Cohomology of higher categories

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$\Rightarrow$ Problem broken to smaller pieces: given a map $f : X \longrightarrow P_n(Y)$, understand all lifts of f to $\overline{f} : X \longrightarrow P_{n+1}(Y)$ (up to homotopy over $P_n(Y)$).

Obstruction theory

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is a **principal fibration** with structure group $K(\pi_{n+1}(Y), n+1)$, classified by the k-**invariant** $k_n \in H^{n+2}(P_n(Y); \pi_{n+1}(Y))$.

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$\Rightarrow$ if $f : X \longrightarrow P_n(Y)$ is a map then f lifts to $P_{n+1}(Y)$ if and only if $f^* k_n \in H^{n+2}(X; \pi_{n+1}(Y))$ vanishes. In this case the possible lifts $\overline{f} : X \longrightarrow P_{n+1}(Y)$ (up to homotopy over $P_n(Y)$) form a torsor under $H^{n+1}(X; \pi_{n+1}(Y))$.

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Bousfield-Kan spectral sequence

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- If Y is not simply connected then $\pi_t(Y)$ (for $t \geq 2$) is a **local system** of abelian groups on Y, and $H^s(X;\pi_t(Y))$ should be interpreted as cohomology with **local coefficients**.
- Given maps $i : A \longrightarrow X$ and $f_0 : A \longrightarrow Y$ one can also use the above machinery to study all the maps $f : X \longrightarrow Y$ which extend f_0 along i (up to homotopy), this time using the **relative cohomology** with local coefficients $H^\bullet(X,A;\pi_\bullet(Y))$.

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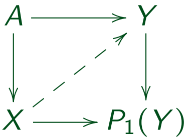

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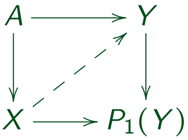

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Corollary (The Hurewicz principle for spaces)

A map $f : A \longrightarrow X$ is an equivalence if and only if

- *f induces an equivalence on fundamental groupoids.*
- *f induces an isomorphism on cohomology for every local coefficient system on X.*

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Towards higher categories

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- Can we get an associated Hurewicz principle?

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Can one generalize the notion of parameterized spectrum outside the realm of spaces?

The cotangent complex formalism (Lurie)

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A **parameterized spectrum** over X is an Ω-spectrum object in the slice ∞-category $\mathcal{D}_{/X}$, i.e., an object of

$$\mathrm{Sp}(\mathcal{D}_{/X}) = \varprojlim \left[\ldots \longrightarrow (\mathcal{D}_{/X})_* \xrightarrow{\Omega} (\mathcal{D}_{/X})_* \xrightarrow{\Omega} \ldots \xrightarrow{\Omega} (\mathcal{D}_{/X})_* \right]$$

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A canonical adjunction:

$$\Sigma^\infty_+ : \mathcal{D}_{/X} \rightleftarrows \mathrm{Sp}(\mathcal{D}_{/X}) : \Omega^\infty.$$

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$$L_X := \Sigma^\infty_+(\mathrm{Id}_X) \in \mathrm{Sp}(\mathcal{D}_{/X})$$

the **abstract cotangent complex** of X.

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When $\mathcal{D} = \mathcal{S}$ is the ∞-category of spaces we may identify $\mathrm{Sp}(\mathcal{S}_{/X}) \simeq \mathrm{Fun}(X, \mathrm{Sp})$ with local systems of spectra, and L_X with the constant local system with value the sphere spectrum $\mathbb{S} \in \mathrm{Sp}$.

Quillen cohomology

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For a parametertized spectrum $M \in \mathrm{Sp}(\mathcal{D}_{/X})$, define the n'th **Quillen cohomology** of X with coefficients in M by

$$\mathrm{H}^n_{\mathrm{Q}}(X; M) = \pi_0 \mathrm{Map}_{\mathrm{Sp}(\mathcal{D}_{/X})}(L_X, M[n])$$

where $M[n]$ is the n'th suspension of M in $\mathrm{Sp}(\mathcal{D}_{/X})$.

Twisted arrow categories

In spaces, the Quillen cohomology groups of a space X with coefficient in $M \in \mathrm{Sp}(\mathcal{S}_{/X}) \simeq \mathrm{Fun}(X, \mathrm{Sp})$ are just the homotopy groups of the limit spectrum $\lim_X M$.

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The twisted arrow category

Recall that for an ∞-category $\mathcal{C}$ the **twisted arrow category** $\mathrm{Tw}(\mathcal{C})$ is the ∞-category whose objects are the morphisms of $\mathcal{C}$ and such that maps from $f : X \longrightarrow Y$ to $g : Z \longrightarrow W$ are given by commutative diagrams of the form

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Such diagrams can be encoded by 3-simplices in $\mathcal{C}$. Similarly, the n-simplicies of $\mathrm{Tw}(\mathcal{C})$ are the $(2n+1)$-simplices of $\mathcal{C}$.

Main result

Theorem (J. Nuiten, M. Prasma, H.)

For an ∞-category $\mathcal{C}$ there is a natural equivalence

$$\mathrm{Sp}((\mathrm{Cat}_\infty)_{/\mathcal{C}}) \simeq \mathrm{Fun}(\mathrm{Tw}(\mathcal{C}), \mathrm{Sp})$$

between parameterized specra over $\mathcal{C}$ and functors from the **twisted arrow category** *of $\mathcal{C}$ to spectra. The* **cotangent complex** *of $\mathcal{C}$ is the constant functor with value $\mathbb{S}[-1]$.*

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This means that Quillen cohomology of ∞-categories can be described as a functor cohomology: its coefficients are functors $M : \mathrm{Tw}(\mathcal{C}) \longrightarrow \mathrm{Sp}$ and the corresponding Quillen cohomology groups are (up to a shift) the homotopy groups of the limit $\lim_{\mathrm{Tw}(\mathcal{C})} M$.

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- To prove this statement for a particular enriched ∞-category $\mathcal{C}$ we show that one can restrict attention to enriched ∞-categories with a **fixed set of objects**.
- Enriched ∞-categories with a fixed set of objects can also be described as ∞-categories of algebras over a suitable colored operad. We then use a previous joint result which involves identifying parameterized spectra over algebras objects with parameterized spectra over module objects.

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Yes. We may identify $\Omega^\infty(M)$ with the ∞-category whose objects are pairs (X, η) with $X \in \mathcal{C}$ and η is a map $\eta : \mathbb{S}[-1] \longrightarrow M(\mathrm{Id}_X)$. Maps from (X, η) to (X', η') are pairs (f, H) where $f : X \longrightarrow X'$ is a map in $\mathcal{C}$ and H is a homotopy between the two resulting maps $f_*\eta, f^*\eta' : \mathbb{S}[-1] \longrightarrow M(f)$.

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We note that $\Omega^\infty(M)$ is naturally an $\mathbb{E}_\infty$-group object in $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$. In particular, we can sum objects which lie in the same fiber: $(X,\eta) \boxplus (X,\eta') = (X,\eta+\eta')$.

Quillen obstruction theory

Back to the abstract setting: $\mathcal{D}$ a presentable ∞-category and $X \in \mathcal{D}$ an object. How can we use Quillen cohomology to do a Bousfield-Kan type obstruction theory and spectral sequence?

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Small extensions

A **small extension** of $X \in \mathcal{D}$ with coefficients in $M \in \mathrm{Sp}(\mathcal{D}_{/X})$ is a Cartesian square in $\mathcal{D}_{/X}$ of the form:

$$\begin{array}{ccc} Y & \longrightarrow & \Omega^{\infty}(0) \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \xrightarrow{\alpha} & \Omega^{\infty}(M[1]) \end{array}$$

where the right vertical map is obtained by applying the functor $\Omega^{\infty} : \mathrm{Sp}(\mathcal{D}_{/X}) \longrightarrow \mathcal{D}_{/X}$ to the 0-map $0 \longrightarrow M[1]$.

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The data of $\alpha : X \longrightarrow \Omega^\infty(M[1])$ is equivalent by adjunction to the data of a map $L_X \longrightarrow M[1]$ and hence determines a class $[\alpha] \in \mathrm{H}^1_{\mathrm{Q}}(X; M)$. We consider p as a geometric incarnation of $[\alpha]$.

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Example

If Y is a space then for every $n \geq 1$ the map $P_{n+1}(Y) \longrightarrow P_n(Y)$ is a small extension with coefficients in the local system of Eilenberg-MacLane spectra $\mathbb{H}K(\pi_{n+1}(Y), n+1)$, whose Quillen cohomology class is

$$k_n \in H^{n+2}(Y; \pi_{n+1}(Y)) = \mathrm{H}^1_{\mathrm{Q}}(Y; \mathbb{H}K(\pi_{n+1}(Y), n+1))$$

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Answer

Yes. We may identify $\mathcal{C}_{\alpha}$ with the ∞-category whose objects are pairs (X, η) with $X \in \mathcal{C}$ and η is a null-homotopy of the Id_X component $\alpha_{\mathrm{Id}_X} : \mathbb{S}[-1] \longrightarrow M(\mathrm{Id}_X)[1]$ of α. Maps from (X, η) to (X', η') are pairs (f, H) where $f : X \longrightarrow X'$ is a map in $\mathcal{C}$ and H is a homotopy between the two resulting null homotopies $f_*\eta, f^*\eta'$ of $\alpha_f : \mathbb{S}[-1] \longrightarrow M(f)[1]$.

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The elements $k_n \in \mathrm{H}^1_{\mathrm{Q}}(\mathrm{Ho}_n(\mathcal{C}), \mathbb{H}K(\pi_{n+1}(\mathcal{C}), n+1))$ classifying the small extensions $\mathrm{Ho}_{n+1}(\mathcal{C}) \longrightarrow \mathrm{Ho}_n(\mathcal{C})$ can be considered as the k-**invariants** of $\mathcal{C}$.

Quillen obstruction theory for ∞-categories

Starting from a map $\mathcal{D} \longrightarrow \mathrm{Ho}_1(\mathcal{C})$ we now obtain an obstruction theory to the existence of a lift $\mathcal{D} \longrightarrow \mathcal{C}$ as well as a spectral sequence starting from

$$E_1^{t,s} = \mathrm{H}^s_{\mathrm{Q}}(\mathcal{D}, \pi_t(\mathcal{C})) = \lim^{s+1}_{\mathrm{Tw}(\mathcal{D})} \pi_t(\mathcal{C}) \quad s \geq -1, t \geq 2$$

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Example (Splitting homotopy idempotents)

Let Idem be the category with one object x_0 and one non-identify morphism $f : x_0 \longrightarrow x_0$ such that $f \circ f = f$. A direct computation shows that the category $\mathrm{Tw}(\mathrm{Idem})$ has cohomological dimension 1. We may then conclude that every map $\mathrm{Idem} \longrightarrow \mathrm{Ho}_1(\mathcal{C})$ admits a non-empty and even simply-connected space of lifts $\mathrm{Idem} \longrightarrow \mathcal{C}$. When $\mathcal{C}$ is idempotent complete we may conclude that a homotopy idempotent $\mathrm{Idem} \longrightarrow \mathrm{Ho}(\mathcal{C})$ splits if and only if it lifts to $\mathrm{Ho}_1(\mathcal{C})$.

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Theorem (Nuiten, Prasma, H., cf. H. K. Nguyen)

Let $\mathcal{C}$ be an (∞, n)-category and let $\mathrm{Ho}_k(\mathcal{C})$ denote the (∞, n)-category obtained from $\mathcal{C}$ by replacing each space of n-morphisms by its k'th Postnikov piece. Then

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Corollary (The Hurewicz principle for (∞, n)-categories)

Let $f : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between (∞, n)-categories. Then f is an equivalence if and only if

- $f_* : \mathrm{Ho}_1(\mathcal{C}) \longrightarrow \mathrm{Ho}_1(\mathcal{D})$ *is an equivalence of $(n+1, n)$-categories.*
- f *induces an isomorphism on Quillen cohomology for any choice of coefficients $M \in \mathrm{Sp}((\mathrm{Cat}_{(\infty,n)})_{/\mathcal{C}})$.*

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Let $\mathcal{C}$ be an $(\infty, 2)$-category. Then there exists a natural equivalence

$$\mathrm{Sp}((\mathrm{Cat}_{\infty,2})_{/C}) \simeq \mathrm{Fun}(\mathrm{Tw}_2(\mathcal{C}), \mathrm{Sp})$$

between parameterized spectra over $\mathcal{C}$ and functors from what we call the **twisted 2-cell category** $\mathrm{Tw}_2(\mathcal{C})$ *of $\mathcal{C}$ to spectra.*

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Conjecturally, there exists an analogue of the above theorem for all n, using a suitable **twisted n-cell category**.

Example: the classification of adjunctions

Let $[1] = \bullet \xrightarrow{f} \bullet$ and let $[1] \longrightarrow \mathrm{Adj}$ be the free 2-category generated from $[1]$ by adding a right adjoint to f.

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The induced map $\mathrm{Tw}_2([1]) \longrightarrow \mathrm{Tw}_2(\mathrm{Adj})$ is coinitial. In particular, the map $[1] \longrightarrow \mathrm{Adj}$ induces an isomorphism on Quillen cohomology for any choice of coefficients $M : \mathrm{Tw}_2(\mathrm{Adj}) \longrightarrow \mathrm{Sp}$.

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Corollary (cf. Riehl and Verity)

Any diagram of the form:

$$\begin{array}{ccc} [1] & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Adj} & \longrightarrow & \mathrm{Ho}_1(\mathcal{C}) \end{array}$$

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This means, in effect, that homotopy coherent adjunctions are uniquely determined by explicit low dimensional data.

Questions?

Thank you!