The cotangent complex formalism

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Let $\mathcal{D}$ be a category with finite limits. The notion of an **abelian group** can be generalized to this abstract setting by defining an **abelian group object in** $\mathcal{D}$ to be an object $X \in \mathcal{D}$ equipped with maps $m : X \times X \longrightarrow X$, $e : * \longrightarrow X$ and $\mathrm{inv} : X \longrightarrow X$ which satisfy (diagramatically) all the axioms of an abelian group. In favorable cases (such as when $\mathcal{D}$ is presentable) the forgetful functor $\mathrm{Ab}(\mathcal{D}) \longrightarrow \mathcal{D}$ admits a left adjoint $\mathcal{D} \longrightarrow \mathrm{Ab}(\mathcal{D})$. For an object $X \in \mathcal{D}$ one should consider $\mathbb{Z}X$ as the **free abelian group object** generated from X. Our starting point is the following notion:

Definition 1. Let $\mathcal{C}$ be a category which admits finite limits and let $X \in \mathcal{C}$ an object. A **Beck module** over X is an abelian group object in the category of $\mathcal{C}_{/X}$ of objects equipped with a map to X. We denote by $\mathrm{Ab}(\mathcal{C}_{/X})$ the category of Beck modules. We will denote by $\mathbb{Z}_{/X} : \mathbb{C} \longrightarrow \mathcal{C}_{/X}$ a left adjoint to the forgetful functor (when exists).

Examples 2.

1. Let Set be the category of sets. Then for every set A the category $\mathrm{Ab}(\mathrm{Set}_{/A})$ is equivalent to the category Ab^X of tuples of abelian groups $(G_a)_{a \in A}$ indexed by A. If $f : B \longrightarrow A$ is a map of sets then $\mathbb{Z}_{/A}(B) = (\mathbb{Z}f^{-1}(a))_{a \in A}$ where $\mathbb{Z} : \mathrm{Set} \longrightarrow \mathrm{Ab}$ is the usual free abelian group functor. More generally, if $\mathcal{T}$ is a topos and $X \in \mathcal{T}$ is an object then $\mathcal{T}_{/X}$ is a topos as well, and one can find a small site $\mathcal{C}_X \subseteq \mathcal{T}_{/X}$ such that $\mathcal{T}_{/X}$ is the category of sheaves of sets on $\mathcal{C}_X$. In this case $\mathrm{Ab}(\mathcal{T}_{/X})$ is the category of sheaves of abelian groups on $\mathcal{C}_X$.

2. Let Ring be the category of associative rings. Given an associative ring A and an A-bimiodule M one may form the **square-zero extension** ring $M \rtimes A$ whose underlying abelian group is $M \oplus A$ and whose multiplcation is given by $(m, a)(n, b) = (mb + an, ab)$. One may then show that $M \rtimes A$, together with the canonical projection $M \rtimes A \longrightarrow A$ possess a natural structure of an abelian group object in the category $\mathrm{Ring}_{/A}$. Furthermore, the formation of square-zero extensions induces an equivalence between the category of A-bimodules and the category of abelian group objects in $\mathrm{Ring}_{/A}$. Under this equivalence the free abelian group functor can be identified with the functor which sends a map $B \longrightarrow A$ of rings to the

A-bimodule $A \otimes_B I_B \otimes_B A$, where I_B is the kernel of the multiplication map $B \otimes B \longrightarrow B$ considered as a map of B-bimodules.

3. Let Ring be the category of commutative rings. Given an commutative ring A and an A-miodule M one may canonically consider M as an A-bimodule and consequently form the square-zero extension ring $M \rtimes A$ as above, which will now be a commutative ring. One can then similarly show that the formation of square-zero extension induces an equivalence between $\mathrm{Mod}(A)$ and $\mathrm{Ab}(\mathrm{CRing}_{/A})$. Under this equivalence the free abelian group functor can be identified with the functor which sends a map $B \longrightarrow A$ of rings to the A-bimodule $(I_B/I_B^2) \otimes_B A = \Omega_B \otimes_B A$, where Ω_B is the module of Kähler differentials, or global 1-forms.

Given a map $f : X \longrightarrow Y$ we may then consider the induced map $f_* : \mathbb{Z}_{/Y}X \longrightarrow \mathbb{Z}_{/Y}Y$ as the abelian "shadow" of f. It turns out that this passage to a suitable abelianization is esspecially useful when one is doing homotopy theory. To fully appriciate this statement one should first understand what will be a good analogue of an abelian group in a higher categorical context. Suppose that $\mathcal{M}$ is a **model category**. A first possibility is to take the category $\mathrm{Ab}(\mathcal{M})$ of abelian group objects in $\mathcal{M}$ (which will carry an induced model structure in favorable cases). However, this construction is not well-behaved from a homotopical point of view. For example, it is possible that $\mathcal{M}$ and $\mathcal{M}'$ are Quillen equivalent model categories but $\mathrm{Ab}(\mathcal{M})$ and $\mathrm{Ab}(\mathcal{M}')$ are not Quillen equivalent. Focusing on the specific case of simplicial sets, it was long recognized that there are objects which should be considered as homotopical abelian groups, such as infinite loop spaces, but which cannot be modelled by simplicial abelian groups (nor by topological abelian groups). As algebraic topology developped it was realized that the best behaved higher analogue of an abelian group is not a space with additional structure, but something that is not a space at all, namely, a **spectrum**, or more precisely, an Ω-spectrum, i.e., a sequence of Kan complexes $X_0, X_1, ...,$ equipped with homotopy equivalences $f_n : X_n \xrightarrow{\simeq} \Omega X_{n+1}$. This was backed by results showing that Ω-spectra classify generalized cohomology theories (which can be considered as all possible well-behaved "abelian" invariants), and that the homotopy category of spectra is a triangulated category, a notion that was indepedently observed as being a good extension of the notion of an abelian category from the point of view of homological algebra.

Now let $\mathcal{M}$ be a model category. We will say that $\mathcal{M}$ is **pointed** if the canonical map $\varnothing \longrightarrow *$ from the initial to the terminal object is a weak equivalence. In favorable cases (e.g., when one has functorial factorizations), one can use the formation of homotopy limits to construct a functor $\Omega : \mathcal{M} \longrightarrow \mathcal{M}$ modelling the loop functor. In general Ω is only unique up to weak equivalence. Choosing such a loop functor one can define the notion of an Ω-spectrum in $\mathcal{M}$ as a sequence of objects $X_0, X_1, ...,$ equipped with homotopy equivalences $f_n : X_n \xrightarrow{\simeq} \Omega X_{n+1}$. The next step is to define a model category $\mathrm{Sp}(\mathcal{M})$ in which one can work with such spectrum objects. As is generally the case when constructing model categories, we do not expect all objects to look like Ω-specra, only the fibrant(-

cofibrat) objects. If one can realize the suspension-loop adjunction as a Quillen adjunction (for example, when $\mathcal{M}$ is a simplicial model category), then one can mimic the classicial construction of spectra (under some assumptions on $\mathcal{M}$) in a straightforward way, i.e., consider sequences $\{X_n\}$ of objects with structure maps $f_n : X_n \longrightarrow \Omega X_{n+1}$ which are not required to be equivalences (but will be so for fibrant objects). In the project [HNP] we offer another model for $\mathrm{Sp}(\mathcal{M})$ which does not require $\mathcal{M}$ to be simplicial. As with most construction, the model $\mathrm{Sp}(\mathcal{M})$ comes equipped with a canonical Quillen adjunction $\Sigma^\infty : \mathcal{M} \rightleftarrows \mathrm{Sp}(\mathcal{M}) : \Omega^\infty$ which is the analogue of the classical suspension infinity-loop infinty adjunction.

Definition 3. Let $\mathcal{M}$ be a pointed model category. We will say that $\mathcal{M}$ is **stable** if the loop functor $\Omega : \mathrm{Ho}(\mathcal{M}) \longrightarrow \mathrm{Ho}(\mathcal{M})$ is an equivalence of categories (equivalently, if the suspension functor is an equivalence of categories).

The passage from $\mathcal{M}$ to $\mathrm{Sp}(\mathcal{M})$ should be considered as a process of **stabilization**. Indeed, one can show that $\mathrm{Sp}(\mathcal{M})$ itself is stable, and that it is, in a suitable homotopy-theoretical sense, universal among stable model categories recieving a left Quillen functor from $\mathcal{M}$ (warning: this is not true on the nose). This leads to the following homotopy theoretic analogue of the notion of a Beck module:

Remark 4.

1. The contruction of $\mathrm{Sp}(\mathcal{M})$ requires that $\mathcal{M}$ be combinatorial and **left proper**. This last conditions can be hard to keep track of and doesn't hold in many examples of interest. It is hence important to note that of $\mathcal{M}$ is just assume to be combinatorial, the construction of [HNP] still works to produce a (left) **semi-model category**. The theory of semi-model categories is less well-known, although almost all of the favorable properties of model categories hold for semi-model categories as well. In what follows we will assume that all model categories in sight are combinatorial, and will use the notation $\mathrm{Sp}(\mathcal{M})$ to denote the construction of [HNP] whether the result is a model category or just a semi-model category.

2. The construction of $\mathrm{Sp}(\mathcal{M})$ requires that $\mathcal{M}$ is pointed. However, if $\mathcal{M}$ is not pointed then we may replace it with the pointed model category $\mathcal{M}_* = \mathcal{M}_{*/}$ of objects equipped with a map from the terminal object. If $*$ is cofibrant or $\mathcal{M}$ is left proper then $\mathcal{M}_*$ is homotopically well-behaved. In this case we will denote $\mathrm{Sp}(\mathcal{M}) \stackrel{\text{def}}{=} \mathrm{Sp}(\mathcal{M}_*)$ without indicating it explicitely. If $\mathcal{M}$ is not left proper or $*$ is not cofibrant then there are other ways to pointify. This pointification can be made functorial if one is working with semi-model categories. We will not elaborate more on this point.

Definition 5. Let $\mathcal{M}$ be a model category and let $X \in \mathcal{M}$ be an object. A **spectral Beck module** is a spectrum object in the model category $\mathcal{M}_{/X}$, i.e., an object of $\mathrm{Sp}(\mathcal{M}_{/X})$.

Let us now consider a classical example. Let $\mathcal{S}$ denote the category of simplcial sets and let $Y \in \mathcal{S}$ be an object. Then the category $\mathrm{Sp}(\mathcal{M}_{/Y})$ is a model for the theory of **parametrized spectra** over Y. In particular, we should think of Ω-spectra in $\mathrm{Sp}(\mathcal{M}_{/Y})$ as encding a family $\{S_y\}_{y \in Y}$ of spectra paramterized by Y. Let $\Sigma^{\infty}_{/Y} : \mathcal{M}_{/Y} \longrightarrow \mathrm{Sp}(\mathcal{M}_{/Y})$ denotes the associated suspension-infinity functor mentioned above. If $f : X \longrightarrow Y$ is a map, consider as an object of $\mathcal{M}_{/Y}$, then $\Sigma^{\infty}_{/Y} \in \mathrm{Sp}(\mathcal{M}_{/Y})$ corresponds under the above identification to the family of spectra $\{\Sigma^{\infty}_{+}(X_y)\}_{y \in Y}$, where X_y denotes the homotopy fiber of f over $y \in Y$, and Σ^{∞}_{+} is the (pointed) suspension-infinity functor of $\mathcal{S}$. To illustrate the interest of this construction, consider the following claim:

Theorem 6. *Let $f : X \longrightarrow Y$ be a 1-connected map of simplicial sets. Then f is a weak equivalence if and only if the map $\Sigma^{\infty}_{/Y}(X) \longrightarrow \Sigma^{\infty}_{/Y}(Y)$ is a weak equivalence in $\mathrm{Sp}(\mathcal{S}_{/Y})$*

Note that the parameterized specturm $\Sigma^{\infty}_{/Y}(Y)$ is just the constant family over Y with value the sphere spectrum. Theorem 6 can be reduced to the fact that if Z is a simply connected simplicial set then the terminal map $Z \longrightarrow *$ is a weak equivalence if and only if the map $\Sigma^{\infty}_{+}(Z) \longrightarrow \Sigma^{\infty}_{+}(*) = \mathbb{S}$ is a weak equivalence, which is iteself a consequence of Huerwicz's theorem relating homotopy and homology groups. One may interpret Theorem 6 as saying that the problem of determining if a map is an equivalence can be reduced to low dimensional computation (to check that f is 1-connected) followed by an analysis of the **abelian** or **stable** object $\Sigma^{\infty}_{/Y}(X)$. Since the homotopy theory of stable model categories is significatly simpler such reductions are truely useful in practice. We will later see another case where a theorem such as 6 holds.

Let $\mathcal{M}$ be a combinatorial model category and $X \in \mathcal{M}$ an object. The model category of spectral Beck modules over X can be informally considered as the **tangent model category** to $\mathcal{M}$ at X. This leads to considering all the model categories $\mathrm{Sp}(\mathcal{M}_{/X})$ as X varies. One may achieve this by defining a suitable category $\int_{X \in \mathcal{M}} \mathrm{Sp}(\mathcal{M}_{/X})$, known as the **Grothendieck construction**. The objects of $\int_{X \in \mathcal{M}} \mathrm{Sp}(\mathcal{M}_{/X})$ are pairs (X, S) where $X \in \mathcal{M}$ is an object and $S \in \mathrm{Sp}(\mathcal{M}_{/X})$ is a spectral Beck module over X, and morphisms are defined in a suitable way. We then have the following result:

Theorem 7. *Let $\mathcal{M}$ be a proper combinatorial model category. Then $\int_{X \in \mathcal{M}} \mathrm{Sp}(\mathcal{M}_{/X})$ can be endowed with a canonical model structure (compatible with each individual $\mathrm{Sp}(\mathcal{M}_{/X})$), such that the projection $\int_{X \in \mathcal{M}} \mathrm{Sp}(\mathcal{M}_{/X}) \longrightarrow \mathcal{M}$ is both a left and a right Quillen functor.*

If we think of each $\mathrm{Sp}(\mathcal{M}_{/X})$ as the tangent model category to $\mathcal{M}$ at X, then the model category $\int_{X \in \mathcal{M}} \mathrm{Sp}(\mathcal{M}_{/X})$ should be considered as the **model categorical tangent bundle** of $\mathcal{M}$. We will consequently denote it by

$$\mathcal{T}\mathcal{M} \stackrel{\text{def}}{=} \int_{X \in \mathcal{M}} \mathrm{Sp}(\mathcal{M}_{/X})$$

To phrase our next theorem we will need to explain the notion of an **operad**. Recall that an operad is a categorical gadget that is meant to encode a type of algebraic theory. To describe an operad (with values in sets) one needs to give for every $n \geq 0$ a set $\mathcal{P}(n)$ equipped with an action of the symmsetric group Σ_n. We think of elements of $\mathcal{P}(n)$ as indexing the n-to-1 operations of our algebraic theory. The action of Σ_n is then interpreted as a permutation of the entries. The objects $\{\mathcal{P}(n)\}$ must then be given a structure which corresponds to all the possible ways in which we can compose various multi-entry operations. For example, if we are given a 3-to-1 operation and three 2-to-1 operations then we should be able to compose them to get a 6-to-1 operation. Furthermore, the actions of the symmsetric groups Σ_2, Σ_3 by permuting the entries should be suitable compatible with the action of Σ_6 on 6-to-1 operations. This be encoded, for example, as a map of sets

$$\left[\mathcal{P}(2)^3 \times \mathcal{P}(3)\right] \times_{(\Sigma_2)^3 \rtimes \Sigma_3} \Sigma_6 \longrightarrow \mathcal{P}(6). \tag{1}$$

Now let $\mathcal{M}$ be a **symmetric monoidal** model category and let $\mathcal{P}$ be an operad as above. An **algebra** over $\mathcal{P}$ is an object $A \in \mathcal{M}$ together with maps $\mu_n : \mathcal{P}(n) \otimes_{\Sigma_n} A^{\otimes n} \longrightarrow A$ suitably compatible with the action maps of $\mathcal{P}$. We denote by $\mathrm{Alg}^{\mathcal{P}}(\mathcal{M})$ the category of $\mathcal{P}$-algebras and algebra maps. Here the tensor $K \otimes X$ of a set K and an object X stands for the coproduct $\coprod_{k \in K} X$. One may now observe that such algebras can be defined for a more general notion of an operad. Using maps asin 1 one may then formally define an operad **with values in** $\mathcal{M}$ to be a sequence of objects $\mathcal{P}(n) \in \mathcal{M}$, with $\mathcal{P}(n)$ is equipped with an action of Σ_n, and such that the entire sequence $\{\mathcal{P}(n)\}$ is equipped with all the maps which encode composition of multi-entry operations (where one replaced Cartesian products of sets as above with the monoidal structure of $\mathcal{M}$), subject that all the natural compatibility and associativity conditions these compositions satisfy. Similarly, one may define what it means for an object $A \in \mathcal{M}$ to have the structure of an algebra over such operad. If $\mathcal{P}$ is an operad in $\mathcal{M}$ then we will denote $\mathrm{Alg}^{\mathcal{P}}(\mathcal{M})$ simply by $\mathrm{Alg}^{\mathcal{P}}$.

We are now ready to describe our main result. Given an operad $\mathcal{P}$ in $\mathcal{M}$, we denote by $\mathcal{P}_{\leq 1}$ the operad whose 0-ary and 1-ary operations are those of $\mathcal{P}$ and such that $\mathcal{P}(n) = \varnothing \in \mathcal{M}$ for $n \geq 2$. There is a natural map of operads $\mathcal{P}_{\leq 1} \longrightarrow \mathcal{P}$. Finally, given any model category $\mathcal{M}$ let us denote by $\mathcal{M}_{\mathrm{aug}} = \mathcal{M}_{/\varnothing}$ the category of objects equipped with a map to the initial object $\varnothing \in \mathcal{M}$ (with its natural model structure). We then have the following theorem:

Theorem 8. *Let $\mathcal{M}$ be a differentiable combinatorial symmetric monoidal model category and let $\mathcal{P}$ be a cofibrant operad in $\mathcal{M}$. If $\mathcal{P}$ and $\mathcal{P}_{\leq 1}$ are both addimissible then the free-forgetful adjunction*

$$\mathcal{F} : \mathrm{Sp}(\mathrm{Alg}^{\mathcal{P}_{\leq 1}}_{\mathrm{aug}}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathrm{Sp}(\mathrm{Alg}^{\mathcal{P}}_{\mathrm{aug}}) : \mathcal{U}$$

is a Quillen equivalence.

Remark 9.

1. In plain terms Theorem 8 says that for the process of stabilizing a category of algebras does not depend on any of the higher oprations of the algebraic theory.

2. In practice, $\mathrm{Alg}_{\mathrm{aug}}^{\mathcal{P}_{\leq 1}}$ is a much simpler object than $\mathrm{Alg}_{\mathrm{aug}}^{\mathcal{P}}$, and its stabilization is much more readily accessible. One should hence consider Theorem 8 as offering a computation of $\mathrm{Alg}_{\mathrm{aug}}^{\mathcal{P}}$ in terms of $\mathrm{Alg}_{\mathrm{aug}}^{\mathcal{P}_{\leq 1}}$.

3. Let $A \in \mathcal{M}$ be a $\mathcal{P}$-algebra object. Then one can form another operad $\mathcal{P}_A$, known as the **enveloping operad** of A, such that $\mathcal{P}_A$ algebras are the same as $\mathcal{P}$-algebras equipped with a map from A. The category $\mathrm{Alg}_{\mathrm{aug}}^{\mathcal{P}_A}$ of augmented $\mathcal{P}_A$-algebras is the same as the category of pointed objects in $\mathrm{Alg}_{/A}^{\mathcal{P}}$, and in particular $\mathrm{Sp}(\mathrm{Alg}_{\mathrm{aug}}^{\mathcal{P}_A}) \simeq \mathrm{Sp}(\mathrm{Alg}_{/A}^{\mathcal{P}})$. One can hence use the above theorem to compute the stabilization not only of categories of algebras but also for algebras over a fixed algebra A.

The following Corollary of Theorem 8 was proven in the setting of ∞-categories and ∞-operads by Lurie (see[Lu14]).

Corollary 10. *Let $\mathcal{M}$ and $\mathcal{P}$ be as in Theorem 8 and assume in addition that $\mathcal{M}$ is pointed, stable and additive (e.g., $\mathcal{M}$ is a category of complexes). The for any cofibrant $\mathcal{P}$-algebra object A the stabilization $\mathrm{Sp}(\mathrm{Alg}_{/A}^{\mathcal{P}})$ is canonically Quillen equivalent to the category of (operadic) A-modules.*

Theorem 8 can also be used to compute stabilizations which were not known before:

Corollary 11. *Let $\mathcal{M}$ be the model category of simplicial (associative) monoids and let $X \in \mathcal{M}$ be a monoid. Then $\mathrm{Sp}(\mathcal{M}_{/X})$ is equivalent to the model category of $X \times X^{\mathrm{op}}$-equivariant parametrized spectra over X.*

Let $\mathcal{S}$ be the model category of simplicial sets and let $\mathcal{T}\mathcal{S} = \int_{X \in \mathcal{S}} \mathrm{Sp}(\mathcal{S}_{/X})$ be its tangent bundle (recall that objects of $\mathcal{T}\mathcal{S}$ are pairs $(X, \overline{S})$ where X is a simplicial set and $\overline{S} \in \mathrm{Sp}(\mathcal{S}_{/X})$ is a parametrized spectrum over X), equipped with the model structure of Theorem 7. It can then be shown that $\mathcal{T}\mathcal{S}$ is furthermore a simplicial model category. A suitable generalization of 8 to the setting of **colored operads** implies the following previously unknown result:

Proposition 12. *Let Cat_{Δ} be a the model category of small simplicial categories and let $\mathcal{C} \in \mathrm{Cat}_{\Delta}$ be a small simplicial category. Then $\mathrm{Sp}((\mathrm{Cat}_{\Delta})_{/\mathcal{C}})$ is equivalent to the category of simplcial dotted lifts*

$$\begin{array}{ccc} & & \mathcal{T}\mathcal{S} \\ & \nearrow & \downarrow \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\mathcal{F}_{\mathcal{C}}} & \mathcal{S} \end{array}$$

where $\mathcal{F}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{S}$ is the mapping space functor $\mathcal{F}_{\mathcal{C}}(X, Y) = \mathrm{Map}_{\mathcal{C}}(X, Y)$.

Let us now elaborate more on Corollary 10 in the case where $\mathcal{M}$ is the model category fo complexes over a field k of characteristic 0, and $\mathcal{P}$ is the commutative operad. In this case Corollary 10 says that for every CDGA A the stabilization

$\mathrm{Sp}(\mathrm{CDGA}_{/A})$ is canonically Quilen equivalent to the category $\mathrm{Mod}(A)$ of A-module complexes. In this case one can track down the suspension-infinity functor $\Sigma^\infty : \mathrm{CDGA}_{/A} \longrightarrow \mathrm{Sp}(\mathrm{CDGA}_{/A}) \cong \mathrm{Mod}(A)$ explicitely. Recall that for a discrete k-algebra A one may consider its **cotangent module** Ω_A, also known as the module of **Kähler differentials**. If A is finitely generated then we consider it as the algebra of functions on an affine variety $X = \mathrm{spec}(A)$ over $\mathbb{C}$. If X is smooth then Ω_A can be identified with the module of (globally defined) differentials 1-forms on X, and is an algebraic incarnation of the **cotangent bundle** of the variety X. If X is not smooth then Ω_A is not such a well-behaved object. It turns out that one way to "fix" Ω_A is to consider A as a CDGA which is concentrated in degree 0. One may then extend the formation of cotangent modules to this context. This yeilds a left Quillen functor, which can consequently be **derived** (by precomposing it with a cofibrant replacement functor). The resulting object $L_A = \Omega_{A^{\mathrm{cof}}}$ is now not a just an A-module, but a complex of A-modules (or, alternatively, an A-module complex), and is known as the **cotangent complex** of A. It turns out that L_A is much better behaved then Ω_A when A is not smooth, and has many applications in deformation theory and related fields. We may now relate the cotangent complex to our previous discussion:

Proposition 13. *Under the identification* $\mathrm{Sp}(\mathrm{CDGA}_{/A}) \simeq \mathrm{Mod}(A)$ *the suspension-infinity functor* $\Sigma^\infty(\mathrm{CDGA}_{/A}) \longrightarrow \mathrm{Sp}(\mathrm{CDGA}_{/A})$ *corresponds to the functor which sends a map of CDGA's* $B \longrightarrow A$ *to the base change* $L_B \otimes_B^L A$ *of the cotangent complex of* B *to* A.

It then turns out that the model category CDGA admits an analogue of Theorem 6: (see [Lu14, 7.4.3.4]):

Theorem 14 (The cotagent complex Whitehead theorem)**.** *Let* $f : A \longrightarrow B$ *be a map of non-negatively graded CDGA's over* $\mathbb{C}$ *such that the induced map* $\pi_0(A) \longrightarrow \pi_0(B)$ *is an isomoprhism of* $\mathbb{C}$*-algebras. Then* f *is a quasi-isomorphism if and only if the induced map* $f_* : L_B \otimes_B^L A \longrightarrow L_A$ *is a quasi-isomorphism of* A*-modules.*

Corollary 10 now allows one to abstractly identify the claims of both Theorem 14 and Theorem 14. In both cases one can reduce the problem of determining if a map is an equivalence to a low dimensional computation followed by an analysis of a **stable** object which is obtained in both cases by applying the suspension-infinity functor $\Sigma^\infty : \mathcal{M}_{/Y} \longrightarrow \mathrm{Sp}(\mathcal{M}_{/Y})$ to the map in question. It is then a very interesting question to understand for what type of model categories one may expect a statement of this kind to hold. For example, using the computation of Proposition 12 one has the formal tools needed to check if one can obtain results such as 6 and 14 for simplicial categories.

References

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