# On the resolution of linear systems

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# Purpose

Solve $AX = b$.

- $A$ is a squared matrix,
- $b$ is a given righthand side, or a family of given righthand sides





# Description

$$\underbrace{\begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & -1 \\ 3 & 11 & 6 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 9 \\ 1 \\ 36 \end{pmatrix}}_{b}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 6 & 36 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 3 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix}}_{M} \underbrace{\left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 6 & 36 \end{array}\right)}_{(A|b)} = \underbrace{\left(\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 1 & 1 \end{array}\right)}_{(U|Mb)}$$

$$Ax = b \iff Ux : MAx = Mb$$

$M$ is a preconditioner

$$M = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix} \longrightarrow L := M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}$$

$$U = MA \iff A = LU, Ax = b \iff LUx = b$$

1. $LU$ decomposition $\mathcal{O}(\frac{2n^3}{3})$ elementary operations.
2. Solve $Ly = b$ $\mathcal{O}(n^2)$ elementary operations.
3. Solve $Ux = y$ $\mathcal{O}(n^2)$ elementary operations.

For $P$ values of the righthand side, $N_{op} \sim \frac{2n^3}{3} + P \times 2n^2$.





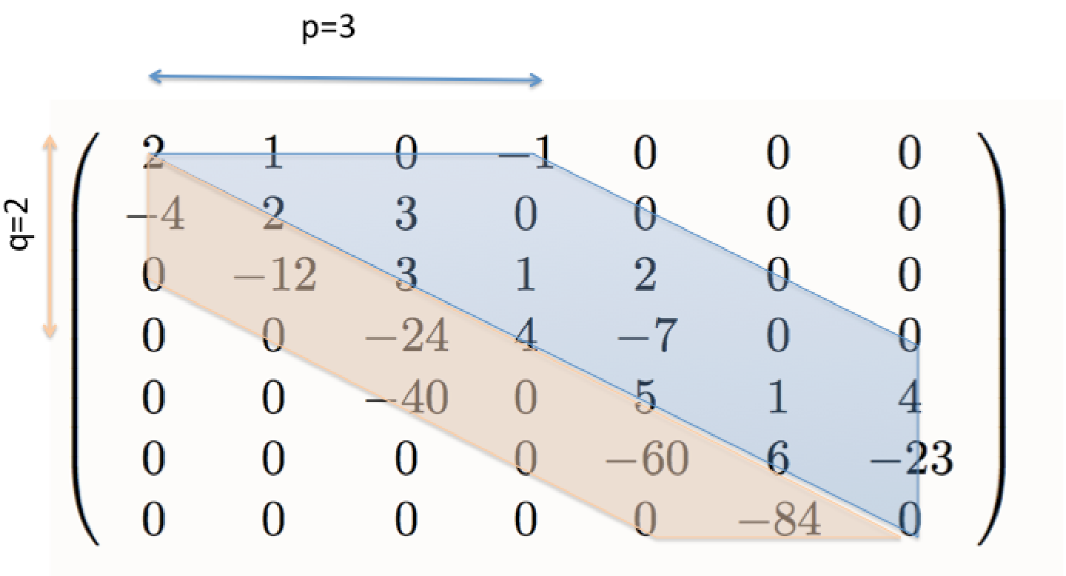
# Theoretical results

**Theorem 1** Let $A$ be an invertible matrix, with principal minors $\neq 0$. Then there exists a unique matrix $L$ lower triangular with $l_{ii} = 1$ for all $i$, and a unique matrix $U$ upper triangular, such that $A = LU$. Furthermore $det\,(A) = \prod_{i=1}^{n} u_{ii}$.

**Theorem 2** Let $A$ be an invertible matrix. There exist a permutation matrix $P$, a matrix $L$ lower triangular with $l_{ii} = 1$ for all $i$, and a matrix $U$ upper triangular, such that

$$PA = LU$$

# Sparse and banded matrices

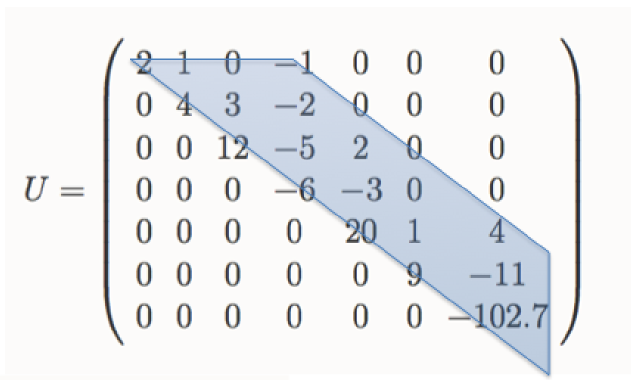


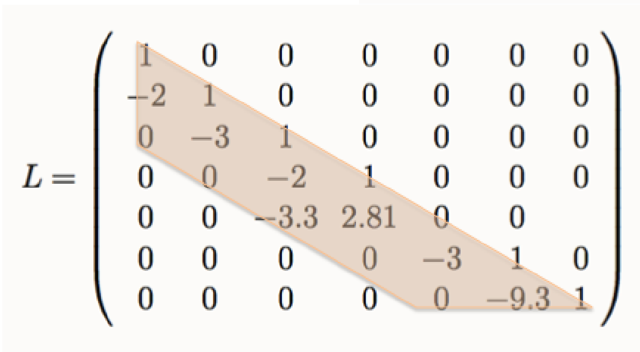

A banded matrix, upper bandwidth $p = 3$ and lower bandwidth $q = 2$, in total $p + q + 1$ nonzero diagonals.





# Sparse and banded matrices




$L$ lowerbanded $q = 2$, and $U$ upperbanded $p = 3$.

# Sparse and banded matrices with pivoting

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -0.5 & -0.17 & -0.05 & -0.21 & 0.025 & 0.0027 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} -4 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & -12 & 3 & 1 & 2 & 0 & 0 \\ 0 & 0 & -40 & 0 & 5 & 1 & 4 \\ 0 & 0 & 0 & 4 & -10 & -0.6 & -2.4 \\ 0 & 0 & 0 & 0 & -60 & 6 & -23 \\ 0 & 0 & 0 & 0 & 0 & -84 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.275 \end{pmatrix}$$





# The permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

# Stationary iterative methods

$$AX = b \; ; \quad A = M - N \; ; \qquad MX = NX + b,$$
$$MX^{m+1} = NX^m + b.$$

Use A = D - E - F.

1. Jacobi : $M = D$ diagonal part of $A$.
2. Gauss-Seidel : $M = D - E$ lower part of $A$.
3. Relaxation : $\hat{U}^{m+1}$ obtained by Gauss-Seidel,
$$X^{m+1} = \omega \hat{U}^{m+1} + (1-\omega) X^m.$$
$$M = \frac{1}{\omega} D - E, \;\; N = F + \frac{1-\omega}{\omega} D - E$$
4. Richardson algorithm
$$X^{m+1} = X^m - \rho r^m = X^m - \rho (AX^m - b)$$
$$M = \frac{1}{\rho} I \quad \rho_{opt} = \frac{2}{\lambda_1 + \lambda_n}$$



Direct methods Stationary iterative methods Non-Stationary iterative methods Preconditioning

# Stationary methods, continue

$$\begin{aligned} MX^{m+1} = NX^m + b \quad &\Longleftrightarrow \quad MX^{m+1} = (M-A)X^m + b \\ &\Longleftrightarrow \quad X^{m+1} = (I - M^{-1}A)X^m + M^{-1}b \\ &\Longleftrightarrow \quad \text{fixed point algorithm to solve } M^{-1}AX = M^{-1}b \end{aligned}$$

Preconditioning

$$\begin{aligned} AX = b \quad &\Longleftrightarrow \quad M^{-1}AX = M^{-1}b \\ &\Longleftrightarrow \quad X = (I - M^{-1}A)X + M^{-1}b \end{aligned}$$

# Stationary methods, continue

$$\text{Error } e^m := X - X^m,$$
$$\text{Residual } \; r^m := b - AX^m = AX - AX^m = Ae^m.$$

$$MX^{m+1} = NX^m + b$$

$$MX = NX + b$$

$$Me^{m+1} = Ne^m$$

$$e^{m+1} = M^{-1}Ne^m$$

$R = M^{-1}N$ is the iteration matrix

Useful alternative formula $R = I - M^{-1}A$.





# Fundamentals tools

$$X^{m+1} = RX^m + \tilde{b}, \quad e^{m+1} = Re^m, \; R = M^{-1}N.$$

**Theorem** The sequence is convergent for any initial guess $X^0$ if and only if $\rho(R) < 1$.
$\rho(R) = \max\{|\lambda|, \lambda \text{ eigenvalue of } A\}$ : convergence factor.
To reduce the initial error by a factor $\epsilon$, we need

$$\frac{\|e^m\|}{\|e^0\|} \leq\sim (\rho(R))^m \sim \epsilon$$

So we have $M \sim \dfrac{\ln \epsilon}{\ln \rho(R)}$.
Convergence rate$= -\ln \rho(R)$ digits per iteration.

# Symmetric positive definite matrices

**Householder-John theorem** : Suppose $A$ is positive. If $M + M^T - A$ is positive definite, then $\rho(R) < 1$.

**Corollary**

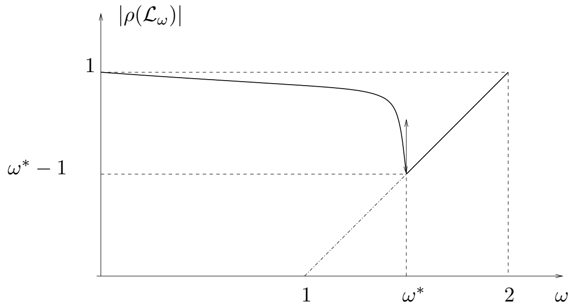
1. If $D + E + F$ is positive definite, then Jacobi converges.
2. If $\omega \in (0, 2)$, then SOR converges.





# Tridiagonale matrices

1. $\rho(\mathcal{L}_1) = (\rho(J))^2$ : Jacobi Gauss-Seidel converge or diverge simultaneously. If convergent, Gauss-Seidel is twice as fast.
2. Suppose the eigenvalues of $J$ are real. Then Jacobi and SOR convergent ou converge or diverge simultaneously for $\omega \in ]0, 2[$.
3. Same assumptions, SOR has an optimal parameter
$$\omega^* = \frac{2}{1 + \sqrt{1 - (\rho(J))^2}}, \quad \rho(\mathcal{L}_{\omega^*}) = \omega^* - 1.$$



# Descent methods

The descent directions $p_m$ are given. Define

$$X^{m+1} = X^m + \alpha_m p^m, \quad e^{m+1} = e^m - \alpha_m p^m, \quad r^{m+1} = r^m - \alpha_m A p^m.$$

**Theorem** $X$ is the solution of $AX = b \iff$ it minimizes over $\mathbb{R}^N$ the functional $J(y) = \frac{1}{2}(Ay, y) - (b, y)$.
This is equivalent to minimizing $G(y) = \frac{1}{2}(A(y - X), y - X)$
At each step minimize $J$ in the direction of $p_m$

$$\boxed{\alpha_m = \frac{(p^m, r^m)}{(Ap^m, p^m)}, \quad (p^m, r^{m+1}) = 0}$$

$$G(x^{m+1}) = G(x^m)(1 - \mu_m), \quad \mu_m = \frac{(r^m, p^m)^2}{(Ap^m, p^m)(A^{-1}r^m, r^m)}$$





# Steepest descent

$p^m = r^m$.

$$X^{m+1} = X^m + \alpha_m r^m, \quad e^{m+1} = e^m - \alpha_m r^m, \quad r^{m+1} = (I - \alpha_m A) p^m.$$

$$\boxed{\alpha_m = \frac{\|r^m\|^2}{(Ar^m, r^m)}, \quad (r^m, r^{m+1}) = 0}$$

$$G(x^{m+1}) = G(x^m)\left(1 - \frac{\|r^m\|^4}{(Ar^m, r^m)(A^{-1}r^m, r^m)}\right) \leq \left(\frac{\kappa(A) - 1}{\kappa(A) + 1}\right)^2 G(x^m)$$

## Conjugate gradient

$$X^{m+1} = X^m + \alpha_m p^m, \quad (r^m, p^{m-1}) = 0.$$

Search $p^m$ as $p^m = r^m + \beta_m p^{m-1}$

$$G(x^{m+1}) = G(x^m)(1 - \mu_m)$$

$$\mu_m = \frac{(r^m, p^m)^2}{(Ap^m, p^m)(A^{-1}r^m, r^m)} = \frac{\|r^m\|^4}{(Ap^m, p^m)(A^{-1}r^m, r^m)}$$

Maximize $\mu_m$, or minimize

$$(Ap^m, p^m) = \beta_m^2(Ap^{m-1}, p^{m-1}) + 2\beta_m(Ap^{m-1}, r^m) + (Ar^m, r^m)$$

$$\beta_m = -\frac{(Ap^{m-1}, r^m)}{Ap^{m-1}, p^{m-1})} \quad \Rightarrow (Ap^{m-1}, p^m) = 0$$

$$(r^m, r^{m+1}) = 0, \quad \beta_m = \frac{\|r^m\|^2}{\|r^{m-1}\|^2}.$$





## Other properties

Choose $p^0 = r^0$. Then $\forall m \geq 1$, if $r^i \neq 0$ for $i < m$.

1. $(r^m, p^i) = 0$ for $i \leq m-1$.
2. $\text{vec}(r^0, \ldots, r^m) = \text{vec}(r^0, Ar^0, \ldots, A^m r^0)$.
3. $\text{vec}(p^0, \ldots, p^m) = \text{vec}(r^0, Ar^0, \ldots, A^m r^0)$.
4. $(p^m, Ap^i) = 0$ for $i \leq m-1$.
5. $(r^m, r^i) = 0$ for $i \leq m-1$.

**Definition** Krylov space $\mathcal{K}_m = \text{vec}(r^0, Ar^0, \ldots, A^{m-1}r^0)$.

**Theorem** $G(x^m) = \inf_{y \in x^0 + \mathcal{K}_m} G(y)$.

# Final properties

**Theorem** Convergence in at most $N$ steps (size of the matrix)

**Theorem** $G(x^m) \leq 4 \left( \frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^2 G(x^m)$





# The algorithm

$$X^0 \text{chosen}, \quad p^0 = r^0 = b - AX^0.$$

While $m < Niter$ or $\|r^m\| \geq tol$, do

$$\begin{aligned}
\alpha_m &= \frac{\|r^m\|^2}{(Ap^m, p^m)}, \\
X^{m+1} &= X^m + \alpha_m p^m, \\
r^{m+1} &= r^m - \alpha_m A p^m, \\
\beta_{m+1} &= \frac{\|r^{m+1}\|^2}{\|r^m\|^2}, \\
p^{m+1} &= r^{m+1} - \beta_{m+1} p^m.
\end{aligned}$$

# 1-D Poisson problem

Poisson equation $-u'' = f$ on $(0,1)$,
Dirichlet boundary conditions $u(0) = g_g, \;\; u(1) = g_d$ .
Second order finite difference stencil.

$$(0,1) = \cup(x_j, x_{j+1}), \quad x_{j+1} - x_j = h = \frac{1}{n+1}, \quad j = 0, \ldots, n.$$

$$-\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} \sim f(x_i), \quad i = 1, \ldots n$$

$$u_0 = g_g, \quad u_{n+1} = g_d.$$

$$|u_i - u(x_i)| \leq h^2 \; \frac{\sup_{x\in[a,b]} |u^{(4)}(x)|}{12}$$

.





# 1-D Poisson problem

Discrete unknowns $U =^t (u_1, \ldots, u_n)$.

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & 0 & \\ & \ddots & \ddots & \ddots & \\ 0 & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \quad b = \begin{pmatrix} f_1 - \frac{g_g}{h^2} \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n - \frac{g_d}{h^2} \end{pmatrix}$$

The matrix $A$ is symmetric definite positive.

Discrete problem to be solved is

$$AX = b$$

# Condition number and error

$$AX = b, \quad A\hat{X} = \hat{b}$$

Define $\kappa(A) = \|A\|_2 \|A^{-1}\|_2$. If $A$ is symmetric $> 0$, $\kappa(A) = \frac{\max \lambda_i}{\min \lambda_i}$.

**Theorem**

$$\frac{\|\hat{X} - X\|_2}{\|X\|_2} \leq \kappa(A) \frac{\|\hat{b} - b\|_2}{\|b\|_2}$$

and there is a $b$ such that it is equal.

Eigenvalues of $A$ $(h \times (n+1) = 1)$.

$$\lambda_k = \frac{2}{h^2}(1 - \cos\frac{k\pi}{n+1}) = \frac{4}{h^2}\sin^2\frac{k\pi h}{2}, \quad V_k = (\sin\frac{jk\pi}{n+1})_{1 \leq i \leq n},$$

$$\kappa(A) = \frac{\sin^2 \frac{n\pi h}{2}}{\sin^2 \frac{\pi h}{2}} = \frac{\cos^2 \frac{\pi h}{2}}{\sin^2 \frac{\pi h}{2}} \sim \frac{4}{\pi^2 h^2}$$





# Comparison of the iterative methods

| **Algorithm** | **spectral radius** $\rho(R)$ | $n = 5$ | $n = 30$ |
|---|---|---|---|
| Jacobi | $\cos \pi h$ | 0.81 | 0.99 |
| Gauss-Seidel | $(\rho(J))^2 = \cos^2 \pi h$ | 0.65 | 0.98 |
| SOR | $\frac{1 - \sin \pi h}{1 + \sin \pi h}$ | 0.26 | 0.74 |
| steepest descent | $\frac{K(A) - 1}{K(A) + 1}$ | 0.81 | 0.99 |
| conjugate gradient | $\frac{\sqrt{K(A)} - 1}{\sqrt{K(A)} + 1}$ | 0.51 | 0.86 |

Reduction factor for one digit $M \sim -\frac{1}{\mathrm{Log}_{10}\rho(R)}$. For $n = 30$,

| $n$ | Jacobi | Gauss-Seidel | SOR | St Des | CG |
|---|---|---|---|---|---|
| 10 | 56 | 28 | 4 | 56 | 8 |
| 100 | 4759 | 2380 | 37 | 4759 | 74 |

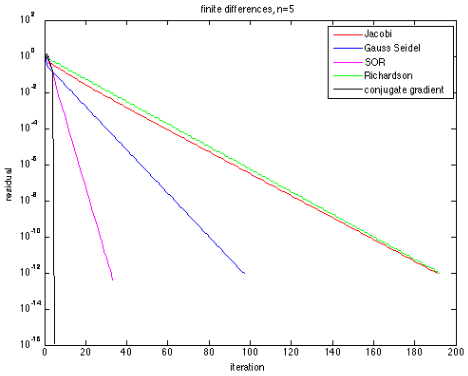
# Asymptotic behavior

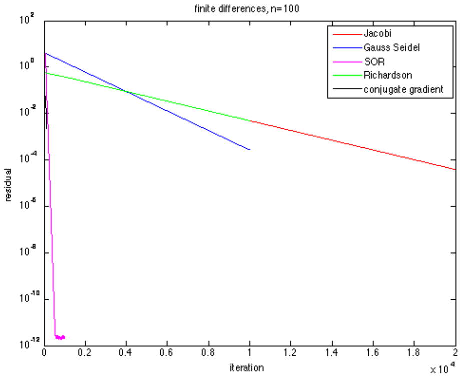
| Algorithm | spectral radius |
|---|---|
| Jacobi | $1 - \frac{\pi^2}{2} h^2,$ |
| Gauss-Seidel | $1 - \pi^2 h^2,$ |
| SOR | $1 - 2\pi h$ |
| gradient | $1 - \pi h,$ |
| conjugate gradient | $1 - \dfrac{\pi h}{2}.$ |





# Convergence history



# Number of elementary operations

| | |
|---|---|
| Gauss elimination | $n^2$ |
| optimal overrelaxation | $n^{3/2}$ |
| FFT | $n \ln_2(n)$ |
| conjugate gradient | $n^{5/4}$ |
| multigrid | $n$ |

Asymptotic order of the number of elementary operations needed to solve the $1-D$ problem as a function of the number of grid points





# Purpose

Take the system $AX = b$, with $A$ symmetric definite positive, and the conjugate gradient algorithm. The speed of convergence of the algorithm deteriorates when $\kappa(A)$. The purpose is to replace the problem by another system, better conditioned. Let $M$ be a symmetric regular matrix. Multiply the system on the left by $M^{-1}$.

$$AX = b \iff M^{-1}AX = M^{-1}b \iff (M^{-1}AM^{-1})MX = M^{-1}b$$

Define

$$\tilde{A} = M^{-1}AM^{-1}, \quad \tilde{X} = MX, \quad \tilde{b} = M^{-1}b,$$

and the new problem to solve $\tilde{A}\tilde{X} = \tilde{b}$. Since $M$ is symmetric, $\tilde{A}$ is symmetric definite positive. Write the conjugate gradient algorithm for this “tilde“ problem.

## The algorithm for $\tilde{A}$

$$\tilde{X}^0 \text{ given}, \quad \tilde{p}^0 = \tilde{r}^0 = \tilde{b} - \tilde{A}\tilde{X}^0.$$

While $m < Niter$ or $\|\tilde{r}^m\| \geq tol$, do

$$\begin{aligned}
\alpha_m &= \frac{\|\tilde{r}^m\|^2}{(\tilde{A}\tilde{p}^m, \tilde{p}^m)}, \\
\tilde{X}^{m+1} &= \tilde{X}^m + \alpha_m \tilde{p}^m, \\
\tilde{r}^{m+1} &= \tilde{r}^m - \alpha_m \tilde{A}\tilde{p}^m, \\
\beta_{m+1} &= \frac{\|\tilde{r}^{m+1}\|^2}{\|\tilde{r}^m\|^2}, \\
\tilde{p}^{m+1} &= \tilde{r}^{m+1} - \beta_{m+1}\tilde{p}^m.
\end{aligned}$$

Now define

$$p^m = M^{-1}\tilde{p}^m, \quad X^m = M^{-1}\tilde{X}^m, \quad r^m = M\tilde{r}^m,$$

and replace in the algorithme above.





## The algorithm for $A$

$$Mp^0 = M^{-1}r^0 = M^{-1}b - M^{-1}AM^{-1}MX^0 \iff \begin{cases} p^0 = M^{-2}r^0, \\ r^0 = b - AX^0. \end{cases}$$

$$\|\tilde{r}^m\|^2 = (M^{-1}r^m, M^{-1}r^m) = (M^{-2}r^m, r^m)$$

Define $\boxed{z^m = M^{-2}r^m}$. Then $\boxed{\beta_{m+1} = \frac{(z^{m+1}, r^{m+1})}{(z^m, r^m)}}$.

$$(\tilde{A}\tilde{p}^m, \tilde{p}^m) = (M^{-1}AM^{-1}Mp^m, Mp^m) = (Ap^m, p^m)$$

$$\Rightarrow \boxed{\alpha_m = \frac{(z^m, r^m)}{(Ap^m, p^m)}}.$$

$$MX^{m+1} = MX^m + \alpha_m Mp^m \iff \boxed{X^{m+1} = X^m + \alpha_m p^m}.$$

$$M^{-1}r^{m+1} = M^{-1}r^m - \alpha_m M^{-1}AM^{-1}Mp^m \iff \boxed{r^{m+1} = r^m - \alpha_m Ap^m}.$$

$$Mp^{m+1} = M^{-1}r^{m+1} - \beta_{m+1}Mp^m \iff \boxed{p^{m+1} = z^{m+1} - \beta_{m+1}p^m}.$$

## The algorithm for $A$

Define $C = M^2$.

$$X^0 \text{ given}, \quad r^0 = b - AX^0, \quad \text{solve } Cz^0 = r^0, \quad p^0 = z^0.$$

While $m < Niter$ or $\|r^m\| \geq tol$, do

$$\begin{aligned}
\alpha_m &= \frac{(z^m, r^m)}{(Ap^m, p^m)}, \\
X^{m+1} &= X^m + \alpha_m p^m, \\
r^{m+1} &= r^m - \alpha_m A p^m, \\
\beta_{m+1} &= \frac{(z^{m+1}, r^{m+1})}{(z^m, r^m)}, \\
\text{solve } Cz^{m+1} &= r^{m+1}, \\
p^{m+1} &= z^{m+1} - \beta_{m+1} p^m.
\end{aligned}$$





## How to choose $C$

$C$ must be chosen such that

1. $\tilde{A}$ is better conditioned than $A$,
2. $C$ is easy to invert.

Use an iterative method such that $A = C - N$ with symmetric $C$. For instance it can be a symmetrized version of SOR, named SSOR, defined for $\omega \in (0, 2)$ by

$$C = \frac{1}{\omega(2-\omega)}(D - \omega E)D^{-1}(D - \omega F).$$

Notice that if $A$ is symmetric definite positive, so is $D$ and its coefficients are positive, then its square root $\sqrt{D}$ is defined naturally as the diagonal matrix of the square roots of the coefficients. Then $C$ can be rewritten as

$$C = SS^T, \quad \text{with } S = \frac{1}{\sqrt{\omega(2-\omega)}}(D - \omega E)D^{-1/2},$$

yielding a natural Cholewski decomposition of $C$.

## How to choose $C$, continue

Define furthermore $M$ as the square root of $C$ ($M$ will never been use), *i.e.* diagonalize $C$ as $P\Lambda P^T$, with $\Lambda_{ii} > 0$ the eigenvalues of $C$, then define the symmetric square root of $C$ as $M = P\sqrt{\Lambda}P^T$. Notice that $\tilde{\lambda}$ is an eigenvalue of $\tilde{A}$ associated to the eigenvector $\tilde{z}$ if and only if

$$M^{-1}AM^{-1}\tilde{z} = \tilde{\lambda}\tilde{z} \iff M^{-2}AM^{-1}\tilde{z} = \tilde{\lambda}M^{-1}\tilde{z}$$

if and only if $\tilde{\lambda}$ is an eigenvalue of $C^{-1}A$ associated to the eigenvector $M^{-1}\tilde{z}$. The speed of convergence of the iterative method is measured by the spectral radius of $C^{-1}N$, $\rho(C^{-1}N) < 1$. Note $\mu_i$ the eigenvalues of $C^{-1}N$. Since $C^{-1}N = I - C^{-1}A$, the eigenvalues of $C^{-1}A$ are equal to $1 - \mu_i \in [1 - \rho(C^{-1}N), 1 + \rho(C^{-1}N)]$.

Therefore $\kappa(\tilde{A}) \leq \dfrac{1 + \rho(C^{-1}N)}{1 - \rho(C^{-1}N)}$, and the smallest $\rho(C^{-1}N)$, the smallest the condition number of $\tilde{A}$.





## Comparison

For the 1-D finite differences matrix and $n = 100$, we compare the convergence of the conjugate gradient and the preconditioning by SSOR with $\omega = 1$ and with the optimal parameter. The gain even with $\omega = 1$ is striking.