Cohomology of higher categories

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Fundamental goal of algebraic topology: understand maps $X \to Y$ between spaces, up to homotopy.
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Classical idea
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**Classical idea**

Filter $Y$ by its **Postnikov tower**

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Here $P_n(Y)$ is $n$-truncated and we have a natural $n$-equivalence $Y \to P_n(Y)$ such that $Y \sim \lim_n P_n(Y)$. 
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Here $P_n(Y)$ is $n$-truncated and we have a natural $n$-equivalence $Y \to P_n(Y)$ such that $Y \xrightarrow{\sim} \lim_n P_n(Y)$.

⇒ Problem broken to smaller pieces: given a map $f : X \to P_n(Y)$, understand all lifts of $f$ to $\bar{f} : X \to P_{n+1}(Y)$ (up to homotopy over $P_n(Y)$).
Second step: understand the small pieces.
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### $k$-invariants

Suppose $Y$ is simply connected (so that $P_1(Y)/\sim$). For $n \geq 1$, $P_n + 1(Y)$ is a principal fibration with structure group $K(\pi_n + 1(Y), n + 1)$, classified by the $k_n$-invariant $k_n \in H^{n+2}(P_n(Y); \pi_n + 1(Y))$.

⇒ If $f : X \to P_n(Y)$ is a map then $f$ lifts to $P_n + 1(Y)$ if and only if $f^* k_n \in H^{n+2}(X; \pi_n + 1(Y))$ vanishes.

In this case the possible lifts $f : X \to P_n + 1(Y)$ (up to homotopy over $P_n(Y)$) form a torsor under $H^{n+1}(X; \pi_n + 1(Y))$. 
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\textit{k}-invariants

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**k-invariants**

Suppose \( Y \) is **simply connected** (so that \( P_1(Y) \simeq \ast \)). For \( n \geq 1 \)

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P_{n+1}(Y) \longrightarrow P_n(Y)
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is a **principal fibration** with structure group \( K(\pi_{n+1}(Y), n + 1) \), classified by the **\( k \)-invariant** \( k_n \in H^{n+2}(P_n(Y); \pi_{n+1}(Y)) \).

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Obstruction theory

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The obstruction theoretic process can be streamlined into a Bousfield-Kan spectral sequence starting from $H^s(X; \pi_t(Y))$ and abutting to $\pi_{t-s}(\text{Map}(X, Y))$. If $Y$ is not simply connected then $\pi_t(Y)$ (for $t \geq 2$) is a local system of abelian groups on $Y$, and $H^s(X; \pi_t(Y))$ should be interpreted as cohomology with local coefficients.

Given maps $i: A \to X$ and $f_0: A \to Y$ one can also use the above machinery to study all the maps $f: X \to Y$ which extend $f_0$ along $i$ (up to homotopy), this time using the relative cohomology with local coefficients $H^\bullet(X, A; \pi^\bullet(Y))$. 

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Remarks

- The obstruction theoretic process can be streamlined into a Bousfield-Kan spectral sequence starting from $H^s(X; \pi_t(Y))$ and abutting to $\pi_{t-s} \text{Map}(X, Y)$.

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The Hurewicz principle

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A & \to & Y \\
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**Corollary (The Hurewicz principle for spaces)**

A map \( f : A \to X \) is an equivalence if and only if

- \( f \) induces an equivalence on fundamental groupoids.
- \( f \) induces an isomorphism on cohomology for every local coefficient system on \( X \).
Towards higher categories

Can we obtain a similar theory when spaces are replaced with $\infty$-categories?

Questions

What should replace cohomology with local coefficients?

What should replace homotopy groups?

Can we make such an obstruction theory accessible and computable?

Can we get an associated Hurewicz principle?
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What is cohomology?

To generalize cohomology outside spaces, we should start from the most general form we know for spaces.
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Question
What kind of cohomologies do we know for spaces?

Answer
- Cohomology with coefficients.
- Generalized cohomology theories - coefficients in a spectrum.
- Twisted generalized cohomology theories - coefficients in a parameterized spectrum (i.e., a local system of spectra).

Can one generalize the notion of parameterized spectrum outside the realm of spaces?
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Can one generalize the notion of parameterized spectrum outside the realm of spaces?
The cotangent complex formalism (Lurie)

\( \mathcal{D} \) - a presentable \( \infty \)-category, \( X \in \mathcal{D} \) an object.

Definition

A parameterized spectrum over \( X \) is an \( \Omega \)-spectrum object in the slice \( \infty \)-category \( \mathcal{D}/X \), i.e., an object of \( \text{Sp}(\mathcal{D}/X) = \lim_{\leftarrow / \text{alt}}^{\text{} \rightarrow / \text{alt}} (\mathcal{D}/X)^* \).

A canonical adjunction:

\[ \Sigma_\infty^+ \vdash \mathcal{D}/X \leftarrow \text{Sp}(\mathcal{D}/X) : \Omega_\infty \].

Definition

\( L_X = \Sigma_\infty^+ (\text{Id}_X) \in \text{Sp}(\mathcal{D}/X) \)

the abstract cotangent complex of \( X \).
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\( \mathcal{L}_X := \Sigma_{\infty}^+ (\text{Id}_X) \in \text{Sp}(\mathcal{D}/X) \) the **abstract cotangent complex** of \( X \).
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Quillen cohomology

The cotangent complex

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Example

When \( \mathcal{D} = \mathcal{S} \) is the \( \infty \)-category of spaces we may identify \( \text{Sp}(\mathcal{S}/\mathcal{L}_X) \simeq \text{Fun}(X, \text{Sp}) \) with local systems of spectra, and \( L_X \) with the constant local system with value the sphere spectrum \( \mathcal{S} \in \text{Sp} \).
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For a parametized spectrum \( M \in \text{Sp}(\mathcal{D}/X) \), define the \( n \)'th **Quillen cohomology** of \( X \) with coefficients in \( M \) by

\[ H^n_{Q}(X; M) = \pi_0 \text{Map}_{\text{Sp}(\mathcal{D}/X)}(L_X, M[n]) \]

where \( M[n] \) is the \( n \)'th suspension of \( M \) in \( \text{Sp}(\mathcal{D}/X) \).
Twisted arrow categories

In spaces, the Quillen cohomology groups of a space $X$ with coefficient in $M \in \text{Sp}(S/X) \simeq \text{Fun}(X, \text{Sp})$ are just the homotopy groups of the limit spectrum $\lim_X M$. 

What about $\infty$-categories?

The twisted arrow category $\text{Tw}(C)$ of an $\infty$-category $C$ is the $\infty$-category whose objects are the morphisms of $C$ and such that maps from $f : X \to Y$ to $g : Z \to W$ are given by commutative diagrams of the form:

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Such diagrams can be encoded by 3-simplices in $C$. Similarly, the $n$-simplices of $\text{Tw}(C)$ are the $(2n+1)$-simplices of $C$. 

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Such diagrams can be encoded by 3-simplices in \( \mathcal{C} \). Similarly, the \( n \)-simplices of \( \text{Tw}(\mathcal{C}) \) are the \( (2n+1) \)-simplices of \( \mathcal{C} \).
Theorem (J. Nuiten, M. Prasma, H.)

For an $\infty$-category $\mathcal{C}$ there is a natural equivalence

$$\text{Sp}((\text{Cat}_\infty)/\mathcal{C}) \simeq \text{Fun}(\text{Tw}(\mathcal{C}), \text{Sp})$$

between parameterized spectra over $\mathcal{C}$ and functors from the twisted arrow category of $\mathcal{C}$ to spectra. The cotangent complex of $\mathcal{C}$ is the constant functor with value $S[-1]$. 
Main result

Theorem (J. Nuiten, M. Prasma, H.)

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between parameterized spectra over $\mathcal{C}$ and functors from the twisted arrow category of $\mathcal{C}$ to spectra. The cotangent complex of $\mathcal{C}$ is the constant functor with value $\mathbb{S}[-1]$.

This means that Quillen cohomology of $\infty$-categories can be described as a functor cohomology: its coefficients are functors $\mathcal{M} : \text{Tw}(\mathcal{C}) \longrightarrow \text{Sp}$ and the corresponding Quillen cohomology groups are (up to a shift) the homotopy groups of the limit $\lim_{\text{Tw}(\mathcal{C})} \mathcal{M}$. 

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A few words about the proof:

The statement that we prove is actually more general, and pertains to enriched ∞-categories. Roughly speaking, we prove that the data of a parameterized spectrum over an enriched ∞-category $C$ is equivalent to data of choosing in a suitably compatible way a parameterized spectrum over each mapping object $\text{Map}_C(x, y)$.

To prove this statement for a particular enriched ∞-category $C$ we show that one can restrict attention to enriched ∞-categories with a fixed set of objects. Enriched ∞-categories with a fixed set of objects can also be described as ∞-categories of algebras over a suitable colored operad. We then use a previous joint result which involves identifying parameterized spectra over algebra objects with parameterized spectra over module objects.
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In light of the theorem above we may identify functors $M : \text{Tw}(\mathcal{C}) \to \text{Sp}$ with parameterized spectra over $\mathcal{C}$.
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**Question**

Given \( M : \text{Tw}(\mathcal{C}) \to \text{Sp} \), can we describe the associated \( \infty \)-category \( \Omega^\infty(M) \to \mathcal{C} \) in explicit terms?
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Question

Given $M : \text{Tw}(\mathcal{C}) \rightarrow \text{Sp}$, can we describe the associated $\infty$-category $\Omega^\infty(M) \rightarrow \mathcal{C}$ in explicit terms?

Answer

Yes. We may identify $\Omega^\infty(M)$ with the $\infty$-category whose objects are pairs $(X, \eta)$ with $X \in \mathcal{C}$ and $\eta$ is a map $\eta : S[-1] \rightarrow M(\text{Id}_X)$. Maps from $(X, \eta)$ to $(X', \eta')$ are pairs $(f, H)$ where $f : X \rightarrow X'$ is a map in $\mathcal{C}$ and $H$ is a homotopy between the two resulting maps $f_*\eta, f_*\eta' : S[-1] \rightarrow M(f)$.
In light of the theorem above we may identify functors $M : \text{Tw}(C) \to \text{Sp}$ with parameterized spectra over $C$.

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We note that $\Omega^\infty(M)$ is naturally an $\mathbb{E}_\infty$-group object in $(\text{Cat}_\infty)_{/C}$. In particular, we can sum objects which lie in the same fiber: $(X, \eta) \boxplus (X, \eta') = (X, \eta + \eta')$. 
Back to the abstract setting: $\mathcal{D}$ a presentable $\infty$-category and $X \in \mathcal{D}$ an object. How can we use Quillen cohomology to do a Bousfield-Kan type obstruction theory and spectral sequence?
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**Small extensions**

A *small extension* of $X \in \mathcal{D}$ with coefficients in $M \in \text{Sp}(\mathcal{D}/X)$ is a Cartesian square in $\mathcal{D}/X$ of the form:

$$
\begin{array}{ccc}
Y & \longrightarrow & \Omega^\infty(0) \\
\downarrow & & \downarrow \\
\downarrow & & \\
X & \xrightarrow{\alpha} & \Omega^\infty(M[1])
\end{array}
$$

where the right vertical map is obtained by applying the functor $\Omega^\infty : \text{Sp}(\mathcal{D}/X) \longrightarrow \mathcal{D}/X$ to the 0-map $0 \longrightarrow M[1]$. 
Quillen obstruction theory

Back to the abstract setting: $\mathcal{D}$ a presentable $\infty$-category and $X \in \mathcal{D}$ an object. How can we use Quillen cohomology to do a Bousfield-Kan type obstruction theory and spectral sequence?

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\downarrow & & \downarrow \\
X & \xrightarrow{\alpha} & \Omega^\infty(M[1])
\end{array}
$$

where the right vertical map is obtained by applying the functor $\Omega^\infty : \text{Sp}(\mathcal{D}/X) \to \mathcal{D}/X$ to the 0-map $0 \to M[1]$.

The data of $\alpha : X \to \Omega^\infty(M[1])$ is equivalent by adjunction to the data of a map $L_X \to M[1]$ and hence determines a class $[\alpha] \in H^1_Q(X; M)$. We consider $p$ as a geometric incarnation of $[\alpha]$. 

Yonatan Harpaz
Small extensions

Let $Y \longrightarrow X$ be a small extension of $X$ with coefficients in $M$ and corresponding class $[\alpha] \in H^1_Q(X; M)$. Then:

1. $\Omega^\infty(M) \rightarrow X$ is an $E^\infty$-group object in $\mathcal{C}/\mathcal{X}$ and $p: Y \rightarrow X$ is a torsor under $\Omega^\infty(M)$.
2. $p: Y \rightarrow X$ has a section if and only if $[\alpha] = 0$, in which case $Y$ is equivalent over $X$ to $\Omega^\infty(M)$.
3. If $f: A \rightarrow X$ is any map then $f$ lifts to $f: A \rightarrow Y$ if and only if the pulled back class $f^* [\alpha] \in H^1_Q(X; f^* M)$ vanishes.

Example: If $Y$ is a space then for every $n \geq 1$ the map $P^n+1(Y) \rightarrow P^n(Y)$ is a small extension with coefficients in the local system of Eilenberg-MacLane spectra $\mathcal{E}_{K}(\pi_{n+1}(Y), n+1)$, whose Quillen cohomology class is $k^n \in H^{n+2}(Y; \pi_{n+1}(Y)) = H^1_Q(Y; \mathcal{E}_{K}(\pi_{n+1}(Y), n+1))$. 

Yonatan Harpaz
Let $Y \rightarrow X$ be a small extension of $X$ with coefficients in $M$ and corresponding class $[\alpha] \in H^1_Q(X; M)$. Then:

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Example: If $Y$ is a space then for every $n \geq 1$ the map $P_{n+1}(Y) \to P_n(Y)$ is a small extension with coefficients in the local system of Eilenberg-MacLane spectra $H^1(K(\pi_n+1(Y), n+1))$, whose Quillen cohomology class is $k_n \in H^{n+2}(Y; \pi_{n+1}(Y)) = H^1_Q(Y; H^1(K(\pi_n+1(Y), n+1))) = H^1_Q(Y; M)$. 

Yonatan Harpaz
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Let \( Y \rightarrow X \) be a small extension of \( X \) with coefficients in \( M \) and corresponding class \( [\alpha] \in H^1_Q(X; M) \). Then:

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**Example**

If \( Y \) is a space then for every \( n \geq 1 \) the map \( P_{n+1}(Y) \rightarrow P_n(Y) \) is a small extension with coefficients in the local system of Eilenberg-MacLane spectra \( \mathbb{H}K(\pi_{n+1}(Y), n + 1) \), whose Quillen cohomology class is

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k_n \in H^{n+2}(Y; \pi_{n+1}(Y)) = H^1_Q(Y; \mathbb{H}K(\pi_{n+1}(Y), n + 1))
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Recall that we may identify parameterized spectra over $\mathcal{C}$ with functors $M : \text{Tw}(\mathcal{C}) \to \text{Sp}$.
Small extensions of $\infty$-categories

Recall that we may identify parameterized spectra over $\mathcal{C}$ with functors $M : \text{Tw}(\mathcal{C}) \longrightarrow \text{Sp}$.

**Question**

Given $\alpha \in \text{Map}(\mathbb{S}[-1], \lim_{\text{Tw}(\mathcal{C})} M[1]) \cong \text{Map}_{\mathcal{C}}(\mathcal{C}, \Omega^\infty M[1])$, can we describe the small extension $p_\alpha : \mathcal{C}_\alpha \longrightarrow \mathcal{C}$ corresponding to $\alpha$ in explicit terms?
Recall that we may identify parameterized spectra over $\mathcal{C}$ with functors $M : \text{Tw}(\mathcal{C}) \rightarrow \text{Sp}$.

**Question**

Given $\alpha \in \text{Map}(\mathbb{S}[-1], \lim_{\text{Tw}(\mathcal{C})} M[1]) \simeq \text{Map}_\mathcal{C}(\mathcal{C}, \Omega^\infty M[1])$, can we describe the small extension $p_\alpha : \mathcal{C}_\alpha \rightarrow \mathcal{C}$ corresponding to $\alpha$ in explicit terms?

**Answer**

Yes. We may identify $\mathcal{C}_\alpha$ with the $\infty$-category whose objects are pairs $(X, \eta)$ with $X \in \mathcal{C}$ and $\eta$ is a null-homotopy of the $\text{Id}_X$ component $\alpha_{\text{Id}_X} : \mathbb{S}[-1] \rightarrow M(\text{Id}_X)[1]$ of $\alpha$. Maps from $(X, \eta)$ to $(X', \eta')$ are pairs $(f, H)$ where $f : X \rightarrow X'$ is a map in $\mathcal{C}$ and $H$ is a homotopy between the two resulting null homotopies $f_* \eta, f_* \eta'$ of $\alpha_f : \mathbb{S}[-1] \rightarrow M(f)[1]$. 
Postnikov towers for $\infty$-categories

$\text{Ho}_n(\mathcal{C})$ - the $\infty$-category obtained from $\mathcal{C}$ by replacing each mapping object with its $n$'th Postnikov piece.
Postnikov towers for ∞-categories

$\text{Ho}_n(C)$ - the $\infty$-category obtained from $C$ by replacing each mapping object with its $n$'th Postnikov piece.

$\pi_{n+1}(C) : \text{Tw}(\text{Ho}_1(C)) \longrightarrow \text{Ab}$ the functor which sends an arrow $f : X \longrightarrow Y$ to $\pi_{n+1}(\text{Map}(X, Y), f)$. 
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**Fact (Goes back to Dwyer and Kan)**

The tower

$$\ldots \rightarrow \text{Ho}_n(\mathcal{C}) \rightarrow \ldots \rightarrow \text{Ho}_1(\mathcal{C})$$

converges to $\mathcal{C}$ and each $\text{Ho}_{n+1}(\mathcal{C}) \rightarrow \text{Ho}_n(\mathcal{C})$ is a small extension with coefficients in $\mathbb{H}K(\pi_{n+1}(\mathcal{C}), n + 1)$. 

Yonatan Harpaz
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The elements $k_n \in H^1_Q(\text{Ho}_n(C), \mathbb{H}K(\pi_{n+1}(C), n + 1))$ classifying the small extensions $\text{Ho}_{n+1}(C) \to \text{Ho}_n(C)$ can be considered as the $k$-invariants of $C$. 
Quillen obstruction theory for ∞-categories

Starting from a map $D \longrightarrow \text{Ho}_1(C)$ we now obtain an obstruction theory to the existence of a lift $D \longrightarrow C$ as well as a spectral sequence starting from

$$E_{1}^{t,s} = H_{Q}^{s}(D, \pi_{t}(C)) = \lim_{s+1}^{\infty} \text{Tw}(D) \pi_{t}(C) \quad s \geq -1, \ t \geq 2$$

and abutting to $\pi_{t-s} \text{Map}_{/\text{Ho}_1(C)}(D, C)$. 

Example (Splitting homotopy idempotents)

Let $\text{Idem}$ be the category with one object $x_0$ and one non-identify morphism $f : x_0 \longrightarrow x_0$ such that $f \circ f = f$. A direct computation shows that the category $\text{Tw}(\text{Idem})$ has cohomological dimension 1.

We may then conclude that every map $\text{Idem} \longrightarrow \text{Ho}_1(C)$ admits a non-empty and even simply-connected space of lifts $\text{Idem} \longrightarrow C$.

When $C$ is idempotent complete we may conclude that a homotopy idempotent $\text{Idem} \longrightarrow \text{Ho}_1(C)$ splits if and only if it lifts to $\text{Ho}_1(C)$. 

Yonatan Harpaz
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Starting from a map $\mathcal{D} \longrightarrow \text{Ho}_1(\mathcal{C})$ we now obtain an obstruction theory to the existence of a lift $\mathcal{D} \longrightarrow \mathcal{C}$ as well as a spectral sequence starting from

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Example (Splitting homotopy idempotents)

Let $\text{Idem}$ be the category with one object $x_0$ and one non-identify morphism $f : x_0 \longrightarrow x_0$ such that $f \circ f = f$. A direct computation shows that the category $\text{Tw}(\text{Idem})$ has cohomological dimension 1. We may then conclude that every map $\text{Idem} \longrightarrow \text{Ho}_1(\mathcal{C})$ admits a non-empty and even simply-connected space of lifts $\text{Idem} \longrightarrow \mathcal{C}$. When $\mathcal{C}$ is idempotent complete we may conclude that a homotopy idempotent $\text{Idem} \longrightarrow \text{Ho}(\mathcal{C})$ splits if and only if it lifts to $\text{Ho}_1(\mathcal{C})$. 
Even higher categories

Can we do something similar for $(\infty, n)$-categories?
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**Theorem (Nuiten, Prasma, H., cf. H. K. Nguyen)**

Let $\mathcal{C}$ be an $(\infty, n)$-category and let $\text{Ho}_k(\mathcal{C})$ denote the $(\infty, n)$-category obtained from $\mathcal{C}$ by replacing each space of $n$-morphisms by its $k$’th Postnikov piece. Then

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Even higher categories

Can we do something similar for \((\infty, n)\)-categories?

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is a tower of small extensions.

**Corollary (The Hurewicz principle for \((\infty, n)\)-categories)**

Let \(f : \mathcal{C} \longrightarrow \mathcal{D}\) be a functor between \((\infty, n)\)-categories. Then \(f\) is an equivalence if and only if

- \(f_* : \text{Ho}_1(\mathcal{C}) \longrightarrow \text{Ho}_1(\mathcal{D})\) is an equivalence of \((n + 1, n)\)-categories.
- \(f\) induces an isomorphism on Quillen cohomology for any choice of coefficients \(M \in \text{Sp}((\text{Cat}(\infty, n))/\mathcal{C})\).
The twisted 2-cell category

But what are parameterized spectra over an \((\infty, n)\)-category?
The twisted 2-cell category

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**Theorem (Nuiten, Prasma, H.)**

Let \(\mathcal{C}\) be an \((\infty, 2)\)-category. Then there exists a natural equivalence

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\text{Sp}((\text{Cat}_{\infty, 2})/\mathcal{C}) \simeq \text{Fun}(\text{Tw}_2(\mathcal{C}), \text{Sp})
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between parameterized spectra over \(\mathcal{C}\) and functors from what we call the **twisted 2-cell category** \(\text{Tw}_2(\mathcal{C})\) of \(\mathcal{C}\) to spectra.
But what are parameterized spectra over an $(\infty, n)$-category? Well, at the moment we only know how to handle $n = 2$.

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Conjecturally, there exists an analogue of the above theorem for all $n$, using a suitable **twisted $n$-cell category**.
Example: the classification of adjunctions

Let $\mathbf{1} = \bullet \xrightarrow{f} \bullet$ and let $\mathbf{1} \to \text{Adj}$ be the free 2-category generated from $\mathbf{1}$ by adding a right adjoint to $f$.

Proposition (Nuiten, Prasma, H.)

The induced map $\text{Tw}_2(\mathbf{1}) \to \text{Tw}_2(\text{Adj})$ is coinitial. In particular, the map $\mathbf{1} \to \text{Adj}$ induces an isomorphism on Quillen cohomology for any choice of coefficients $M : \text{Tw}_2(\text{Adj}) \to \text{Sp}$.

Corollary (cf. Riehl and Verity)

Any diagram of the form:

\[
\begin{array}{ccc}
\mathbf{1} & \downarrow & \mathbf{1} \\
\downarrow & & \downarrow \\
\Rightarrow & & \Rightarrow \\
\Rightarrow & & \Rightarrow \\
\text{C} & \to & \text{Adj} \\
\Rightarrow & & \Rightarrow \\
\text{Ho}_1(\text{C}) & \to & \text{Ho}_1(\text{Adj})
\end{array}
\]

admits a contractible space of lifts. This means, in effect, that homotopy coherent adjunctions are uniquely determined by explicit low dimensional data.
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Thank you!