \partial_{n_{\widetilde{\Omega}}}w, \tag{7}$$

and introduce the initial boundary value problem in Ω:

$$\begin{cases}\partial_{tt}u-\Delta u=0 \quad \text{in } (0,T)\times\Omega,\\ u(0,\cdot)=u_0, \quad \partial_t u(0,\cdot)=u_1 \quad \text{in } \Omega,\\ \partial_{n_\Omega}u+N_{\widetilde{\Omega}}u=0 \quad \text{on } (0,T)\times\Gamma.\end{cases} \tag{8}$$

Proposition 1. *For any u_0 in $H^1(\mathbb{R}^2)$ and u_1 in $L^2(\mathbb{R}^2)$, problem* (8) *has a unique solution $u\in H^1_{\mathrm{loc}}(\mathbb{R}_t\times\mathbb{R}^2)\cap H^2_{\mathrm{loc}}(\mathbb{R}_t,L^2(\mathbb{R}^2))$, equal to the solution U of* (1).

Proof. For U a regular solution to (1), we have $[U]=[\partial_{n_\Omega}U]=0$ on Γ, and therefore $\partial_{n_\Omega}U=-\partial_{n_{\widetilde{\Omega}}}U=-N_{\widetilde{\Omega}}U$. Thus U is a solution to problem (8), which therefore has at least one solution, namely U. For the uniqueness, suppose that the initial values be zero. Defining $h=u|_\Gamma$, and w the solution of (6), we construct energy estimates for u and w: first, we multiply the first equation in (8) by $\partial_t u$ an integrate in Ω. Second, we multiply the first equation in (6) by $\partial_t w$ and integrate in $\widetilde{\Omega}$. By using Green's formula, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\partial_t u\|^2_\Omega+\|\nabla u\|^2_\Omega]-\int_\Gamma \partial_{n_\Omega}u\partial_t u\,\mathrm{d}s=0,\\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|\partial_t w\|^2_{\widetilde{\Omega}}+\|\nabla w\|^2_{\widetilde{\Omega}}]-\int_\Gamma \partial_{n_{\widetilde{\Omega}}}w\partial_t w\,\mathrm{d}s=0.\end{aligned} \tag{9}$$

We add the two equations, taking into account that $u=w$ and $\partial_{n_\Omega}u=-N_{\widetilde{\Omega}}u=-\partial_{n_{\widetilde{\Omega}}}w$, and get

$$\frac{\mathrm{d}}{\mathrm{d}t}[\|\partial_t u\|^2_\Omega+\|\nabla u\|^2_\Omega+\|\partial_t w\|^2_{\widetilde{\Omega}}+\|\nabla w\|^2_{\widetilde{\Omega}}]=0.$$

Since the initial conditions vanish, u vanishes on $(0,T)$. □

In [5,9], the symbol of $N_{\widetilde{\Omega}}$ is obtained by a high frequency expansion. Here, we use a Laplace transform to compute the solution w of (6). We extend Eq. (6) for $t\in\mathbb{R}$. By the causality principle, $N_{\widetilde{\Omega}}h$ is the restriction to $(0,T)$ of the solution of (6) on $\mathbb{R}$. After a Laplace transform in time, we obtain a Helmholtz equation

$$\begin{cases}\Delta W+k^2W=0 \quad \text{in } \widetilde{\Omega},\\ W=H \quad \text{on } \Gamma,\end{cases} \tag{10}$$

with $W(x,k)=\int_{\mathbb{R}}w(x,t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t$, $H(x,k)=\int_{\mathbb{R}}h(x,t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t$, where k is a complex number, with $\mathscr{R}e\,k\neq 0$ and $\mathscr{I}m\,k<0$, according to the Paley–Wiener–Schwartz theorem. Throughout this paper, we make this assumption on k. Furthermore, we assume that W is a distribution with traces on Γ.

Coupling the exterior problem to 0 in Ω, by the jump formula we can write for any φ in $\mathscr{D}(\mathbb{R}^2)$

$$\langle(\Delta+k^2)W,\varphi\rangle=\int_\Gamma W\partial_{n_{\widetilde{\Omega}}}\varphi\,\mathrm{d}S-\int_\Gamma\partial_{n_{\widetilde{\Omega}}}W\varphi\,\mathrm{d}S.$$

Defining the distributions δ_Γ and δ'_Γ by their action on φ in $\mathscr{D}(\mathbb{R}^2)$,

$$\langle \delta_\Gamma, \varphi\rangle = \int_\Gamma \varphi \,\mathrm{d}S,$$

$$\langle \delta'_\Gamma, \varphi\rangle = -\langle \delta_\Gamma, \nabla\varphi . n_{\tilde{\Omega}}\rangle = -\int_\Gamma \partial_{n_{\tilde{\Omega}}}\varphi \,\mathrm{d}S, \tag{11}$$

we obtain, for $W|_\Gamma \in \mathscr{D}'(\Gamma)$ and $\partial_{n_{\tilde{\Omega}}} W \in \mathscr{D}'(\Gamma)$,

$$W = -(\Delta + k^2)^{-1}(\partial_{n_{\tilde{\Omega}}} W \otimes \delta_\Gamma + W \otimes \delta'_\Gamma). \tag{12}$$

Taking the limit on Γ in (12) whenever it is possible, gives the Neumann to Dirichlet map, while taking the normal derivative yields the Dirichlet to Neumann map, as we describe now in the more difficult case of a wedge.

3. Computation of the solution of the Helmholtz equation in the exterior wedge

We are back to the situation depicted in Fig. 1. Let $\mathscr{T}$ be the linear operator from $L^2(\Gamma)$ to $(L^2(\mathbb{R}_+))^2$ defined as follows: for each φ in $L^2(\Gamma)$,

$$\varphi_+(s) = \varphi(s, 0), \quad \varphi_-(s) = \varphi(s\cos\gamma, -s\sin\gamma), \quad \mathscr{T}\varphi = (\varphi_+, \varphi_-). \tag{13}$$

The Dirac distributions on the faces of the corner are defined for a $\varphi \in \mathscr{D}(\mathbb{R}^2)$ as

$$\langle \delta_+, \varphi\rangle = \int_0^{+\infty} \varphi(s, 0)\,\mathrm{d}s,$$

$$\langle \delta'_+, \varphi\rangle = -\langle \delta_+, \varphi'\rangle = \int_0^{+\infty} \partial_y\varphi(s, 0)\,\mathrm{d}s,$$

$$\langle \delta_-, \varphi\rangle = \int_{-\infty}^0 \varphi\left(-\frac{y}{\tan\gamma}, y\right)\frac{\mathrm{d}y}{\sin\gamma} = \int_0^{+\infty} \varphi(s\cos\gamma, -s\sin\gamma)\,\mathrm{d}s,$$

$$\begin{aligned}\langle \delta'_-, \varphi\rangle &= -\langle \delta_-, \nabla\varphi\cdot n_-\rangle = -\int_{-\infty}^0 \varphi\left(-\frac{y}{\tan\gamma}, y\right)\frac{\mathrm{d}y}{\sin\gamma}\\ &= -\int_0^{+\infty} \nabla\varphi\cdot n_-(s\cos\gamma, -s\sin\gamma)\,\mathrm{d}s.\end{aligned} \tag{14}$$

Lemma 2. *Let W be the solution of problem* (10). *Then we have*

$$W = -(\Delta + k^2)^{-1}(f_+ \otimes \delta_{\Gamma_+} + f_- \otimes \delta_{\Gamma_-} + h_+ \otimes \delta'_{\Gamma_+} + h_- \otimes \delta'_{\Gamma_-}), \tag{15}$$

with

$$\begin{aligned}h_+(s) &= W(s, 0),\\ h_-(s) &= W(s\cos\gamma, -s\sin\gamma),\\ f_+(s) &= (\partial_{n_{\tilde{\Omega}}} W)_+(s) = -\partial_y u(s, 0),\\ f_-(s) &= (\partial_{n_{\tilde{\Omega}}} W)_-(s) = \nabla W\cdot n_{\Gamma_-}(s\cos\gamma, -s\sin\gamma).\end{aligned} \tag{16}$$

Proof. If W is the solution of problem (10), the distribution $S = W + (\Delta + k^2)^{-1}(\partial_{n_{\tilde{\Omega}}} W \otimes \delta_\Gamma + W \otimes \delta'_\Gamma)$ which exists, see [7] is a solution of the homogeneous Helmholtz equation in $\mathbb{R}^2$. Therefore, it is a sum of Hankel and Bessel functions, and its singularity on the corner is at most given by the Hankel function, which is equivalent on the corner to

$(1/2\pi)\ln(1/\sqrt{x^2+y^2})$, and $\nabla \ln r \notin L^2$ at the corner. By the jump formula, S is supported on the corner $(0,0)$, and thus is a Dirac distribution. Assuming that $W|_\Gamma$ is in $H^1(\Gamma)$ and $\partial_{n_{\tilde{\Omega}}} W$ is in $L^2(\Gamma)$, $(\Delta+k^2)^{-1}(\partial_{n_{\tilde{\Omega}}} W \otimes \delta_\Gamma + W \otimes \delta'_\Gamma) \in$ is in $H^{(1/2)-\varepsilon}(\mathbb{R}^2)$, hence necessarily $S=0$. $\square$

We start with a representation of W. The following calculation has been performed for a curved corner in [11]. Define the function

$$g(\xi,\eta) = \xi\cos\gamma - \eta\sin\gamma. \tag{17}$$

For $\xi \in \mathbb{R}$, denote by ξ_0 the principal part of the square root of $k^2-\xi^2$ such that $\mathscr{I}m\,\xi_0 > 0$. For h in $L^2(\mathbb{R}_+)$, we define by extension the Fourier transform

$$\hat{h}(\xi) = \int_0^{+\infty} h(s)\mathrm{e}^{-\mathrm{i}s\xi}\,\mathrm{d}\xi. \tag{18}$$

Theorem 3. *We use the notation of Lemma* 2. *Let W be the solution of problem* (10). *Then, for $|y|>0$ and $|Y|>0$, the equality $W = W_+ + W_-$ holds, with*

$$W_+(x,y) = \frac{\mathrm{i}}{4\pi}\int_{\mathbb{R}} \frac{1}{\xi_0}\left(\hat{f}_+(\xi) - \mathrm{i}\xi_0\frac{y}{|y|}\hat{g}_+(\xi)\right)\mathrm{e}^{\mathrm{i}x\xi+\mathrm{i}|y|\xi_0}\,\mathrm{d}\xi,$$

$$W_-(x,y) = \frac{\mathrm{i}}{4\pi}\int_{\mathbb{R}} \frac{1}{\xi_0}\left(\hat{f}_-(\xi) + \mathrm{i}\xi_0\frac{Y}{|Y|}\hat{g}_-(\xi)\right)\mathrm{e}^{\mathrm{i}X\xi+\mathrm{i}|Y|\xi_0}\,\mathrm{d}\xi, \tag{19}$$

where $X = x\cos\gamma - y\sin\gamma$, $Y = x\sin\gamma + y\cos\gamma$.

Remark 4. The above integrals are defined as absolutely convergent integrals for $\mathrm{Im}\,k<0$ and non vanishing y and Y, whereas for $y=0$ or for $Y=0$, they are defined only as oscillatory integrals.

Proof. Define the fundamental solution of the Helmholtz equation as $E = (\Delta+k^2)^{-1}\delta$. Then we have $W = W_+ + W_-$, with $W_+ = -E * (f_+\otimes\delta_{\Gamma_+} + h_+\otimes\delta'_{\Gamma_+})$ and $W_- = -E * (f_-\otimes\delta_{\Gamma_-} + h_-\otimes\delta'_{\Gamma_-})$. Denoting by $E_{x,y}$ the associated Green's function, i.e., $E_{x,y}(s,u) = E(x-s, y-u)$, we have

$$W_+(x,y) = -\int_0^{+\infty} f_+(s)E_{x,y}(s,0)\,\mathrm{d}s - \int_0^{+\infty} h_+(s)\partial_u E_{x,y}(s,0)\,\mathrm{d}s.$$

The Fourier transform of E being $\hat{E}(\xi,\eta) = (k^2-\xi^2-\eta^2)^{-1}$, we obtain

$$W_+(x,y) = -\frac{1}{(2\pi)^2}\int_0^{+\infty}\int_{\mathbb{R}^2} \frac{f_+(s) - \mathrm{i}\eta h_+(s)}{k^2-\xi^2-\eta^2}\mathrm{e}^{\mathrm{i}(x-s)\xi+\mathrm{i}\eta y}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}s.$$

As $\mathscr{I}m\,k^2 \neq 0$, we may use the Cauchy formula for the integral in the η variable, using the pole η_0 which verifies $\mathscr{R}e\,\mathrm{i}\eta_0 y < 0$, i.e., $\eta_0 = (y/|y|)\xi_0$:

$$W_+(x,y) = -\frac{\mathrm{i}}{(2\pi)}\int_0^{+\infty}\int_{\mathbb{R}} \frac{f_+(s) - \mathrm{i}\xi_0(y/|y|)h_+(s)}{-2\xi_0}\mathrm{e}^{\mathrm{i}(x-s)\xi+\mathrm{i}\xi_0|y|}\,\mathrm{d}\xi\,\mathrm{d}s.$$

We finally use the Fourier transforms of f_+ and h_+ to obtain the formula for W_+ in Theorem 3. For the second formula, we write in the same fashion

$$W_-(x,y) = -\frac{1}{(2\pi)^2}\int_0^{+\infty}\int_{\mathbb{R}^2} \frac{f_-(s) + \mathrm{i}(\xi\sin\gamma + \eta\cos\gamma)h_-(s)}{k^2-\xi^2-\eta^2}\mathrm{e}^{\mathrm{i}(x-s\cos\gamma)\xi+\mathrm{i}\eta(y+t\sin\gamma)}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}s.$$

In the double integral, we perform the change of variables

$$\tilde{\tilde{\xi}} = \xi\cos\gamma - \eta\sin\gamma, \quad \tilde{\eta} = \xi\sin\gamma + \eta\cos\gamma.$$

With the notation in Theorem 3, we get

$$W_-(x,y) = -\frac{1}{(2\pi)^2}\int_0^{+\infty}\int_{\mathbb{R}^2}\frac{f_-(s)+\mathrm{i}\tilde{\eta}h_-(s)}{k^2-\tilde{\tilde{\xi}}^2-\tilde{\eta}^2}\,\mathrm{e}^{\mathrm{i}(X-s)\tilde{\tilde{\xi}}-\mathrm{i}\tilde{\eta}Y}\,\mathrm{d}\tilde{\tilde{\xi}}\,\mathrm{d}\tilde{\eta}\,\mathrm{d}s.$$

Applying again the Cauchy formula, we have

$$W_-(x,y) = -\frac{\mathrm{i}}{(2\pi)}\int_0^{+\infty}\int_{\mathbb{R}}\frac{f_-(s)+\mathrm{i}\xi_0(y/|y|)h_-(s)}{-2\xi_0}\,\mathrm{e}^{\mathrm{i}(X-s)\xi+\mathrm{i}\xi_0|Y|}\,\mathrm{d}\xi\,\mathrm{d}s,$$

where we replaced $\tilde{\tilde{\xi}}$ by ξ, which leads to expression (19). □

4. Computation of the Dirichlet to Neumann map for the Helmholtz equation in a wedge

We define the operators S, R_0, R_1, T_0, T_1 for $h \in L^2_{\mathrm{comp}}(\mathbb{R}_+)$ (compactly supported functions of $L^2(\mathbb{R}_+)$) by

$$Sh(s) = \frac{1}{2\pi}1_{s\geqslant 0}\int_{\mathbb{R}}\frac{g(\xi_0,\xi)}{\xi_0}\,\mathrm{e}^{\mathrm{i}sg(\xi,-\xi_0)}\hat{h}(\xi)\,\mathrm{d}\xi,$$

$$R_0h(s) = \frac{\mathrm{i}k^2}{2\pi}1_{s>0}\int_{\mathbb{R}}\frac{1}{\xi_0}\,\mathrm{e}^{\mathrm{i}s\xi}\hat{h}(\xi)\,\mathrm{d}\xi,$$

$$R_1h(s) = -\frac{1}{2\pi}1_{s>0}\int_{\mathbb{R}}\frac{\xi}{\xi_0}\,\mathrm{e}^{\mathrm{i}s\xi}\hat{h}(\xi)\,\mathrm{d}\xi,$$

$$T_0h(s) = \frac{\mathrm{i}k^2\cos\gamma}{2\pi}1_{s>0}\int_{\mathbb{R}}\frac{1}{\xi_0}\,\mathrm{e}^{\mathrm{i}sg(\xi,-\xi_0)}\hat{h}(\xi)\,\mathrm{d}\xi,$$

$$T_1h(s) = -\frac{1}{2\pi}1_{s>0}\int_{\mathbb{R}}\frac{g(\xi,-\xi_0)}{\xi_0}\,\mathrm{e}^{\mathrm{i}sg(\xi,\xi_0)}\hat{h}(\xi)\,\mathrm{d}\xi. \tag{20}$$

These operators map $L^2_{\mathrm{comp}}(\mathbb{R}_+)$ into itself. This is a consequence of $\mathscr{I}m\,g(\xi,-\xi_0)>0$ and of the boundedness of ξ/ξ_0 and $g(\xi,-\xi_0)/\xi_0$ for $\mathscr{I}m\,k^2\neq 0$.

Remark 5. The functions we consider have to be compactly supported because a L^∞ kernel (as it is the case of ξ/ξ_0) behaves properly when acting on a Fourier series. Whenever one chooses a time T, the finite speed of propagation of waves imposes that the solution of (21) is compactly supported, hence its traces are also compactly supported.

Theorem 6. *The Dirichlet to Neumann operator $N_{\tilde{\Omega}}$ for the wedge in Fig. 1 is the operator from $H^1(\Gamma)$ to $L^2(\Gamma)$ defined by*

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \begin{pmatrix} I & -S \\ -S & I \end{pmatrix}^{-1}\mathscr{M}\begin{pmatrix} h_+ \\ h_- \end{pmatrix}, \tag{21}$$

with $\mathscr{T}h=(h_+,h_-)$, and $\mathscr{T}N_{\tilde{\Omega}}h=(f_+,f_-)$. The operators in (21) are given by $\mathscr{M}=\begin{pmatrix} R & -T \\ -T & R\end{pmatrix}$ and for $h\in L^2(\mathbb{R}_+)$, $Rh=R_0h+R_1h'$, $Th=T_0h+T_1h'$.

Proof. We compute the normal derivatives of $W_\pm$ on $\Gamma_\pm$, using Theorem 3. Note that in a neighborhood of Γ_+ in $\tilde{\Omega}$, we have $y\geqslant 0$ and $Y\geqslant 0$, whereas in a neighborhood of Γ_- in $\tilde{\Omega}$, we have $y\leqslant 0$ and $Y\leqslant 0$. We start with Γ_+, and differentiate W_+ and W_- with respect to y.

$$\begin{aligned}\vec{n}_+.\nabla W_+(x,y) &= -\partial_y W_+(x,y)\\ &= \frac{1}{4\pi}\int_{\mathbb{R}} (\hat{f}_+(\xi) - \mathrm{i}\xi_0 \hat{h}_+(\xi))\mathrm{e}^{\mathrm{i}x\xi+\mathrm{i}y\xi_0}\,\mathrm{d}\xi,\\ \vec{n}_+.\nabla W_-(x,y) &= \frac{1}{4\pi}\int_{\mathbb{R}} \frac{g(\xi_0,\xi)}{\xi_0}(\hat{f}_-(\xi) + \mathrm{i}\xi_0 \hat{h}_-(\xi))\mathrm{e}^{\mathrm{i}X\xi+\mathrm{i}Y\xi_0}\,\mathrm{d}\xi.\end{aligned} \tag{22}$$

Since the Fourier transform of an element h_+ in $H^1(\mathbb{R}_+)$ is merely in $L^2(\mathbb{R})$ in general and in particular does not verify $(1+\xi^2)^{1/2}\hat{h}_+ \in L^2(\mathbb{R})$ (more regularity occurs in the case $h_+(0)=0$), the distribution $\xi_0\hat{h}_+(\xi)$ is not in $L^2(\mathbb{R})$, which is the fundamental difference with the regular boundary case, and makes the problem significantly harder. We introduce the operators

$$\begin{aligned}\tilde{R}h &= \lim_{y\to 0+} \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{i}\xi_0\hat{h}(\xi)\mathrm{e}^{\mathrm{i}x\xi+\mathrm{i}y\xi_0}\,\mathrm{d}\xi,\\ \tilde{T}h &= \lim_{y\to 0_+} \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\xi_0}\,\mathrm{i}g(\xi_0,\xi)\hat{h}(\xi)\mathrm{e}^{\mathrm{i}X\xi+\mathrm{i}Y\xi_0}\,\mathrm{d}\xi. \qquad \square\end{aligned} \tag{23}$$

Lemma 7. *For h in $\mathscr{S}(\mathbb{R})$, and $h_+ = h1_{x\geqslant 0}$, $h'_+ = h'1_{x\geqslant 0}$, the operators $\tilde{R}$ and $\tilde{T}$ introduced in* (23) *are well defined and equal to*

$$\begin{aligned}\tilde{R}h_+(x) &= R_0h_+(x) + R_1h'_+(x) - h(0)\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\xi}{\xi_0}\,\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi,\\ \tilde{T}h_+(x) &= T_0h_+(x) + T_1h'_+(x) - h(0)\frac{1}{2\pi}\int_{\mathbb{R}}\frac{g(\xi,-\xi_0)}{\xi_0}\mathrm{e}^{\mathrm{i}sg(\xi,-\xi_0)}\,\mathrm{d}\xi.\end{aligned} \tag{24}$$

Proof. For h in $\mathscr{S}(\mathbb{R})$, since the kernels of the integrals are in $L^2(\mathbb{R})$, we can take limits in (23) as $y \to 0_+$, which proves the existence of $\tilde{R}$ and $\tilde{T}$ as maps from $\mathscr{S}(\mathbb{R})$ into $\mathscr{S}'(\mathbb{R})$. We now transform the right-hand side in the definition of $\tilde{R}h_+$. We first use the relation $\mathrm{i}\xi_0 = \mathrm{i}(k^2/\xi_0) - (\xi/\xi_0)\mathrm{i}\xi$ to obtain

$$\mathrm{i}\xi_0\hat{h}_+(\xi) = \mathrm{i}\,\frac{k^2}{\xi_0}\,\hat{h}_+(\xi) - \frac{\xi}{\xi_0}(\mathrm{i}\xi\hat{h}_+(\xi)),$$

and next we use the identity (in which $\hat{h}'_\pm$ denotes the Fourier transform of the restriction to $\mathbb{R}_+$ of the derivative $h'_\pm$)

$$\mathrm{i}\xi\hat{h}_\pm(\xi) = \hat{h}'_\pm(\xi) + h_\pm(0). \tag{25}$$

We then split the integral

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{i}\xi_0\hat{h}_+(\xi)\mathrm{e}^{\mathrm{i}x\xi+\mathrm{i}y\xi_0}\,\mathrm{d}\xi = &-\frac{1}{4\pi}h(0)\int_{\mathbb{R}}\frac{\xi}{\xi_0}\,\mathrm{e}^{\mathrm{i}x\xi+\mathrm{i}y\xi_0}\,\mathrm{d}\xi\\ &+\frac{1}{4\pi}\int_{\mathbb{R}}\left(\mathrm{i}\,\frac{k^2}{\xi_0}\,\hat{h}_+(\xi) - \frac{\xi}{\xi_0}\,\hat{h}'_+(\xi)\right)\mathrm{e}^{\mathrm{i}x\xi+\mathrm{i}y\xi_0}\,\mathrm{d}\xi.\end{aligned} \tag{26}$$

By the Lebesgue's dominated convergence Theorem in $L^1(\mathbb{R})$, we can take the limit of each term on the right-hand side, with the particular case of the function $\int_{\mathbb{R}}(\xi/\xi_0)\,\mathrm{e}^{\mathrm{i}x\xi+\mathrm{i}y\xi_0}\,\mathrm{d}\xi$, whose limit is the derivative of the inverse Fourier transform of $1/\xi_0$ which is a distribution in $L^2(\mathbb{R})$. With the operators defined in (20), we obtain the first equation in (24).

We treat $\tilde{T}$ the same way, using (25) and the identity

$$g(\xi_0,\xi) = \cos\gamma\xi_0 - \sin\gamma\xi = \cos\gamma\,\frac{k^2}{\xi_0} - \xi\,\frac{g(\xi,-\xi_0)}{\xi_0},$$

and we obtain

$$\tilde{T}h_+(s) = T_0 h_+(s) + T_1 h'_+(s) - h(0)\frac{1}{2\pi}\int_{\mathbb{R}} \frac{g(\xi,-\xi_0)}{\xi_0}\,\mathrm{e}^{\mathrm{i}sg(\xi,-\xi_0)}\,\mathrm{d}\xi. \tag{27}$$

Now for $\gamma \neq \pi/2$, we have by a contour transform

$$\int_{\mathbb{R}} \frac{g(\xi,-\xi_0)}{\xi_0}\,\mathrm{e}^{\mathrm{i}sg(\xi,-\xi_0)}\,\mathrm{d}\xi = \int_{\mathbb{R}} \frac{\xi}{\xi_0}\,\mathrm{e}^{\mathrm{i}s\xi}\,\mathrm{d}\xi, \tag{28}$$

which proves the second equality in (24), and concludes the proof of the lemma. $\square$

Taking the limit in (22) as y tends to 0, we obtain

$$\begin{aligned} f_+(x) &= \frac{1}{2} f_+(x) + \frac{1}{4\pi}\int_{\mathbb{R}} \frac{g(\xi_0,\xi)}{\xi_0}\,\hat{f}_-(\xi)\mathrm{e}^{\mathrm{i}xg(\xi,-\xi_0)}\,\mathrm{d}\xi \\ &\quad - \frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{i}\xi_0 \hat{h}_+(\xi)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi + \frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{i}g(\xi_0,\xi)\hat{h}_-(\xi)\mathrm{e}^{\mathrm{i}xg(\xi,-\xi_0)}\,\mathrm{d}\xi, \end{aligned}$$

which can be rewritten as

$$f_+ - Sf_- = -\tilde{R}h_+ + \tilde{T}h_-.$$

We can perform the same calculation on Γ_-, using the symmetry of the problem, and write

$$\begin{aligned} f_-(x) &= \frac{1}{2} f_-(x) + \frac{1}{4\pi}\int_{\mathbb{R}} \frac{g(\xi_0,\xi)}{\xi_0}\,\hat{f}_+(\xi)\mathrm{e}^{\mathrm{i}xg(\xi,-\xi_0)}\,\mathrm{d}\xi \\ &\quad - \frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{i}\xi_0 \hat{h}_-(\xi)\mathrm{e}^{\mathrm{i}x\xi}\,\mathrm{d}\xi + \frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{i}g(\xi_0,\xi)\hat{h}_+(\xi)\mathrm{e}^{\mathrm{i}xg(\xi,-\xi_0)}\,\mathrm{d}\xi, \end{aligned}$$

and we obtain altogether

$$f_+ - Sf_- = -\tilde{R}h_+ + \tilde{T}h_-, \quad f_- - Sf_+ = -\tilde{R}h_- + \tilde{T}h_+. \tag{29}$$

We can now express the right-hand side in (29) in the form

$$\begin{aligned} -\tilde{R}h_+ + \tilde{T}h_- &= -Rh_+ + Th_- - (h_+(0) - h_-(0))\int_{\mathbb{R}} \frac{\xi}{\xi_0}\,\mathrm{e}^{\mathrm{i}s\xi}\,\mathrm{d}\xi, \\ -\tilde{R}h_- + \tilde{T}h_+ &= -Rh_- + Th_+ + (h_+(0) - h_-(0))\int_{\mathbb{R}} \frac{\xi}{\xi_0}\,\mathrm{e}^{\mathrm{i}s\xi}\,\mathrm{d}\xi. \end{aligned}$$

Since h is in $H^1(\Gamma)$, it is continuous at the corner, and $h_+(0) - h_-(0) = 0$. We finally have

$$f_+ - Sf_- = -Rh_+ + Th_-, \quad f_- - Sf_+ = -Rh_- + Th_+, \tag{30}$$

which is formally (21). The last step of the proof is to use a Neumann expansion, using the following property of the singular operator S: it maps a function whose Fourier transform is holomorphic in the half-plane $\mathscr{I}m\,\xi < 0$ to a function whose Fourier transform is holomorphic in the half-plane $\mathscr{I}m(\xi\mathrm{e}^{-\mathrm{i}\gamma\theta}) < 0$. The Neumann expansion is then computed until $n\gamma > \pi$ (see [6]). This gives a sense to the operator $\begin{pmatrix} I & -S \\ -S & I \end{pmatrix}^{-1}$ from $(L^2(\mathbb{R}_+))^2$ to $(L^2(\mathbb{R}_+))^2$, hence the result in the theorem.

Remark 8. In the particular case $\gamma = \pi$ we recover the usual Dirichlet to Neumann map for a straight boundary, using the splitting of a function in $H^1(\mathbb{R})$ into its part on $\mathbb{R}_+$ and its part on $\mathbb{R}_-$, with $g(\xi,\xi_0) = -\xi$ in the phase function (taking into account the symmetry).

5. Another formulation of the Dirichlet to Neumann map

In this section we use the following notation:

- The operator K is given by

$$Kf(s) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{f}}{\xi_0 - k}\,\mathrm{e}^{\mathrm{i}s\xi}\,\mathrm{d}\xi, \quad \left(\mathscr{F}(Kf) = \frac{\hat{f}}{\xi_0 - k}\right).$$

- The distributions $D_{R,0}$, $D_{T,0}$, $D_{R,1}$, $D_{T,1}$ are defined by

$$D_{R,1} = \frac{\mathrm{i}}{2\pi}\int \frac{\mathrm{e}^{\mathrm{i}s\xi}}{\xi_0 - k}\,\mathrm{d}\xi, \quad D_{T,1} = \int_{\mathbb{R}} \frac{g(\xi_0,\xi)}{\xi_0(\xi_0 - k)}\,\mathrm{e}^{\mathrm{i}sg(\xi,-\xi_0)}\,\mathrm{d}\xi,$$

$$D_{R,0} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{k\xi}{\xi_0(\xi_0 - k)}\,\mathrm{e}^{\mathrm{i}s\xi}\,\mathrm{d}\xi, \quad D_{T,0} = \frac{\sin\gamma}{2\pi}\int_{\mathbb{R}} \frac{k}{\xi_0}\,\mathrm{e}^{\mathrm{i}s\xi}\,\mathrm{d}\xi + \frac{\cos\gamma}{2\pi}\int_{\mathbb{R}} \frac{k\xi}{\xi_0(\xi_0 - k)}\,\mathrm{e}^{\mathrm{i}sg(\xi,-\xi_0)}\,\mathrm{d}\xi.$$

We present now another formulation of the Dirichlet to Neumann operator, which is useful to make the link with the case of the half space.

Theorem 9. *The DtN operator given in Theorem* 6 *can be written in the form*

$$\begin{pmatrix} I & -S \\ -S & I \end{pmatrix}^{-1} \mathscr{M}\begin{pmatrix} h_+ \\ h_- \end{pmatrix} = -\mathrm{i}k\begin{pmatrix} h_+ \\ h_- \end{pmatrix} + \mathrm{i}\begin{pmatrix} K(h_+'') \\ K(h_-'') \end{pmatrix} + \mathscr{R}\begin{pmatrix} h_+ \\ h_- \end{pmatrix}, \tag{31}$$

where

$$\begin{pmatrix} I & -S \\ -S & I \end{pmatrix}\mathscr{R}\begin{pmatrix} h_+ \\ h_- \end{pmatrix} = h(0)\begin{pmatrix} D_{R,0} & D_{T,0} \\ D_{T,0} & D_{R,0} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} D_{R,1} & D_{T,1} \\ D_{T,1} & D_{R,1} \end{pmatrix}\begin{pmatrix} h_+'(0) \\ h_-'(0) \end{pmatrix}.$$

Proof. This is a consequence of the identity

$$\xi_0 = \xi_0 + k - k = \frac{-\xi^2}{\xi_0 - k} - k$$

and of relation (25) applied to the functions $h_\pm$ and their derivatives. We obtain

$$\xi_0\hat{h}_+(\xi) = -k\hat{h}_+(\xi) + \frac{1}{\xi_0 - k}(\hat{h}_+'' + h_+'(0) + \mathrm{i}\xi h_+(0)),$$

$$\frac{g(\xi_0,\xi)}{\xi_0}(\mathrm{i}\xi_0\hat{h}_-(\xi)) = -k\frac{g(\xi_0,\xi)}{\xi_0}\hat{h}_-(\xi) + \frac{g(\xi_0,\xi)}{\xi_0}\frac{\hat{h}_-''(\xi) + h_-'(0) + \mathrm{i}\xi h_-(0)}{\xi_0 - k},$$

which can be rewritten in the form

$$\xi_0\hat{h}_+(\xi) = \mathscr{F}(-kh_+ + Kh_+'') + \frac{h_+'(0) + \mathrm{i}\xi h_+(0)}{\xi_0 - k},$$

$$\frac{g(\xi_0,\xi)}{\xi_0}(\mathrm{i}\xi_0\hat{h}_-(\xi)) = \frac{g(\xi_0,\xi)}{\xi_0}(\mathscr{F}(-kh_- + Kh_-'')) + \frac{g(\xi_0,\xi)}{\xi_0(\xi_0 - k)}(h_-'(0) + \mathrm{i}\xi h_-(0)).$$

Hence the result follows, using

$$\frac{1}{2\pi}\int \frac{g(\xi_0,\xi)}{\xi_0}\mathscr{F}(-kh_- + K(h_-''))\mathrm{e}^{\mathrm{i}sg(\xi,-\xi_0)}\,\mathrm{d}\xi = S(-kh_- + K(h_-'')). \qquad \square$$

Remark 10. The usual first order absorbing boundary condition can be recovered from (31) by replacing K and $D_{R,j}$ by 0 for $j=0, 1$. Moreover, the second order absorbing boundary condition (4) can be derived by using the approximation

$\xi_0 \simeq -k$ for $\xi \simeq 0$ in the symbol of K, leading to the approximation of K given by $Kf \simeq -(1/2k)f$. On the other hand, numerical computations using the full operator K seem tractable.

For the first order approximation of the DtN in the wedge, we have a well-posedness result:

Proposition 11. *Assume the operator $\mathscr{B}$ in* (2) *to be $\partial_t + \partial_n$ on each face of the wedge. Then problem* (2) *has a unique solution u in $\mathscr{C}([0,T], H^s(\mathbb{R}^2)) \cap H^1(]0,T[, H^{s-1}(\mathbb{R}^2))$ with $s > 3/2$. This solution satisfies $u|_\Gamma \in H^1(\Gamma)$.*

The proof of this proposition is given in [11] as a particular case of the impedance boundary condition with the impedance coefficient equal to 1 on each face after Fourier transform in time.

6. Conclusion

We have given here an explicit form of the Dirichlet to Neumann map, regardless of the regularity of the solution and the angle at the corner, except that the trace of u on the whole boundary Γ be in $H^1(\Gamma)$. We derived, for $u|_\Gamma$ in $H^2(\Gamma)$, an approximate boundary condition

$$\mathscr{B} = \partial_{tn} + \partial^2_{t^2} - \frac{1}{2}\partial^2_{\tau^2} \tag{32}$$

(where ∂_τ is the tangential derivative along the boundary defined in $H^2(\Gamma)$). For further use, we propose, for any angle, to use either the exact Dirichlet to Neumann map in the form (31), or the boundary operator (32) with the regularity condition $u|_\Gamma \in H^2(\Gamma)$ as "corner condition" (it is equivalent to the continuity of $\nabla u|_\Gamma$ at the corner). The well-posedness for the problem with second order approximate boundary conditions is still work in progress.

References

[1] B. Alpert, L. Greengard, T. Hagstrom, Rapid evaluation of nonreflecting boundary kernels for time-domain wave propagation, SIAM J. Numer. Anal. 37 (4) (2000) 1138–1164.

[2] A. Bamberger, P. Joly, J.E. Roberts, Second order absorbing boundary conditions for the wave equation: a solution for the corner problem, INRIA Report RR-0644, 1987.

[3] J.P. Bérenger, Three-dimensional perfectly matched layer for the absorption of electromagnetic waves, J.C.P. 114 (1994) 185–200.

[4] M. Costabel, M. Dauge, General edge asymptotics of solutions of second order elliptic boundary value problems I and II, Proc. Edinburgh Math. Soc. 123A(2), 109–155, 157–184.

[5] B. Engquist, A. Majda, Absorbing boundary conditions for the numerical simulation of waves, Math. Comp. 31 (139) (1977) 629–651.

[6] P. Gérard, G. Lebeau, Diffusion d'une onde par un coin, J. Amer. Math. Soc. 6 (2) (1993).

[7] P. Grisvard, Elliptic problems in nonsmooth domains, Monographs and Studies in Mathematics, vol. 24, Pitman, New York, 1985.

[8] T. Ha Duong, P. Joly, On the stability analysis of boundary conditions for the wave equation by energy methods. I. The homogeneous case, Math. Comp. 62 (206) (1994) 539–563.

[9] L. Halpern, J. Rauch, Absorbing boundary conditions for diffusion equations, Numer. Math. 71 (1995) 185–224.

[10] P. Joly, S. Lohrengel, O. Vacus, Un résultat d'existence et d'unicité pour l'équation de Helmholtz avec conditions aux limites absorbantes d'ordre 2, C. R. Acad. Sci. Paris Sér. 1 Math. 329 (3) (1999) 193–198.

[11] O. Lafitte, The wave diffracted by a wedge with mixed boundary condition, Mémoires de la SMF, 88, Nelle Série (2002).

[12] L.N. Trefethen, L. Halpern, Well-posedness of one-way wave equations and absorbing boundary conditions, Math. Comp. 47 (176) (1986) 421–435.