From the Quasi-static to the Dynamic Maxwell's Model in Micromagnetism

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Abstract

A commonly used model for ferromagnetic materials in the quasistatic regime is the Landau-Lifshitz system coupled with the so-called quasistatic Maxwell's equations. By an appropriate scaling, we justify this approach and we propose a new asymptotic expansion. This suggest a new numerical method.

1 The micromagnetism model

The magnetic material fills a bounded domain Ω in $\mathbb{R}^3$. The evolution of the magnetization field is governed by the Landau-Lifshitz system

$$\frac{\partial \mathbf{M}}{\partial T} = -\gamma\mu_0 \left(\mathbf{M} \times \mathbf{H}_T + \frac{\alpha}{|\mathbf{M}|}\mathbf{M} \times (\mathbf{M} \times \mathbf{H}_T) \right) \text{ in } \Omega, \tag{1}$$

with initial condition $\mathbf{M}^{(0)}$. $\mathbf{M}$ is the magnetization field ; it vanishes outside Ω, and has a prescribed length in Ω

$$|\mathbf{M}(\mathbf{X}, T)| = |\mathbf{M}(\mathbf{X}, 0)| = M_S \text{ a.e in } \Omega. \tag{2}$$

μ_0 is the magnetic permeability, γ the Larmor precession factor, and α a dimensionless dumping factor. They are all positive factors. The total magnetic field $\mathbf{H}_T$ is a linear function of $\mathbf{M}$. It is the sum of three magnetic contributions (we consider the external field to be zero) : the exchange field $\mathbf{H}_{ex} = A\Delta\mathbf{M}$, the anisotropy field $\mathbf{H}_a = -K\,\mathbf{u} \times (\mathbf{M} \times \mathbf{u})$, where $\mathbf{u}$ is the direction of anisotropy, and K and A are physical positive constants. Finally the Maxwell's field $\mathbf{H}$ solves the system of equations whose unknowns are the magnetic field $\mathbf{H}$, the electric field $\mathbf{E}$, and the electrostatic charge ρ, the magnetization field $\mathbf{M}$ being given

$$\begin{aligned} &\varepsilon(\mathbf{X})\frac{\partial \mathbf{E}}{\partial T} + \sigma(\mathbf{X})\mathbf{E} - \mathbf{rot}\ \mathbf{H} = \mathbf{0}, \\ &\mu_0 \frac{\partial}{\partial T}(\mathbf{H} + \mathbf{M}) + \mathbf{rot}\ \mathbf{E} = \mathbf{0}, \end{aligned} \tag{3}$$

with prescribed initial conditions. Furthermore we have for all time the following constraints

$$\mathbf{div}\ (\varepsilon(\mathbf{X})\mathbf{E}) = \rho, \quad \mathbf{div}\ (\mu_0(\mathbf{H} + \mathbf{M})) = 0. \tag{4}$$

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ε_0 is the permittivity in the vacuum, ε_r the relative permittivity of the material, and the value of $\varepsilon(x)$ is $\varepsilon_0\varepsilon_r$ in Ω, ε_0 in the exterior. $\sigma(x)$ is the conductivity of the material; it vanishes outside Ω.

The total field is thus given by

$$\mathbf{H}_T(\mathbf{M}) = -K\,\mathbf{u}\times(\mathbf{M}\times\mathbf{u}) + A\Delta\mathbf{M} + \mathbf{H}(\mathbf{M}). \tag{5}$$

The system we consider here is composed of (1,...,5), with the mandatory constraints (4) and initial conditions.

2 Two scalings for the micromagnetism model

We perform the following scaling

$$\mathbf{H} = \bar{h}\hat{\mathbf{h}},\ \mathbf{E} = \bar{e}\hat{\mathbf{e}},\ \rho = \bar{\rho}\hat{R},\ \mathbf{M} = \bar{m}\hat{\mathbf{m}}, \tag{6}$$

and the change of variables

$$\mathbf{X} = \bar{x}\,\mathbf{x},\ T = \bar{t}\,t, \tag{7}$$

where $\bar{x}$ and $\bar{t}$ are the characteristic length and time. By homogeneity, we have the following relations :

$$\bar{m} = \bar{h},\ \mu_0\frac{\bar{h}}{\bar{t}} = \frac{\bar{e}}{\bar{x}},\ \frac{\varepsilon_0\bar{e}}{\bar{x}} = \bar{\rho}. \tag{8}$$

The dimensionless Landau and Lifchitz system is

$$\frac{\partial\hat{\mathbf{m}}}{\partial t} = -\gamma\mu_0\bar{m}\left(\hat{\mathbf{m}}\times\hat{\mathbf{h}}_T + \frac{\alpha}{|\hat{\mathbf{m}}|}\hat{\mathbf{m}}\times(\hat{\mathbf{m}}\times\hat{\mathbf{h}}_T)\right) \text{ in } \omega, \tag{9}$$

with the constraint $|\hat{\mathbf{m}}(\mathbf{x},t)| = \frac{M_S}{\bar{m}}$ a.e in ω.

By homogeneity in the Landau-Lifshitz system, and linearity in $\mathbf{H}_T$, there appears a new scale $\zeta = \bar{t}\gamma\mu_0$. Applying the new scaling to all variables,

$$\hat{\mathbf{h}} = \zeta\mathbf{h},\ \hat{R} = \zeta R,\ \hat{\mathbf{e}} = \zeta\mathbf{e},\ \hat{\mathbf{m}} = \zeta\mathbf{m}, \tag{10}$$

and choosing $\bar{m} = \bar{t}\gamma\mu_0 M_S$, system (9) becomes

$$\frac{\partial\mathbf{m}}{\partial t} = -\mathbf{m}\times\mathbf{h}_T - \alpha\mathbf{m}\times(\mathbf{m}\times\mathbf{h}_T) \text{ in } \omega, \tag{11}$$

with the constraint

$$|\hat{\mathbf{m}}(\mathbf{x},t)| = |\hat{\mathbf{m}}(\mathbf{x},0)| = 1 \text{ a.e in } \omega. \tag{12}$$

The total field $\mathbf{h}_T$ is given by the three contributions $\mathbf{h}_a = -K\,\mathbf{u}\times(\mathbf{m}\times\mathbf{u})$, $\mathbf{h}_{ex} = \bar{A}\Delta\mathbf{m} = \frac{A}{\bar{x}^2}\Delta\mathbf{m}$ and $\mathbf{h}$:

$$\mathbf{h}_T(\mathbf{m}) = -K\,\mathbf{u}\times(\mathbf{m}\times\mathbf{u}) + \bar{A}\Delta\mathbf{m} + \mathbf{h}(\mathbf{m}). \tag{13}$$

We set $\eta = \dfrac{\bar{x}}{c\bar{t}}$ where c is the speed of light . In our context, the length of Ω is supposed to be small with respect to the wavelengths. Thus the parameter η is small. With these notations, the Maxwell's system of equations becomes

$$
\begin{aligned}
&\eta^2\tilde{\varepsilon}\frac{\partial \mathbf{e}}{\partial t} + \eta\tilde{\sigma}\mathbf{e} - \mathbf{rot}\ \mathbf{h} = \mathbf{0},\\
&\frac{\partial}{\partial t}(\mathbf{h}+\mathbf{m}) + \mathbf{rot}\ \mathbf{e} = \mathbf{0},\\
&\mathbf{div}\ (\mathbf{h}+\mathbf{m}) = 0, \quad \mathbf{div}\ (\tilde{\varepsilon}\mathbf{e}) = R,
\end{aligned}
\tag{14}
$$

with *ad hoc* initial values. The problem is now ready for asymptotic expansion. Note that it is a kind of singular perturbation for the electric and magnetic fields, in the time variable.

3 Asymptotic expansion for the Maxwell's system

We place ourselves in the linear case, where the magnetization field $\mathbf{m}$ is given, and we consider the system (14). For other asymptotic expansions and scaling concerning Maxwell's equations see [1] and [3]. The well-posedness can be shown using the theory of semi-groups.

We expand now R and $\mathbf{m}$ as functions of η,

$$
R = \sum_{i=0}^{\infty} \eta^i\ R_i, \quad \mathbf{m} = \sum_{i=0}^{\infty} \eta^i\ \mathbf{m}_i, \tag{15}
$$

and we search for $\mathbf{e}$ and $\mathbf{h}$ such that

$$
\mathbf{e} = \sum_{i=0}^{\infty} \eta^i\ \mathbf{e}_i, \quad \mathbf{h} = \sum_{i=0}^{\infty} \eta^i\ \mathbf{h}_i. \tag{16}
$$

Inserting these expansions into the system (14), we obtain first the so-called quasi-static Maxwell's system

$$
\left\{
\begin{array}{l}
\mathbf{div}\ (\mathbf{h}_0+\mathbf{m}_0) = 0,\ \mathbf{rot}\ \mathbf{h}_0 = \mathbf{0},\\
\mathbf{rot}\ \mathbf{e}_0 = -\dfrac{\partial}{\partial t}(\mathbf{m}_0+\mathbf{h}_0),\ \mathbf{div}\ (\tilde{\varepsilon}\mathbf{e}_0) = R_0,
\end{array}
\right.
\tag{17}
$$

and a sequence of systems for $k \geq 1$

$$
\left\{
\begin{array}{l}
\mathbf{div}\ (\mathbf{h}_k+\mathbf{m}_k) = 0,\ \mathbf{rot}\ \mathbf{h}_k = \tilde{\varepsilon}\dfrac{\partial \mathbf{e}_{k-2}}{\partial t} + \tilde{\sigma}\mathbf{e}_{k-1},\\
\mathbf{rot}\ \mathbf{e}_k = -\dfrac{\partial}{\partial t}(\mathbf{h}_k+\mathbf{m}_k),\ \mathbf{div}\ (\tilde{\varepsilon}\mathbf{e}_k) = R_k.
\end{array}
\right.
\tag{18}
$$

(with the convention $\mathbf{e}_{-2} = \mathbf{e}_{-1} = \mathbf{0}$). Using the Helmholtz decomposition in weighted Sobolev spaces, we proved

THEOREM 3.1. *Suppose $\mathbf{m}_k$ belongs to $\mathcal{C}^{p-k+1}(\mathbb{R}^+;\mathbb{L}^2(\omega))$ and R_k belongs to $\mathcal{C}^{p-k}(\mathbb{R}^+,\mathbb{L}^2(\omega))$ for $0 \leq k \leq p$. Then problems (17)and (18) have a unique solution $(\mathbf{h}_k,\mathbf{e}_k)$ in $\mathcal{C}^{p-k+1}(\mathbb{R}^+,\mathbb{L}^2(\mathbb{R}^3)) \times \mathcal{C}^{p-k}(\mathbb{R}^+,\mathbb{L}^2(\mathbb{R}^3))$ for $0 \leq k \leq p$.*

We verify now that the asymptotic expansions really approximate the fields. Let $\check{\mathbf{h}}_p$ and $\check{\mathbf{e}}_p$ be the partial sums, $\tilde{\mathbf{h}}_p$ and $\tilde{\mathbf{e}}_p$ denote the errors, *i.e.*

$$
\check{\mathbf{e}}_p = \sum_{i=0}^{p} \eta^i\ \mathbf{e}_i, \tilde{\mathbf{e}}_p = \mathbf{e} - \check{\mathbf{e}}_p; \quad \check{\mathbf{h}}_p = \sum_{i=0}^{p} \eta^i\ \mathbf{h}_i, \tilde{\mathbf{h}}_p = \mathbf{h} - \check{\mathbf{h}}_p. \tag{19}
$$

The errors satisfy, for any $p \geq 0$, a system of the type

$$\left\{\begin{array}{l}\eta^2\tilde{\varepsilon}\dfrac{\partial\tilde{\mathbf{e}}_p}{\partial t}+\eta\tilde{\sigma}\tilde{\mathbf{e}}_p-\quad \mathbf{rot}\ \tilde{\mathbf{h}}_p=\ 0(\eta^{p+1}),\\ \dfrac{\partial\tilde{\mathbf{h}}_p}{\partial t}+\mathbf{rot}\ \tilde{\mathbf{e}}_p=\ 0(\eta^{p+1}),\\ \tilde{\mathbf{h}}_p(\mathbf{x},0)=\tilde{\mathbf{e}}_p(\mathbf{x},0)=\mathbf{0}.\end{array}\right. \tag{20}$$

We obtain hyperbolic estimates by multiplying the first equation by $\tilde{\mathbf{e}}$, the second by $\tilde{\mathbf{h}}$, and using Green's formula. The Gronwall lemma leads to the conclusion

THEOREM 3.2. *For any $p \geq 1$, the following error estimates hold : for any positive time τ, there exists a constant C such that*

$$\begin{array}{l}||\tilde{\mathbf{h}}_p||_{\mathbb{L}^\infty(0,\tau;\mathbb{L}^2(\mathbb{R}^3))}\leq C\eta^p,\\ ||\tilde{\mathbf{e}}_p||_{\mathbb{L}^\infty(0,\tau;\mathbb{L}^2(\mathbb{R}^3))}\leq C\eta^{p-1},\ ||\tilde{\mathbf{e}}_p||_{\mathbb{L}^2(0,\tau;\mathbb{L}^2(\omega))}\leq C\eta^{p-\frac{1}{2}}.\end{array} \tag{21}$$

For $p = 0$, the error estimates are weaker : for any positive time τ, there exists a constant C such that

$$\begin{array}{l}||\tilde{\mathbf{h}}_0||_{\mathbb{L}^\infty(0,\tau;\mathbb{L}^2(\mathbb{R}^3))}\leq C\sqrt{\eta},\ |\tilde{\mathbf{h}}_0|_{\mathbb{L}^\infty(0,\tau;\mathbb{H}^1(\mathbb{R}^3))}\leq C\eta,\\ ||\mathbf{rot}\ (\tilde{\varepsilon}\tilde{\mathbf{e}}_0)||_{\mathbb{L}^2(0,\tau;\mathbb{L}^2(\omega))}\leq C\eta.\end{array} \tag{22}$$

4 Asymptotic developpement for the Micromagnetism system coupled with the Maxwell's model

We come back now to the Landau-Lifshitz system (11). Theorems of existence and comments on uniqueness can be found in [5].

The magnetization field $\mathbf{m}$ is now an unknown, with initial value independent of η. Inserting expansions (15) and (16) into (11) and (14), we obtain the first term

$$\begin{array}{l}\dfrac{\partial\mathbf{m}_0}{\partial t}=-\mathbf{m}_0\times\mathbf{h}_{T,0}-\alpha\mathbf{m}_0\times(\mathbf{m}_0\times\mathbf{h}_{T,0})\text{ in }\omega,\\ |\mathbf{m}_0|=1,\mathbf{m}_0(\mathbf{x},0)=\mathbf{m}^{(0)}(\mathbf{x}),\text{a.e. in }\omega,\end{array} \tag{23}$$

and

$$\mathbf{h}_{T,0}=-K\,\mathbf{u}\times(\mathbf{m}_0\times\mathbf{u})+\bar{A}\Delta\mathbf{m}_0+\mathbf{h}_0 \tag{24}$$

where $\mathbf{h}_0$ is given by the quasi-static Maxwell's system in (17).

The problem (23) is proved to be well-posed in [4]. We first give an energy estimate on the solution to (11) and (14).

THEOREM 4.1. *Let $(\mathbf{m},\mathbf{e},\mathbf{h})$ solve the equations (11) and (14). The following energy estimate holds*

$$\begin{array}{l}\dfrac{1}{2}\dfrac{d}{dt}[\eta^2\displaystyle\int_{\mathbb{R}^3}\tilde{\varepsilon}|\mathbf{e}|^2\,dx\quad+\int_{\mathbb{R}^3}|\mathbf{h}|^2\,dx+\int_{\mathbb{R}^3}\tilde{A}|\mathbf{grad}\ \mathbf{m}|^2\,dx+\int_{\mathbb{R}^3}K|\mathbf{u}\cdot\mathbf{m}|^2\,dx]\\ \qquad\qquad+\eta\displaystyle\int_\omega\tilde{\sigma}|\mathbf{e}|^2\,dx+\int_\omega|\mathbf{m}\times\mathbf{h}_T|^2\,dx=0.\end{array} \tag{25}$$

With these estimates, we can prove convergence

THEOREM 4.2. *The solution $(\mathbf{m},\mathbf{h})$ to (11) (14) converges weak-* to the solution $(\mathbf{m}_0,\mathbf{h}_0)$ of the quasistatic model (23) in $\mathbb{L}^\infty(0,\tau;\mathbb{H}^1(\mathbb{R}^3))$ as η tends to 0(modulo the extraction of a subsequence).*

The proof mimics the proof by Carbou in [2] for the convergence of the complete system towards the quasistatic system as the permittivity ε_0 tends to 0. But we still do not approximate the electric field. Therefore we introduce the other terms in the expansion.

They are given for any $n \geq 0$ by

$$\begin{aligned} &\frac{\partial \mathbf{m}_n}{\partial t} = -\sum_{k+l=n} \mathbf{m}_k \times \mathbf{h}_{T,l} - \alpha \sum_{k+l=n} \sum_{i+j=k} \mathbf{m}_l \times (\mathbf{m}_i \times \mathbf{h}_{T,j}) \text{ in } \omega, \\ &\sum_{k+l=n} \mathbf{m}_k \cdot \mathbf{m}_l = 0, \mathbf{m}_n(\mathbf{x}, 0) = 0, \text{a.e. in } \omega \end{aligned} \tag{26}$$

and

$$\mathbf{h}_{T,j} = -K\,\mathbf{u} \times (\mathbf{m}_j \times \mathbf{u}) + \bar{A}\Delta \mathbf{m}_j + \mathbf{h}_j \tag{27}$$

where $\mathbf{h}_j$ is given by (18).

It is a linear equation which can be shown to be well-posed. There is no proof of convergence today.

5 A dynamical method of simulation using finite volume

The idea is to compute the partial sums $(\check{\mathbf{m}}_n, \check{\mathbf{h}}_n, \check{\mathbf{e}}_n)$. Using the fact that for all i in $\mathbb{N}$, $(\mathbf{e}_i, \mathbf{h}_i, \mathbf{m}_i)$ depend only on $(\mathbf{e}_j, \mathbf{h}_j, \mathbf{m}_j)$ for $j \leq i$, we compute the $(\mathbf{e}_j, \mathbf{h}_j, \mathbf{m}_j)$ successively. At each level, we use the same finite volume method in space, but a different scheme in time to compute $(\mathbf{e}_j^n, \mathbf{h}_j^n, \mathbf{m}_j^n)$, approximation of $(\mathbf{e}_j, \mathbf{h}_j, \mathbf{m}_j)$ at time t_n.

For each time step t_n, $(\mathbf{e}_0^n, \mathbf{h}_0^n, \mathbf{m}_0^n)$ is first computed, by an explicit second order Taylor scheme in time for the system (26). It is proved in [4] that there exists a unique time step Δt_n such that the scheme is stable and has optimal convergence. $\mathbf{e}_0^n$ and $\mathbf{h}_0^n$ are obtained by solving a Laplace equation in ω.

Then $(\mathbf{e}_k^n, \mathbf{h}_k^n, \mathbf{m}_k^n)$ for $k > 0$ are computed successively by the following algorithm :

1. Prediction of $\mathbf{h}_k^n$ using $\mathbf{e}_{k-1}^n$, $\mathbf{e}_{k-2}^n$ and $\mathbf{m}_k^{n-1}$ with a first order implicit scheme in (18).
2. Computation of $\mathbf{m}_k^n$ using $\mathbf{h}_k^n$ with a first order implicit scheme in (26).
3. Correction of $\mathbf{h}_k^n$ using $\mathbf{m}_k^n$ in (18) .
4. Computation of $\mathbf{e}_k^n$ by solving (18) .

All the computations in space amount to solving a Poisson equation, for which we have fast solvers (see [4]). The presented algorithm provides an accurate method to compute the solutions to Landau-Lifshitz coupled with Maxwell's equations in ferromagnets.

References

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