# OPTIMAL SCHWARZ WAVEFORM RELAXATION FOR THE ONE DIMENSIONAL WAVE EQUATION

MARTIN J. GANDER*, LAURENCE HALPERN†, AND FRÉDÉRIC NATAF‡

**Abstract.** We introduce a non-overlapping variant of the Schwarz waveform relaxation algorithm for wave propagation problems with variable coefficients in one spatial dimension. We derive transmission conditions which lead to convergence of the algorithm in a number of iterations equal to the number of subdomains, independently of the length of the time interval. These optimal transmission conditions are in general non local, but we show that the non-locality depends on the time interval under consideration and we introduce time windows to obtain optimal performance of the algorithm with local transmission conditions in the case of piecewise constant wave speed. We show that convergence in two iterations can be achieved independently of the number of subdomains in that case. The algorithm thus scales optimally with the number of subdomains, provided the time windows are chosen appropriately. For continuously varying coefficients we prove convergence of the algorithm with local transmission conditions using energy estimates. We then introduce a finite volume discretization which permits computations on non matching grids and we prove convergence of the fully discrete Schwarz waveform relaxation algorithm. We finally illustrate our analysis with numerical experiments.

**Key words.** Domain Decomposition, Waveform Relaxation, Schwarz Methods

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*Dept. of Mathematics and Statistics, McGill University, Montreal, Canada. mgander@math.mcgill.ca

†Département de Mathématiques, Université Paris XIII, 93430 Villetaneuse, and CMAP, Ecole Polytechnique, 91128 Palaiseau, France. halpern@math.univ-paris13.fr

‡CMAP, CNRS UMR 7641, Ecole Polytechnique, 91128 Palaiseau, France. nataf@cmap.polytechnique.fr

**1. Introduction.** Domain decomposition methods have been mainly developed and analyzed for elliptic coercive problems and their convergence theory is well understood, see [41, 8, 37, 36] and references therein. When treating evolution problems, the classical approach consists of discretizing the time dimension uniformly on the whole domain by an implicit scheme and then treating the obtained problems at each time step by a classical domain decomposition method for steady problems. For parabolic problems see for example [6, 29, 7], and for hyperbolic problems [2, 40].

This approach has two disadvantages. First one needs to impose a uniform time discretization for all subdomains and thus looses one of the main features of domain decomposition algorithms, namely to adapt the solution process to the physical properties of the subdomain. It is still possible to refine in space, but for evolution problems this is not sufficient, since the space and time discretization are linked in general by stability constraints and conditions on the dispersion of the numerical scheme. Second the algorithm needs to communicate small amounts of information at each time step. Each communication involves in addition to the cost for the data transmitted a startup cost independent of the amount of data transmitted. It can thus be of interest to communicate larger packages of data at once over several time steps instead of many small packages to save communication time. This factor can become important if the algorithm runs on an existing network of workstations without special high performance links.

To avoid the above disadvantages, we propose in this paper an approach different from the classical one. We decompose the original domain into subdomains like in the classical case, but we do not discretize the time dimension. Instead we solve time dependent subproblems on each subdomain. This approach is related to waveform relaxation algorithms for ordinary differential equations and has first been considered for partial differential equations by Bjørhus in [5, 4] where first order hyperbolic problems were analyzed, in which case only incoming characteristic information can be imposed on subdomain interfaces. An overlapping Schwarz algorithm of this type has been analyzed for the heat equation in [13, 19, 18], and for more general parabolic problems in [20, 14], which led to a new understanding of the performance of the waveform relaxation algorithm when applied to parabolic partial differential equations; in particular a new and faster asymptotic convergence rate is obtained with overlapping subdomain splitting compared to the classical waveform relaxation rate for Jacobi splittings. For overlapping and non-overlapping Schwarz waveform relaxation methods for the wave equation and convection reaction diffusion equation see [15].

We are focusing in this paper on wave propagation phenomena in the presence of variable and discontinuous coefficients. We first perform an analysis at the continuous level and derive transmission conditions for non overlapping Schwarz waveform relaxation algorithms which lead to optimal convergence. The optimal transmission conditions involve linear operators $\mathcal{S}_j$ related to the Dirichlet to Neumann maps at the artificial interfaces. For elliptic problems, results of this type have been studied in [9, 33, 32, 12, 17]. These optimal transmission conditions are nonlocal in general, but we show that the non-locality depends on the time interval under consideration in the wave equation case. We introduce then time windows to obtain optimal performance of the algorithm with local transmission conditions for piecewise constant wave speed. We show that convergence in two iterations can be achieved independently of the number of subdomains in that case. The algorithm thus scales optimally with the number of subdomains, without any additional mechanism like a coarse grid. For

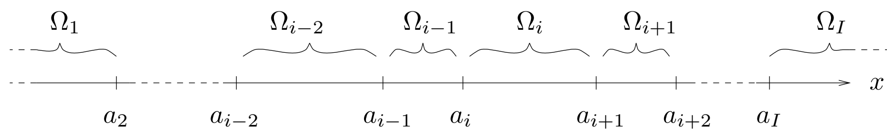


FIG. 2.1. *Domain decomposition into $I$ non-overlapping subdomains.*

continuously varying coefficients we prove convergence of the algorithm with local transmission conditions using energy estimates. We then introduce a finite volume discretization and analyze the fully discrete Schwarz waveform relaxation algorithm. This algorithm allows us to use non-matching grids both in space and time on different subdomains, so that the resolution can be adapted to the underlying physical properties of the problem. For piecewise constant wave speed we analyze the convergence of the algorithm using discrete Laplace transforms, a tool introduced for the continuous analysis of waveform relaxation algorithms by Miekkala and Nevanlinna in [30] and later used by Nevanlinna in [34, 35]. For an analysis of waveform relaxation algorithms discretized in time, see also Janssen and Vandewalle [22] and references therein. For continuously varying wave speed we prove stability of the subdomain problems and convergence on non-matching grids using energy estimates. Our approach is an alternative to the mortar method for non-matching grids [3], see also [1]. We finally illustrate the analysis with numerical experiments for model problems and a simulation for a typical underwater sound speed profile from an application. For a different approach of a space time decomposition for evolution problems using virtual controls see [27] and for other ways of grid refinement in space and time see [2, 10, 25].

**2. The Optimal Schwarz Waveform Relaxation Algorithm.** We consider the second order wave equation with variable wave speed in one dimension,

$$\mathcal{L}(u) = \frac{1}{c^2(x)}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f \tag{2.1}$$

on the domain $\mathbb{R} \times (0,T)$ with initial conditions

$$u(x,0) = p(x), \quad \frac{\partial u}{\partial t}(x,0) = q(x).$$

For $0 < \underline{c} \leq c(x) \leq \overline{c} < \infty$ there exists a unique weak solution $u$ of (2.1) on any bounded time interval $t \in [0,T]$, see [26].

**2.1. A General Non-Overlapping Schwarz Waveform Relaxation Algorithm.** We decompose the domain $\mathbb{R}$ into $I$ non overlapping subdomains $\Omega_i = (a_i, a_{i+1})$, $a_j < a_i$ for $j < i$ and $a_1 = -\infty$, $a_{I+1} = \infty$ as given in Figure 2.1. We introduce a general non overlapping Schwarz waveform relaxation algorithm

$$\begin{array}{rcll} \mathcal{L}(u_i^{k+1}) & = & f & \text{in } \Omega_i \times (0,T) \\ \mathcal{B}_i^-(u_i^{k+1})(a_i,t) & = & \mathcal{B}_i^-(u_{i-1}^k)(a_i,t) & t \in (0,T) \\ \mathcal{B}_i^+(u_i^{k+1})(a_{i+1},t) & = & \mathcal{B}_i^+(u_{i+1}^k)(a_{i+1},t) & t \in (0,T) \\ u_i^{k+1}(x,0) & = & p(x) & x \in \Omega_i \\ \frac{\partial u_i^{k+1}}{\partial t}(x,0) & = & q(x) & x \in \Omega_i \end{array} \tag{2.2}$$

where $\mathcal{B}_i^{\pm}$ are linear transmission operators which we will determine to get optimal performance of the algorithm. For ease of notation we defined here

$$u_0^k := 0, \quad u_{I+1}^k := 0 \tag{2.3}$$

so that the index $i$ in (2.2) ranges from $i = 1, 2, \dots, I$. Note that we call this algorithm a waveform relaxation algorithm because time dependent problems are solved on subdomains like in the waveform relaxation algorithm for large systems of ordinary differential equations [24]. The algorithm we consider here is a Jacobi type or additive Schwarz algorithm, since all the subdomains are treated in parallel. A Gauss Seidel or multiplicative Schwarz algorithm could be considered as well. But since the analysis would be similar, we focus in this paper on the additive version of the algorithm only.

**2.2. Transmission Conditions for Optimal Convergence.** For elliptic problems, the Dirichlet to Neumann map has been used in [31] to define optimal transmission conditions. For wave propagation, it is more convenient to introduce the linear operator $\mathcal{S}_1(x_0)$ defined by

$$\mathcal{S}_1(x_0) : g(t) \mapsto \frac{\partial v}{\partial x}(x_0, t) \tag{2.4}$$

where $v(x,t)$ is the solution of

$$\begin{array}{rcll} \mathcal{L}(v) & = & 0, & \text{in } (-\infty, x_0) \times (0,T) \\ \frac{\partial v}{\partial t}(x_0,t) & = & g(t) & t \in (0,T) \\ v(x,0) = \frac{\partial v}{\partial t}(x,0) & = & 0 & x \in (-\infty, x_0). \end{array} \tag{2.5}$$

and the linear operator $\mathcal{S}_2(x_0)$ defined by

$$\mathcal{S}_2(x_0) : g(t) \mapsto \frac{\partial v}{\partial x}(x_0, t) \tag{2.6}$$

where $v(x,t)$ is the solution of

$$\begin{array}{rcll} \mathcal{L}(v) & = & 0, & \text{in } (x_0, \infty) \times (0,T) \\ \frac{\partial v}{\partial t}(x_0,t) & = & g(t) & t \in (0,T) \\ v(x,0) = \frac{\partial v}{\partial t}(x,0) & = & 0 & x \in (x_0, \infty). \end{array} \tag{2.7}$$

The operators $\mathcal{S}_j$ defined in (2.5) and (2.7) are the key ingredients to obtain an optimal Schwarz Waveform Relaxation algorithm. This algorithm is obtained by choosing the transmission operators $\mathcal{B}_i^{\pm}$ in the algorithm (2.2) to be

$$\mathcal{B}_i^- := \mathcal{S}_1(a_i)\partial_t - \partial_x, \quad \mathcal{B}_i^+ := \mathcal{S}_2(a_{i+1})\partial_t - \partial_x. \tag{2.8}$$

This choice is not arbitrary. The absorption property of these transmission operators allows the algorithm to compute subdomain solutions which do not see the interfaces and hence are exact; for the steady convection diffusion case, see [33].

THEOREM 2.1 (Convergence in $I$ steps). *The non-overlapping Schwarz waveform relaxation algorithm (2.2) with transmission operators defined by (2.8) converges in $I$ iterations where $I$ denotes the number of subdomains.*

*Proof.* First note that convergence in less than $I$ iterations, where $I$ denotes the number of subdomains, is not possible over long time intervals, since the solution on each subdomain depends on the data on all the other subdomains and information is only propagated locally to neighboring subdomains. To show that the algorithm with the transmission operators (2.8) achieves convergence in $I$ iterations and therefore is optimal we rewrite the algorithm (2.2) in substructured form on the interfaces only. By linearity it suffices to consider the homogeneous case (the error equations) only,

$f = p = q = 0$, and to prove convergence to zero. We denote the interface values which subdomain $\Omega_i$ obtains from its neighbors $\Omega_{i-1}$ and $\Omega_{i+1}$ by

$$g_i^{k^-}(t) := \mathcal{B}_i^-(u_{i-1}^k)(a_i, t), \quad g_i^{k^+}(t) := \mathcal{B}_i^+(u_{i+1}^k)(a_{i+1}, t), \quad \forall t \in (0, T)$$

and put all these values $g_i^{k^-}(t)$ and $g_i^{k^+}(t)$ together into the vector-valued function $g^k := (g_1^{k^+}, g_2^{k^-}, g_2^{k^+}, \ldots, g_I^{k^-})$. Note that there is only one element in $g^k$ for the leftmost and rightmost subdomain, since they both extend to infinity. One step of the Schwarz waveform relaxation algorithm (2.2) can now be seen as a linear map taking a vector-valued function $g^k$ as input and producing a new vector-valued function $g^{k+1}$ as output. On each subdomain $\Omega_i$ for interior subdomains, $i = 2, \ldots I-1$, there are two linear mappings, both taking as input arguments the values of the neighboring subdomains and one producing a new value on the left boundary

$$A_i^- : (g_i^{k^-}, g_i^{k^+}) \mapsto \mathcal{B}_{i-1}^+ \left(u_i^{k+1}\right)(a_i, \cdot)$$

and the other one a new value on the right boundary

$$A_i^+ : (g_i^{k^-}, g_i^{k^+}) \mapsto \mathcal{B}_{i+1}^- \left(u_i^{k+1}\right)(a_{i+1}, \cdot).$$

For the outermost subdomains there is only one linear map each, taking one input argument only,

$$A_1^+ : g_1^{k^+} \mapsto \mathcal{B}_2^- \left(u_1^{k+1}\right)(a_2, \cdot) \quad \text{and} \quad A_I^- : g_I^{k^-} \mapsto \mathcal{B}_{I-1}^+ \left(u_I^{k+1}\right)(a_I, \cdot).$$

Note that by the definition of the operators $\mathcal{S}_j$ both $A_1^+$ and $A_I^-$ map any argument to zero: for any function $g(t)$ we have

$$A_1^+(g) \equiv A_I^-(g) \equiv 0. \tag{2.9}$$

This can be seen for $A_1^+$ for example by

$$A_1^+(g) = \mathcal{B}_2^-(v)(a_2, \cdot) = (\mathcal{S}_1(a_2)\partial_t - \partial_x)v = v_x(a_2) - v_x(a_2) = 0$$

where we have used that by the definition of $\mathcal{S}_1$ the function $v$ is solution of (2.5). Similarly for interior subdomains we have for any function $g(t)$

$$A_i^-(g, 0) \equiv A_i^+(0, g) \equiv 0. \tag{2.10}$$

But this implies by linearity that $A_i^+(g, h)$ does only depend on $g$ and $A_i^-(g, h)$ does only depend on $h$, since

$$\begin{aligned} A_i^+(g, h) &= A_i^+(g, 0) + A_i^+(0, h) =: \widetilde{A}_i^+(g), \\ A_i^-(g, h) &= A_i^-(g, 0) + A_i^-(0, h) =: \widetilde{A}_i^-(h). \end{aligned} \tag{2.11}$$

Using these linear mappings on each subdomain, a complete step of the non overlapping Schwarz waveform relaxation algorithm can be described by the linear map $\mathcal{A} : g^k \mapsto g^{k+1}$ where $\mathcal{A}g^k$ is defined by

$$\begin{aligned} \mathcal{A}g^k &= \mathcal{A}(g_1^{k^+}, g_2^{k^-}, g_2^{k^+}, \ldots, g_I^{k^-}) \\ &= (A_2^-(g_2^{k^-}, g_2^{k^+}), A_1^+(g_1^{k^+}), A_3^-(g_3^{k^-}, g_3^{k^+}), A_2^+(g_2^{k^-}, g_2^{k^+}), \ldots, A_I^-(g_I^{k^-}), A_{I-1}^+(g_{I-1}^{k^-}, g_{I-1}^{k^+})) \\ &= (\widetilde{A}_2^-(g_2^{k^+}), 0, \widetilde{A}_3^-(g_3^{k^+}), \widetilde{A}_2^+(g_2^{k^-}), \ldots, 0, \widetilde{A}_{I-1}^+(g_{I-1}^{k^-})) \end{aligned}$$

or written in matrix form

$$\mathcal{A}=\begin{bmatrix} 0 & 0 & \widetilde{A}_2^- & & & & & & \\ 0 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & 0 & \widetilde{A}_3^- & & & & \\ & \widetilde{A}_2^+ & 0 & 0 & 0 & \ddots & & & \\ & & 0 & 0 & 0 & & 0 & & \\ & & & \widetilde{A}_3^+ & 0 & & 0 & \widetilde{A}_{I-1}^- & \\ & & & & \ddots & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & & \widetilde{A}_{I-1}^+ & 0 & 0 \end{bmatrix}.$$

Now proving that the non-overlapping Schwarz waveform relaxation algorithm (2.2) converges in $I$ iterations is equivalent to showing that the Schwarz iteration map satisfies $\mathcal{A}^{I-1}=0$ since then after $I-1$ iterations all the interface values are zero and thus after one more iteration the solution will be converged to zero everywhere. We prove this by showing that $\mathcal{A}^{I-1}$ applied to an arbitrary vector valued function $e(t)=(e_1(t),e_2(t),\ldots,e_{2I-2}(t))$ equals zero. The first application of $\mathcal{A}$ will delete the second and second last entry in $e$. The second application therefore will delete the fourth and fourth last entry in $e$ because of the structure of $\mathcal{A}$. This process continues until the $I-1$ application of $\mathcal{A}$ deleted the $2I-2$ and the $2I-2$ last entry, which is the first entry in $e$. Thus now $\mathcal{A}^{I-1}e$ is the zero vector and in the next step the solution is zero everywhere. □

Note that the proof only uses the fundamental property of the linear operators $\mathcal{S}_j$ leading to the transparent transmission conditions, and no special properties of the wave equation. The result is therefore valid for other partial differential equations as well where the appropriate operators $\mathcal{S}_j$ can be defined.

**3. Optimal Convergence with Local Transmission Conditions for Piecewise Constant Wave Speed.** We now consider the wave equation (2.1) with piecewise constant wave speed to investigate the optimal transmission operators further. This will lead to the interesting result of convergence in less iterations than the number of subdomains on certain bounded time intervals. We consider first the wave equation (2.1) with two physical domains

$$O_1=\mathbb{R}^- \text{ with } c(x)=c_1 \text{ and } O_2=\mathbb{R}^+ \text{ with } c(x)=c_2. \tag{3.1}$$

In this case we can compute the linear operators $\mathcal{S}_j$ explicitly and from them we gain more insight in the optimal Schwarz waveform relaxation algorithm. In Subsection 3.3 we generalize the results to an arbitrary number of discontinuities.

**3.1. Identification of the Optimal Non-Local Transmission Conditions.** We define the ratio $r$ by

$$r:=\frac{c_2-c_1}{c_2+c_1}. \tag{3.2}$$

LEMMA 3.1. *In the case of piecewise constant wave speed (3.1), the linear operators $\mathcal{S}_j$ in (2.4), (2.6) are given by*

$$(\mathcal{S}_1(x_0))\,g(t)=\begin{cases} \frac{1}{c_1}g(t) & x_0\in O_1 \\ \frac{1}{c_2}\left(g(t)+2\sum_{k=1}^{\lfloor\frac{c_2 t}{2x_0}\rfloor} r^k g(t-2kx_0/c_2)\right) & x_0\in O_2 \end{cases} \tag{3.3}$$

*and*

$$(3.4)\qquad (\mathcal{S}_2(x_0))\, g(t) = \begin{cases} \frac{1}{c_1}\left( g(t) + 2\sum_{k=1}^{\lfloor \frac{c_1 t}{2x_0}\rfloor} r^k g(t+2kx_0/c_1)\right) & x_0 \in O_1 \\ \frac{1}{c_2} g(t) & x_0 \in O_2 \end{cases}.$$

*Proof.* This result can be obtained by explicitly computing the solutions, for details, see [16]. ▯

In this special case one can see why the operators $\mathcal{S}_j$ are non local in general: they have to include reflections stemming from the discontinuity in the wave speed between the two different physical domains. Using Theorem 2.1 the non overlapping Schwarz waveform relaxation algorithm for the wave equation with a discontinuity in the wave speed converges in $I$ steps where $I$ denotes the number of subdomains. However an implementation of these non-local transmission conditions is rather complicated and we do not recommend this, especially if several discontinuities occur in the physical domain. But the result indicates a better approach already, and we develop it in the next subsection.

**3.2. Local Transmission Conditions Using the Time Evolution.** The optimal transmission operators (2.8) depend on the time interval under consideration, since the linear operators $\mathcal{S}_j$ in (3.3), (3.4) contain a sum with a number of terms proportional to the length of the time interval. In particular for a given time interval $[0,T]$ we have at most

$$\max\left(\lfloor \frac{c_1 T}{2x_0}\rfloor, \lfloor \frac{c_2 T}{2x_0}\rfloor\right)$$

terms in the sum to obtain optimal convergence. We thus obtain the important corollary of Theorem 2.1:

Corollary 3.2. *In the case of piecewise constant wave speed (3.1), if the discontinuity at $x=0$ lies within the subdomain $\Omega_l$, $a_l < 0 < a_{l+1}$, then the non overlapping Schwarz waveform relaxation algorithm (2.2) with local transmission operators*

$$(3.5)\qquad \mathcal{B}_i^- := \frac{1}{c(a_i)}\partial_t - \partial_x, \quad \mathcal{B}_i^+ := \frac{1}{c(a_{i+1})}\partial_t + \partial_x$$

*converges in $I$ iterations where $I$ denotes the number of subdomains, if the computation is restricted to the time interval $t \in [0,T]$ with*

$$(3.6)\qquad T \le T_1 = 2\min\left(\frac{|a_l|}{c(a_l)}, \frac{|a_{l+1}|}{c(a_{l+1})}\right).$$

*Proof.* If we choose $T$ such that

$$(3.7)\qquad \max_{1<j\le I} \lfloor \frac{c(a_j)T}{2|a_j|}\rfloor \equiv 0$$

then there are no terms left in the sum of the operators $\mathcal{S}_j$ in (3.3), (3.4) and thus the optimal transmission operators become the local operators (3.5). But the maximum in condition (3.7) can only be attained for either $j=l$ or $j=l+1$ because the discontinuity lies in subdomain $\Omega_l$ and thus (3.7) is equivalent to the condition (3.6). ▯

This corollary suggests to avoid the costly non-local transmission conditions by cutting the given time domain $[0, T]$ into time windows of length $T_1$ given in (3.6). Then the algorithm can employ local transmission conditions and will still converge in at most $I$ iterations.

But condition (3.6) can impose very small time windows if $a_l$ or $a_{l+1}$ are very close to the discontinuity at $x = 0$. At first glance this suggests that it is best to place the subdomains so that the discontinuities lie inside the subdomains, away from its boundaries. The optimal location for $a_l < 0 < a_{l+1}$ would be such that

$$\frac{|a_l|}{c(a_l)} = \frac{|a_{l+1}|}{c(a_{l+1})} \tag{3.8}$$

to maximize the time interval (3.6) one can use with the algorithm and local transmission conditions. There is however a better choice: taking the limit of the operators $\mathcal{S}_j$ in (3.3,3.4) as $x_0$ goes to zero we find

$$(\mathcal{S}_1(0))g(t) = \frac{1}{c_1}g(t), \quad (\mathcal{S}_2(0))g(t) = \frac{1}{c_2}g(t) \tag{3.9}$$

and thus the operators $\mathcal{S}_j$ become local operators in that case. This suggests that aligning physical domains with computational ones is an advantage for the transmission conditions. Defining

$$c(x^-) := c(x - 0), \quad c(x^+) := c(x + 0)$$

to include the correct limits when the discontinuity lies exactly at an interface between two subdomains, we obtain

COROLLARY 3.3. *In the case of piecewise constant wave speed (3.1), if the discontinuity lies on the interface between the two subdomains $\Omega_l$ and $\Omega_{l+1}$, $a_{l+1} = 0$, then the non overlapping Schwarz waveform relaxation algorithm (2.2) with local transmission operators*

$$\mathcal{B}_i^- := \frac{1}{c(a_i^-)}\partial_t - \partial_x, \quad \mathcal{B}_i^+ := \frac{1}{c(a_{i+1}^+)}\partial_t + \partial_x, \tag{3.10}$$

*converges in $I$ iterations where $I$ denotes the number of subdomains, if the computation is restricted to the time interval $t \in [0, T]$ with*

$$T \leq T_2 = 2\min\left(\frac{|a_l|}{c(a_l)}, \frac{|a_{l+2}|}{c(a_{l+2})}\right). \tag{3.11}$$

*Proof.* If we place the discontinuity directly between the two subdomains $\Omega_l$ and $\Omega_{l+1}$, then the optimal transmission conditions between $\Omega_l$ and $\Omega_{l+1}$ are local as seen in (3.9). Therefore the largest time interval we can choose for local transmission conditions depends now only on the total width of $\Omega_l$ and $\Omega_{l+1}$ which leads to condition (3.11). ☐

Hence with the discontinuity at $x = 0$ aligned with a subdomain boundary, say at $a_{l+1} = 0$, one would choose the subdomain boundaries $a_l$ and $a_{l+2}$ such that

$$\frac{|a_l|}{c(a_l)} = \frac{|a_{l+2}|}{c(a_{l+2})} \tag{3.12}$$

to maximize the possible time interval in (3.11) where the algorithm can be used with local transmission conditions. This choice leads to a longer time interval than the choice with the discontinuity within one subdomain (3.8) since $a_l < a_{l+1} < a_{l+2}$.

**3.3. Convergence in 2 Iterations Independent of the Number of Subdomains.** In more realistic situations there will be more than one discontinuity in the computational domain which seems to complicate the situation, because for the global optimal transmission conditions of the type (3.3,3.4) one would need to track more and more reflections from the various discontinuities in the wave speed. But due to the finite speed of propagation in the wave equation, the previous analysis can be applied locally using time windows again. In addition with time windows, not every subdomain solution depends on the solution on all the other subdomains if the time interval is short enough. Neighboring information suffices in that case and it is thus possible to reduce the number of iterations below $I$ for $I$ subdomains.

Suppose we have $J$ physical domains $O_j = (d_j, d_{j+1})$ with constant wave speed per physical domain, $c(x) = c_j$ for $d_j < x < d_{j+1}$, $j = 1, \ldots, J$, $d_1 = -\infty$ and $d_{J+1} = \infty$. We decompose the physical domain $\mathbb{R}$ into $I$ computational subdomains $\Omega_i = (a_i, a_{i+1})$ as before. We denote by $n_i$ the number of discontinuities within each subdomain $\Omega_i$ and we exclude for the moment the case where a discontinuity is aligned precisely between two computational subdomains. We also denote by $m_i$ the index of the first physical domain $O_{m_i}$ which intersects the computational subdomain $\Omega_i$. We define the transmission time $t_i$ of a signal across subdomain $\Omega_i$ by

$$t_i := \begin{cases} \dfrac{a_{i+1} - a_i}{c_{m_i}} & \text{if } n_i = 0 \\ \dfrac{d_{m_i+1} - a_i}{c_{m_i}} + \displaystyle\sum_{k=1}^{n_i-1} \dfrac{d_{m_i+k+1} - d_{m_i+k}}{c_{m_i+k}} + \dfrac{a_{i+1} - d_{m_i+n_i}}{c_{m_i+n_i}} & \text{if } n_i > 0. \end{cases} \tag{3.13}$$

We also define the reflection time at each interface $a_i$ of the computational subdomains by

$$\tau_i := 2 \min_j \frac{|a_i - d_j|}{c(a_i)}. \tag{3.14}$$

These two time constants allow us to formulate conditions for convergence in less than $I$ steps.

Theorem 3.4. *The non overlapping Schwarz waveform relaxation algorithm (2.2) with local transmission conditions (3.10) and any discontinuities strictly in the interior of the computational subdomains converges in 2 iterations independently of the number of subdomains if the time interval* $[0, T]$ *is chosen such that*

$$T \leq T_3 = \min\left(\min_i t_i, \min_i \tau_i\right) \tag{3.15}$$

*where $t_i$ is defined in (3.13) and $\tau_i$ is defined in (3.14).*

*Proof.* Consider one of the computational domains $\Omega_i$. The solution on that domain depends only on the solution of the neighboring domains $\Omega_{i+1}$ and $\Omega_{i-1}$ determined by their initial conditions, because the time interval $[0, T]$ given by (3.15) is too short for any signal to reach domain $\Omega_i$ across the neighboring subdomains due to condition (3.15). So after one iteration, the exact boundary conditions for domain $\Omega_i$ are available, if the transmission conditions employed at the boundary of $\Omega_i$ are exact absorbing boundary conditions. But this is ensured by condition (3.15) as well because $T$ is smaller than any reflection time $\tau_i$ so that the local transmission conditions (3.10) are indeed exactly absorbing. Thus the second iteration produces the exact solution on subdomain $\Omega_i$. Since this argument holds for all computational subdomains, the result is established. □

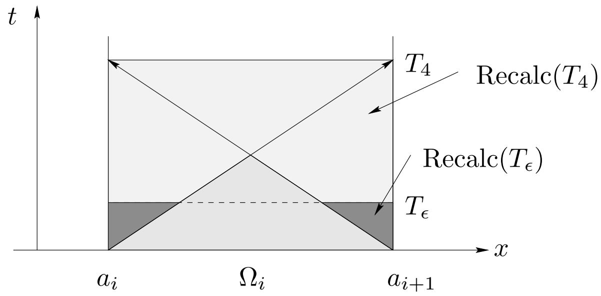


FIG. 3.1. *Towards the optimal algorithm: unknowns to recompute for the time window $T_4$ and $T_\epsilon$ respectively.*

As in Corollary 3.2 this result can require very small time intervals, since the reflection times $\tau_i$ can be very small when a discontinuity approaches a subdomain boundary. This can however be avoided as before by aligning physical discontinuities with the boundaries of the subdomains, a natural approach for domain decomposition. Doing this for all discontinuities, the minimal transmission time $\min_i t_i$ becomes necessarily smaller than the minimal reflection time $\min_i \tau_i$ because the reflection time requires the signal to go twice across a subdomain (there are no discontinuities any more within subdomains). In addition the transmission times formula (3.13) simplifies greatly, it becomes

$$t_i = \frac{a_{i+1} - a_i}{c(a_i^+)}.$$

We therefore obtain the following

THEOREM 3.5. *The non overlapping Schwarz waveform relaxation algorithm (2.2) with local transmission conditions (3.10) and any discontinuities aligned with the computational subdomain boundaries converges in 2 iterations independently of the number of subdomains if the time interval $[0, T]$ is chosen such that*

$$T \leq T_4 = \min_i \frac{a_{i+1} - a_i}{c(a_i^+)}. \tag{3.16}$$

*Proof.* The argument is the same as in the previous theorem. □

Further computation can be saved by noting that only values above the characteristics in each subdomain need to be recomputed during the second iteration, as shown in Figure 3.1. If the time window is chosen to be $[0, T_4]$ from Theorem 3.5 then convergence will be achieved in two iterations and in the second iteration only the variables in the region denoted by Recalc($T_4$) need to be recalculated. If we choose however an even smaller time window $T_\epsilon$, then much less variables need to be recalculated in the second iteration, namely the ones denoted by Recalc($T_\epsilon$). Thus with our algorithm the solution of the wave equation can be optimally parallelized: the parallel algorithm run on a sequential machine will run at a cost $1+\epsilon$ of the optimal sequential code, provided the cost is linear in the number of unknowns. The optimal choice of $T_\epsilon$ depends on the latency time of the network linking the computational nodes. If the latency time is small, then a short $T_\epsilon$ will lead to the best performance, since almost no values need to be recomputed. If the latency time is important however, it is better to communicate larger amounts of data each time a communication needs to be done. This can be achieved by choosing a larger $T_\epsilon$ and will lead to faster solution times even if more values need to be recomputed in the second iteration.

**4. Convergence with Local Transmission Conditions for Continuous Wave Speed.** Discontinuous wave speeds allowed us to use local transmission conditions in the Schwarz waveform relaxation algorithm and still get optimal performance. If the wave speed is varying continuously, such a result can not hold any more, because reflections become relevant immediately. Nevertheless the algorithm is well defined with local transmission conditions and we prove that it converges using energy estimates. Energy estimates are useful tools for proving well-posedness of boundary or initial boundary value problems, in particular for variable coefficients. They have been used to analyze the convergence of Schwarz algorithms in the stationary case before, see for example [28], [11] or [32]. We extend these techniques here to time dependent problems.

Let $u$ be a solution of the wave equation in the interval $[a,b]$, for $t \geq 0$,

$$\frac{1}{c^2(x)}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{4.1}$$

where $c(x)$ is now any continuous function. We define the kinetic and potential energies by

$$E_K(u)(t) := \frac{1}{2}\int_a^b \frac{1}{c^2(x)}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 dx,\ \ E_P(u)(t) := \frac{1}{2}\int_a^b \left(\frac{\partial u}{\partial x}(x,t)\right)^2 dx \tag{4.2}$$

and the total energy by the sum $E := E_K + E_P$. Multiplying (4.1) by $\frac{\partial u}{\partial t}$ and integrating on the interval $[a,b]$ yields

Theorem 4.1 (Continuous Energy Identity). *The energy identity*

$$\frac{d}{dt}[E(u)(t)] + \frac{\partial u}{\partial t}(a,t)\frac{\partial u}{\partial x}(a,t) - \frac{\partial u}{\partial t}(b,t)\frac{\partial u}{\partial x}(b,t) = 0 \tag{4.3}$$

*holds for any positive time $t$.*

**4.1. Well-Posedness of the Continuous Subdomain Problems.** Introducing the general progressive and regressive transport operators

$$\mathcal{T}_\alpha^+ = \frac{1}{\alpha}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \quad \mathcal{T}_\alpha^- = \frac{1}{\alpha}\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \tag{4.4}$$

where $\alpha$ is a positive real number, we can rewrite (4.3) for any positive $\alpha$ and $\beta$ as

$$\frac{d}{dt}[E(u)(t)] + \frac{\alpha}{4}\left[\mathcal{T}_\alpha^+ u(a,t)\right]^2 + \frac{\beta}{4}\left[\mathcal{T}_\beta^- u(b,t)\right]^2 = \frac{\alpha}{4}\left[\mathcal{T}_\alpha^- u(a,t)\right]^2 + \frac{\beta}{4}\left[\mathcal{T}_\beta^+ u(b,t)\right]^2. \tag{4.5}$$

Suppose that the boundary conditions (4.4) are given by

$$\mathcal{T}_\alpha^- u(a,t) = g^-(t), \quad \mathcal{T}_\beta^+ u(b,t) = g^+(t). \tag{4.6}$$

Then we get a bound on the energy on any finite time interval.

Theorem 4.2. *For the wave equation (4.1) on $[a,b]$ with boundary conditions (4.6), the energy $E(u)(t)$ on $[a,b]$ stays bounded for all finite time $t$,*

$$E(u)(t) \leq E(u)(0) + \int_0^t [\frac{\alpha}{4}|g^-(\tau)|^2 + \frac{\beta}{4}|g^+(\tau)|^2]d\tau. \tag{4.7}$$

By standard techniques, see for example [26], the well-posedness is then established.

**4.2. Convergence with Local Transmission Conditions.** Consider now the domain decomposition algorithm (2.2) for continuously variable wave speed $c(x)$. By linearity it suffices to consider the homogeneous wave equation with homogeneous initial conditions and prove convergence to zero. The local transmission operators (3.10) can be expressed in terms of the transport operators (4.4),

$$\mathcal{B}_i^- = \mathcal{T}_{c(a_i)}^-, \quad \mathcal{B}_i^+ = \mathcal{T}_{c(a_{i+1})}^+ \tag{4.8}$$

where $i = 1, \ldots, I$.

THEOREM 4.3. *Suppose the velocity is continuous on the interfaces $a_i$. Then on any time interval $[0, T]$, the non-overlapping Schwarz waveform relaxation algorithm with local transmission conditions*

$$\begin{aligned} \mathcal{T}_{c(a_i)}^-(u_i^{k+1})(a_i, \cdot) &= \mathcal{T}_{c(a_i)}^-(u_{i-1}^k)(a_i, \cdot), && \text{on } (0,T) \\ \mathcal{T}_{c(a_{i+1})}^+(u_i^{k+1})(a_{i+1}, \cdot) &= \mathcal{T}_{c(a_{i+1})}^+(u_{i+1}^k)(a_{i+1}, \cdot) && \text{on } (0,T) \end{aligned} \tag{4.9}$$

*converges in the energy norm,*

$$\sum_{i=1}^{I} E(u_i^k)(T) \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

*Proof.* We can write (4.5) on the interval $[a_i, a_{i+1}]$, with $\alpha = c(a_i)$ and $\beta = c(a_{i+1})$, which gives

$$\begin{aligned} \tfrac{d}{dt}[E(u_i^{k+1})(\cdot)] &+ \tfrac{c(a_i)}{4}\left[\mathcal{T}_{c(a_i)}^+(u_i^{k+1})(a_i, \cdot)\right]^2 + \tfrac{c(a_{i+1})}{4}\left[\mathcal{T}_{c(a_{i+1})}^-(u_i^{k+1})(a_{i+1}, \cdot)\right]^2 \\ &= \tfrac{c(a_i)}{4}\left[\mathcal{T}_{c(a_i)}^-(u_i^{k+1})(a_i, \cdot)\right]^2 + \tfrac{c(a_{i+1})}{4}\left[\mathcal{T}_{c(a_{i+1})}^+(u_i^{k+1})(a_{i+1}, \cdot)\right]^2 \end{aligned} \tag{4.10}$$

for $1 \le i \le I$, with the convention

$$\mathcal{T}_{c(a_1)}^+(u_1^{k+1})(a_1, \cdot) = 0, \quad \mathcal{T}_{c(a_{I+1})}^-(u_I^{k+1})(a_{I+1}, \cdot) = 0, \tag{4.11}$$

and by using the boundary conditions, we obtain

$$\begin{aligned} \tfrac{d}{dt}[E(u_i^{k+1})(\cdot)] &+ \tfrac{c(a_i)}{4}\left[\mathcal{T}_{c(a_i)}^+(u_i^{k+1})(a_i, \cdot)\right]^2 + \tfrac{c(a_{i+1})}{4}\left[\mathcal{T}_{c(a_{i+1})}^-(u_i^{k+1})(a_{i+1}, \cdot)\right]^2 \\ &= \tfrac{c(a_i)}{4}\left[\mathcal{T}_{c(a_i)}^-(u_{i-1}^k)(a_i, \cdot)\right]^2 + \tfrac{c(a_{i+1})}{4}\left[\mathcal{T}_{c(a_{i+1})}^+(u_{i+1}^k)(a_{i+1}, \cdot)\right]^2. \end{aligned} \tag{4.12}$$

Summing these equations for $1 \le i \le I$ and shifting the indices of the two sums on the right-hand side, we find

$$\begin{aligned} \sum_{i=1}^{I} \frac{d}{dt}[E(u_i^{k+1})(\cdot)] + \sum_{i=1}^{I} \frac{c(a_i)}{4}\left[\mathcal{T}_{c(a_i)}^+(u_i^{k+1})(a_i, \cdot)\right]^2 + \sum_{i=1}^{I} \frac{c(a_{i+1})}{4}\left[\mathcal{T}_{c(a_{i+1})}^-(u_i^{k+1})(a_{i+1}, \cdot)\right]^2 \\ = \sum_{i=0}^{I-1} \frac{c(a_{i+1})}{4}\left[\mathcal{T}_{c(a_{i+1})}^-(u_i^k)(a_{i+1}, \cdot)\right]^2 + \sum_{i=2}^{I+1} \frac{c(a_i)}{4}\left[\mathcal{T}_{c(a_i)}^+(u_i^k)(a_i, \cdot)\right]^2. \end{aligned} \tag{4.13}$$

Now note that by (2.3) we have $u_{I+1}^k = u_0^k = 0$ and thus $\mathcal{T}^+_{c(a_{I+1})}(u_{I+1}^k)(a_{I+1},\cdot) = \mathcal{T}^-_{c(a_1)}(u_0^k)(a_1,\cdot) = 0$. Together with (4.11), we obtain the energy equality

$$
\begin{aligned}
\sum_{i=1}^{I}\frac{d}{dt}[E(u_i^{k+1})(\cdot)] + \sum_{i=2}^{I}\frac{c(a_i)}{4}\left[\mathcal{T}^+_{c(a_i)}(u_i^{k+1})(a_i,\cdot)\right]^2 + \sum_{i=1}^{I-1}\frac{c(a_{i+1})}{4}\left[\mathcal{T}^-_{c(a_{i+1})}(u_i^{k+1})(a_{i+1},\cdot)\right]^2 \\
= \sum_{i=2}^{I}\frac{c(a_i)}{4}\left[\mathcal{T}^+_{c(a_i)}(u_i^{k})(a_i,\cdot)\right]^2 + \sum_{i=1}^{I-1}\frac{c(a_{i+1})}{4}\left[\mathcal{T}^-_{c(a_{i+1})}(u_i^{k})(a_{i+1},\cdot)\right]^2.
\end{aligned}
\tag{4.14}
$$

Now we have the same terms on the boundary, on the left for iteration step $k+1$ and on the right for iteration step $k$. Defining

$$
\begin{aligned}
\hat{E}^k(t) &:= \sum_{i=1}^{I} E(u_i^{k+1})(t) \\
\hat{E}_B^k(t) &:= \sum_{i=2}^{I}\frac{c(a_i)}{4}\left[\mathcal{T}^+_{c(a_i)}(u_i^k,\cdot)(a_i)\right]^2 + \sum_{i=1}^{I-1}\frac{c(a_{i+1})}{4}\left[\mathcal{T}^-_{c(a_{i+1})}(u_i^k)(a_{i+1},\cdot)\right]^2
\end{aligned}
$$

we find the energy equality

$$
\frac{d}{dt}\hat{E}^{k+1} + \hat{E}_B^{k+1} = \hat{E}_B^k \quad \text{on } (0,T)
$$

and thus summing up over all iteration steps $k = 0 \dots K$ and denoting the sum of the energies $\hat{E}^k$ at each step by

$$
E^K := \sum_{k=0}^{K} \hat{E}^k
$$

we find by cancellation of the $\hat{E}_B^k$ terms

$$
\frac{d}{dt}E^K + \hat{E}_B^K = \hat{E}_B^0 \quad \text{on } (0,T).
$$

Since $\hat{E}_B^K \geq 0$ we obtain

$$
\frac{d}{dt}E^K \leq \hat{E}_B^0 \quad \text{on } (0,T).
$$

Now integrating over $(0,T)$ and noting that $E^K(0) = 0$ we find

$$
E^K(T) \leq \int_0^T \hat{E}_B^0(t)dt
$$

and thus the total energy $E^K$ is uniformly bounded independently of the number of iterations $K$. Hence the energy at each iteration must go to zero and the algorithm converges. □

**5. A Finite Volume Discretization.** We discretize the wave equation (2.1) on each subdomain $\Omega_i \times (0,T)$, $i = 1,\dots,I$ separately using a finite volume discretization on rectangular grids. For simplicity we set $f = 0$. We allow non-matching grids on different subdomains, with $J_i + 2$ points in space numbered from 0 up to $J_i + 1$ and

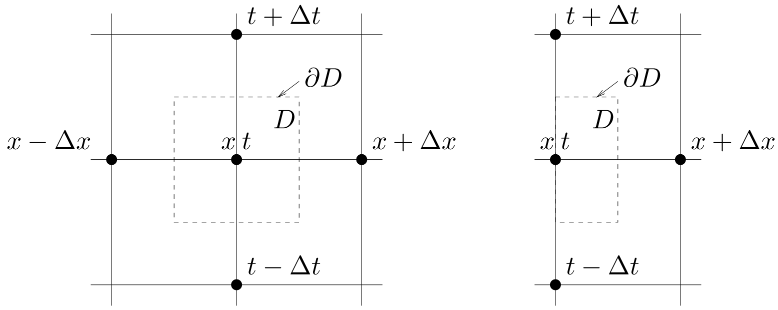


FIG. 5.1. *Control volume of an interior grid point and a boundary grid point.*

$\Delta x_i = (a_{i+1} - a_i)/(J_i + 1)$ and $N_i + 1$ grid points in time with $\Delta t_i = T/N_i$ numbered from 0 up to $N_i$. Note that we chose for the exposition here uniform spacing in time per subdomain, but the techniques developed are not limited to this special case. We denote the numerical approximation to $u_i^k(a_i + j\Delta x_i, n\Delta t_i)$ on $\Omega_i$ at iteration step $k$ by $U_i^k(j,n)$. To simplify the notation we omit the index $i$ on quantities depending on the subdomain and the index $k$ referring to the iteration as long as we are discussing one subdomain only.

### 5.1. Discretization of the Subdomain Problem.

**5.1.1. Interior Points.** Denoting by $D$ the volume around a grid point $(x = a_i + j\Delta x_i, t = n\Delta t_i)$ in the interior of subdomain $\Omega_i \times (0,T)$, as shown in Figure 5.1 on the left, we obtain the finite volume scheme by integrating the equation over the volume $D$ and applying the divergence theorem,

$$\begin{aligned} 0 &= \int_{x-\Delta x/2}^{x+\Delta x/2} \int_{t-\Delta t/2}^{t+\Delta t/2} \left[ \frac{1}{c^2(\xi)} \frac{\partial^2 u}{\partial t^2}(\xi,\tau) - \frac{\partial^2 u}{\partial x^2}(\xi,\tau) \right] d\tau d\xi \\ &= \int_{x-\Delta x/2}^{x+\Delta x/2} \frac{1}{c^2(\xi)} \frac{\partial u}{\partial t}(\xi, t+\Delta t/2) d\xi - \int_{x-\Delta x/2}^{x+\Delta x/2} \frac{1}{c^2(\xi)} \frac{\partial u}{\partial t}(\xi, t-\Delta t/2) d\xi \\ &\quad - \int_{t-\Delta t/2}^{t+\Delta t/2} \frac{\partial u}{\partial x}(x+\Delta x/2, \tau) d\tau + \int_{t-\Delta t/2}^{t+\Delta t/2} \frac{\partial u}{\partial x}(x-\Delta x/2, \tau) d\tau. \end{aligned} \tag{5.1}$$

Now we approximate the remaining derivatives by finite differences on the grid,

$$\begin{aligned} D_t^+(U)(j,n) &:= \tfrac{U(j,n+1)-U(j,n)}{\Delta t} \approx \tfrac{\partial u}{\partial t}(\xi, t+\Delta t/2), \\ D_t^-(U)(j,n) &:= \tfrac{U(j,n)-U(j,n-1)}{\Delta t} \approx \tfrac{\partial u}{\partial t}(\xi, t-\Delta t/2), \\ D_x^+(U)(j,n) &:= \tfrac{U(j+1,n)-U(j,n)}{\Delta x} \approx \tfrac{\partial u}{\partial x}(x+\Delta x/2, \tau), \\ D_x^-(U)(j,n) &:= \tfrac{U(j,n)-U(j-1,n)}{\Delta x} \approx \tfrac{\partial u}{\partial x}(x-\Delta x/2, \tau), \end{aligned} \quad \begin{aligned} & x - \tfrac{\Delta x}{2} \le \xi \le x + \tfrac{\Delta x}{2}, \\ & t - \tfrac{\Delta t}{2} \le \tau \le t + \tfrac{\Delta t}{2}. \end{aligned} \tag{5.2}$$

We introduce a discrete speed function $C_i(j)$ which approximates $c(a_i + j\Delta x_i)$ through the integral relation

$$\frac{\Delta x_i}{C_i^2(j)} := \int_{x-\Delta x_i/2}^{x+\Delta x_i/2} \frac{1}{c^2(\xi)} d\xi \tag{5.3}$$

and we will omit the subdomain index $i$ as long as we are on one subdomain. We thus obtain from (5.1) the discrete scheme

$$0 = \frac{\Delta x}{C^2(j)} (D_t^+ - D_t^-)(U)(j,n) - \Delta t (D_x^+ - D_x^-)(U)(j,n).$$

which yields on using the identities $\Delta t D_t^+ D_t^- = D_t^+ - D_t^-$ and $\Delta x D_x^+ D_x^- = D_x^+ - D_x^-$ the well known finite difference scheme

$$\left(\frac{1}{C^2(j)} D_t^+ D_t^- - D_x^+ D_x^-\right)(U)(j,n) = 0, \quad 1 \le j \le J, \tag{5.4}$$

for points in the interior of subdomains.

**5.1.2. Boundary Points.** So far the finite volume scheme led to a similar discretization as a finite difference scheme. On the boundary however the finite volume scheme leads automatically to a consistent discretization of the transmission conditions, whereas a finite difference discretization would require a special treatment. In addition the finite volume scheme leads naturally to the correct transmission operators when using non-matching grids in different subdomains.

Suppose the point $(x = a_i, t = n\Delta t_i)$ is on the left boundary of subdomain $\Omega_i \times (0, T)$, as shown in Figure 5.1 on the right. Then we have only half a volume $D$ to integrate over. Proceeding as before, we obtain

$$\begin{aligned} 0 = &\int_x^{x+\Delta x/2} \frac{1}{c^2(\xi)} \frac{\partial u}{\partial t}(\xi, t + \Delta t/2) d\xi - \int_x^{x+\Delta x/2} \frac{1}{c^2(\xi)} \frac{\partial u}{\partial t}(\xi, t - \Delta t/2) d\xi \\ &- \int_{t-\Delta t/2}^{t+\Delta t/2} \frac{\partial u}{\partial x}(x + \Delta x/2, \tau) d\tau + \int_{t-\Delta t/2}^{t+\Delta t/2} \frac{\partial u}{\partial x}(x, \tau) d\tau. \end{aligned}$$

Again we can approximate $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ by the finite differences given in (5.2), except on the left side of the control volume where we can not approximate $\frac{\partial u}{\partial x}(x, \tau)$ by a finite difference, since we are on the boundary and the point at $x - \Delta x$ is not available. We approximate only on the three other sides by finite differences and obtain

$$0 = \left(\frac{\Delta x}{2C^2(0)}(D_t^+ - D_t^-) - \Delta t D_x^+\right)(U)(0,n) + \int_{t-\Delta t/2}^{t+\Delta t/2} \frac{\partial u}{\partial x}(x, \tau) d\tau. \tag{5.5}$$

Note that this equation defines the spatial derivative along the boundary, once all the grid values are known. But to compute the grid values, we need to use the transmission condition imposed on the left boundary which also defines the spatial derivative at the boundary, since it is of the form

$$\mathcal{B}^-(u)(x,t) = \left(\frac{1}{c(x^-)} \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}\right)(x,t) = g^-(t) \tag{5.6}$$

where $g^-(t)$ is a given boundary condition. Solving for $\frac{\partial u}{\partial x}$ and integrating we find

$$\int_{t-\Delta t/2}^{t+\Delta t/2} \frac{\partial u}{\partial x}(x, \tau) d\tau = \int_{t-\Delta t/2}^{t+\Delta t/2} \frac{1}{c(x^-)} \frac{\partial u}{\partial t}(x, \tau) d\tau - \int_{t-\Delta t/2}^{t+\Delta t/2} g^-(\tau) d\tau \tag{5.7}$$

which gives us the missing expression for the spatial derivative in the discrete scheme (5.5). The newly introduced time derivative on the right can be approximated again by finite differences as in (5.2), on the upper part of the integral by $D_t^+$ and on the lower part by $D_t^-$. Summing those contributions we obtain a centered finite difference,

$$D_t^0(U)(j,n) := \frac{U(j, n+1) - U(j, n-1)}{2\Delta t} \tag{5.8}$$

and we get on denoting by $C^- := c(x^-)$ for the integral of $\frac{\partial u}{\partial x}$

$$\int_{t-\Delta t/2}^{t+\Delta t/2} \frac{\partial u}{\partial x}(x,\tau)d\tau = \frac{\Delta t}{C^-} D_t^0(U)(0,n) - \int_{t-\Delta t/2}^{t+\Delta t/2} g^-(\tau)d\tau \tag{5.9}$$

which we insert into our scheme (5.5). Denoting the integral over the boundary condition $g^-(t)$ by

$$\Delta t G^-(n) := \int_{t-\Delta t/2}^{t+\Delta t/2} g^-(\tau)d\tau, \quad t = n\Delta t,$$

we obtain the discretization

$$0 = \left(\frac{\Delta x}{2C^2(0)} D_t^+ D_t^- - D_x^+ + \frac{1}{C^-} D_t^0\right)(U)(0,n) - G^-(n). \tag{5.10}$$

This result also defines the discrete transmission operator $B^-$. Comparing with (5.6) we find (where we add now the subdomain index $i$ for completeness)

$$B_i^-(U_i)(0,n) := \left(\frac{\Delta x_i}{2C_i^2(0)} D_t^+ D_t^- - D_x^+ + \frac{1}{C_{i-1}(J_{i-1}+1)} D_t^0\right)(U_i)(0,n) \tag{5.11}$$

where we used the fact that $C^- = C_{i-1}(J_{i-1}+1)$. Similarly for a point $(x,t)$ on the right boundary of a subdomain with imposed transmission condition

$$\mathcal{B}^+(u)(x,t) = \left(\frac{1}{c(x^+)}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right)(x,t) = g^+(t) \tag{5.12}$$

one obtains on defining

$$\Delta t G^+(n) := \int_{t-\Delta t/2}^{t+\Delta t/2} g^+(\tau)d\tau, \quad t = n\Delta t$$

and $C^+ := c(x^+)$ the discrete scheme

$$0 = \left(\frac{\Delta x}{2C^2(J+1)} D_t^+ D_t^- + D_x^- + \frac{1}{C^+} D_t^0\right)(U)(J+1,n) - G^+(n) \tag{5.13}$$

and thus the definition of the discrete transmission operator for subdomain $i$

$$B_i^+(U_i)(J_i+1,n) := \left(\frac{\Delta x_i}{2C_i^2(J_i+1)} D_t^+ D_t^- + D_x^- + \frac{1}{C_{i+1}(0)} D_t^0\right)(U_i)(J_i+1,n) \tag{5.14}$$

where we used that $C^+ = C_{i+1}(0)$.

**5.1.3. Points on the Initial Line.** Suppose $(x = a_i + j\Delta x_i, 0)$ is a grid point on the interior of the initial line of subdomain $\Omega_i \times (0,T)$. We have again half a volume $D$ to integrate over, as shown in Figure 5.2 on the left. Integrating as before we obtain

$$\begin{aligned} 0 = &\int_{x-\Delta x/2}^{x+\Delta x/2} \frac{1}{c^2(\xi)}\frac{\partial u}{\partial t}(\xi,\Delta t/2)d\xi - \int_{x-\Delta x/2}^{x+\Delta x/2} \frac{1}{c^2(\xi)}\frac{\partial u}{\partial t}(\xi,0)d\xi \\ &- \int_0^{\Delta t/2} \frac{\partial u}{\partial x}(x+\Delta x/2,\tau)d\tau + \int_0^{\Delta t/2} \frac{\partial u}{\partial x}(x-\Delta x/2,\tau)d\tau. \end{aligned}$$

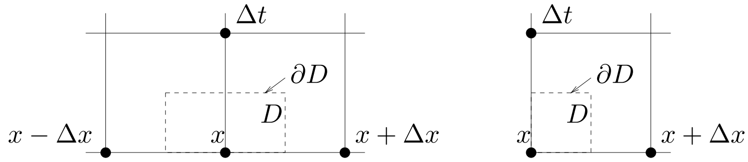


FIG. 5.2. *Control volume of a grid point on the initial line and in a corner.*

Now the remaining derivatives can be approximated by finite differences (5.2), except $\frac{\partial u}{\partial t}(\xi, 0)$. But this derivative is given explicitly by the initial condition, and approximating it on one grid cell by

$$\Delta x \frac{U_t(j)}{C^2(j)} := \int_{x-\Delta x/2}^{x+\Delta x/2} \frac{1}{c^2(\xi)} \frac{\partial u}{\partial t}(\xi, 0) d\xi,$$

we obtain the scheme

$$\left( \frac{1}{C^2(j)} D_t^+ - \frac{\Delta t}{2} D_x^+ D_x^- \right)(U)(j, 0) - \frac{1}{C^2(j)} U_t(j) = 0. \tag{5.15}$$

**5.1.4. Corner Points.** For the corner points on the initial line, there is only a quarter of the original finite volume left to integrate over. For example on the left corner we obtain according to Figure 5.2

$$\begin{aligned} 0 = & \int_0^{\Delta x/2} \frac{1}{c^2(\xi)} \frac{\partial u}{\partial t}(\xi, \Delta t/2) d\xi - \int_0^{\Delta x/2} \frac{1}{c^2(\xi)} \frac{\partial u}{\partial t}(\xi, 0) d\xi \\ & - \int_0^{\Delta t/2} \frac{\partial u}{\partial x}(\Delta x/2, \tau) d\tau + \int_0^{\Delta t/2} \frac{\partial u}{\partial x}(0, \tau) d\tau. \end{aligned}$$

Here two of the remaining derivatives can be approximated by the finite differences (5.2), whereas $\frac{\partial u}{\partial t}(\xi, 0)$ is given by the initial condition and $\frac{\partial u}{\partial x}(0, \tau)$ has to be obtained from the transmission condition by proceeding as before along the boundary. We obtain the discrete scheme

$$0 = \left( \frac{\Delta x}{2C^2(0)} D_t^+ - \frac{\Delta t}{2} D_x^+ + \frac{\Delta t}{2C^-} D_t^+ \right)(U)(0, 0) - \frac{\Delta x}{2C^2(0)} U_t(0) - \frac{\Delta t}{2} G^-(0)$$

and thus the discrete transmission operator $B^-$ on the initial line on the left is obtained by dividing through by $\Delta t/2$,

$$B_i^-(U_i)(0,0) = \left( \frac{\Delta x_i}{\Delta t C_i^2(0)} D_t^+ - D_x^+ + \frac{1}{C_{i-1}(J_{i-1}+1)} D_t^+ \right)(U_i)(0, 0) - \frac{\Delta x_i}{\Delta t C_i^2(0)} U_{t,i}(0). \tag{5.16}$$

Similarly for the corner point on the right, we get

$$0 = \left( \frac{\Delta x}{2C^2(J+1)} D_t^+ + \frac{\Delta t}{2} D_x^- + \frac{\Delta t}{2C^+} D_t^+ \right)(U)(J+1, 0) - \frac{\Delta x}{2C^2(J+1)} U_t(J+1) - \frac{\Delta t}{2} G^+(0)$$

and thus the discrete transmission operator $B^+$ on the initial line on the right is

$$\begin{aligned} B_i^+(U_i)(J_i+1, 0) = & \left( \frac{\Delta x_i}{\Delta t C_i^2(J_i+1)} D_t^+ + D_x^- + \frac{1}{C_{i+1}(0)} D_t^+ \right)(U_i)(J_i+1, 0) \\ & - \frac{\Delta x_i}{\Delta t C_i^2(J_i+1)} U_{t,i}(J_i+1). \end{aligned} \tag{5.17}$$

For given discrete transmission conditions $G^-(n)$ and $G^+(n)$, $n = 0 \dots N$, the above discrete scheme describes a numerical method to solve one subproblem on one subdomain.

**5.2. Extraction of the Transmission Conditions from Neighboring Subdomains.** Now the boundary values $g_i^-(t)$ and $g_i^+(t)$ imposed through the transmission conditions on subdomain $\Omega_i$ have to come from the neighboring subdomains $\Omega_{i-1}$ and $\Omega_{i+1}$. We thus need to calculate from our discrete scheme above on the neighboring subdomains the integrals

$$\int_{t_i-\Delta t_i/2}^{t_i+\Delta t_i/2} g_i^-(\tau)d\tau = \int_{t_i-\Delta t_i/2}^{t_i+\Delta t_i/2} \left[\frac{1}{c(x^-)}\frac{\partial u_{i-1}}{\partial t}(x,\tau) - \frac{\partial u_{i-1}}{\partial x}(x,\tau)\right] d\tau \tag{5.18}$$

where $(x,t)$ is on the right of subdomain $\Omega_{i-1}$ and similarly

$$\int_{t_i-\Delta t_i/2}^{t_i+\Delta t_i/2} g_i^+(\tau)d\tau = \int_{t_i-\Delta t/2}^{t_i+\Delta t_i/2} \left[\frac{1}{c(x^+)}\frac{\partial u_{i+1}}{\partial t}(x,\tau) + \frac{\partial u_{i+1}}{\partial x}(x,\tau)\right] d\tau \tag{5.19}$$

where $(x,t)$ is on the left of the subdomain $\Omega_{i+1}$. Lets take for example the subdomain to the right, $\Omega_{i+1}$. To perform the integration (5.19) we note that the numerical approximation to $\frac{\partial u_{i+1}}{\partial t}$ in the finite volume scheme is piecewise constant and according to (5.2) given by $D_t^+(U_{i+1})(0,n)$ for $t \in [n\Delta t_{i+1},(n+1)\Delta t_{i+1})$. Similarly the numerical approximation to $\frac{\partial u_{i+1}}{\partial x}$ is piecewise constant in the finite volume scheme. According to (5.5) it is given for $t \in [(n-\frac{1}{2})\Delta t_{i+1},(n+\frac{1}{2})\Delta t_{i+1})$ by

$$\left(-\frac{\Delta x_{i+1}}{2\Delta t_{i+1}C_{i+1}^2(0)}(D_t^+ - D_t^-) + D_x^+\right)(U_{i+1})(0,n).$$

Inserting these two numerical approximations into (5.19) we obtain on one grid cell of $\Omega_{i+1}$

$$\int_{(n-1/2)\Delta t_{i+1}}^{(n+1/2)\Delta t_{i+1}} g_i^+(\tau)d\tau = \left(-\frac{\Delta x_{i+1}\Delta t_{i+1}}{2C_{i+1}^2(0)}D_t^+D_t^- + \Delta t_{i+1}D_x^+ + \frac{\Delta t_{i+1}}{C_{i+1}(0)}D_t^0\right)(U_{i+1})(0,n)$$

and thus the definition of the discrete transmission operator $\widetilde{B}_i^+$ is

$$\widetilde{B}_i^+(U_{i+1})(0,n) := \left(-\frac{\Delta x_{i+1}}{2C_{i+1}^2(0)}D_t^+D_t^- + D_x^+ + \frac{1}{C_{i+1}(0)}D_t^0\right)(U_{i+1})(0,n) = \widetilde{G}_i^+(n). \tag{5.20}$$

Similarly we find on the left subdomain $\Omega_{i-1}$ the discrete transmission operator $\widetilde{B}_i^-$ to be

$$\begin{aligned}\widetilde{B}_i^-(U_{i-1})(J_{i-1}+1,n) :=& \left(-\frac{\Delta x_{i-1}}{2C_{i-1}^2(J_{i-1}+1)}D_t^+D_t^- - D_x^- + \frac{1}{C_{i-1}(J_{i-1}+1)}D_t^0\right)(U_{i-1})(J_{i-1}+1,n)\\ &= \widetilde{G}_i^-(n).\end{aligned} \tag{5.21}$$

Note that in the discrete case $B_i^\pm$ and $\widetilde{B}_i^\pm$ are different operators, whereas in the continuous case we found the identical operator $\mathcal{B}_i^\pm$. On the initial line we find accordingly

$$
\begin{aligned}
\widetilde{B}_i^+(U_{i+1})(0,0) &:= \left(-\tfrac{\Delta x_{i+1}}{\Delta t_{i+1}C_{i+1}^2(0)}D_t^+ + D_x^+ + \tfrac{1}{C_{i+1}(0)}D_t^+\right)(U_{i+1})(0,0)\\
&\quad + \tfrac{\Delta x_{i+1}}{\Delta t_{i+1}C_{i+1}^2(0)}U_t(0),\\
\widetilde{B}_i^-(U_{i-1})(J_{i-1}+1,0) &:= \left(-\tfrac{\Delta x_{i-1}}{\Delta t_{i-1}C_{i-1}^2(J_{i-1}+1)}D_t^+ - D_x^- + \tfrac{1}{C_{i-1}(J_{i-1}+1)}D_t^+\right)(U_{i-1})(J_{i-1}+1,0)\\
&\quad + \tfrac{\Delta x_{i-1}}{\Delta t_{i-1}C_{i-1}^2(J_{i-1}+1)}U_t(J_{i-1}+1)
\end{aligned}
$$

(5.22)

and we have

$$\widetilde{B}_i^+(U_{i+1})(0,0) = \widetilde{G}_i^+(0), \quad \widetilde{B}_i^-(U_{i-1})(J_{i-1}+1,0) = \widetilde{G}_i^-(0).$$

**5.3. Projections for Different Grids.** If different grids are used on different subdomains, the extracted transmission condition $\widetilde{G}_i^+$ is a vector in $\mathbb{R}^{N_{i+1}+1}$ and $\widetilde{G}_i^-$ is a vector in $\mathbb{R}^{N_{i-1}+1}$ which both represent step functions on their corresponding grids and what we need to impose on the boundary on $\Omega_i$ are vectors $G_i^\pm$ in $\mathbb{R}^{N_i+1}$. We thus need to introduce a projection operation to transfer the boundary values onto the grid of $\Omega_i$. Suppose we are given a vector $\boldsymbol{v} = (v_0, \ldots, v_N) \in \mathbb{R}^{N+1}$ which represents the values of a step function on the corresponding intervals $I_n = (t_n, t_{n+1})$ where $t_0 = 0$, $t_{N+1} = T$ and $\cup_{n=0}^N \overline{I_n} = [0,T]$ and the intervals do not overlap. Then we define the scalar product on $\mathbb{R}^{N+1}$ by

$$(\boldsymbol{v}, \boldsymbol{w})_{N+1} := \sum_{n=0}^{N} |I_n| v_n w_n$$

where $|I_n|$ denotes the length of the interval $I_n$. We thus obtain the induced norm on $\mathbb{R}^{N+1}$

$$||\boldsymbol{v}||_{N+1}^2 := (\boldsymbol{v}, \boldsymbol{v})_{N+1}.$$

We first define the operator $\mathbb{F} : \mathbb{R}^{N+1} \longrightarrow L^2(0,T)$ which constructs a piecewise constant function on the intervals $I_n$ from the vector $\boldsymbol{v}$,

$$\mathbb{F} : \boldsymbol{v} \longmapsto f(t) := v_n, \quad t \in I_n$$

Then we define the operator $\mathbb{E} : L^2(0,T) \longrightarrow \mathbb{R}^{N+1}$ which projects a given function $f(t)$ onto a vector $\boldsymbol{v} \in \mathbb{R}^{N+1}$ corresponding to a piecewise constant function in the intervals $I_n$

$$\mathbb{E} : f(t) \longmapsto v_n := \frac{1}{|I_n|}\int_{I_n} f(t)dt.$$

Denoting by $\mathbb{F}_i$ and $\mathbb{E}_i$ the corresponding operators using the grid of $\Omega_i$, we define the operator $\mathbb{P}_{i,j} : \mathbb{R}^{N_i+1} \longrightarrow \mathbb{R}^{N_j+1}$ by

$$\mathbb{P}_{i,j} := \mathbb{E}_j \circ \mathbb{F}_i. \tag{5.23}$$

A direct calculation shows that for any $u$ in $\mathbb{R}^{N_i+1}$ we have

$$||\mathbb{P}_{i,j}u||_{N_j+1} \le ||u||_{N_i+1} \tag{5.24}$$

which is a natural consequence of the $L^2$ projection on piecewise constant functions. To perform the projection $\mathbb{P}_{i,j}$ between arbitrary grids is a nontrivial task, since one needs to find the intersections of corresponding arbitrary grid cells. For one dimension however, there is a short, concise algorithm, see Appendix A.

**5.4. The Discrete Schwarz Waveform Relaxation Algorithm.** We obtain the discrete Schwarz waveform relaxation algorithm on subdomains $\Omega_i$, $i = 1 \dots I$ with non-matching grids

$$
\begin{aligned}
\left(\tfrac{1}{C_i^2(j)} D_t^+ D_t^- - D_x^+ D_x^-\right)(U_i^{k+1})(j,n) &= 0, \quad 1 \le j \le J_i,\ 1 \le n \le N_i,\\
\left(\tfrac{1}{C_i^2(j)} D_t^+ - \tfrac{\Delta t_i}{2} D_x^+ D_x^-\right)(U_i^{k+1})(j,0) - \tfrac{1}{C_i^2(j)} U_{t,i}(j) &= 0, \quad 1 \le j \le J_i,\\
B_i^-(U_i^{k+1})(0,\cdot) &= \mathbb{P}_{i-1,i}\widetilde{B}_i^-(U_{i-1}^k)(J_{i-1}+1,\cdot),\\
B_i^+(U_i^{k+1})(J_i+1,\cdot) &= \mathbb{P}_{i+1,i}\widetilde{B}_i^+(U_{i+1}^k)(0,\cdot),
\end{aligned}
\tag{5.25}
$$

where the operators $\mathbb{P}_{i\pm1,i}$ are defined in (5.23), the discrete transmission operators $B_i^\pm$ for $n \ge 1$ are given in (5.11), (5.14) and the extraction operators $\widetilde{B}_i^\pm$ are for $n \ge 1$ given in (5.20), (5.21). For $n = 0$ the corresponding operators are given in (5.16), (5.17) and (5.22). For conforming grids with these transmission conditions, the solution obtained at convergence satisfies the finite volume discretization scheme without decomposition, as one can see by taking the difference of $B_i^\pm$ and $\widetilde{B}_i^\pm$. For example for constant wave speed across the interface we obtain

$$
B_i^+ - \widetilde{B}_i^+ = \frac{\Delta x}{C^2} D_t^+ D_t^- + D_x^- - D_x^+ = \Delta x\left(\frac{1}{C^2} D_t^+ D_t^- - D_x^- D_x^+\right)
$$

which is the discretized wave operator (5.4). For non-conforming grids (5.25) defines the solution when converged. For a different definition of a solution on non-matching grids see [10].

**6. Normal Mode Analysis and Convergence Proof for Piecewise Constant Wave Speed and Two Subdomains.** We consider two sub-domains $\Omega_i, i = 1,2$ with piecewise constant velocity $c_i$ per subdomain. We discretize the problem on each subdomain in space with spatial discretization parameter $\Delta x_i$ and we keep the time discretization uniform across the subdomains with discretization parameter $\Delta t$. Then there is no projection in the transmission operators in (5.25).We denote by $\gamma_i$ the CFL number in the corresponding subdomain $\Omega_i$,

$$
\gamma_i = c_i \frac{\Delta t}{\Delta x_i}
\tag{6.1}
$$

For the stability of the Cauchy problem, we suppose that $\gamma_i < 1$ [38]. By linearity it suffices to analyze algorithm (5.25) for homogeneous initial conditions and to prove convergence to zero. To avoid the special case of the interface conditions for $n = 0$ in the analysis, we set $U(j,0) = U(j,1) = 0$ which corresponds to initial conditions $u(x,0) = u_t(x,0) = 0$.

**6.1. Discrete Laplace Transforms.** The discrete Laplace transform of a grid function $v = \{v_n\}_{n\ge0}$ on a regular grid with time step $\Delta t$ is defined for $\eta > 0$ by [38]

$$
\mathcal{L}v(s) = \hat{v}(s) = \frac{1}{\sqrt{2\pi}} \Delta t \sum_{n\ge0} e^{-sn\Delta t} v_n, \quad s = \eta + i\tau,\ |\tau| \le \frac{\pi}{\Delta t},
\tag{6.2}
$$

and the inversion formula is given by

$$
v_n = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{\Delta t}}^{\frac{\pi}{\Delta t}} e^{sn\Delta t} \hat{v}(s) d\tau = -\frac{i}{\sqrt{2\pi}} \int_{|z|=e^{\eta\Delta t}} z^{n-1} \hat{v}(z) dz.
$$

The corresponding norms are

$$(6.3) \qquad ||v||_{\eta,\Delta t} = (\Delta t \sum_{n\geq 0} e^{-2\eta n\Delta t}|v_n|^2)^{\frac{1}{2}}, \qquad ||\hat{v}||_\eta = (\int_{-\frac{\pi}{\Delta t}}^{\frac{\pi}{\Delta t}} |\hat{v}(\eta + i\tau)|^2 d\tau)^{\frac{1}{2}}$$

and we have Parseval's equality

$$(6.4) \qquad ||v||_{\eta,\Delta t} = ||\hat{v}||_\eta.$$

Suppose $U(j,n)$ is a solution of the difference equation

$$(6.5) \qquad \left(\frac{1}{C^2}D_t^+ D_t^- - D_x^+ D_x^-\right)(U)(j,n) = 0$$

with the initial condition $U(j,0) = U(j,1) = 0$. We denote by $\hat{U}(j,s)$ the discrete Laplace transform in time of $U(j,n)$. Equation (6.5) becomes the difference equation

$$(6.6) \qquad \gamma^2\hat{U}(j-1,s) - 2(\gamma^2 + h(z))\hat{U}(j,s) + \gamma^2\hat{U}(j+1,s) = 0,$$

with $z = e^{s\Delta t}$, $h(z) = \frac{1}{2}(z + \frac{1}{z}) - 1$ and $\gamma = c\Delta t/\Delta x$. The solutions of (6.6) are formed by powers of the roots of the second order equation

$$(6.7) \qquad \gamma^2 r^2 - 2(\gamma^2 + h(z))r + \gamma^2 = 0.$$

We need several technical Lemmas about these roots.

LEMMA 6.1. *For $\eta = 0$ and $\tau = 0$, equation (6.7) has one double root $r_\pm = 1$. For $\eta = 0$ and $|\sin(\frac{\tau\Delta t}{2})| = \gamma$, equation (6.7) has one double root $r_\pm = -1$.*

*Proof.* One can do the analysis on a case by case basis. □

LEMMA 6.2. *For $|z| > 1$ (i.e. $\eta > 0$), equation (6.7) has one root $r_-$ whose modulus is strictly less than 1 and one root $r_+$ whose modulus is strictly bigger than 1.*

*Proof.* The discriminant of the equation (6.7) is

$$(6.8) \qquad \Delta = h(z)(2\gamma^2 + h(z))$$

and for it to vanish we have the sequence of necessary and sufficient conditions

$$\begin{aligned} \Delta = 0 \quad &\Longleftrightarrow \quad h(z) = 0 \quad \text{or} \quad 2\gamma^2 + h(z) = 0 \\ &\Longleftrightarrow \quad z = 1 \quad \text{or} \quad z + \tfrac{1}{z} - 2 + 4\gamma^2 = 0 \\ &\Longleftrightarrow \quad z = 1 \quad \text{or} \quad z = 1 - 2\gamma^2 \pm 2i\gamma\sqrt{1-\gamma^2}. \end{aligned}$$

In both cases $|z| = 1$, as a short computation in the second case shows, and $|z| = 1$ is excluded in this Lemma and treated in Lemma 6.3. Hence for $|z| > 1$ there are two distinct roots whose product equals 1. They are therefore either complex conjugate of modulus 1 or one is of modulus strictly bigger than 1 whereas the other is of modulus strictly less than 1. It thus remains to exclude the complex conjugate case. We find

$$\bar{r} = \frac{1}{r} \iff r + \bar{r} = r + \frac{1}{r} \iff \frac{\gamma^2 + h(z)}{\gamma^2} \in \mathbb{R} \iff h(z) \in \mathbb{R}$$

$$\bar{r} = \frac{1}{r} \iff z + \frac{1}{z} \in \mathbb{R} \iff \eta\Delta t = 0 \text{ or } \tau\Delta t = 0, \pm\pi.$$

If $\eta = 0$, we have $|z| = 1$, which is again excluded by the conditions of the Lemma. If on the other hand $\tau\Delta t = 0$, we compute the real part of the root $r$ and obtain

$$\Re(r) = 1 + \frac{h(z)}{\gamma^2} = 1 + \frac{2}{\gamma^2}\sinh^2(\frac{\eta\Delta t}{2}) \geq 1,$$

and if $\tau\Delta t = \pm\pi$, we have

$$\Re(r) = 1 - \frac{2}{\gamma^2}\cosh^2(\frac{\eta\Delta t}{2}) \leq -1,$$

and in both cases $|r| > 1$, a contradiction which excludes this case as well and hence proves the Lemma. □

Lemma 6.3. *For $|z| = 1$ (i.e. $\eta = 0$), $|\sin(\frac{\tau\Delta t}{2})|$ different from $0$ and $\gamma$, equation (6.7) has two distinct roots $r_-$ and $r_+$,*

$$r_\pm = \begin{cases} \frac{1}{\gamma^2}\left[\gamma^2 - 2\sin^2(\frac{\tau\Delta t}{2}) \pm 2i\sin(\frac{\tau\Delta t}{2})\sqrt{\gamma^2 - \sin^2(\frac{\tau\Delta t}{2})}\right] & \textit{if } |\sin(\frac{\tau\Delta t}{2})| < \gamma, \\ \frac{1}{\gamma^2}\left[\gamma^2 - 2\sin^2(\frac{\tau\Delta t}{2}) \mp 2\,|\sin(\frac{\tau\Delta t}{2})|\sqrt{-\gamma^2 + \sin^2(\frac{\tau\Delta t}{2})}\right] & \textit{if } |\sin(\frac{\tau\Delta t}{2})| > \gamma. \end{cases}$$

*Proof.* For $\eta = 0$, one can compute the roots directly. The only difficulty is the determination of the signs in front of the square root. In the case $|\sin(\frac{\tau\Delta t}{2})| < \gamma$, the roots are complex conjugate. We compute the roots $r_\pm(z)$ for $z = (1+\epsilon)z_0$ with $z_0 = e^{i\theta}$ and let $\epsilon$ tend to zero. The sign is then defined by continuity. For $|\sin(\frac{\tau\Delta t}{2})| > \gamma$, there are two real roots and the sign is determined by the fact that $|r_+| > 1$ and $|r_-| < 1$. □

Lemma 6.4. *For $z$ real positive, which corresponds to $\tau\Delta t = 0$, we have*

$$r_\pm = \frac{1}{\gamma^2}[\gamma^2 + 2\sinh^2(\frac{\eta\Delta t}{2}) \pm 2\sinh(\frac{\eta\Delta t}{2})\sqrt{\gamma^2 + \sinh^2(\frac{\eta\Delta t}{2})}],$$

*and $0 < r_- < 1$, $r_+ > 1$.*
*For $z$ real negative, which corresponds to $\tau\Delta t = \pi$, we have*

$$r_\pm = \frac{1}{\gamma^2}[\gamma^2 - 2\cosh^2(\frac{\eta\Delta t}{2}) \mp 2\cosh(\frac{\eta\Delta t}{2})\sqrt{-\gamma^2 + \cosh^2(\frac{\eta\Delta t}{2})}]$$

*and $-1 < r_- < 0$, $r_+ < -1$.*

*Proof.* For $\tau\Delta t = 0, \pm\pi$, which means $z$ real, one can do the analysis on a case by case basis. □

In all cases except for Lemma 6.1 there are functions $a_+(s)$ and $a_-(s)$ such that for all $j$ the solution of (6.6) is given by

$$\hat{U}(j,s) = a_+(s)r_+^j + a_-(s)r_-^j. \tag{6.9}$$

In the case of Lemma 6.1 there exists functions $a(s)$ and $b(s)$ such that for all $j$

$$\hat{U}(j,s) = (a(s)j + b(s))r_\pm^j.$$

**6.2. The Discrete Homogeneous Subdomain Problem.** We consider for example the problem posed in $\Omega_1$, with a non-homogeneous boundary condition of type $B_1^+$,

$$
\begin{array}{rcll}
\left(\frac{1}{C^2}D_t^+D_t^- - D_x^+D_x^-\right)(U)(j,n) & = & 0, & -\infty < j < 0,\ n \geq 1 \\
U(j,0) & = & U(j,1) = 0, & -\infty < j < 0, \\
\left(\frac{\Delta x}{2C^2}D_t^+D_t^- + D_x^- + \frac{1}{\alpha C}D_t^0\right)(U)(0,n) & = & g(n) & n \geq 1
\end{array}
\tag{6.10}
$$

where $\alpha$ is a given, strictly positive real number. Applying the discrete Laplace transform we obtain with the results of the previous subsection that every solution bounded in space is of the form

$$\hat{U}(j,s) = a_+(s)r_+^j.$$

Applying the discrete Laplace transform to the boundary condition, we get

$$\left(\frac{\Delta x}{C^2\Delta t^2}h(z) + \frac{1}{\Delta x}(1-r_-) + \frac{1}{\alpha C\Delta t}(z - \frac{1}{z})\right)\hat{U}(0,s) = \hat{g}(s)$$

and introducing the notation

$$k(z) = \frac{1}{2}\left(z - \frac{1}{z}\right), \quad E(z,\gamma,\alpha) = \frac{1}{\gamma^2}h(z) + 1 - r_- + \frac{1}{\alpha\gamma}k(z) \tag{6.11}$$

the boundary condition becomes $E(z,\gamma,\alpha)a_+(s) = \Delta x\hat{g}(s)$. We call the problem well posed in the sense of GKS (Gustafsson, Kreiss and Sundstrom) if the preceding equation is invertible for all $z$ with $|z| \geq 1$. If $z_0$ is such that $E(z_0,\gamma,\alpha) = 0$, we call $z_0$ a generalized eigenvalue [38].

THEOREM 6.5. *If $\gamma < 1$ and $|z| \geq 1$ with $z \neq 1$ then for any strictly positive $\alpha$, $E(z,\gamma,\alpha) \neq 0$. The only generalized eigenvalues are $z = 1$ and if $\gamma = 1$, $z = -1$.*

*Proof.* Using the relation $r_+ + r_- = \frac{2}{\gamma^2}(\gamma^2 + h(z))$ satisfied by the roots of (6.7) we find

$$E(z,\gamma,\alpha) = \frac{1}{2}(r_+ - r_-) + \frac{k(z)}{\alpha\gamma}. \tag{6.12}$$

For $z = 1$ we obtain by Lemma 6.1 that $E(1,\gamma,\alpha) = 0$. If $\gamma = 1$ we also get for $z = -1$ by Lemma 6.1 that $E(-1,\gamma,\alpha) = 0$. We have to show now that there are no other generalized eigenvalues. For any generalized eigenvalue $z$, we must have $E(z,\gamma,\alpha) = 0$, which means

$$\frac{r_+ - r_-}{2} = -\frac{k(z)}{\alpha\gamma}. \tag{6.13}$$

Squaring both sides and using the relations of the roots $r_+$ and $r_-$ of the quadratic equation (6.7) to the coefficients of that equation, we obtain

$$\frac{k^2(z)}{\alpha^2\gamma^2} = \frac{1}{4}(r_+ - r_-)^2 = \frac{1}{4}\left((r_+ + r_-)^2 - 4r_+r_-\right) = \frac{(\gamma^2 + h(z))^2}{\gamma^4} - 1 = \frac{h(z)\cdot(2\gamma^2 + h(z))}{\gamma^4}.$$

Inserting the definitions of $h(z)$ from (6.6) and $k(z)$ from (6.11) and factoring, we obtain the equation $z$ has to satisfy to be a generalized eigenvalue,

$$(z-1)^2\left((\gamma^2 - \alpha^2)z^2 + 2(\alpha^2 - 2\alpha^2\gamma^2 + \gamma^2)z + \gamma^2 - \alpha^2\right) = 0. \tag{6.14}$$

The first factor contains the generalized eigenvalue $z = 1$ we have found earlier. For $\gamma = 1$ the second factor contains the generalized eigenvalue $z = -1$ and for $\alpha = 1$ it contains again the generalized eigenvalue $z = 1$. It remains to show that for $\gamma < 1$ and $\alpha \neq 1$ the solutions of the second factor are introduced by the squaring and they are not solutions to the original equation (6.13) and therefore not generalized eigenvalues. To do this, we perform the change of variables

$$a^2 = \frac{1}{\gamma^2} - 1, \quad \epsilon b^2 = \frac{1}{\alpha^2} - 1$$

with $\epsilon = \pm 1$, $\epsilon(\frac{1}{\alpha^2} - 1) > 0$, $a > 0$, $b > 0$ in the second factor of (6.14). Note that we exclude the case $b = 0 \Leftrightarrow \alpha = 1$ because then the only solution is $z = 1$. We obtain after the change of variables for the second factor of (6.14)

$$(\epsilon b^2 - a^2)z^2 + 2(\epsilon b^2 + a^2)z + \epsilon b^2 - a^2 = 0$$

or equivalently

$$\epsilon b^2(z+1)^2 = a^2(z-1)^2.$$

Now if $\epsilon = +1$, the only root with modulus greater or equal to 1 is

$$z_1 = \frac{a+b}{a-b}.$$

Using Lemma 6.4 we see that the signs of $(r_+ - r_-)(z_1)$ and $k(z_1)$ are equal, which contradicts (6.13) and thus $z_1$ is not a generalized eigenvalue, $E(z_1) \neq 0$. If $\epsilon = -1$ there are two complex conjugate roots of modulus 1,

$$z_1 = \frac{a - ib}{a + ib} = e^{i\tau\Delta t}, \quad z_2 = \frac{a + ib}{a - ib} = e^{-i\tau\Delta t}.$$

To apply Lemma 6.3 we need to check that $|\sin(\tau\Delta t/2)|$ is different from 0 and $\gamma$. To do so, note that the real part of both $z_1$ and $z_2$ is given by $\cos\tau\Delta t = \frac{a^2-b^2}{a^2+b^2}$ and thus we obtain for $|\sin(\tau\Delta t/2)|$

$$\sin^2\frac{\tau\Delta t}{2} = \frac{1}{2} - \frac{1}{2}\cos(\tau\Delta t) = \frac{b^2}{a^2+b^2} > 0.$$

Now since $\epsilon = -1$ we have $b^2 < 1$ and therefore

$$\sin^2\frac{\tau\Delta t}{2} = \frac{b^2}{a^2+b^2} < \frac{1}{a^2+1} = \gamma^2.$$

Hence the first case of Lemma 6.3 applies and we obtain

$$r_+ - r_- = \frac{4i}{\gamma^2}\sin(\frac{\tau\Delta t}{2})\sqrt{\gamma^2 - \sin^2\frac{\tau\Delta t}{2}}, \quad k(z) = 2i\sin\tau\Delta t$$

which again contradicts (6.13) because of the sign. The results for $z_2$ are the same with a sign change. $\square$

The values $z = 1$ (and $z = -1$ if $\gamma = 1$) are the generalized eigenvalues in the sense of GKS. Following the analysis of Trefethen [39], they correspond to stationary solutions which propagate at the same time towards the left and the right. We will see that this does not affect the convergence of the Schwarz method.

**6.3. Convergence Rate.** We denote by $\hat{U}_i^k(j,s)$, $i=1,2$ the discrete Laplace transforms in time of the iterates $U_i^k(j,n)$, $i=1,2$ in algorithm (5.25). We obtain with the results from Subsection 6.1

$$\hat{U}_1^k(j,s) = \hat{U}_1^k(0,s) r_{1,+}^j \qquad \hat{U}_2^k(j,s) = \hat{U}_2^k(0,s) r_{2,-}^j \tag{6.15}$$

where $r_{i,+}$and $r_{i,-}$ are the roots of the second order equation (6.7) defined in each subdomain,

$$\gamma_i^2 r^2 - 2(\gamma_i^2 + h(z))r + \gamma_i^2 = 0, \quad \text{with } \gamma_i = c_i \tfrac{\Delta t}{\Delta x_i}.$$

The coefficients $\hat{U}_1^k(0,s)$ and $\hat{U}_2^k(0,s)$ are determined iteratively by the Laplace transform of the transmission conditions in (5.25). The discrete transmission operators in the Laplace transformed domain are given by

$$\begin{aligned} b_1^+(z) &= \frac{1}{\gamma_1^2} h(z) + 1 - r_{1,-} + \frac{c_1}{c_2\gamma_1} k(z), \\ \widetilde{b}_1^+(z) &= -\frac{1}{\gamma_2^2} h(z) - 1 + r_{2,-} + \frac{1}{\gamma_2} k(z), \\ b_2^-(z) &= \frac{1}{\gamma_2^2} h(z) + 1 - r_{2,-} + \frac{c_2}{c_1\gamma_2} k(z), \\ \widetilde{b}_2^-(z) &= -\frac{1}{\gamma_1^2} h(z) - 1 + r_{1,-} + \frac{1}{\gamma_1} k(z). \end{aligned} \tag{6.16}$$

The transmission conditions impose therefore

$$\begin{aligned} \frac{1}{\Delta x_1} b_1^+(z) \hat{U}_1^{k+1}(0,s) &= \frac{1}{\Delta x_2} \widetilde{b}_1^+(z) \hat{U}_2^k(0,s) \\ \frac{1}{\Delta x_2} b_2^-(z) \hat{U}_2^{k+1}(0,s) &= \frac{1}{\Delta x_1} \widetilde{b}_2^-(z) \hat{U}_1^k(0,s) \end{aligned}$$

Inserting the second equation at iteration $k$ into the first one, we find

$$\hat{U}_1^{k+1}(0,s) = \frac{\widetilde{b}_1^+(z)}{b_1^+(z)} \frac{\widetilde{b}_2^-(z)}{b_2^-(z)} \hat{U}_1^{k-1}(0,s)$$

and a similar relation for $\hat{U}_2^{k+1}(0,s)$. Defining

$$\sigma(z,\gamma) = \frac{1}{\gamma^2} h(z) + 1 - r_- = \frac{1}{2}(r_+ - r_-), \quad \rho(z,\gamma,q) = \frac{-\sigma(z,\gamma) + \frac{1}{\gamma}k(z)}{\sigma(z,\gamma) + \frac{q}{\gamma}k(z)} \tag{6.17}$$

we obtain for the convergence rate of the discrete Schwarz waveform relaxation algorithm

$$R(z,\gamma_1,\gamma_2,c_1/c_2) := \frac{\widetilde{b}_1^+(z)}{b_1^+(z)} \frac{\widetilde{b}_2^-(z)}{b_2^-(z)} = \rho(z,\gamma_2,c_2/c_1)\rho(z,\gamma_1,c_1/c_2) \tag{6.18}$$

and by induction we find

$$\hat{U}_i^{2k}(0,s) = R^k \hat{U}_i^0(0,s), \quad i=1,2. \tag{6.19}$$

Lemma 6.6. *The convergence rate $R(z,\gamma_1,\gamma_2,c_1/c_2)$ is an analytic function of $z$ for $|z| \geq 1$.*

*Proof.* By Theorem 6.5, for $\gamma < 1$, $z = 1$ is the only root of $E(z, \gamma, \alpha)$ such that $|z| \geq 1$. Furthermore a Taylor expansion shows that it is a simple root, for $z$ close to 1 we have

$$E(z, \gamma, \alpha) \approx \frac{\alpha + 1}{\alpha\gamma}(z - 1).$$

With equations (6.16) and (6.18) we see that $z = 1$ is only an apparent pole for $R$ which concludes the proof. $\square$

Since $R$ is analytic for $|z| \geq 1$ which corresponds to $\eta \geq 0$, $R$ satisfies a maximum principle for $\eta \geq 0$ and hence attains its maximum on the boundary $\eta = 0$. It therefore suffices to study the behavior of $R$ for $\eta = 0$ and we do this by studying the factors $\rho(z, \gamma, q)$ for $\eta = 0$. Setting $\omega := \frac{\tau \Delta t}{2}$, we consider $\omega$ varying between 0 and $\frac{\pi}{2}$ (the same computations apply for negative $\omega$). For $\eta = 0$ we have the explicit formulas

$$\rho(z, \gamma, q) = \begin{cases} \dfrac{-\sqrt{\gamma^2 - \sin^2\omega} + \gamma\cos\omega}{\sqrt{\gamma^2 - \sin^2\omega} + q\gamma\cos\omega}, & \text{if } \sin\omega < \gamma, \\ \dfrac{\sqrt{\sin^2\omega - \gamma^2} + i\gamma\cos\omega}{-\sqrt{\sin^2\omega - \gamma^2} + iq\gamma\cos\omega}, & \text{if } \sin\omega > \gamma, \\ \dfrac{1}{q}, & \text{if } \sin\omega = \gamma. \end{cases}$$

To find a first necessary condition for convergence of the Schwarz method, we choose $\omega_1$ such that $\sin\omega_1 = \gamma_1$, $\omega_1 \in (0, \frac{\pi}{2})$, and $q = \dfrac{c_1}{c_2}$. We obtain for the convergence rate at $\eta = 0$

$$R = \frac{1}{q}\begin{cases} \dfrac{\gamma_2\cos\omega_1 - \sqrt{\gamma_2^2 - \gamma_1^2}}{\gamma_2\cos\omega_1 + q\sqrt{\gamma_2^2 - \gamma_1^2}}, & \text{if } \gamma_1 < \gamma_2, \\ \dfrac{i\gamma_2\cos\omega_1 + \sqrt{\gamma_2^2 - \gamma_1^2}}{i\gamma_2\cos\omega_1 - q\sqrt{\gamma_2^2 - \gamma_1^2}}, & \text{if } \gamma_1 > \gamma_2. \end{cases}$$

In the first case, $R$ is a real number strictly between 0 and 1, in the second case if $q < 1$, $|R| > 1$. We therefore have the

THEOREM 6.7. *If the convergence rate $R$ given in (6.18) of the discrete Schwarz waveform relaxation algorithm is bounded by 1 for all $z$ of modulus bigger or equal to 1, then*

$$(c_1 - c_2)(\gamma_1 - \gamma_2) \geq 0, \tag{6.20}$$

*in other words, $c_1 > c_2$ implies $\gamma_1 \geq \gamma_2$.*

We now study the variations of $|\rho(z, \gamma, q)|$, for $\eta = 0$ as a function of $\omega$. An example is shown in Figure 6.1 for the case $\gamma_1 < \gamma_2$ and $q = c_1/c_2 < 1$. The complete results are obtained by explicitly computing the derivatives and are summarized in Table 6.1. They rely on

$$\frac{d}{dz}\rho(z, \gamma, q) = \frac{(q + 1)(\gamma^2 - 1)}{\gamma^5}\frac{h(z)^2}{z\sigma(z)(\sigma(z) + \frac{q}{\gamma}k(z))^2} \tag{6.21}$$

where $h$ is given in (6.6), $k$ in (6.11) and $\sigma$ in (6.17). For $\omega \leq \arcsin(\gamma_2)$, we have

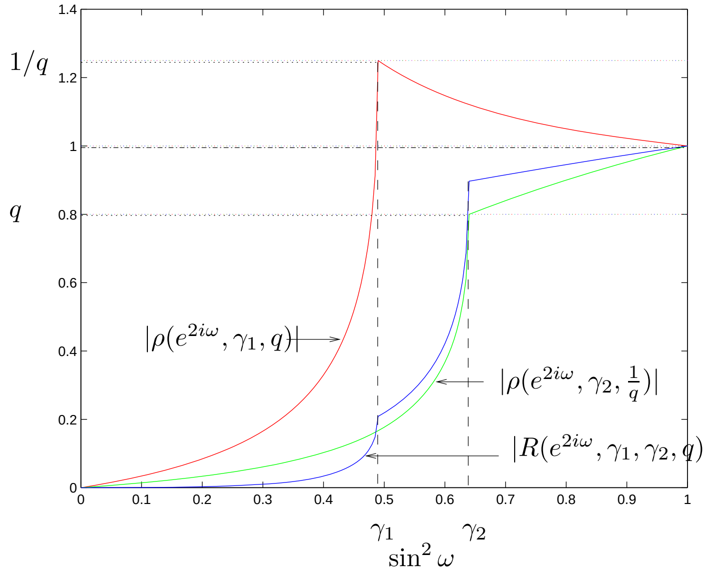


FIG. 6.1. *Example of the dependence of* $|\rho(e^{2i\omega},\gamma,q)|$ *as function of* $\omega$ *at* $\eta=0$, $q=c_1/c_2$.

| | $\omega=0$ | $0<sin(\omega)<\gamma$ | $sin(\omega)=\gamma$ | $\gamma<\sin(\omega)<1$ | $sin(\omega)=1$ |
|---|---|---|---|---|---|
| $q<1$ | 0 | $\nearrow$ | $1/q$ | $\searrow$ | 1 |
| $q>1$ | 0 | $\nearrow$ | $1/q$ | $\nearrow$ | 1 |

TABLE 6.1
*Behavior of* $|\rho(e^{2i\omega},\gamma,q)|$ *as function of* $\omega>0$ *for* $\eta=0$.

$|\rho(e^{2i\omega},\gamma_1,q)|\leq 1/q$ and $|\rho(e^{2i\omega},\gamma_2,\frac{1}{q})|\leq q$. By (6.21), a final explicit computation shows that the modulus of $R$ is an increasing function of $\omega$ for $\omega\geq\arcsin(\gamma_2)$ and

$$\sup_{\omega\in[0,\frac{\pi}{2}]}|R(e^{2i\omega},\gamma_1,\gamma_2,q)|=|R(-1,\gamma_1,\gamma_2,q)|=1. \tag{6.22}$$

We therefore have

THEOREM 6.8. *For* $(c_1-c_2)(\gamma_1-\gamma_2)\geq 0$ *the convergence rate* $R(z,\gamma_1,\gamma_2,c_1/c_2)$ *satisfies*

$$\sup_{|z|=1}|R(z,\gamma_1,\gamma_2,c_1/c_2)|=1,$$

*For purely propagating modes,* $\eta=0$*, the convergence rate equals* 1.

For $\eta>0$ however, we have the convergence result

THEOREM 6.9. *For* $(c_1-c_2)(\gamma_1-\gamma_2)\geq 0$ *and* $\eta>0$ *fixed, there exists a constant* $K$ *strictly positive such that, for* $\eta\Delta t$ *sufficiently small but non-zero the convergence rate satisfies*

$$\sup_{|z|=e^{\eta\Delta t}}|R(z,\gamma_1,\gamma_2,c_1/c_2)|\leq 1-K\eta\Delta t.$$

*Proof.* By (6.21) we can calculate the derivative of $|\rho|$ with respect to $\eta\Delta t$ and get

$$\frac{d}{d(\eta\Delta t)}(|\rho(z,\gamma,q)|^2)\bigg|_{\eta\Delta t=0} = 2\frac{(q+1)(\gamma^2-1)}{\gamma^5}\Re\frac{h(z)^2\bar{\rho}(z)}{\sigma(z)(\sigma(z)+\frac{q}{\gamma}k(z))^2}.$$

For $\eta\Delta t = 0$, $h$ is real, $k$ is purely imaginary and by Lemma 6.3 we have the explicit value of $\sigma$ which is purely imaginary for $sin(\omega) < \gamma$ and real negative for $sin(\omega) > \gamma$. We thus have

$$\begin{cases} \frac{d}{d(\eta\Delta t)}(|\rho(z,\gamma,q)|^2)\Big|_{\eta\Delta t=0} = 0 & \text{if } \sin(\omega) < \gamma, \\ \frac{d}{d(\eta\Delta t)}(|\rho(z,\gamma,q)|^2)\Big|_{\eta\Delta t=0} < 0 & \text{if } \sin(\omega) > \gamma, \end{cases}$$

which together with (6.22) gives the desired result. $\square$

**6.4. Convergence of the Discrete Schwarz Waveform Relaxation Algorithm.** We introduce the discrete norms in space and time

$$||U||_{\Omega_i,\eta,\Delta t} = (\Delta t\Delta x_i \sum_{j\in\Omega_i}\sum_{n\geq 0} e^{-2\eta n\Delta t}|U(j,n)|^2)^{\frac{1}{2}}. \tag{6.23}$$

THEOREM 6.10. *Let $U_i^p$ be the iterates of algorithm (5.25). For $(c_1-c_2)(\gamma_1-\gamma_2) \geq 0$ there exists a positive constant $K$ such that for $\eta\Delta t$ sufficiently small but non-zero, we have*

$$||U_i^p||_{\Omega_i,\eta,\Delta t} \leq (1-K\eta\Delta t)^{\lfloor\frac{p}{2}\rfloor} max_{i=1,2}||U_i^0||_{\Omega_i,\eta,\Delta t}.$$

*Proof.* By (6.15) and (6.19), we have $\hat{U}_i^{2k}(j,s) = R^k\hat{U}_i^0(j,s)$ for any $j,s$ and therefore

$$\begin{aligned} ||U_i^{2k}||^2_{\Omega_i,\eta,\Delta t} = \int_{|z|=e^{\eta\Delta t}} |R(z)|^{2k}||\hat{U}_i^0(z)||^2_{\Omega_i}dz \leq \sup_{|z|=e^{\eta\Delta t}}|R(z)|^{2k}\int_{|z|=e^{\eta\Delta t}}||\hat{U}^0(z)||^2_{\Omega_1}dz \\ \leq \sup_{|z|=e^{\eta\Delta t}}|R(z)|^{2k}||U_i^0||^2_{\Omega_i,\eta,\Delta t} \leq (1-K\eta\Delta t)^{2k}||U_i^0||^2_{\Omega_i,\eta,\Delta t}. \end{aligned}$$

A similar argument holds for $U_i^{2k+1}$. $\square$

**7. Energy Estimates and Convergence Proof for Continuous Wave Speed.** We consider here the case of $I$ subdomains, with a continuous velocity, and non uniform grids in space and time. We use the same approach as in the continuous case to prove convergence of the discrete domain decomposition algorithm. Such estimates have been used in [21] in the context of discrete absorbing boundary conditions for the wave equation, and in [10] to prove stability for a non uniform scheme.

**7.1. Stability for the Discrete Subdomain Problem.** Let $U(j,n)$ for $0 \leq j \leq J+1$ and $0 \leq n \leq N$ solve the leap-frog scheme

$$\frac{1}{C^2(j)}D_t^+D_t^-(U)(j,n) - D_x^+D_x^-(U)(j,n) = 0, \quad 1 \leq j \leq J. \tag{7.1}$$

Here $n$ stands for the discrete time variable, and $j$ for the discrete space variable. We define a discrete energy. First we denote by $V = \{V(j)\}_{0\leq j\leq J+1}$ a sequence in $\mathbb{R}^{J+2}$, and we define for $V, W \in \mathbb{R}^{J+2}$ a bilinear form on $\mathbb{R}^{J+2}$ by

$$a_h(V,W) = \frac{\Delta x}{2} \sum_{j=1}^{J+1} D_x^-(V)(j) \cdot D_x^-(W)(j). \tag{7.2}$$

Accordingly, for any positive $n$, $V(n)$ stands for the sequence $\{V(j,n)\}_{0\le j\le J+1}$. The discrete energy $E_n$ at time step $n$, global in space, is defined as the sum of a discrete kinetic energy $E_{K,n}$ and a discrete potential energy $E_{P,n}$ given by

$$\begin{aligned}
E_{K,n} &= \frac{\Delta x}{2}\left[\frac{1}{2C^2(0)}(D_t^-(V)(0,n))^2+\sum_{j=1}^{J}\frac{1}{C^2(j)}(D_t^-(V)(j,n))^2+\frac{1}{2C^2(J+1)}(D_t^-(V)(J+1,n))^2\right],\\
E_{P,n} &= a_h(V(n),V(n-1)),\\
E_n &= E_{K,n} + E_{P,n}.
\end{aligned} \tag{7.3}$$

The quantity $E_{K,n}$ is clearly a discrete kinetic energy. It is less evident to identify $E_n$ as an energy. The following lemma gives a lower bound for $E_n$ under a CFL condition and hence shows that $E_n$ is indeed an energy.

LEMMA 7.1. *For any $n \ge 1$, we have*

$$E_n \ge \left(1 - \left(C\frac{\Delta t}{\Delta x}\right)^2\right) E_{K,n}, \tag{7.4}$$

*where $C$ is defined by $C = \sup_{1\le j\le J+1} C(j)$. Hence, under the CFL condition*

$$C\frac{\Delta t}{\Delta x} < 1, \tag{7.5}$$

*$E_n$ is bounded from below by an energy.*

*Proof.* For any $V, W \in \mathbb{R}^{J+2}$ we have

$$a_h(V,W) = \frac{1}{4}A_h(V+W) - \frac{1}{4}A_h(V-W) \tag{7.6}$$

with $A_h$ defined by $A_h(V) = a_h(V,V)$. Since $a_h$ is a positive bilinear form, the first term on the right-hand side is positive, which gives a first lower bound on $E_n$,

$$E_n \ge E_{K,n} - \frac{1}{4}A_h(V(n) - V(n-1)). \tag{7.7}$$

It remains to estimate the second term on the right-hand side. Using the well known inequality

$$(a+b)^2 \le 2(a^2+b^2) \tag{7.8}$$

we obtain

$$\begin{aligned}
A_h(V(n)-V(n-1)) &= \frac{\Delta x}{2}\sum_{j=1}^{J+1}\left[D_x^-(V)(j,n) - D_x^-(V)(j,n-1)\right]^2\\
&= \frac{\Delta x}{2}\sum_{j=1}^{J+1}\frac{\Delta t^2}{\Delta x^2}\left[D_t^-(V)(j,n) - D_t^-(V)(j-1,n)\right]^2\\
&\le C^2\frac{\Delta t^2}{\Delta x^2}\left[\Delta x\sum_{j=1}^{J+1}\frac{1}{C^2(j)}(D_t^-(V)(j,n))^2+\Delta x\sum_{j=0}^{J}\frac{1}{C^2(j)}(D_t^-(V)(j,n))^2\right].
\end{aligned}$$

Thus

$$A_h(V(n) - V(n-1)) \leq 4C^2 \frac{\Delta t^2}{\Delta x^2} E_{K,n} \tag{7.9}$$

which, together with (7.7), gives the desired result (7.4). □

Having defined a discrete energy, we obtain the discrete energy identity

THEOREM 7.2 (Discrete Energy Identity). *For any $n \geq 1$, if $U(j,n)$ is a solution of (7.1), we have the energy identity*

$$\begin{aligned} E_{n+1} - E_n \quad &+ \quad \Delta t D_t^0(U)(0,n) \cdot (D_x^+ - \frac{\Delta x}{2C^2(0)} D_t^+ D_t^-)(U)(0,n) \\ &= \quad \Delta t D_t^0(U)(J+1,n) \cdot (D_x^- + \frac{\Delta x}{2C^2(J+1)} D_t^+ D_t^-)(U)(J+1,n). \end{aligned} \tag{7.10}$$

*Furthermore if $U(j,0)$ is a solution of (5.15), we have the energy identity*

$$\begin{aligned} E_{K,1} + E_1 \quad &+ \quad \tfrac{\Delta t}{2} D_t^+(U)(0,0) \cdot (D_x^+ - \frac{1}{C^2(0)} D_t^+)(U)(0,0) \\ &= \quad \frac{\Delta t}{2} D_t^+(U)(J+1,0) \cdot (D_x^- + \frac{1}{C^2(J+1)} D_t^+)(U)(J+1,0) \\ &+ \quad \Delta x \sum_{j=1}^{J} \frac{1}{C^2(j)} U_t(j) D_t^+(U)(j,0) + \frac{\Delta x}{2} \sum_{j=1}^{J+1} (D_x^-(U)(j,0))^2. \end{aligned} \tag{7.11}$$

*Proof.* The proof is the discrete analog to the proof in the continuous case. The problem here is that there is no canonical translation of the derivatives and the integrals. For $n \geq 1$, the appropriate choice is to multiply equation (7.1) by the centered finite differences $D_t^0(U)(j,n)$. Then we sum up for $1 \leq j \leq J$. We obtain for the derivatives in time denoted by $I_1$

$$\begin{aligned} I_1 &= \sum_{j=1}^{J} \frac{1}{C^2(j)} \left(D_t^+ D_t^-(U)(j,n)\right) \left(D_t^0(U)(j,n)\right) \\ &= \frac{1}{2\Delta t} \sum_{j=1}^{J} \frac{1}{C^2(j)} \left((D_t^+ - D_t^-)(U)(j,n)\right) \left((D_t^+ + D_t^-)(U)(j,n)\right) \\ &= \frac{1}{2\Delta t} \sum_{j=1}^{J} \frac{1}{C^2(j)} \left[\left(D_t^+(U)(j,n)\right)^2 - \left(D_t^+(U)(j,n-1)\right)^2\right] \end{aligned}$$

where we used $D_t^-(U)(j,n) = D_t^+(U(j,n-1))$, and for the derivatives in space denoted by $I_2$

$$\begin{aligned} I_2 &= \sum_{j=1}^{J} D_x^+ D_x^-(U)(j,n) \cdot D_t^0(U)(j,n) \\ &= \frac{1}{\Delta x} \left[\sum_{j=1}^{J} D_x^+(U)(j,n) \cdot D_t^0(U)(j,n) - \sum_{j=1}^{J} D_x^-(U)(j,n) \cdot D_t^0(U)(j,n)\right]. \end{aligned}$$

By a translation of indices in the first sum of $I_2$ using $D_x^+(U)(j,n) = D_x^-(U(j+1,n))$ we get

$$I_2 = \frac{1}{\Delta x} \sum_{j=1}^{J+1} D_x^-(U)(j,n) \cdot (D_t^0(U)(j-1,n) - D_t^0(U)(j,n))$$

$$+\frac{1}{\Delta x}\left(-D_x^+(U)(0,n)\cdot D_t^0(U)(0,n)+D_x^-(U)(J+1,n))\cdot D_t^0(U)(J+1,n)\right)$$
$$=\frac{1}{2\Delta t}\left[-\sum_{j=1}^{J+1}D_x^-(U)(j,n+1)\cdot D_x^-(U)(j,n)+\sum_{j=1}^{J+1}D_x^-(U)(j,n)\cdot D_x^-(U)(j,n-1)\right]$$
$$+\frac{1}{\Delta x}\left(-D_x^+(U(0,n))\cdot D_t^0(U)(0,n)+D_x^-(U(J+1,n))\cdot D_t^0(U)(J+1,n)\right).$$

We now compute the difference $I_1-I_2$ and find

$$0=\frac{1}{\Delta t}\left[\frac{1}{2}\sum_{j=1}^{J}\frac{1}{C^2(j)}(D_t^-(U)(j,n+1))^2-\frac{1}{2}\sum_{j=1}^{J}\frac{1}{C^2(j)}(D_t^-(U)(j,n))^2\right]$$
$$+\frac{1}{\Delta t\Delta x}\left(a_h(U(n+1),U(n))-a_h(U(n),U(n-1))\right)$$
$$+\frac{1}{\Delta x}\left(D_x^+(U)(0,n)\cdot D_t^0(U)(0,n)-D_x^-(U)(J+1,n)\cdot D_t^0(U)(J+1,n)\right).$$

Using the definition of $E_n$, we finally obtain

$$0=\frac{1}{\Delta t\Delta x}(E_{n+1}-E_n)+\frac{1}{4C^2(0)\Delta t}\left[-(D_t^+(U)(0,n))^2+(D_t^-(U)(0,n))^2\right]$$
$$+\frac{D_x^+(U)(0,n)\cdot D_t^0(U)(0,n)}{\Delta x}+\frac{-(D_t^+(U)(J{+}1,n))^2+(D_t^-(U)(J{+}1,n))^2}{4C^2(J{+}1)\Delta t}$$
$$-\frac{1}{\Delta x}D_x^-(U)(J+1,n)\cdot D_t^0(U)(J+1,n)$$

which gives (7.10) using the identities $D_t^+-D_t^-=\Delta t D_t^+D_t^-$ and $D_t^++D_t^-=2D_t^0$. For $n=0$, the appropriate choice is to multiply equation (5.15) by the forward finite difference $D_t^+(U)(j,0)$ and to perform the same computations. □

We define the discrete boundary operators

$$\begin{aligned}T_{\alpha,C}^- &:= \frac{1}{\alpha}D_t^0-D_x^++\frac{\Delta x}{2C^2}D_t^+D_t^- &\qquad \widetilde{T}_{\alpha,C}^- &:= \frac{1}{\alpha}D_t^0-D_x^--\frac{\Delta x}{2C^2}D_t^+D_t^-\\ T_{\alpha,C}^+ &:= \frac{1}{\alpha}D_t^0+D_x^-+\frac{\Delta x}{2C^2}D_t^+D_t^- &\qquad \widetilde{T}_{\alpha,C}^+ &:= \frac{1}{\alpha}D_t^0+D_x^+-\frac{\Delta x}{2C^2}D_t^+D_t^-\end{aligned}\tag{7.12}$$

to be applied to $U(j,n)$ for $n\geq 1$, where $\alpha$ is a positive real number. For $n=0$, $D_t^+D_t^-/2$ above is replaced by $D_t^+/\Delta t$ and $D_t^0$ is replaced by $D_t^+$. Using the identity $ab=\frac{\alpha}{4}\left((\frac{1}{\alpha}a+b)^2-(\frac{1}{\alpha}a-b)^2\right)$ for $\alpha>0$, the energy identities (7.10,7.11) can be rewritten for any positive $\alpha$ and $\beta$ as

$$\begin{aligned}E_{n+1}-E_n &+ \frac{\Delta t}{4}\left(\alpha\left(\widetilde{T}_{\alpha,C(0)}^+(U)(0,n)\right)^2+\beta\left(\widetilde{T}_{\beta,C(J+1)}^-(U)(J+1,n)\right)^2\right)\\ &= \frac{\Delta t}{4}\left(\alpha\left(T_{\alpha,C(0)}^-(U)(0,n)\right)^2+\beta\left(T_{\beta,C(J+1)}^+(U)(J+1,n)\right)^2\right).\end{aligned}\tag{7.13}$$

$$\begin{aligned}2E_{K,1}+2E_1 &+ \frac{\Delta t}{4}\left(\alpha\left(\widetilde{T}_{\alpha,C(0)}^+(U)(0,0)\right)^2+\beta\left(\widetilde{T}_{\beta,C(J+1)}^-(U)(J+1,0)\right)^2\right)\\ &= \frac{\Delta t}{4}\left(\alpha\left(T_{\alpha,C(0)}^-(U)(0,0)\right)^2+\beta\left(T_{\beta,C(J+1)}^+(U)(J+1,0)\right)^2\right)\\ &\quad +2\Delta x\sum_{j=1}^{J}\frac{1}{C^2(j)}U_t(j)D_t^+(U)(j,0)+\Delta x\sum_{j=1}^{J+1}(D_x^-(U)(j,0))^2.\end{aligned}\tag{7.14}$$

Suppose now the discrete boundary condition to be given for $n \geq 0$ by

$$T^-_{\alpha,C(0)}(U)(0,n) = G^-(n), \quad T^+_{\beta,C(J+1)}(U)(J+1,n) = G^+(n) \tag{7.15}$$

and the initial conditions to be given by $\{U(j,0)\}$, $\{U_t(j)\}$. Summing (7.13) in time and adding (7.14) we get

$$\begin{aligned} E_{n+1} + 2E_{K,1} + E_1 \leq\ & \frac{1}{4}\Delta t \sum_{p=0}^{n} \left(\alpha(G^-(p))^2 + \beta(G^+(p))^2\right) \\ & + 2\Delta x \sum_{j=1}^{J} \frac{1}{C^2(j)} U_t(j) D_t^+(U)(j,0) + \Delta x \sum_{j=1}^{J+1} (D_x^-(U)(j,0))^2 \end{aligned}$$

Using the discrete Cauchy Schwartz inequality on the right-hand side, we get stability for the numerical scheme.

THEOREM 7.3 (Stability). *Suppose $U(j,n)$ is solution of (7.1), together with initial conditions and boundary conditions (7.15), with $\alpha$ and $\beta$ positive. For any positive time step $n$ one has*

$$E_{n+1} + E_1 \leq \frac{1}{4}\Delta t \sum_{p=0}^{n} (\alpha(G^-(p))^2 + \beta(G^+(p))^2) + \Delta x \sum_{j=1}^{J+1} (D_x^-(U)(j,0))^2 + \Delta x \sum_{j=1}^{J} \frac{1}{C^2(j)} (U_t(j))^2. \tag{7.16}$$

*Thus, under the CFL condition $\sup_{1\leq j\leq J} C(j)\frac{\Delta t}{\Delta x} < 1$ required in Lemma 7.1, the scheme is stable.*

**7.2. Convergence of the Discrete Schwarz Waveform Relaxation Algorithm.** Corresponding to the continuous analysis we take the velocity to be continuous at the interfaces. To shorten the notation we denote by $T_i^-$ the operator $T^-_{C(a_i),C(a_i)}$ and the others accordingly. To analyze convergence of the discretized domain decomposition algorithm (5.25) it suffices to consider homogeneous initial conditions in (5.25) and to prove convergence to zero.

THEOREM 7.4. *Assume that the velocity is continuous on the interfaces $a_i$. If the CFL condition (7.5) is satisfied by the discretization in each subdomain, then the non-overlapping discrete Schwarz waveform relaxation algorithm (5.25) with homogeneous initial condition converges to zero on any time interval $[0,T]$ in the energy norm,*

$$\sum_{i=1}^{I} E_{N_i}(U_i^k) \to 0 \text{ as } k \to \infty.$$

*Proof.* The energy estimates (7.13) and (7.14 ) can be rewritten as

$$\begin{aligned} E_{n+1}^{k+1} - E_n^{k+1} + \frac{\Delta t_i}{4} & \left( c(a_i) \left(\widetilde{T}_i^+(U_i^{k+1})(0,n)\right)^2 + c(a_{i+1}) \left(\widetilde{T}_{i+1}^-(U_i^{k+1})(J_i+1,n)\right)^2 \right) \\ &= \frac{\Delta t_i}{4} \left( c(a_i) \left(T_i^-(U_i^{k+1})(0,n)\right)^2 + c(a_{i+1}) \left(T_{i+1}^+(U_i^{k+1})(J_i+1,n)\right)^2 \right). \end{aligned} \tag{7.17}$$

$$\begin{aligned} 2E_{K,1} + 2E_1 + \frac{\Delta t_i}{4} & \left( c(a_i) \left(\widetilde{T}_i^+(U_i^{k+1})(0,0)\right)^2 + c(a_{i+1}) \left(\widetilde{T}_{i+1}^-(U_i^{k+1})(J_i+1,0)\right)^2 \right) \\ &= \frac{\Delta t_i}{4} \left( c(a_i) \left(T_i^-(U_i^{k+1})(0,0)\right)^2 + c(a_{i+1}) \left(T_{i+1}^+(U_i^{k+1})(J_i+1,0)\right)^2 \right). \end{aligned} \tag{7.18}$$

We define the boundary energies

$$\widetilde{F}_{i,n}^{k,+} = \tfrac{\Delta t_i}{4} c(a_i)(\widetilde{T}_i^{+}(U_i^k)(0,n))^2, \qquad \widetilde{F}_{i,n}^{k,-} = \tfrac{\Delta t_{i-1}}{4} c(a_i)(\widetilde{T}_i^{-}(U_{i-1}^k)(J_{i-1}+1,n))^2,$$
$$F_{i,n}^{k,+} = \tfrac{\Delta t_{i-1}}{4} c(a_i)(T_i^{+}(U_{i-1}^k)(J_{i-1}+1,n))^2, \quad F_{i,n}^{k,-} = \tfrac{\Delta t_i}{4} c(a_i)(T_i^{-}(U_i^k)(0,n))^2,$$
(7.19)

and rewrite (7.17) and (7.18) as

$$\begin{aligned} [E_{n+1} - E_n](U_i^{k+1}) + \widetilde{F}_{i,n}^{k+1,+} + \widetilde{F}_{i+1,n}^{k+1,-} &= F_{i+1,n}^{k+1,+} + F_{i,n}^{k+1,-}, \\ [2E_{K,1} + 2E_1]\,(U_i^{k+1}) + \widetilde{F}_{i,0}^{k+1,+} + \widetilde{F}_{i+1,0}^{k+1,-} &= F_{i+1,0}^{k+1,+} + F_{i,0}^{k+1,-}. \end{aligned} \tag{7.20}$$

Summing these equations in every subdomain for $1 \le n \le N_i$ , we find

$$[E_{N_i+1} + 2E_{K,1} + E_1](U_i^{k+1}) + \sum_{n=1}^{N_i} \widetilde{F}_{i,n}^{k+1,+} + \sum_{n=1}^{N_i} \widetilde{F}_{i+1,n}^{k+1,-} = \sum_{n=1}^{N_i} F_{i,n}^{k+1,+} + \sum_{n=1}^{N_i} F_{i+1,n}^{k+1,-}.$$
(7.21)

Using now the transmission conditions and the fact that the projection is a contraction in $L^2$, we get

$$[E_{N_i+1} + 2E_{K,1} + E_1](U_i^{k+1}) + \sum_{n=1}^{N_i} \widetilde{F}_{i,n}^{k+1,+} + \sum_{n=1}^{N_i} \widetilde{F}_{i+1,n}^{k+1,-} \le \sum_{n=1}^{N_{i+1}} \widetilde{F}_{i+1,n}^{k,-} + \sum_{n=1}^{N_{i-1}} \widetilde{F}_{i,n}^{k,+}.$$
(7.22)

Note now that by definition we have as in the continuous case $F_1^{k,\pm} = F_{I+1}^{k,\pm} = 0$. Thus, defining the total boundary energy at iteration k by

$$\widetilde{F}^k = \sum_{i=1}^{I} \sum_{n=0}^{N_i} [\widetilde{F}_{i,n}^{k,-} + \widetilde{F}_{i,n}^{k,+}]$$

we have, by summing in $i$ and shifting the indices, the inequality

$$\sum_{i=1}^{I} [E_{N_i+1} + 2E_{K,1} + E_1](U_i^{k+1}) + \widetilde{F}^{k+1} \le \widetilde{F}^k. \tag{7.23}$$

Thus the same arguments as in the continuous case proves that $\sum_{i=1}^{I} E_{N_i}(U_i^k) \to 0$ as $k \to \infty$. $\square$

**8. Numerical Experiments.** We perform the numerical experiments on the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2(x) \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\ 0 < t < T, \tag{8.1}$$

where we truncate the spatial domain at 0 and $L$ using absorbing boundary conditions, so that the results obtained correspond to the analysis on an infinite domain. We discretize the wave equation and the optimal transmission conditions using the finite volume scheme presented in Section 5.

**8.1. Optimal Global Transmission Conditions.** First we test the convergence result proved in Theorem 2.1 for a two subdomain problem with $L = 2$, $T = 2$ and a constant wave speed $c(x) = 1$. The domain is partitioned at $x = 1$ and the optimal transmission conditions at the continuous level are local. We use the initial conditions

$$\begin{aligned} u(x,0) &= 0, \\ \frac{\partial u}{\partial t}(x,0) &= -100(0.5-x)e^{-50(0.5-x)^2}, \end{aligned}$$

and we start the iteration with the initial guess zero. Table 8.1 shows for various mesh parameters the difference of the domain decomposition algorithm result after 2 iterations and the numerical solution on the whole domain and compares this value to the truncation error, the difference between the numerical solution on the whole domain and the exact solution. One can see that the discretization of the optimal

| grid | error after 2 iterations | discretization error |
|---|---|---|
| 50 x 50 | 2.6128e-04 | 2.1515e-02 |
| 100 x 100 | 2.7305e-05 | 4.9472e-03 |
| 200 x 200 | 3.2361e-06 | 1.2218e-03 |
| 400 x 400 | 3.9852e-07 | 3.0321e-04 |
| 800 x 800 | 4.9548e-08 | 7.5567e-05 |

TABLE 8.1
*Convergence in two iterations to below the accuracy of the discretization.*

local transmission conditions leads to an algorithm which converges in two iterations to well below the accuracy of the numerical scheme.

For the next model problem, we choose $L = 6$, $T = 8$ and a speed function $c(x)$ with a discontinuity at $x = 1$,

$$c(x) = \begin{cases} 1 & 1 < x < 6 \\ 2 & 0 < x < 1 \end{cases}.$$

We decompose the domain into three subdomains, $\Omega_1 = [0,2]\times[0,6]$, $\Omega_2 = [2,4]\times[0,6]$ and $\Omega_3 = [4,6]\times[0,6]$ and we use the initial conditions

$$\begin{aligned} u(x,0) &= 0 \\ \frac{\partial u}{\partial t}(x,0) &= -20(5-x)e^{-10(5-x)^2}. \end{aligned}$$

We use again a discretization of the optimal transmission conditions, which are nonlocal in this case. We start the Schwarz waveform relaxation with a zero initial guess. Table 8.2 shows that the algorithm converges at the third iteration to the discretization error level and illustrates Theorem 2.1, which states that we should find convergence in a number of iterations identical to the number of subdomains. More accuracy is achieved as the iteration progresses further. The nonlocal transmission conditions (2.8) require for this value of $T$ to include three terms of the sum of the operators $\mathcal{S}_j$ in (3.3), (3.4) on the middle subdomain and one term on the right subdomain. We computed the solution on a uniform grid with $\Delta x = 1/20$ and $\Delta t = 1/40$ for this example.

| Iteration | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\|\|u-u_k\|\|_\infty$ | 5.0230e-01 | 5.0164e-01 | 5.0065e-01 | 4.0289e-03 | 3.9800e-03 | 3.9535e-05 |

Table 8.2

*Convergence for a 3 subdomain problem and a discontinuity in the left subdomain.*

**8.2. Optimal Local Transmission Conditions.** Now we introduce time windows for the same example to be able to use optimal local transmission conditions. We cut the time domain into four equal pieces $[0, 2]$, $[2, 4]$, $[4, 6]$ and $[6, 8]$ such that the condition (3.15) according to Theorem 3.4 is satisfied. We solve the problem consecutively on the four time subdomains and in each time window we expect the algorithm to converge in two iterations. We show the convergence results for the first time window only, $[0, T = 2]$. Table 8.3 shows the error in the infinity norm over five iterations for the same mesh parameters as before. The algorithm converges now

| Iteration | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\|\|u-u_k\|\|_\infty$ | 5.0200e-01 | 5.0135e-01 | 1.2089e-03 | 9.0654e-06 | 1.4775e-06 | 1.2551e-06 |

Table 8.3

*Convergence with local transmission conditions over a shorter time interval.*

already at the second iteration as predicted by Theorem 3.4 to the discretization error level and more accuracy is achieved as the iteration progresses.

**8.3. Non-Conforming Grids.** As an illustration of Theorem 3.5 on non-conforming grids we consider a problem with a layered medium of six layers and we decompose the domain into six subdomains corresponding to the different layers, $\Omega_i = [i - 1, i]$ with corresponding wave speeds $c_i \in \{1, 2/3, 1/2, 3/4, 1, 4/5\}$. We discretize each subdomain with a grid in space using $\Delta x_i = 1/50$ and in time using an appropriate time step satisfying the CFL condition $c_i \frac{\Delta t_i}{\Delta x_i} < 1$ but close to 1 which is important for accuracy in the propagation properties of the solution, so different time steps are essential in different subdomains. This leads to non-conforming grids between subdomains. Since we have no algorithm that computes the entire solution over non-conforming grids to compare with, we choose to compute the zero solution to the homogeneous problem with zero initial conditions. We start with a non zero initial guess on the artificial interfaces, $g^{\pm}(t) = 1$. According to Theorem 3.5 the algorithm will converge in two iterations if $T \leq 1$. Table 8.4 shows that this is also observed numerically. After two iterations the Schwarz waveform relaxation has converged

| Iteration | 0 | 1 | 2 |
|---|---|---|---|
| $\|\|u-u_k\|\|_\infty$ | 5.0234e+00 | 5.0234e+00 | 1.1738e-02 |

Table 8.4

*Convergence with local transmission conditions in 2 iterations to the level of the truncation error for a problem with five discontinuities and six subdomains aligned with the discontinuities.*

to the precision of the numerical scheme. Figure 8.1 shows a solution computed on non-matching grids with the optimal Schwarz waveform relaxation algorithm.

**8.4. Variable Wave Speed and Local Transmission Conditions.** Now we consider a variable propagation speed $c(x)$ for which convergence of the Schwarz waveform relaxation algorithm with local transmission conditions was proved in Theorem

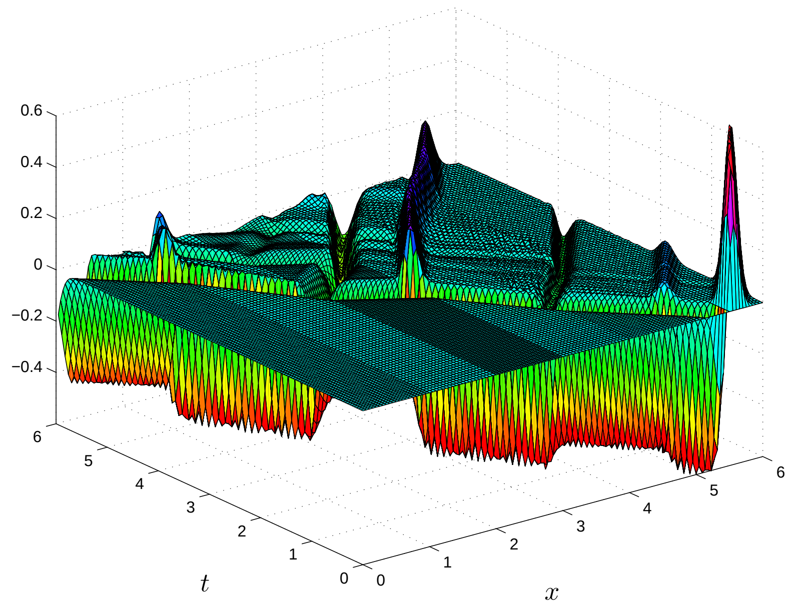


FIG. 8.1. *Computation with non-matching grids on a layered medium with five discontinuities, a pulse created in the surface layer and propagating downward.*

4.3 using energy estimates. The speed profile, which is a typical underwater profile, was obtained from [23] and it is given as a function of depth by

$$c(x) = \begin{cases} 300 & \frac{m}{s} & x < 0 \text{ (above ground)} \\ 1500 - x/12 & \frac{m}{s} & 0 < x < 120 \\ 1480 + x/12 & \frac{m}{s} & 120 < x < 240 \\ 1505 & \frac{m}{s} & x > 240. \end{cases}$$

We decompose the domain into two subdomains $\Omega_1 = [0, 300]$ and $\Omega_2 = [300, 600]$ and we apply the domain decomposition algorithm with the local transmission conditions (3.10) which would be exact if the sound speed was identically constant over both subdomains and equal to the sound speed at the artificial interface at $x = 300$. Table 8.5 shows the convergence of the algorithm for the variable sound speed for a time interval $[0, 1/2]$. The algorithm converges again to the accuracy of the scheme in two

| Iteration | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\|\|u - u_k\|\|_\infty$ | 5.0316e+00 | 5.0316e+00 | 9.9024e-03 | 6.2439e-04 |

TABLE 8.5
*Convergence behavior of the algorithm for the variable sound speed profile from an application.*

iterations, even though the sound speed is variable in this example. This is because the variation is small in scale and thus the local approximations to the transmission conditions are sufficiently accurate to lead to the convergence in two steps. Note also that continuing the iteration, the error is further reduced.

**9. Conclusions.** We have presented and analyzed a non-overlapping Schwarz waveform relaxation algorithm for the one dimensional wave equation with variable coefficients, both at the continuous and the discrete level. The algorithm permits the use of grid CFL conditions adapted to the local wave speeds, non-matching grids on different subdomains and it has optimal scalability when implemented on a parallel computer. The formulation of the algorithm is quite general, it can be applied to the wave equation in higher dimensions and even to other types of evolution equations. The convergence result with the optimal transmission conditions also holds in these more general situations: specific results for the wave equation in higher dimensions will be presented elsewhere. The convergence analysis for the discretized algorithm with approximate transmission conditions however is specific to the one dimensional wave equation with variable coefficients. Although the ideas can be generalized to higher dimensions, the discrete energy estimates present a real challenge, already notation-wise. The advantage of the continuous analysis is however that convergence results similar to the continuous ones can be expected to hold when the algorithm is consistently discretized.

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**Appendix A. A Projection Algorithm for Non-Matching Grids.**

The projection operation between non-conforming grids as given in (5.23) is not an easy task in an algorithm, since one cell can intersect with an arbitrary number of neighboring ones or even not intersect at all, if it is fully contained in the neighboring one. In one dimension, the following short algorithms in Matlab performs this task in an efficient manner:

```
function b=transfer(a,ta,tb);
% TRANSFER integrates a stepfunction between given intervals
%   b=transfer(a,ta,tb); computes the integral of the
%   stepfunction with values a(j) in [ta(j),ta(j+1)] in the
%   intervals [tb(i),tb(i+1)] and stores the result in b(i).
%   Note that the first and last entry in ta and tb must be equal.

n=length(tb);            % n-1 is the length of b
ta(length(ta))=tb(n);    % numerical equality for proper termination
j=1;
for i=1:n-1,
  b(i)=0;
  m=ta(j+1)-tb(i);
  while ta(j+1)<tb(i+1),
    b(i)=b(i)+m*a(j);
    j=j+1;
    m=ta(j+1)-ta(j);
  end;
  m=m-(ta(j+1)-tb(i+1));
  b(i)=b(i)+m*a(j);
end;
```

Given a vector $\texttt{ta} = [0, t_a(1), \ldots, T]$ of arbitrary grid points in time and a piecewise constant function on the intervals between the grid points $ta$ whose values are given in the vector $a$, the algorithm computes the integrals of $a$ on the intervals between the grid points of a second grid given in the vector $\texttt{tb} = [0, t_b(1), \ldots, T]$. The algorithm does a single pass without any special cases using the fact that the grid points are sorted in time. It advances automatically on whatever side the next cell boundary is coming and handles any possible cases of non-matching grids at a one dimensional interface.

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