HERMITIAN K-THEORY FOR STABLE ∞-CATEGORIES III: GROTHENDIECK-WITT GROUPS OF RINGS

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ABSTRACT. We establish a fibre sequence relating the classical Grothendieck-Witt theory of a ring R to the homotopy C_2-orbits of its K-theory and Ranicki's original (non-periodic) symmetric L-theory. We use this fibre sequence to remove the assumption that 2 is a unit in R from various results about Grothendieck-Witt groups. For instance, we solve the homotopy limit problem for Dedekind rings whose fraction field is a number field, calculate the various flavours of Grothendieck-Witt groups of $\mathbb{Z}$, show that the Grothendieck-Witt groups of rings of integers in number fields are finitely generated, and that the comparison map from quadratic to symmetric Grothendieck-Witt theory of coherent rings of global dimension d is an equivalence in degrees $\geq d+3$. As an important tool, we establish the hermitian analogue of Quillen's localisation-dévissage sequence for Dedekind rings and use it to solve a conjecture of Berrick-Karoubi.

CONTENTS

INTRODUCTION

This paper investigates the hermitian K-theory spectra of non-degenerate symmetric and quadratic forms over a ring R and their homotopy groups: the *higher Grothendieck-Witt groups* of R. Many structural and computational features of the higher Grothendieck-Witt groups of rings R in which 2 is a unit are well understood, prevalently due to extensive work of Karoubi [Kar71d, Kar71b, Kar71c, Kar71a, Kar80] and Schlichting [Sch10a, Sch10b, Sch17, Sch19a]. Previously, in Paper [II] we have used the categorical framework of Poincaré ∞-categories to establish some fundamental properties of the higher Grothendieck-Witt groups of rings in which 2 is not necessarily a unit, most notably a form of Karoubi periodicity [II].4.3.4 and the existence of a fibre sequence relating the Grothendieck-Witt theory of any Poincaré ∞-category with its algebraic K-theory and L-theory. This allows for the separation of K-theoretic and L-theoretic arguments, and the theme of this paper is to deduce results about Grothendieck-Witt theory from their counterparts in L-theory.

Date: May 21, 2021.

Main results. Let D be a duality on the category $\mathrm{Proj}(R)$ of finitely generated projective R-modules, that is, an equivalence of categories $\mathrm{D}\colon \mathrm{Proj}(R)^{\mathrm{op}} \to \mathrm{Proj}(R)$. Then D is necessarily of the form $\mathrm{D}P = \hom_R(P, M)$, where $M := \mathrm{D}(R)$ is an invertible module with involution (see Definition R.1). An M-valued unimodular symmetric form on P is then a self-dual isomorphism $\varphi\colon P \to \mathrm{D}P$. Together with their isomorphisms, these form a groupoid $\mathrm{Unimod}(R; M)$, symmetric monoidal under orthogonal direct sum. The classical symmetric Grothendieck-Witt theory of R is its group-completion:

$$\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R; M) = (\mathrm{Unimod}(R; M), \oplus)^{gp}.$$

By construction, $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R; M)$ is a group-like $\mathcal{E}_\infty$-space, which we view equivalently as a connective spectrum. Its homotopy groups are the higher symmetric Grothendieck-Witt groups $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},*}(R; M)$ of R. Similarly, one can consider $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(R; M)$, the variant of $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R; M)$ where M-valued symmetric bilinear forms are replaced by M-valued quadratic forms, whose higher homotopy groups are the higher quadratic Grothendieck-Witt groups.

After inverting 2, it turns out that $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(R; M)[\frac{1}{2}] \simeq \mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R; M)[\frac{1}{2}]$, and the study of the higher Grothendieck-Witt groups of R reduces to the study of the K-groups and Witt-groups, as by work of Karoubi there is a natural splitting

$$\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},*}(R; M)[\tfrac{1}{2}] \cong (\mathrm{K}_*(R; M)[\tfrac{1}{2}])^{\mathrm{C}_2} \oplus (\mathrm{W}_*(R; M)[\tfrac{1}{2}]),$$

see also [BF85]. Here $\mathrm{K}(R; M)$ is the K-theory spectrum of R with C_2-action induced by sending P to its dual $\mathrm{D}P$, and the first summand is the subgroup of invariants of its homotopy groups with 2 inverted. The second summand consists of the Witt groups of symmetric forms and formations, which are 4-periodic by definition. The first main result of the present paper combines the general fibre sequence of Paper [II] with Ranicki's algebraic surgery to obtain an integral version of this result.

Theorem 1. *For every ring R and duality $\mathrm{D} = \hom_R(-, M)$ on $\mathrm{Proj}(R)$, there is a fibre sequence of spectra*

$$\mathrm{K}(R; M)_{\mathrm{hC}_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R; M) \longrightarrow \mathrm{L}^{\mathrm{short}}(R; M)$$

where $\mathrm{L}^{\mathrm{short}}(R; M)$ is a canonical connective spectrum whose homotopy groups are Ranicki's original (non-4-periodic) symmetric L-groups from [Ran80].

After inverting 2, this fibre sequence recovers Karoubi's splitting of $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R; M)$, but it also allows to efficiently treat the behaviour of Grothendieck-Witt theory at the prime 2, as we will explain below. Without inverting 2 on the outside, but when 2 is a unit in R, Ranicki's L-groups $\mathrm{L}^{\mathrm{short}}_*(R; M)$ are still 4-periodic and isomorphic to the Witt groups $\mathrm{W}_*(R; M)$, and in this case the sequence of Theorem 1 is due to Schlichting [Sch17, §7]. However, if 2 is not invertible in R, there are several variants of L-spectra in addition to $\mathrm{L}^{\mathrm{short}}(R; M)$, most notably the 4-periodic symmetric L-theory $\mathrm{L}^{\mathrm{s}}(R; M)$ used by Ranicki in later work [Ran92]. Our insight is that it is the non-periodic classical symmetric L-theory of Ranicki [Ran80] which makes Theorem 1 true for all rings.

Coming back to the 2-local behaviour of Grothendieck-Witt theory, we note that sending a symmetric bilinear form to its underlying finitely generated projective module leads to a canonical map $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R; M) \to \mathrm{K}(R; M)^{\mathrm{hC}_2}$. The question whether this map is a 2-adic equivalence in positive degrees is known as Thomason's homotopy limit problem [Tho83], which admits a positive solution for many rings in which 2 is invertible, notably by work of Hu, Kriz and Ormsby [HKO11], Bachmann and Hopkins [BH20], and Berrick, Karoubi, Schlichting and Østvær [BKSØ15]. In §3.1 we will show:

Theorem 2. *Let R be a Dedekind ring whose fraction field is a number field. Then the canonical map $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R; M) \to \mathrm{K}(R; M)^{\mathrm{hC}_2}$ is a 2-adic equivalence in non-negative degrees.*

To the best of our knowledge this is the first general result on the homotopy limit problem for a class of rings which are not fields and in which 2 is not assumed to be a unit. The strategy we adopt to prove Theorem 2 is to use Theorem 1 to reduce it to the case of $R[\frac{1}{2}]$, where it holds by [BKSØ15]. For a general ring R, we further observe that the failure of 4-periodicity of $\mathrm{L}^{\mathrm{short}}(R; M)$ in high degrees provides a purely L-theoretic obstruction for the homotopy limit problem map $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R; M) \to \mathrm{K}(R; M)^{\mathrm{hC}_2}$ to be a 2-adic equivalence in positive degrees; see Proposition 3.1.12.

The polarisation of a quadratic form induces a comparison map $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(R;M) \to \mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R;M)$, and in §1.3 we show that for coherent rings of finite global dimension, this map is an equivalence in high degrees:

Theorem 3. *Suppose R is a coherent ring of finite global dimension d. Then the map $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl},n}(R;M) \to \mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},n}(R;M)$ is injective for $n \geq d+2$ and an isomorphism for $n \geq d+3$. Moreover, if R is 2-torsion free and $M = R$, the map is injective for $n \geq d$ and an isomorphism for $n \geq d+1$.*

Here, by slight abuse of terminology, by a coherent ring we mean a left-coherent ring, and likewise finite global dimension refers to left-global dimension. The same result is, however, true for right-coherent rings of finite right-global dimension, see Remark 1.3.10. In addition, similar statements hold for Grothendieck-Witt groups associated to any form parameter in the sense of Bak in place of quadratic forms in the above theorem, see Remark 1.3.11 for details.

Theorem 1 does not only provide a conceptual description of symmetric Grothendieck-Witt spectra, but it can also be used for explicit calculations. For instance, when $R = \mathbb{Z}$ there are two dualities on $\mathrm{Proj}(\mathbb{Z})$, leading to the symmetric and symplectic Grothendieck-Witt groups of $\mathbb{Z}$, respectively. In §3.2 we explicitly calculate these groups in a range of degrees < 20000, and beyond that conditionally on the Kummer-Vandiver conjecture in the following sense: Of some Grothendieck–Witt groups, we can only determine the order, and the Kummer-Vandiver conjecture implies that these groups are cyclic.

Theorem 4. *The symmetric and symplectic Grothendieck-Witt groups of $\mathbb{Z}$ are given in the table of Theorem 3.2.1.*

Finally, using in addition Theorem 3 and explicit low dimensional calculations, we also obtain the quadratic and skew-quadratic Grothendieck-Witt groups of $\mathbb{Z}$ in Theorems 3.2.9 and 3.2.13.

Proof strategy and further results. We approach $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}$ by investigating Grothendieck-Witt theory in the general context of Poincaré ∞-categories, as defined by Lurie [Lur11] and further developed in Paper [I], Paper [II]. We briefly recall that a Poincaré ∞-category consists of a small stable ∞-category $\mathcal{C}$ equipped with a Poincaré structure, that is a functor $\Omega\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ which is quadratic and satisfies a non-degeneracy condition, which allows to extract an induced duality $\mathrm{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. We refer to Paper [I] for a general introduction to Poincaré ∞-categories, and to Paper [II] for the construction of their Grothendieck-Witt and L-spectra and their universal properties. In [II].4.4.14, we showed that for any Poincaré ∞-category $(\mathcal{C}, \Omega)$ there is a natural fibre sequence

$$\mathrm{K}(\mathcal{C},\Omega)_{\mathrm{hC}_2} \longrightarrow \mathrm{GW}(\mathcal{C},\Omega) \longrightarrow \mathrm{L}(\mathcal{C},\Omega). \tag{1}$$

where $\mathrm{K}(\mathcal{C},\Omega)$ is the K-theory spectrum of $\mathcal{C}$ with the C_2-action induced by D. To connect this general fibre sequence to Theorem 1, we will be concerned with studying appropriate Poincaré structures on the derived ∞-category of perfect complexes $\mathcal{D}^{\mathrm{p}}(R)$. Some immediate examples of Poincaré structures on $\mathcal{D}^{\mathrm{p}}(R)$ are the quadratic and symmetric Poincaré structures given at a perfect complex X by the formulae

$$\Omega^{\mathrm{q}}_M(X) = \hom_{R\otimes R}(X\otimes X, M)_{\mathrm{hC}_2} \quad\text{and}\quad \Omega^{\mathrm{s}}_M(X) = \hom_{R\otimes R}(X\otimes X, M)^{\mathrm{hC}_2}, \tag{2}$$

where M is an invertible module with involution over R (see Definition R.1), and the C_2-action is given by conjugating the flip action on $X \otimes X$ and the C_2-action on M. These two Poincaré structures are the homotopy theoretic analogues of quadratic and symmetric forms in algebra, which on a finitely generated projective R-module P are respectively the groups of coinvariants and invariants

$$\mathrm{Hom}_{R\otimes R}(P\otimes P, M)_{\mathrm{C}_2} \quad\text{and}\quad \mathrm{Hom}_{R\otimes R}(P\otimes P, M)^{\mathrm{C}_2} \tag{3}$$

for the same C_2-action as above. One insight in our series of papers is that the abstract framework of Paper [I], Paper [II] allows us to work with Poincaré structures on $\mathcal{D}^{\mathrm{p}}(R)$ which are more intimately related to algebra than the naive homotopy theoretic constructions of (2). These are the non-abelian derived functors of the algebraic constructions of (3), which we call the *genuine* quadratic and *genuine* symmetric Poincaré structures and that we denote respectively by Ω^{gq}_M and Ω^{gs}_M. There are canonical comparison maps

$$\Omega^{\mathrm{q}}_M \longrightarrow \Omega^{\mathrm{gq}}_M \longrightarrow \Omega^{\mathrm{gs}}_M \longrightarrow \Omega^{\mathrm{s}}_M$$

relating these Poincaré structures. When 2 is a unit in R, all of them are equivalences, and we showed in §[II].B that the corresponding Grothendieck-Witt spectra coincide with previous constructions of Grothendieck-Witt spectra due to Schlichting and Spitzweck [Sch17, Spi16]. In general, when 2 is not necessarily a unit, the fourth and ninth authors [HS21] relate the genuine Grothendieck-Witt spectra $\mathrm{GW}^{\mathrm{gs}}(R;M) := \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R);\Omega^{\mathrm{gs}}_M)$ and $\mathrm{GW}^{\mathrm{gq}}(R;M) := \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R);\Omega^{\mathrm{gq}}_M)$ to the classical ones, by providing natural equivalences

$$\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R;M) \xrightarrow{\simeq} \tau_{\geq 0}\,\mathrm{GW}^{\mathrm{gs}}(R;M) \quad \text{and} \quad \mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(R;M) \xrightarrow{\simeq} \tau_{\geq 0}\,\mathrm{GW}^{\mathrm{gq}}(R;M),$$

where $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(R;M)$ denotes, similary to $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R;M)$, the group completion of the category of unimodular quadratic forms, and $\tau_{\geq 0}$ denotes the connective cover. Writing similarly $\mathrm{L}^{\mathrm{gs}}(R;M)$ for $\mathrm{L}(\mathcal{D}^{\mathrm{p}}(R);\Omega^{\mathrm{gs}}_M)$, we therefore obtain a fibre sequence $\mathrm{K}(R;M)_{\mathrm{hC}_2} \to \mathrm{GW}^{\mathrm{gs}}(R;M) \to \mathrm{L}^{\mathrm{gs}}(R;M)$, and Theorem 1 is then implied by the following result, see Theorem 1.2.18.

Theorem 5. *For any ring R and non-negative integer n, the genuine symmetric L-groups $\mathrm{L}^{\mathrm{gs}}_n(R;M)$ are canonically isomorphic to Ranicki's original symmetric L-groups from* [Ran80]. *Thus, in the notation of Theorem 1, we have $\mathrm{L}^{\mathrm{short}}(R;M) = \tau_{\geq 0}\,\mathrm{L}^{\mathrm{gs}}(R;M)$.*

We recall that the original symmetric L-groups of Ranicki are defined so that elements of the n'th L-group are represented by Poincaré chain complexes of length at most n, for $n \geq 0$. Ranicki then defines negative symmetric L-groups in an ad hoc manner, and we show that these negative L-groups are also canonically isomorphic to the corresponding negative genuine symmetric L-groups: Concretely they are given by

$$\mathrm{L}^{\mathrm{gs}}_n(R;M) = \begin{cases} \mathrm{L}^{\mathrm{ev}}_{n+2}(R;-M) & \text{if } n = -2,-1 \\ \mathrm{L}^{\mathrm{q}}_n(R;M) & \text{if } n \leq -3, \end{cases}$$

where $\mathrm{L}^{\mathrm{ev}}_*$ and $\mathrm{L}^{\mathrm{q}}_*$ are respectively the even and quadratic L-groups of [Ran80]. In particular, Theorem 5 and the described addendum show that the classical symmetric L-groups can be realised as the homotopy groups of the non-connective spectrum $\mathrm{L}^{\mathrm{gs}}(R;M)$. The general form of Karoubi periodicity of Paper [II], which we review in Theorem R.8 below, relates the Poincaré structures Ω^{gs}_M and Ω^{gq}_M and their GW and L-spectra, in particular showing that $\Sigma^4\,\mathrm{L}^{\mathrm{gs}}(R;M) \simeq \mathrm{L}^{\mathrm{gq}}(R;M)$; see Corollary R.10. From the fibre sequence for general Poincaré ∞-categories, we therefore also obtain a quadratic version of Theorem 1, given by the fibre sequence

$$\mathrm{K}(R;M)_{\mathrm{hC}_2} \xrightarrow{\mathrm{hyp}} \mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(R;M) \longrightarrow \tau_{\geq 0}(\Sigma^4\,\mathrm{L}^{\mathrm{gs}}(R;M)).$$

We prove Theorem 5 in § 1.2 using Ranicki's procedure of algebraic surgery, which allows us to compare the L-groups of various Poincaré structures in a range of degrees. We discuss this technique also for connective ring spectra in Corollary 1.2.24, and in § 1.3, to obtain the following comparison result. We will write $\mathrm{GW}^{\mathrm{s}}(R;M) := \mathrm{GW}^{\mathrm{s}}(\mathcal{D}^{\mathrm{p}}(R);\Omega^{\mathrm{s}}_M)$ for the homotopy symmetric Grothendieck-Witt theory, and write likewise $\mathrm{L}^{\mathrm{s}}(R;M) := \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R);\Omega^{\mathrm{s}}_M)$ for periodic symmetric L-theory.

Theorem 6. *Suppose R is a coherent ring of finite global dimension d. Then:*

i) the map $\mathrm{L}^{\mathrm{gs}}_n(R;M) \to \mathrm{L}^{\mathrm{s}}_n(R;M)$ is injective for $n \geq d-2$ and an isomorphism for $n \geq d-1$,
ii) the map $\mathrm{L}^{\mathrm{gq}}_n(R;M) \to \mathrm{L}^{\mathrm{gs}}_n(R;M)$ is injective for $n \geq d+2$ and an isomorphism for $n \geq d+3$.

Part i) of Theorem 6, together with Theorem 5, improve a similar comparison result of Ranicki [Ran80, Proposition 4.5], where he proves injectivity for non-negative $n \geq 2d-3$ and bijectivity for non-negative $n \geq 2d-2$ for Noetherian rings of finite global dimension d. Combining Theorem 1 and Theorem 6, we obtain Theorem 3 from above, see also Corollary 1.3.8 and Remark 1.3.15.

Furthermore, part i) of Theorem 6 implies that the map $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R;M) \to \tau_{\geq 0}\,\mathrm{GW}^{\mathrm{s}}(R;M)$ is an equivalence if R is a Dedekind domain. Thus in order to study the classical Grothendieck-Witt groups of Dedekind rings, it suffices to study the homotopy symmetric Grothendieck-Witt theory GW^{s}. This is an interesting invariant in its own right which enjoys pleasant properties not shared with the genuine variant $\mathrm{GW}^{\mathrm{gs}}$. Most notably, we prove in Theorem 2.1.8 that 4-periodic symmetric L-theory L^{s}, and hence also GW^{s}, satisfies a dévissage theorem. In particular, we obtain the hermitian analogue of Quillen's famous localisation-dévissage fibre sequence [Qui73], see Corollary 2.1.9:

Theorem 7. *Let R be a Dedekind ring, $T \subset R$ a multiplicative subset, and $\mathbb{F}_{\mathfrak{p}}$ the residue field $R/\mathfrak{p}$ at a maximal ideal $\mathfrak{p} \subseteq R$. Then restriction, localisation, and a choice of uniformiser for every $\mathfrak{p}$ induce a fibre* *sequence of spectra*

$$\bigoplus_{\mathfrak{p}\cap T\neq\emptyset} \mathrm{GW}^{\mathrm{s}}(\mathbb{F}_{\mathfrak{p}};(M/\mathfrak{p})[-1]) \longrightarrow \mathrm{GW}^{\mathrm{s}}(R;M) \longrightarrow \mathrm{GW}^{\mathrm{s}}(R[T^{-1}];R[T^{-1}]\otimes_R M),$$

where $R[T^{-1}]$ is obtained from R by inverting the elements of T.

We in fact construct a more general fibre sequence for localisations of R away from a set of non-empty prime ideals of R, see Corollary 2.1.9. This result establishes a conjecture of Berrick and Karoubi which asserts that the map $\mathbb{Z} \to \mathbb{Z}[\frac{1}{2}]$ induces an equivalence on the positive, 2-localised Grothendieck-Witt groups [BK05]. In fact, this result holds for general rings of integers in number fields as we observe in Proposition 3.1.11. In § 3 we then combine Theorem 7 with work of Berrick, Karoubi, Schlichting and Østvær [BKSØ15] to deduce Theorem 2, as well as the calculations for the integers of Theorem 4.

Finally, we also use Theorem 1, together with a calculation of the symmetric and quadratic L-groups of Dedekind rings to deduce the following finiteness result; see Corollary 2.2.18. When $M = R$ with the involution given by multiplication by $\epsilon = \pm 1$, we write $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},n}(R;\epsilon)$ for $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},n}(R;M)$, and similarly for $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl},n}(R;\epsilon)$.

Corollary 8. *Let $\mathcal{O}$ be a number ring, that is, a localisation of the ring of integers in a number field away from finitely many primes, and $\epsilon = \pm 1$. Then its classical ϵ-symmetric and ϵ-quadratic Grothendieck-Witt groups $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},n}(\mathcal{O};\epsilon)$ and $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl},n}(\mathcal{O};\epsilon)$ are finitely generated.*

In the quadratic case, one can prove this result also through homological stability, but in the generality presented here the argument is not known to carry over to the symmetric case, as we explain in Remark 2.2.19.

Remark. Some of the results presented above have also been announced in [Sch19b]: The calculations of the Grothendieck-Witt groups of the integers of Theorem 4 in the symmetric, symplectic and quadratic cases (although the results are not quite correct away from the prime 2, see Remark 3.2.5), the localisation-dévissage sequence of Theorem 7 in non-negative degrees, and Theorem 3 for the ring $R = \mathbb{Z}$ with the trivial involution.

Notation and Conventions. All tensor products appearing without further explanation are derived tensor products over $\mathbb{Z}$, and will be denoted by $\otimes$ rather than $\otimes^{\mathrm{L}}_{\mathbb{Z}}$. We always denote by $\mathrm{D} = \hom_R(-,M)$ the dualities on $\mathrm{Proj}(R)$ and $\mathcal{D}^{\mathrm{p}}(R)$ determined by an invertible module with involution M.

Acknowledgements. For useful discussions about our project, we heartily thank Tobias Barthel, Clark Barwick, Lukas Brantner, Mauricio Bustamante, Denis-Charles Cisinski, Dustin Clausen, Uriya First, Rune Haugseng, André Henriques, Lars Hesselholt, Gijs Heuts, Geoffroy Horel, Marc Hoyois, Max Karoubi, Daniel Kasprowski, Ben Knudsen, Manuel Krannich, Achim Krause, Henning Krause, Sander Kupers, Wolfgang Lück, Ib Madsen, Cary Malkiewich, Mike Mandell, Akhil Matthew, Lennart Meier, Irakli Patchkoria, Nathan Perlmutter, Andrew Ranicki, Oscar Randal-Williams, George Raptis, Marco Schlichting, Peter Scholze, Stefan Schwede, Graeme Segal, Markus Spitzweck, Jan Steinebrunner, Georg Tamme, Ulrike Tillmann, Maria Yakerson, Michael Weiss, and Christoph Winges.

Furthermore, we owe a tremendous intellectual debt to Jacob Lurie for creating the framework we exploit here, and to Søren Galatius for originally spotting the overlap between various separate projects of ours; his insight ultimately lead to the present collaboration.

The authors would also like to thank the Hausdorff Center for Mathematics at the University of Bonn, the Newton Institute at the University of Cambridge, the University of Copenhagen and the Mathematical Research Institute Oberwolfach for hospitality and support while parts of this project were undertaken.

BC was supported by the French National Centre for Scientific Research (CNRS) through a "délégation" at LAGA, University Paris 13. ED was supported by the German Research Foundation (DFG) through the priority program "Homotopy theory and Algebraic Geometry" (DFG grant no. SPP 1786) at the University of Bonn and WS by the priority program "Geometry at Infinity" (DFG grant no. SPP 2026) at the University of Augsburg. YH and DN were supported by the French National Research Agency (ANR) through the grant "Chromatic Homotopy and K-theory" (ANR grant no. 16-CE40-0003) at LAGA, University of Paris

13. FH is a member of the Hausdorff Center for Mathematics at the University of Bonn (DFG grant no. EXC 2047 390685813) and TN of the cluster “Mathematics Münster: Dynamics-Geometry-Structure” at the University of Münster (DFG grant no. EXC 2044 390685587). FH, TN and WS were further supported by the Engineering and Physical Sciences Research Council (EPSRC) through the program “Homotopy harnessing higher structures” at the Isaac Newton Institute for Mathematical Sciences (EPSRC grants no. EP/K032208/1 and EP/R014604/1). FH was also supported by the European Research Council (ERC) through the grant “Moduli spaces, Manifolds and Arithmetic” (ERC grant no. 682922) and KM by the grant “K-theory, L^2-invariants, manifolds, groups and their interactions” (ERC grant no. 662400). ML and DN were supported by the collaborative research centre “Higher Invariants” (DFG grant no. SFB 1085) at the University of Regensburg. ML was further supported by the research fellowship “New methods in algebraic K-theory” (DFG grant no. 424239956) and by the Danish National Research Foundation (DNRF) through the Center for Symmetry and Deformation (DNRF grant no. 92) and the Copenhagen Centre for Geometry and Topology (DNRF grant GeoTop) at the University of Copenhagen. KM was also supported by the K&A Wallenberg Foundation.

Recollection

In this section we recall some of the material from Paper [I] and Paper [II] on Poincaré structures on the perfect derived ∞-category of a ring and their Grothendieck-Witt and L spectra, as we rely on this framework in the rest of the paper. We will also review the general form of Karoubi's periodicity Theorem [II].4.3.4, which does not require that 2 is a unit in the base ring.

In Paper [II], we view Grothendieck-Witt theory as an invariant of what we call a *Poincaré ∞-category*. A Poincaré ∞-category is a pair $(\mathcal{C}, Ϙ)$ consisting of a small stable ∞-category $\mathcal{C}$ equipped with a *Poincaré structure* Ϙ, that is a functor $Ϙ\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ which is reduced and 2-excisive in the sense of Goodwillie's functor calculus, and whose symmetric cross-effect $\mathrm{B}\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}^{\mathrm{op}} \to \mathcal{S}p$ is of the form $\mathrm{B}(X,Y) = \hom_{\mathcal{C}}(X, \mathrm{D}Y)$ for some equivalence of categories $\mathrm{D}\colon \mathcal{C}^{\mathrm{op}}\to\mathcal{C}$. Poincaré ∞-categories were introduced by Lurie as a novel framework for Ranicki's L-theory (see [Lur11] and Paper [II]). A Poincaré structure provides a formal notion of “hermitian form” on the objects of $\mathcal{C}$. Indeed, there is as a space of Poincaré objects $\mathrm{Pn}(\mathcal{C}, Ϙ)$ which consists of pairs (X, q) where X is an object of $\mathcal{C}$ and $q \in \Omega^\infty Ϙ(X)$ is such that a certain canonical map $X \to \mathrm{D}X$ is an equivalence (see Definition [I].2.1.3). There are then canonical transformations

$$\mathrm{Pn}(\mathcal{C}, Ϙ) \to \Omega^\infty \mathrm{GW}(\mathcal{C}, Ϙ) \quad \text{and} \quad \mathrm{Pn}(\mathcal{C}, Ϙ) \to \Omega^\infty \mathrm{L}(\mathcal{C}, Ϙ)$$

which exhibit the Grothendieck-Witt and L-theory functors as the universal approximation of Pn by a Verdier localising, respectively a bordism invariant, functor (see Observation [II].4.1.2 and Theorem [II].4.4.12). These universal properties are similar to the universal property of the map from the groupoid core to K-theory $\mathrm{Cr}\,\mathcal{C} \to \mathrm{K}(\mathcal{C})$ of a small stable ∞-category provided by [BGT13].

In the present paper we will be concerned with the perfect derived ∞-category $\mathcal{D}^{\mathrm{p}}(R)$ of a ring R, which is the ∞-categorical localisation of the category of bounded chain complexes of finitely generated projective (left) R-modules at the quasi-isomorphisms, or equivalently the ∞-category of compact objects of the localisation $\mathcal{D}(R)$ of all chain complexes at the quasi-isomorphisms. Given a Poincaré structure $Ϙ\colon \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \to \mathcal{S}p$, we will denote the corresponding Grothendieck-Witt spectrum by

$$\mathrm{GW}(R; Ϙ) := \mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R), Ϙ).$$

We are going to consider a specific collection of Poincaré structures on $\mathcal{D}^{\mathrm{p}}(R)$ associated to a module with involution, which we now introduce. For what follows, $\otimes$ denotes the underived tensor product of rings over $\mathbb{Z}$, but see below for a relation with the derived tensor product. Given an $R \otimes R$-module M, we let M^{op} denote the $R \otimes R$-module defined by M with the module action $r \otimes s \cdot m := s \otimes r \cdot m$ for all r, s in R and m in M.

R.1. **Definition.** A *module with involution over* R is an $R \otimes R$-module M together with an $R \otimes R$-module map $\bar{\bullet}\colon M^{op} \to M$ such that $\bar{\bar{m}} = m$. We say that M is *invertible* if it is finitely generated projective for either of its R-module structures, and the map

$$R \longrightarrow \hom_R(M, M)$$

which sends 1 to $\bar{\bullet}$ is an isomorphism, where M is regarded as an R-module via the first R-factor in the source, and the second one in the target.

We warn the reader that the modules with involution over R of Definition R.1 correspond to the modules with involution over the Eilenberg-MacLane spectrum of R in the sense of Definition [I].3.1.1 (defined using the derived tensor product), which are moreover discrete. To see this, we first note that there is a map of $\mathrm{H}\mathbb{Z}$-algebras $R \otimes^{\mathrm{L}}_{\mathbb{Z}} R \to R \otimes R$ so that any module over the latter is canonically one over the former. Furthermore, this map is the canonical 0-truncation map which any connective $\mathrm{H}\mathbb{Z}$-algebra has. If M is a discrete $\mathrm{H}\mathbb{Z}$-module, we deduce from the coconnectivity of $\hom_{\mathbb{Z}}(M, M)$ that any $R \otimes R$-module structure extends essentially uniquely to an $R \otimes^{\mathrm{L}}_{\mathbb{Z}} R$-module structure.

R.2. **Example.**

i) When R is commutative, any line bundle L over R gives rise to an invertible module with involution over R, with $M = L$ and $\overline{\bullet} = \mathrm{id}$.
ii) Let $\epsilon \in R$ be a unit. We recall that an ϵ-involution on R consists of a ring isomorphism $\overline{\bullet}\colon R \to R^{\mathrm{op}}$ such that $\bar{\bar{r}} = \epsilon r \epsilon^{-1}$ and $\bar{\epsilon} = \epsilon^{-1}$. In this case $M = R$ equipped with the $R \otimes R$-module structure $r \otimes s \cdot x = r x \bar{s}$ and the involution $\epsilon(\overline{\bullet})$ is an invertible module with involution over R, that we denote by $R(\epsilon)$. This is the structure commonly used by Ranicki as input for L-theory [Ran80].
iii) Given a module with involution M over R, we can define a new module with involution over R denoted $-M$, with the same underlying $R \otimes R$-module M but with involution $-(\overline{\bullet})$. In the case where $M = R$ we have by definition that $-R = R(-1)$.
iv) If M is an invertible module with involution over R, then $M^{\vee} = \hom_R(M, R)$ is canonically an invertible module with involution over R^{op}, see also Remark R.11.

For every pair of objects X and Y of $\mathcal{D}^{\mathrm{p}}(R)$, we may form the mapping spectrum

$$\mathrm{B}(X, Y) := \hom_{R\otimes R}(X \otimes Y, M)$$

in the stable ∞-category $\mathcal{D}^{\mathrm{p}}(R \otimes R)$. Then B is a symmetric bilinear functor, so the spectrum $\mathrm{B}(X, X)$ inherits a C_2-action by conjugating the flip action on $X \otimes X$ and the involution of M; see §[I].3.1.

Given spectrum with C_2-action $X\colon B\mathrm{C}_2 \to \mathcal{S}p$, we denote by X^{hC_2} and X_{hC_2} its homotopy fixed points and homotopy orbits, respectively. Similarly, we let X^{tC_2} denote its Tate construction, defined as the cofibre of the norm map $N\colon X_{\mathrm{hC}_2} \to X^{\mathrm{hC}_2}$ as defined in [Lur17, §6.1.6], see also [NS18, I.1.11]. As in §[I].4.2 we make the following definition.

R.3. **Definition.** Let M be an invertible module with involution over R. For every $m \in \mathbb{Z} \cup \{\pm\infty\}$, we define a functor $Ϙ^{\geq m}_M\colon \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \to \mathcal{S}p$ as the pullback

$$\begin{array}{ccc} Ϙ^{\geq m}_M(X) & \longrightarrow & \hom_R(X, \tau_{\geq m} M^{\mathrm{tC}_2}) \\ \downarrow & & \downarrow \\ \hom_{R\otimes R}(X \otimes X, M)^{\mathrm{hC}_2} & \longrightarrow & \hom_R(X, M^{\mathrm{tC}_2}). \end{array}$$

Here, the right hand vertical map is induced by the m-connective cover $\tau_{\geq m} M^{\mathrm{tC}_2} \to M^{\mathrm{tC}_2}$, and the bottom horizontal map is induced by a canonical equivalence

$$\hom_{R\otimes R}(X \otimes X, M)^{\mathrm{tC}_2} \simeq \hom_R(X, M^{\mathrm{tC}_2}),$$

see Lemma [I].3.2.4. In the special cases where $m = \pm\infty$ we will denote these functors by

$$Ϙ^{\mathrm{q}}_M := Ϙ^{\geq \infty}_M = \hom_{R\otimes R}(X \otimes X, M)_{\mathrm{hC}_2} \quad \text{and} \quad Ϙ^{\mathrm{s}}_M := Ϙ^{\geq -\infty}_M = \hom_{R\otimes R}(X \otimes X, M)^{\mathrm{hC}_2}.$$

The functors $Ϙ^{\geq m}_M$ are indeed Poincaré structures by Examples [I].3.2.7. By construction, they all share the same underlying duality

$$\mathrm{D}X = \hom_R(X, M),$$

where the mapping spectrum acquires a residual R-module structure from the $R\otimes R$-module structure of M. The canonical connective cover maps $\tau_{\geq m+1} \to \tau_{\geq m}$ define an infinite sequence of natural transformations

$$Ϙ^{\mathrm{q}}_M = Ϙ^{\geq \infty}_M \to \cdots \to Ϙ^{\geq (m+1)}_M \to Ϙ^{\geq m}_M \to Ϙ^{\geq (m-1)}_M \to \cdots \to Ϙ^{\geq -\infty}_M = Ϙ^{\mathrm{s}}_M,$$

and hence analogous sequences between the corresponding Grothendieck-Witt and L spectra.

R.4. **Remark.** Let $\hat{\mathrm{H}}^m(\mathrm{C}_2; M) = \pi_{-m} M^{\mathrm{tC}_2}$ denote the Tate cohomology of C_2 with coefficients in the underlying $\mathbb{Z}[\mathrm{C}_2]$-module of M. When $\hat{\mathrm{H}}^{-m}(\mathrm{C}_2; M) = 0$ the map $\tau_{\geq m+1} M^{\mathrm{tC}_2} \to \tau_{\geq m} M^{\mathrm{tC}_2}$ is an equivalence. Therefore, in this case, $Ϙ_M^{\geq m+1} \to Ϙ_M^{\geq m}$ is an equivalence, and it induces equivalences on the corresponding Grothendieck-Witt and L spectra

$$\mathrm{GW}(R; Ϙ_M^{\geq(m+1)}) \xrightarrow{\sim} \mathrm{GW}(R; Ϙ_M^{\geq m}) \quad \text{and} \quad \mathrm{L}(R; Ϙ_M^{\geq(m+1)}) \xrightarrow{\sim} \mathrm{L}(R; Ϙ_M^{\geq m}).$$

Moreover, $\hat{\mathrm{H}}^*(\mathrm{C}_2; M)$ is 2-periodic, so if this happens for m it also does for all $m + 2k$. In particular, if $2 \in R$ is a unit all the natural transformations $Ϙ_M^{\geq m+1} \to Ϙ_M^{\geq m}$ are equivalences. If 2 is not invertible however, the Grothendieck-Witt and L spectra for different m are not generally equivalent, for instance this is the case for $R = \mathbb{Z}$.

R.5. **Remark.** Among the Poincaré structures of Definition R.3, $Ϙ_M^{\geq 2}, Ϙ_M^{\geq 1}$ and $Ϙ_M^{\geq 0}$ are the ones which send finitely generated projective R-modules P (regarded as chain complexes concentrated in degree zero) to abelian groups (regarded as discrete spectra). The values of $Ϙ_M^{\geq 2}$ and $Ϙ_M^{\geq 0}$ are the abelian groups of strict coinvariants and invariants, respectively,

$$Ϙ_M^{\geq 2}(P) = \hom_{R\otimes R}(P \otimes P, M)_{\mathrm{C}_2} \quad \text{and} \quad Ϙ_M^{\geq 0}(P) = \hom_{R\otimes R}(P \otimes P, M)^{\mathrm{C}_2},$$

which are canonically isomorphic to the usual abelian groups of M-valued quadratic and symmetric forms on P, respectively, see §[I].4.2. Moreover, the group $Ϙ_M^{\geq 1}(P)$ is the image of the norm (or symmetrization) map $Ϙ_M^{\geq 2}(P) \to Ϙ_M^{\geq 0}(P)$. The functors $Ϙ_M^{\geq 2}, Ϙ_M^{\geq 1}$ and $Ϙ_M^{\geq 0}$ are the *non-abelian derived functors* of these functors of classical forms on modules, as shown in Proposition [I].4.2.19. We call them the *genuine* quadratic, *genuine* even, and *genuine* symmetric Poincaré structures respectively, and we denote them by

$$Ϙ_M^{\mathrm{gq}} := Ϙ_M^{\geq 2} \quad , \quad Ϙ_M^{\mathrm{ge}} := Ϙ_M^{\geq 1} \quad \text{and} \quad Ϙ_M^{\mathrm{gs}} := Ϙ_M^{\geq 0}.$$

The connective covers of the associated Grothendieck-Witt spectra are the group-completions of the corresponding spaces of forms

$$\tau_{\geq 0}\,\mathrm{GW}(R; Ϙ_M^{\mathrm{gq}}) \simeq \mathrm{GW}_{\mathrm{cl}}^{\mathrm{q}}(R; M) \ , \quad \tau_{\geq 0}\,\mathrm{GW}(R; Ϙ_M^{\mathrm{ge}}) \simeq \mathrm{GW}_{\mathrm{cl}}^{\mathrm{ev}}(R; M) \ , \quad \tau_{\geq 0}\,\mathrm{GW}(R; Ϙ_M^{\mathrm{gs}}) \simeq \mathrm{GW}_{\mathrm{cl}}^{\mathrm{s}}(R; M)$$

by the main result of [HS21]. In particular if $M = R(\epsilon)$ is the module with involution defined from an ϵ-involution on R these are the classical Grothendieck-Witt spaces of ϵ-quadratic, ϵ-even, and ϵ-symmetric forms on R.

There is a periodicity phenomenon that relates the Poincaré structures $Ϙ_M^{\geq m}$, that we now review. We recall that a hermitian morphism of Poincaré ∞-categories $(\mathcal{C}, Ϙ) \to (\mathcal{C}', Ϙ')$ consists of an exact functor $f : \mathcal{C} \to \mathcal{C}'$ and a natural transformation

$$\eta : Ϙ \to f^*Ϙ' = Ϙ' \circ f.$$

We say that a hermitian morphism (f, η) is a Poincaré morphism if a canonical induced map $f\mathrm{D} \to \mathrm{D}f$ is an equivalence; see §[I].1.2. A Poincaré morphism (f, η) is an equivalence of Poincaré ∞-categories precisely when f is an equivalence of categories and η is a natural equivalence. We recall the following proposition (see Proposition [I].3.4.2 and Corollary [II].4.3.4), first observed by Lurie in the cases where $m = \pm\infty$.

R.6. **Proposition.** *For every invertible module with involution M over R and $m \in \mathbb{Z} \cup \{\pm\infty\}$, the loop functor $\Omega : \mathcal{D}^{\mathrm{p}}(R) \to \mathcal{D}^{\mathrm{p}}(R)$ extends to an equivalence of Poincaré ∞-categories*

$$(\mathcal{D}^{\mathrm{p}}(R), (Ϙ_M^{\geq m})^{[2]}) \xrightarrow{\sim} (\mathcal{D}^{\mathrm{p}}(R), Ϙ_{-M}^{\geq m+1}),$$

where $Ϙ^{[k]} := \Sigma^k Ϙ$ denotes the k-fold shift of a Poincaré structure, and $-M$ is the twist by a sign of Example R.2.

R.7. **Remark.** For a commutative ring R, we may apply Proposition R.6 with $M = R$. If we set $\mathrm{GW}^{[n]}(R) = \tau_{\geq 0}\,\mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R); (Ϙ_R^{\geq 0})^{[n]})$, we obtain from [HS21] the equivalences

$$\mathrm{GW}^{[0]}(R) \simeq \mathrm{GW}_{\mathrm{cl}}^{\mathrm{s}}(R) \quad , \quad \mathrm{GW}^{[2]}(R) \simeq \mathrm{GW}_{\mathrm{cl}}^{-\mathrm{ev}}(R) \quad \text{and} \quad \mathrm{GW}^{[4]}(R) \simeq \mathrm{GW}_{\mathrm{cl}}^{\mathrm{q}}(R).$$

These equivalences were also announced by Schlichting, see [Sch19b, Theorem 3.1], where $\mathrm{GW}^{[2]}(R)$ is described in terms of symplectic forms. Concretely, symplectic forms are those skew-symmetric forms

$b\colon P\otimes P\to R$ which vanish on the diagonal, i.e. $b(x,x)=0$ for all x in P, compare [Sch19a, Definition 3.8 & Example 3.11]. This condition is in fact equivalent to admitting a (-1)-quadratic refinement, so symplectic forms are precisely the (-1)-even forms. To see this, we claim that $\hat{\mathrm{H}}^0(\mathrm{C}_2;\hom_{R\otimes R}(P\otimes P,R(-1)))$ is isomorphic to $\hom_R(P,R_2)$ where R_2 denotes the 2-torsion in R. Combining this isomorphism with the canonical map from ordinary cohomology to Tate cohomology gives a map

$$\mathrm{H}^0(\mathrm{C}_2;\hom_{R\otimes R}(P\otimes P,R(-1)))\longrightarrow \hom_R(P,R_2).$$

Elements of the domain are skew-symmetric forms b, and they are sent under this map to the map $x\mapsto b(x,x)$. Note that this is an additive map which indeed takes values in the 2-torsion of R if b is skew-symmetric. Hence the obstruction to lifting a skew-symmetric form b along the norm map

$$\mathrm{H}_0(\mathrm{C}_2;\hom_{R\otimes R}(P\otimes P,R(-1)))\longrightarrow \mathrm{H}^0(\mathrm{C}_2;\hom_{R\otimes R}(P\otimes P,R(-1))),$$

is given by the vanishing of b on the diagonal as claimed. Of course, one can also give a direct argument for the existence of a quadratic refinement under the assumption $b(x,x)=0$.

The shifted quadratic functor relates to that of the original Poincaré ∞-category by means of the Bott-Genauer sequence, which we now recall. Given a Poincaré ∞-category $(\mathcal{C},\Omega)$ we can functorially form an ∞-category $\mathrm{Met}(\mathcal{C},\Omega)$ whose underlying ∞-category is the ∞-category of arrows in $\mathcal{C}$, where the Poincaré structure is defined by

$$\Omega_{\mathrm{met}}(f\colon L\to X)=\mathrm{fib}\,(\Omega(f)\colon \Omega(X)\to\Omega(L)),$$

and with underlying duality $\mathrm{D}(f\colon L\to X)=(\mathrm{D}(X/L)\to \mathrm{D}X)$, see Definition [I].2.3.5. The Poincaré objects of $\mathrm{Met}(\mathcal{C},\Omega)$ are given by Poincaré objects X of $(\mathcal{C},\Omega)$ equipped with a Lagrangian L (see §[I].2.3); classically, forms equipped with a Lagrangian are called metabolic forms, hence the notation $\mathrm{Met}(\mathcal{C},\Omega)$. The Bott-Genauer sequence is the sequence of Poincaré ∞-categories

$$(\mathcal{C},\Omega[-1])\longrightarrow \mathrm{Met}(\mathcal{C},\Omega)\longrightarrow(\mathcal{C},\Omega)$$

where the underlying functors send an object L of $\mathcal{C}$ to the arrow $L\to 0$, and an object $f\colon L\to X$ in the arrow category to its target X, respectively; see Lemma [I].2.3.7. The Bott-Genauer sequence is both a fibre and a cofibre sequence of Poincaré ∞-categories, that is a Poincaré-Verdier sequence in the terminology of Paper [II], see Example [II].1.2.5. One of the main results of Paper [II] is that Grothendieck-Witt theory is Verdier localising, that is that it sends Poincaré-Verdier sequences to fibre sequences of spectra. There is moreover a natural equivalence

$$\mathrm{GW}(\mathrm{Met}(\mathcal{C},\Omega))\simeq \mathrm{K}(\mathcal{C})$$

established in Corollary [II].4.3.1. By combining these ingredients with the periodicity of Proposition R.6 we obtain the following general form of Karoubi's periodicity theorem. Let $\mathrm{hyp}\colon\ \mathrm{K}(R)\to\mathrm{GW}(R;\Omega)$ and $\mathrm{fgt}\colon\ \mathrm{GW}(R;\Omega)\to\mathrm{K}(R)$ denote the hyperbolic and forgetful map, respectively, from and to the K-theory spectrum of R, and let $\mathrm{U}(\mathcal{C},\Omega)$ and $\mathrm{V}(\mathcal{C},\Omega)$ be their respective fibres.

R.8. **Theorem** ([II].4.3.4)**.** *Let R be a ring and M an invertible module with involution over R. Then there is a natural equivalence*

$$\mathrm{V}(R;\Omega^{\geq m}_M)\simeq \Omega\,\mathrm{U}(R;\Omega^{\geq (m+1)}_{-M})$$

for every $m\in\mathbb{Z}$, where $-M$ is the $R\otimes R$-module M with the involution $(\overline{\bullet})$ replaced by $-(\overline{\bullet})$.

R.9. **Remark.** If $2\in R$ is a unit, Theorem R.8 is due to Karoubi [Kar80]. Since in this case the Poincaré structures $\Omega^{\geq m}_M$ are all equivalent, it takes the form

$$\mathrm{V}(R;\Omega^{\mathrm{s}}_M)\simeq\Omega\,\mathrm{U}(R;\Omega^{\mathrm{s}}_{-M}).$$

There is another case where this theorem simplifies, but where 2 does not need to be invertible. Let R be a commutative ring which is 2-torsion free, for instance the ring of integers in a number field, and let $M=R$ with the trivial involution. In this case $\hat{\mathrm{H}}^0(\mathrm{C}_2;-R)=0$ and $\hat{\mathrm{H}}^{-1}(\mathrm{C}_2;R)=0$, and by Remark R.4 we have that $\Omega^{\mathrm{ge}}_{-R}=\Omega^{\mathrm{gs}}_{-R}$ and $\Omega^{\mathrm{gq}}_R=\Omega^{\mathrm{ge}}_R$. Therefore the periodicity Theorem gives us that

$$\mathrm{V}(R;\Omega^{\mathrm{gs}}_R)\simeq\Omega\,\mathrm{U}(R;\Omega^{\mathrm{gs}}_{-R})\quad,\quad \mathrm{V}(R;\Omega^{\mathrm{gs}}_{-R})\simeq\Omega\,\mathrm{U}(R;\Omega^{\mathrm{gq}}_R)\quad\text{and}\quad \mathrm{V}(R;\Omega^{\mathrm{gq}}_R)\simeq\Omega\,\mathrm{U}(R;\Omega^{\mathrm{gq}}_{-R}).$$

Curiously, $\mathrm{V}(R;\Omega^{\mathrm{gq}}_{-R})\simeq\Omega\,\mathrm{U}(R;\Omega^{\geq 3}_R)$, and to the best of our knowledge $\Omega^{\geq 3}_R$ cannot be expressed in terms of classical forms.

Given any invariant $\mathcal{F}$ of Poincaré ∞-categories which is Verdier localising, the Bott-Genauer sequence induces a fibre sequence upon applying $\mathcal{F}$. If in addition $\mathcal{F}(\mathrm{Met}(\mathcal{C}, Ϙ)) = 0$ for any Poincaré ∞-category $(\mathcal{C}, Ϙ)$ (i.e. in the terminology of Paper [II] $\mathcal{F}$ is in addition bordism invariant), one obtains a canonical equivalence

$$\mathcal{F}(\mathcal{C}, Ϙ^{[n]}) \simeq \Sigma^n \mathcal{F}(\mathcal{C}, Ϙ)$$

for every $n \in \mathbb{Z}$ (see [Lur11] and Proposition [II].3.5.8). Examples of bordism invariant functors are L-theory $\mathrm{L}(\mathcal{C}, Ϙ)$ and the Tate construction on K-theory $\mathrm{K}(\mathcal{C}, Ϙ)^{\mathrm{tC}_2}$, where $\mathrm{K}(\mathcal{C}, Ϙ)$ denotes the K-theory spectrum of $\mathcal{C}$ with the C_2-action induced by the duality underlying $Ϙ$. In fact, for $\mathrm{K}(\mathcal{C}, Ϙ)^{\mathrm{tC}_2}$ this is an immediate consequence of classical additivity: $\mathrm{K}(\mathrm{Met}(\mathcal{C}, Ϙ))$ is equivalent to $\mathrm{C}_2 \otimes \mathrm{K}(\mathcal{C})$, so that its Tate construction vanishes. We can again combine these results with the periodicity of Proposition R.6 to obtain the following.

R.10. **Corollary.** *Let R be a ring and M an invertible module with involution over R. Then there are natural equivalences*

$$\mathrm{L}(R; Ϙ^{\geq m}_M) \simeq \Omega^2\, \mathrm{L}(R; Ϙ^{\geq (m+1)}_{-M}) \quad \textit{and} \quad \mathrm{K}(R; Ϙ^{\geq m}_M)^{\mathrm{tC}_2} \simeq \Omega^2\, \mathrm{K}(R; Ϙ^{\geq (m+1)}_{-M})^{\mathrm{tC}_2}.$$

In particular, the spectra $\mathrm{L}(R; Ϙ^{\mathrm{s}}_M)$, $\mathrm{L}(R; Ϙ^{\mathrm{q}}_M)$ and $\mathrm{K}(R; M)^{\mathrm{tC}_2}$ are 4-periodic, and 2 periodic if R is an $\mathbb{F}_2$-algebra.

The last observation on the periodicity of the quadratic and symmetric L-spectra is of course due to Ranicki, and it has been reworked in the present language by Lurie [Lur11].

R.11. **Remark.** Let R be a ring. We note that the association $X \mapsto \hom_R(X, R)$ refines to an equivalence of stable ∞-categories

$$\hom_R(-, R) \colon \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{D}^{\mathrm{p}}(R^{\mathrm{op}})^{\mathrm{op}}.$$

Now let $Ϙ$ be a Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$. By pulling back $Ϙ$ along its induced duality D, we obtain a Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}}$, and further pulling back along the above equivalence a Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R^{\mathrm{op}})$ which we denote by $Ϙ^\vee$. That is, $Ϙ^\vee$ is the composite

$$Ϙ^\vee \colon \mathcal{D}^{\mathrm{p}}(R^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{D}^{\mathrm{p}}(R) \xrightarrow{\mathrm{D}} \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \xrightarrow{Ϙ} \mathcal{S}p.$$

By construction, there is therefore an equivalence of Poincaré ∞-categories

$$(\mathcal{D}^{\mathrm{p}}(R), Ϙ) \simeq (\mathcal{D}^{\mathrm{p}}(R^{\mathrm{op}}), Ϙ^\vee).$$

In particular, we have $\mathrm{GW}(R^{\mathrm{op}}; Ϙ^\vee) \simeq \mathrm{GW}(R; Ϙ)$ and likewise $\mathrm{L}(R^{\mathrm{op}}; Ϙ^\vee) \simeq \mathrm{L}(R; Ϙ)$. For a ring with involution R, the invertible module $M = R$ is self dual, so that one finds $\mathrm{GW}(R) \simeq \mathrm{GW}(R^{\mathrm{op}})$ and likewise $\mathrm{L}(R) \simeq \mathrm{L}(R^{\mathrm{op}})$.

R.12. **Notation.** Let $\mathcal{F}$ be a functor, such as GW or L, from the category of Poincaré ∞-categories to spectra. We introduce the following compact notation for the value of $\mathcal{F}$ at the perfect derived ∞-category of R with one of the Poincaré structures $Ϙ^\alpha_M$ discussed above:

$$\mathcal{F}^\alpha(R; M) := \mathcal{F}(\mathcal{D}^{\mathrm{p}}(R), Ϙ^\alpha_M)$$

If $M = R(\epsilon)$ is the module with involution associated to an ϵ-involution on R as in Example R.2, we write

$$Ϙ^\alpha_\epsilon := Ϙ^\alpha_{R(\epsilon)} \quad \text{and} \quad \mathcal{F}^\alpha(R; \epsilon) := \mathcal{F}^\alpha(R; R(\epsilon))$$

for any of the decorations α above. In the special cases where $\epsilon = \pm 1$ we will further write

$$\begin{aligned} Ϙ^\alpha &:= Ϙ^\alpha_1 = Ϙ^\alpha_R \\ Ϙ^\alpha_- &:= Ϙ^\alpha_{-1} = Ϙ^\alpha_{R(-1)} \\ \mathcal{F}^\alpha(R) &:= \mathcal{F}^\alpha(R; 1) = \mathcal{F}^\alpha(R; R) \\ \mathcal{F}^{-\alpha}(R) &:= \mathcal{F}^\alpha(R; -1) = \mathcal{F}^\alpha(R; R(-1)). \end{aligned}$$

The homotopy groups of any of these spectra will be denoted by adding a subscript $\mathcal{F}^\alpha_n(R; M) := \pi_n \mathcal{F}^\alpha(R; M)$ for every $n \in \mathbb{Z}$.

1. L-theory and algebraic surgery

This section is devoted to exploring L-theory in the context of modules with involution. In §1.1 we recall the generators and relations description of the L-groups, and an important construction which allows to manipulate representatives in such L-groups (without changing the class in L-theory) called *algebraic surgery*.

In §1.2, we prove a surgery result for Poincaré structures which we call m-quadratic, for $m \in \mathbb{Z}$, and use this to represent L-theory classes by Poincaré objects which satisfy certain connectivity bounds. In particular this allows us to show that the L-groups $\mathrm{L}^{\mathrm{gs}}_n(R; M)$ coincide with Ranicki's original definition of symmetric L-theory of short complexes, Theorem 5 from the introduction.

Finally, in §1.3 we prove a surgery result for Poincaré structures which we call r-symmetric, for $r \in \mathbb{Z}$, in case the ring under consideration is coherent of finite global dimension. We will use this to show that the genuine symmetric L-groups are isomorphic to the symmetric L-groups in sufficiently high degrees, and consequently the analogous statement for the Grothendieck-Witt groups, which are Theorem 6 and Theorem 3 of the introduction.

1.1. L-theoretic preliminaries.

For the whole section we let R be a ring, M an invertible module with involution over R, and $\mathrm{D} = \hom_R(-, M)$ the corresponding duality on $\mathcal{D}^{\mathrm{p}}(R)$. We recall that $Ϙ^{\mathrm{q}}_M$ denotes the quadratic Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$, defined as the homotopy coinvariants $Ϙ^{\mathrm{q}}_M(X) = \hom_{R\otimes R}(X \otimes X, M)_{\mathrm{hC}_2}$, and that the symmetric Poincaré structure $Ϙ^{\mathrm{s}}_M$ is defined in an analogous way by taking homotopy invariants.

1.1.1. **Remark.** If a Poincaré structure $Ϙ\colon \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \to \mathcal{S}p$ has underlying duality D, we will say that $Ϙ$ is *compatible with* M. In this case, the canonical map $Ϙ^{\mathrm{q}}_M \to Ϙ^{\mathrm{s}}_M$ factors as in Construction [I].3.2.5 into a pair of natural transformations

$$Ϙ^{\mathrm{q}}_M \longrightarrow Ϙ \longrightarrow Ϙ^{\mathrm{s}}_M,$$

exhibiting $Ϙ^{\mathrm{q}}_M$ and $Ϙ^{\mathrm{s}}_M$ respectively as the initial and the final Poincaré structure compatible with M, see Corollary [I].1.3.6.

We recall that a spectrum E is m-connective for some integer $m \in \mathbb{Z}$ if $\pi_k E = 0$ for all $k < m$, and m-truncated if $\pi_k E = 0$ for all $k > m$.

1.1.2. **Definition.** For every $r \in \mathbb{Z}$ we will say that $Ϙ$ is *r-symmetric* if for every finitely generated projective module $P \in \mathrm{Proj}(R)$ the fibre of $Ϙ(P[0]) \to Ϙ^{\mathrm{s}}_M(P[0])$ is $(-r)$-truncated. Dually, for $m \in \mathbb{Z}$ we will say that $Ϙ$ is *m-quadratic* if the cofibre of $Ϙ^{\mathrm{q}}_M(P[0]) \to Ϙ(P[0])$ is m-connective for every $P \in \mathrm{Proj}(R)$.

1.1.3. **Remark.** Note that the fibre of $Ϙ \to Ϙ^{\mathrm{s}}_M$ and the cofibre of $Ϙ^{\mathrm{q}}_M \to Ϙ$ are exact (contravariant) functors. It thus suffices to check the conditions in the definition for m-quadratic and r-symmetric Poincaré structures only in the case where $P = R$.

It also follows that the collection of $X \in \mathcal{D}^{\mathrm{p}}(R)$ for which the above fibre is $(-r)$-truncated for a given $r \in \mathbb{Z}$ is closed under suspensions and extensions. In particular, if $Ϙ$ is r-symmetric then the fibre of $Ϙ(X) \to Ϙ^{\mathrm{s}}_M(X)$ is $(-r-k)$-truncated for every k-connective X.

Dually, for a given $m \in \mathbb{Z}$ the collection of $\mathrm{D}X \in \mathcal{D}^{\mathrm{p}}(R)$ for which the above cofibre is m-connective is closed under suspensions and extensions. In particular, if $Ϙ$ is m-quadratic then the cofibre of $Ϙ^{\mathrm{q}}_M(X) \to Ϙ(X)$ is $(m+k)$-connective whenever $\mathrm{D}X$ is k-connective.

1.1.4. **Example.** The symmetric Poincaré structure $Ϙ^{\mathrm{s}}_M$ is r-symmetric for every r and the quadratic Poincaré structure $Ϙ^{\mathrm{q}}_M$ is m-quadratic for every m. More generally, from the exact sequences

$$\tau_{\leq m-2}\Omega M^{\mathrm{tC}_2} \to Ϙ^{\geq m}_M(R) \to Ϙ^{\mathrm{s}}_M(R) \quad \text{and} \quad Ϙ^{\mathrm{q}}_M(R) \to Ϙ^{\geq m}_M(R) \to \tau_{\geq m} M^{\mathrm{tC}_2}$$

we find that the Poincaré structure $Ϙ^{\geq m}_M$ is m-quadratic and $(2-m)$-symmetric. In particular, $Ϙ^{\mathrm{gs}}_M$ is 2-symmetric and 0-quadratic, $Ϙ^{\mathrm{ge}}_M$ is 1-symmetric and 1-quadratic and $Ϙ^{\mathrm{gq}}_M$ is 0-symmetric and 2-quadratic.

1.1.5. **Example.** Let R be a ring and $Ϙ$ an M-compatible r-symmetric and m-quadratic Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$. Then $Ϙ^{\vee}$ is an $M^{\vee}$-compatible r-symmetric and m-quadratic Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R^{\mathrm{op}})$. To see that $Ϙ^{\vee}$ is m-quadratic, it suffices to show that the cofibre of the map

$$Ϙ^{\mathrm{q}}_{M^{\vee}}(R^{\mathrm{op}}) \longrightarrow Ϙ^{\vee}(R^{\mathrm{op}})$$

is m-connective. By Remark R.11, this map is given by the map

$$(\Qoppa^{\mathrm{q}}_M)(M[0]) \longrightarrow \Qoppa(M[0])$$

whose cofibre is m-connective by the assumption that $\Qoppa$ is m-quadratic and that M is finitely generated projective. A similar argument shows that $\Qoppa^\vee$ is r-symmetric.

As explained earlier, one goal of this paper is to show that the genuine symmetric L-groups coincide with Ranicki's classical symmetric L-groups of [Ran80], for which elements can be represented by chain complexes X which are concentrated in a specific range of degrees. The following lemma shows that this can be equivalently phrased in terms of connectivity estimates for X and $\mathrm{D}X$. The latter will be more convenient to work with for us.

1.1.6. **Lemma.** *Let $X \in \mathcal{D}^{\mathrm{p}}(R)$ a perfect R-module and $k \leq l$ integers. Then the following conditions are equivalent:*

i) X can be represented by a chain complex of the form

$$\cdots \to 0 \to P_l \to P_{l-1} \to \cdots \to P_k \to 0 \to \cdots$$

where each P_i is a finitely generated projective R-module concentrated in homological degree i.

ii) X is k-connective and $\mathrm{D}X$ is $(-l)$-connective.

Proof. The implication i) ⇒ ii) is clear. For the other implication, let C be a complex of finitely generated projective R-modules of minimum length representing X. We claim that C is concentrated in the range $[k,l]$. Let i be the minimal integer such that $C_i \neq 0$. We claim that $i \geq k$. Indeed, suppose that $i < k$. Since X is k-connective we have that $\mathrm{H}_i(C) = 0$ and so the differential $C_{i+1} \to C_i$ is a surjection of projective modules, hence a split surjection of projective modules, hence a surjection whose kernel $N := \ker(C_{i+1} \to C_i)$ is projective. Removing C_i and replacing C_{i+1} with N thus yields a shorter complex representing X, contradicting the minimality of C. We may hence conclude that C is concentrated in degrees $\geq k$.

Let now $\mathrm{D}C$ be the complex given by $(\mathrm{D}C)_i := \mathrm{D}(C_{-i})$. Since M is finitely generated projective $\mathrm{D}C = \hom_R(C, M) \in \mathrm{Ch}^{\mathrm{b}}(R)$ represents $\mathrm{D}X \in \mathcal{D}^{\mathrm{p}}(R)$, and is thus also a complex of minimal length representing $\mathrm{D}X$. Since $\mathrm{D}X$ is assumed to be $(-l)$-connective, the same argument as above shows that $\mathrm{D}C$ is concentrated in degrees $\geq -l$. It then follows that C is concentrated in degrees $\leq l$, and hence in the range $[k,l]$, as desired. □

1.1.7. **Remark.** Lemma 1.1.6 does not really require a duality. In its absence the statement still holds if we treat $\mathrm{D}X = \hom_R(X, R)$ as an object of $\mathcal{D}^{\mathrm{p}}(R^{\mathrm{op}})$. For later use, we also remark that our proof also shows that X is k-connective if and only if it can be represented by a chain complex of finitely generated projective modules which are trivial below degree k.

1.1.8. **Remark.** If M is moreover free as an R-module, the proof of Lemma 1.1.6 works verbatim to show that for $X \in \mathcal{D}^{\mathrm{f}}(R)$, condition ii) above is equivalent to X being representable by a complex as in Lemma 1.1.6 with each P_i a finitely generated *stably free* R-module. More generally, if $\mathrm{Free}(R) \subseteq \mathcal{C} \subseteq \mathrm{Proj}(R)$ is any intermediate full subcategory closed under the duality and under direct sums, and X can be represented by a bounded complex valued in $\mathcal{C}$, then the argument in the proof below yields that condition ii) above is equivalent to X being representable by a complex as in i) with each P_i stably in $\mathcal{C}$ (that is, such that there exist $Q_i \in \mathcal{C}$ with $P_i \oplus Q_i \in \mathcal{C}$).

L-theory and surgery. The purpose of this subsection is to recall some fundamental properties of L-theory. For the construction of the L-theory spectra, we refer to [Lur11] and §[II].4.4. However, a key feature of the L-spectrum $\mathrm{L}(\mathcal{C}, \Qoppa)$ is that its homotopy groups have a very simple presentation: They are given by cobordism groups of Poincaré objects. Let us explain what this means precisely, as we rely on this construction throughout the section. We recall that the space $\mathrm{Pn}(\mathcal{C}, \Qoppa)$ is the space of Poincaré objects, that is of pairs (X, q) of an object X in $\mathcal{C}$ and a point $q \in \Omega^\infty\Qoppa(X)$ such that a canonically associated map

$$q_\sharp \colon X \longrightarrow \mathrm{D}X$$

is an equivalence. Likewise, there is the space $\mathrm{Pn}^\partial(\mathcal{C}, \Qoppa)$ of Poincaré pairs, that is of triples $(f \colon L \to X, q, \eta)$ with $q \in \Omega^\infty\Qoppa(X)$ and η a nullhomotopy of $f^*(q)$, such that the canonically associated map

$$\eta_\sharp \colon X/L \longrightarrow \mathrm{D}L$$

induced on the quotient X/L by η is an equivalence. In this case we say that L is a Lagrangian in X (or also that L is a nullcobordism of X). It turns out that $\mathrm{Pn}^\partial(\mathcal{C}, \Upsilon) = \mathrm{Pn}(\mathrm{Met}(\mathcal{C}, \Upsilon))$ where $\mathrm{Met}(\mathcal{C}, \Upsilon)$ is the metabolic category associated to Υ as in §[I].2.3. We find that forgetting the Lagrangian provides a map $\mathrm{Pn}^\partial(\mathcal{C}, \Upsilon) \to \mathrm{Pn}(\mathcal{C}, \Upsilon)$, which is induced from the Poincaré functor $\mathrm{Met}(\mathcal{C}, \Upsilon) \to (\mathcal{C}, \Upsilon)$ sending $L \to X$ to X.

1.1.9. **Definition.** We say that Poincaré objects (X, q) and (X', q') are cobordant if $(X \oplus X', q \oplus (-q'))$ admits a Lagrangian, i.e. is nullcobordant. We define the n'th L-group $\mathrm{L}_n(\mathcal{C}, \Upsilon)$ as the group of cobordism classes of Poincaré objects (X, q) for the Poincaré structure $\Upsilon^{[-n]} := \Omega^n \Upsilon$.

We remark that the cobordism relation really is a congruence relation with respect to $\oplus$, and that the diagonal $X \to X \oplus X$ is a canonical Lagrangian for $(X \oplus X, q \oplus (-q))$, so that $\mathrm{L}_n(\mathcal{C}, \Upsilon)$ is indeed an abelian group.

1.1.10. **Notation.** In the case of the category $\mathcal{C} = \mathcal{D}^{\mathrm{p}}(R)$, we will denote the L-groups and L-spectra respectively by

$$\mathrm{L}_n(R; \Upsilon) := \mathrm{L}_n(\mathcal{D}^{\mathrm{p}}(R), \Upsilon) \quad \text{and} \quad \mathrm{L}(R; \Upsilon) := \mathrm{L}(\mathcal{D}^{\mathrm{p}}(R), \Upsilon).$$

When $\Upsilon = \Upsilon^\alpha$ is one of the genuine functors associated to an invertible module with involution M analysed in the previous section, we use the notation $\mathrm{L}^\alpha(R; M)$ established in Notation R.12 for the corresponding L-groups.

1.1.11. **Remark.** It is immediate from the definition that $\mathrm{L}^{\mathrm{q}}(R; M)$ and $\mathrm{L}^{\mathrm{s}}(R; M)$ are respectively the usual quadratic and symmetric L-theory spectra of R of [Lur11] which also agree with the L-spectra of Ranicki [Ran92]. The other variants are, however, more mysterious, and their study is the focus of this section.

1.1.12. **Remark.** It is immediate from the definition that the L-groups fit into an exact sequence of monoids

$$\pi_0(\mathrm{Pn}^\partial(\mathcal{C}, \Upsilon^{[-n]})) \xrightarrow{\partial} \pi_0(\mathrm{Pn}(\mathcal{C}, \Upsilon^{[-n]})) \longrightarrow \mathrm{L}_n(\mathcal{C}, \Upsilon) \longrightarrow 0,$$

so that they agree with Definition [I].2.3.11. We remark that the quotient monoid of $\pi_0(\mathrm{Pn}(\mathcal{C}, \Upsilon^{[-n]}))$ by the submonoid of metabolic objects is a priori smaller than $\mathrm{L}_n(\mathcal{C}, \Upsilon)$, since two Poincaré objects are identified in the quotient if they become isomorphic after adding metabolic forms.

Let us see directly that $\mathrm{L}_n(\mathcal{C}, \Upsilon)$ is indeed the quotient monoid. We need to verify that if a Poincaré object (X, q) in $\pi_0(\mathrm{Pn}(\mathcal{C}, \Upsilon^{[-n]}))$ is zero in the quotient, that is if there are metabolic Poincaré objects (X', q') and (X'', q'') and an isomorphism $(X \oplus X', q \oplus q') \cong (X'', q'')$, then (X, q) is itself metabolic. If $L' \to X'$ and $L'' \to X''$ are Lagrangians for q' and q'' respectively, then it is straightforward to verify that $L' \times_{X'} L'' \to X$ is a Lagrangian for q, where the pullback is formed via the second projection $L'' \to X'' \cong X \oplus X' \to X'$, and it maps to X via the first projection $L'' \to X'' \cong X \oplus X' \to X$.

We note that given a Lagrangian for (X, q), i.e. a Poincaré object of the metabolic category, through the eyes of L-theory, we may replace (X, q) by 0. Such a procedure in fact works more generally if we start with only a hermitian object for the metabolic category and is the content of algebraic surgery. We recall that the hermitian objects of the metabolic category consist of triples $(f : L \to X, q, \eta)$ such that $q \in \Omega^\infty \Upsilon(X)$ and η is a nullhomotopy of $f^*(q)$. In many cases of interest, the object (X, q) is Poincaré, and in this situation we will refer to L, or more precisely to (f, η), as a *surgery datum* on (X, q). The non-degeneracy condition for this triple to be a Poincaré object for the metabolic category is that the map $\eta_\sharp : X/L \longrightarrow \mathrm{D}L$ defined above is an equivalence, i.e. if its fibre X' is 0. In general, X' need not vanish, but nevertheless acquires a canonical Poincaré form q' induced from (f, q, η). In fact, we have the following result; see §[II].2.4 for a general discussion of algebraic surgery.

1.1.13. **Proposition.** *Let (X, q) be a Poincaré object for Υ with surgery datum $(f : L \to X, \eta)$. Then the object X' carries a canonical Poincaré form q' such that (X, q) and (X', q') are cobordant.*

1.1.14. **Remark.** The underlying object of X' and the of the cobordism $\chi(f)$ between X and X' are summarised in the following surgery diagram consisting of horizontal and vertical fibre sequences.

$$\begin{array}{ccccc} L & = & L & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \chi(f) & \longrightarrow & X & \xrightarrow{\mathrm{D}f\circ q_\sharp} & \mathrm{D}L \\ \downarrow & & \downarrow & & \| \\ X' & \longrightarrow & X/L & \xrightarrow{\eta_\sharp} & \mathrm{D}L \end{array} \tag{4}$$

We can use this to perform the following construction, which we will refer to as *Lagrangian surgery*.

1.1.15. **Construction.** Let $(L \to X, q, \eta)$ be a Lagrangian for a Poincaré object (X, q). Equivalently, we may view $(L \to X, q, \eta)$ as a Poincaré object of the metabolic category $\mathrm{Met}(\mathcal{C}, \Qoppa)$. Now, given a surgery datum for this Poincaré object, i.e. a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & W \\ \downarrow & & \downarrow \\ L & \longrightarrow & X \end{array}$$

and a null-homotopy of $\Phi^*(q,\eta)$ in $\Qoppa_{\mathrm{met}}(Z \to W)$, we may thus perform surgery by Proposition 1.1.13 to obtain a new Poincaré object $(L' \to X', q', \eta')$ of $\mathrm{Met}(\mathcal{C}, \Qoppa)$. We observe that the map $W \to X$ is canonically a surgery datum on (X, q) and that (X', q') is the result of surgery with this surgery datum. In particular, if $W = 0$, then (X', q') is canonically equivalent to (X, q). Moreover, by diagram (4) the new Lagrangian L' sits inside a fibre sequence

$$L' \longrightarrow L/Z \longrightarrow \Omega \mathrm{D}Z.$$

We will refer to such surgery data as *Lagrangian surgery data* and refer to the surgery as a *Lagrangian surgery*. For future reference, we notice that the underlying map of a Lagrangian surgery datum is equivalently described by a map $Z \to N = \mathrm{fib}(L \to X)$. If we denote by N' the fibre of the map $L' \to X$, then we obtain likewise a fibre sequence $N' \to N/Z \to \Omega \mathrm{D}Z$.

1.2. Surgery for *m*-quadratic structures.

In this section we will show how to apply algebraic surgery to Poincaré structure which are sufficiently quadratic and use this to show that the genuine symmetric L-groups coincide with Ranicki's symmetric L-groups of short complexes; see Theorem 1.2.18. We also show that in sufficiently small degrees, the L-groups of an m-quadratic functor coincide with the quadratic L-groups; see Corollary 1.2.8. The surgery arguments we present below are designed to replace (shifted) Poincaré objects and Lagrangians by cobordant counterparts which are suitably connective. The following definition summarises the kind of connectivity we seek:

1.2.1. **Definition.** Let M be an invertible module with involution over R, $\Qoppa$ a Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$ compatible with M. Let $n, a, b \in \mathbb{Z}$ be such that $a, b \geq -1$, $b \geq a-1$, and $(n+a)$ is even.

i) We denote by $\mathrm{Pn}^a_n(R, \Qoppa) \subseteq \mathrm{Pn}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{[-n]})$ the subspace spanned by those Poincaré objects (X, q) such that X is $(\frac{-n-a}{2})$-connective.
ii) We denote by $\mathrm{Pn}^\partial(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{[-n]})$ the subspace spanned by those Poincaré pairs $(L \to X, q, \eta)$ such that X is $(\frac{-n-a}{2})$-connective, L is $\lceil\frac{-n-1-b}{2}\rceil$-connective and $N := \mathrm{fib}(L \to X) \simeq \Omega^{n+1}\mathrm{D}L$ is $\lfloor\frac{-n-1-b}{2}\rfloor$-connective. We refer to such an L as an *allowed Lagrangian* for (X, q).

Finally, we define

$$\mathrm{L}^{a,b}_n(R; \Qoppa) = \mathrm{coker}\left(\pi_0 \mathcal{M}^{a,b}_n(R, \Qoppa) \to \pi_0 \mathrm{Pn}^a_n(R, \Qoppa)\right)$$

as the cokernel in the category of monoids of the map that forgets the Lagrangian.

1.2.2. **Remark.** If $b \geq a$, the diagonal inside $(X, q) \oplus (X, -q)$ is an allowed Lagrangian, so that $\mathrm{L}^{a,b}_n(R; \Qoppa)$ is in fact a group. In Proposition 1.2.3, we will show that this is also the case for $b = a-1$, under additional hypotheses on the Poincaré structure $\Qoppa$.

Furthermore, in case i), $\mathrm{D}X \simeq X[n]$ is $(\frac{n-a}{2})$-connective so that X can be represented by a complex of length a concentrated in degrees $[\frac{-n-a}{2}, \frac{-n+a}{2}]$ by Lemma 1.1.6. In particular, $\mathrm{Pn}^{-1}_n(R, \Qoppa) \simeq *$. In case ii), X

can be represented by a complex concentrated in degrees $[\frac{-n-a}{2}, \frac{-n+a}{2}]$, L by a complex concentrated in degrees $[\lceil\frac{-n-1-b}{2}\rceil, \lceil\frac{-n-1+b}{2}\rceil]$ and N by a complex concentrated in degrees $[\lfloor\frac{-n-1-b}{2}\rfloor, \lfloor\frac{-n-1+b}{2}\rfloor]$. Notice that for the conclusion that L is concentrated in a certain range of degrees, we have used both the connectivity of L and N, as $\mathrm{D}L = N[n+1]$.

For the remainder of the section we fix an invertible module with involution M over R, and we consider only Poincaré structures $Ϙ$ on $\mathcal{D}^{\mathrm{p}}(R)$ which are compatible with M, and we denote the underlying duality by $\mathrm{D} = \hom_R(-, M)$. To put the assumptions of the next result into context, recall that the Poincaré structure $Ϙ^{\geq m}_M$ is m-quadratic.

1.2.3. **Proposition** (Surgery for m-quadratic Poincaré structures)**.** *Let $Ϙ$ be an m-quadratic Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$. Fix an $n \in \mathbb{Z}$ and let $a, b \geq 0$ be two non-negative integers with $b \geq a-1$, and such that*

- $(n+a) \equiv (n+1+b) \equiv 0$ *modulo 2, and*
- $a \geq n - 2m$ *and* $b \geq n - 2m + 1$.

Then the map $\mathrm{L}^{a,b}_n(R;Ϙ) \to \mathrm{L}_n(R;Ϙ)$ is an isomorphism.

The proof of Proposition 1.2.3 will require the following connectivity estimate:

1.2.4. **Lemma.** *Suppose that $Ϙ$ is an m-quadratic Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$. Then for every projective module $P \in \mathrm{Proj}(R)$ and every $k \in \mathbb{Z}$ the spectrum $Ϙ(P[k])$ is $\min(-2k, m-k)$-connective.*

Proof. The cofibre of the map $Ϙ^{\mathrm{q}}_M(P[k]) \to Ϙ(P[k])$ is $(m-k)$-connective by the assumption that $Ϙ$ is m-quadratic, see Remark 1.1.3. Furthermore, $Ϙ^{\mathrm{q}}_M(P[k]) = (\hom_{R\otimes R}(P \otimes P, M)[-2k])_{\mathrm{hC}_2}$ and is thus $(-2k)$-connective as P is projective and homotopy orbits preserve connectivity. □

Proof of Proposition 1.2.3. We start with the surjectivity of the map in question. For this, it suffices to show that every Poincaré object $(X, q) \in \mathrm{Pn}(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{[-n]})$ is cobordant to one whose underlying object is $(\frac{-n-a}{2})$-connective. If X itself is $(\frac{-n-a}{2})$-connective then we are done. Otherwise, since X is perfect there exists some $k < \frac{-n-a}{2}$ such that X is k-connective. By Lemma 1.1.6 the object X can be represented by a chain complex of projectives concentrated in degrees $\geq k$ and so there exists a projective module P and a map $f\colon P[k] \to X$ which is surjective on H_k. By Lemma 1.2.4, the spectrum $Ϙ(P[k])$ is $\min(-2k, m-k)$-connective and since $k < \frac{-n-a}{2}$ we have that

$$\min(-2k, m-k) > \min\left(n+a, \frac{2m+n+a}{2}\right) \geq n$$

by the inequalities in our assumptions. It then follows that $\Omega^{\infty+n}Ϙ(P[k])$ is connected and hence q restricted to $P[k]$ is null-homotopic, so that any nullhomotopy η, makes $(P[k] \to X, q, \eta)$ a hermitian form for the metabolic category. We may therefore apply Proposition 1.1.13 and perform surgery along $f\colon P[k] \to X$ to obtain a cobordant Poincaré object X', given by the fibre of the induced map $X/P[k] \to \mathrm{D}(P)[-k-n]$. Since $-2k-1 > n+a \geq n$ (here we use that $n+a$ is even) we have that $-k-n > k+1$ and so

$$\mathrm{H}_{k'}(X') = \mathrm{H}_{k'}(X) = 0 \quad \text{for } k' < k \quad \text{and} \quad \mathrm{H}_k(X') \cong \operatorname{coker}[P \to \mathrm{H}_k(X)] = 0,$$

which means that X' is $(k+1)$-connective. Proceeding inductively we may thus obtain a Poincaré object (X'', q'') which is cobordant to (X, q) and which is $(\frac{-n-a}{2})$-connective. It then follows that the class $[X, q]$ is in the image of $\mathrm{L}^{a,b}_n(R;Ϙ) \to \mathrm{L}_n(R;Ϙ)$, so we have established surjectivity.

To prove injectivity, we represent an element of $\mathrm{L}^{a,b}_n(R, Ϙ)$ by a Poincaré complex (X, q) such that X is $(\frac{-n-a}{2})$-connective, and we suppose that it represents zero in $\mathrm{L}_n(R;Ϙ)$. We need to verify that in this case (X, q) already represents the zero element in $\mathrm{L}^{a,b}_n(R;Ϙ)$. By Remark 1.1.12 it admits a Lagrangian $(L \to X, q, \eta)$. Let N be the fibre of the map $L \to X$. If N is $\frac{-n-1-b}{2}$-connective, since $b \geq a-1$ then so is L and we are done. Otherwise, let $l < \frac{-n-1-b}{2}$ be such that N is l-connective. We can then find a projective module P and a map $P[l] \to N$ which is surjective on H_l. We may view the map $P[l] \to N$ equivalently as a map $(P[l] \to 0) \to (L \to X)$ in the metabolic category. We claim that this map extends to a Lagrangian surgery datum in the sense of Construction 1.1.15, for which it suffices to see that $Ϙ_{\mathrm{met}}(P[l] \to 0) \simeq \Omega Ϙ(P[l])$ is

$(n+1)$-connective. By Lemma 1.2.4, the spectrum $Ϙ(P[l])$ is then $\min(-2l, m-l)$-connective and since $l < \frac{-n-1-b}{2}$ we have that

$$\min(-2l, m-l) > \min\left(n+1+b, \frac{2m+n+1+b}{2}\right) \geq n+1$$

by the inequalities in our assumptions. We may therefore perform Lagrangian surgery along $P[l] \to L$, see Construction 1.1.15, to obtain a new Lagrangian $L' \to X$ such that the fibre N' of the map $L' \to X$ fits in a fibre sequence

$$N' \longrightarrow N/P[l] \longrightarrow \mathrm{D}(P)[-l-n-1].$$

Since $2l < -n-1-b$ and $n+1+b$ is even, we have that $-2l-1 > n+1+b \geq n+1$. Thus $-l-n-1 > l+1$, and so

$$\mathrm{H}_{l'}(N') = \mathrm{H}_{l'}(N) = 0 \quad \text{for } l' < l \quad \text{and} \quad \mathrm{H}_l(N') \cong \mathrm{coker}[P \to \mathrm{H}_l(N)] = 0,$$

which means that N' is $(l+1)$-connective. Proceeding inductively we may thus obtain a Lagrangian $L'' \to X$ for which N'', and thus L'', is $(\frac{-n-1-b}{2})$-connective. This shows that

i) the kernel of the map $\mathrm{L}_n^{a,b}(R; Ϙ) \to \mathrm{L}_n(R; Ϙ)$ is trivial, and
ii) $\mathrm{L}_n^{a,b}(R; Ϙ)$ is a group.

Indeed, to see ii), we note that $(X \oplus X, q \oplus (-q))$ admits a Lagrangian, and hence by the above argument also a Lagrangian with suitable connectivity properties, which shows that $(X, -q)$ is an inverse of (X, q) in the monoid $\mathrm{L}_n^{a,b}(R; Ϙ)$. This shows the proposition. □

1.2.5. **Remark.** The definition of $\mathrm{L}_n^{a,b}(R, Ϙ)$ as a cokernel in the category of commutative monoids means in particular that a class $[X, q] \in \pi_0 \mathrm{Pn}_n^a(R, Ϙ)$ maps to zero in $\mathrm{L}_n^{a,b}(R, Ϙ)$ if and only if there exists metabolic classes $[L' \to X', q', \eta'], [L'' \to X'', q'', \eta''] \in \pi_0 \mathcal{M}_n^{a,b}(R, Ϙ)$ such that $[X, q] + [X', q'] = [X'', q'']$ in $\pi_0 \mathrm{Pn}_n^a(R, Ϙ)$. When $Ϙ$ is m-quadratic for some $m \in \mathbb{Z}$ and a, b, n satisfy the assumptions of Proposition 1.2.3 then the surgery argument in the proof of that proposition allows us to slightly refine this statement: if (X, q) is a $(\frac{-n-a}{2})$-connective Poincaré object in $(\mathcal{D}^{\mathrm{p}}(R), Ϙ^{[-n]})$ which represents zero in $\mathrm{L}_n^{a,b}(R, Ϙ)$ then (X, q) itself admits a Lagrangian $L \to X$ such that L and $\mathrm{fib}(L \to X)$ are $(\frac{-n-1-b}{2})$-connective. In other words, the sequence

$$\pi_0 \mathcal{M}_n^{a,b}(R, Ϙ) \longrightarrow \pi_0 \mathrm{Pn}_n^a(R, Ϙ) \longrightarrow \mathrm{L}_n^{a,b}(R; Ϙ)$$

is exact in the middle, just as in the case of ordinary L-groups; see Remark 1.1.12.

1.2.6. **Example.** The quadratic Poincaré structure $Ϙ_M^{\mathrm{q}}$ is m-quadratic for every m. Given $n = 2k \in \mathbb{Z}$, we may apply Proposition 1.2.3 to $Ϙ_M^{\mathrm{q}}$ with $(a, b) = (0, 1)$ and get that every class in $\mathrm{L}_n^{\mathrm{q}}(R; M)$ can be represented by a Poincaré object which is concentrated in degree $-k$, and that such a Poincaré object represents zero in $\mathrm{L}_n^{\mathrm{q}}(R; M)$ if and only if it admits a Lagrangian which is concentrated in degrees $[-k-1, -k]$. On the other hand, if $n = 2k+1$ is odd we may apply Proposition 1.2.3 to $Ϙ_M^{\mathrm{q}}$ with $(a, b) = (1, 0)$ and get that every class in $\mathrm{L}_n^{\mathrm{q}}(R; M)$ can be represented by a Poincaré object which is concentrated in degree $[-k-1, -k]$, and that such a Poincaré object represents zero in $\mathrm{L}_n^{\mathrm{q}}(R; M)$ if and only if it admits a Lagrangian which is concentrated in degree $-k-1$. This is often referred to in the literature as *surgery below the middle dimension*. In fact, Proposition 1.2.3 gives this statement for any m-quadratic Poincaré structure, as long as we take $n \leq 2m$.

1.2.7. **Corollary.** *For any $n \geq 0$, the canonical map $\mathrm{L}_n^{n,n+1}(R; Ϙ_M^{\mathrm{gs}}) \to \mathrm{L}_n(R; Ϙ_M^{\mathrm{gs}}) = \mathrm{L}_n^{\mathrm{gs}}(R; M)$ is an isomorphism.*

Proof. The genuine symmetric Poincaré structure $Ϙ_M^{\mathrm{gs}}$ is 0-quadratic. Given $n \geq 0 \in \mathbb{Z}$ we may thus apply Proposition 1.2.3 to $Ϙ_M^{\mathrm{gs}}$ with $(a, b) = (n, n+1)$ so that $\mathrm{L}(R; Ϙ_M^{\mathrm{gs}}) \cong \mathrm{L}_n^{n,n+1}(R; Ϙ_M^{\mathrm{gs}})$. □

That is, every class in $\mathrm{L}_n^{\mathrm{gs}}(R; M)$ can be represented by a Poincaré object which is concentrated in degrees $[-n, 0]$, and such a Poincaré object represents zero in $\mathrm{L}_n^{\mathrm{gs}}(R; M)$ if and only if it admits a Lagrangian which is concentrated in degrees $[-n-1, 0]$.

1.2.8. **Corollary.** *If $Ϙ$ is m-quadratic (e.g., $Ϙ = Ϙ^{\geq m}_M$) then the natural map*

$$\mathrm{L}^{\mathrm{q}}_n(R;M) = \mathrm{L}_n(R;Ϙ^{\mathrm{q}}_M) \longrightarrow \mathrm{L}_n(R;Ϙ)$$

is surjective for $n \leq 2m-2$ and bijective for $n \leq 2m-3$.

Proof. Let $a, b \in \{0,1\}$ be such that $n+a$ and $n+1+b$ are even. Notice that $n+a \leq 2m-2$. By Proposition 1.2.3, we have $\mathrm{L}_n(R,Ϙ^{\mathrm{q}}_M) \cong \mathrm{L}^{a,b}_n(R,Ϙ^{\mathrm{q}}_M)$ and $\mathrm{L}_n(R,Ϙ) \cong \mathrm{L}^{a,b}_n(R,Ϙ)$ whenever $n \leq 2m$, and so to prove the surjectivity statement it will suffice to show that the monoid map

$$\pi_0\mathrm{Pn}^a_n(R,Ϙ^{\mathrm{q}}_M) \longrightarrow \pi_0\mathrm{Pn}^a_n(R,Ϙ)$$

is surjective when $n \leq 2m-2$. So let X be $(\frac{-n-a}{2})$-connective and equipped with a Poincaré form q for $Ϙ^{[-n]}$. It will suffice to prove that the map $\pi_0\Omega^nϘ^{\mathrm{q}}_M(X) \to \pi_0\Omega^nϘ(X)$ is surjective. As $Ϙ$ is m- quadratic, $\mathrm{D}X$ is $(\frac{n-a}{2})$-connective by Remark 1.2.2, so that the cofibre of the map $\Omega^nϘ^{\mathrm{q}}_M(X) \to \Omega^nϘ(X)$ is $m-n+(\frac{n-a}{2})$-connective; see Remark 1.1.3. Since $m-n+(\frac{n-a}{2}) = \frac{2m-(n+a)}{2} \geq 1$ the claim follows.

To prove injectivity, let us now assume that $n \leq 2m-3$. In light of Remark 1.2.5, it will suffice to show that if (X,q) is a $(\frac{-n-a}{2})$-connective Poincaré object in $(\mathcal{D}^{\mathrm{p}}(R),(Ϙ^{\mathrm{q}}_M)^{[-n]})$ whose associated Poincaré object in $(\mathcal{D}^{\mathrm{p}}(R),Ϙ^{[-n]})$ admits a Lagrangian $L \to X$ such that L is $(\frac{-n-1-b}{2})$-connective and $N = \mathrm{fib}(L \to X)$ is also $(\frac{-n-1-b}{2})$-connective, then L can be refined to a Lagrangian of (X,q) with respect to $Ϙ^{\mathrm{q}}_M$. For this, it will suffice to show that in this situation, the map

$$\Omega^nϘ^{\mathrm{q}}_M(L) \longrightarrow \Omega^nϘ(L)$$

is injective on π_0 and surjective on π_1, which follows if the cofibre of this map has trivial π_1. Now, $\mathrm{D}L \simeq \Sigma^{n+1}N$ is $(\frac{n+1-b}{2})$-connective. Hence by Remark 1.1.3 and the assumption that $Ϙ$ is m-quadratic we deduce that $\Omega^nϘ(L)$ is $\frac{2m-n+1-b}{2}$-connective. Since $2m \geq n+3$, we can then estimate $\frac{2m-n+1-b}{2} \geq \frac{4-b}{2} > 1$, which finishes the proof. □

1.2.9. **Remark.** The range in which the map of Corollary 1.2.8 is an isomorphism is essentially optimal. For example for $R = \mathbb{Z}$ and $m = 0$, the map

$$\mathbb{Z}/2 \cong \mathrm{L}^{\mathrm{q}}_{-2}(\mathbb{Z}) \longrightarrow \mathrm{L}^{\mathrm{gs}}_{-2}(\mathbb{Z}) = 0$$

is not an isomorphism, see Example 2.2.12 for the calculations of the groups.

A particular case of 0-quadratic Poincaré structures $Ϙ$ on $\mathcal{D}^{\mathrm{p}}(R)$ which is of special interest is the one where $Ϙ(P[0])$ is discrete (i.e. is the Eilenberg-MacLane spectrum of an abelian group) for every $P \in \mathrm{Proj}(R)$. In this case the restriction of $Ϙ$ to $\mathrm{Proj}(R)$ can be regarded as a functor

$$Ϙ_0 \colon \mathrm{Proj}(R) \longrightarrow \mathcal{A}b$$

which is quadratic in the sense of Eilenberg-MacLane [EM54, §9], that is, its second cross effect is additive in each variable separately, and $Ϙ$ can be identified with the non-abelian derived functor of $Ϙ_0$ as discussed in Propositions [I].4.2.15 and [I].4.2.19. The data of a Poincaré object in $\mathcal{D}^{\mathrm{p}}(R)$ which is concentrated in degree 0 can then be defined purely in terms of $\mathrm{Proj}(R)$ and $Ϙ_0$, namely as pairs (P,q) where $P \in \mathrm{Proj}(R)$ and $q \in Ϙ_0(P)$ is such that the induced map $q_\sharp \colon P \to \mathrm{D}P$ is an isomorphism (where we note that the association $q \mapsto q_\sharp$ depends only on $Ϙ_0$). Let us then write $\mathrm{Pn}(\mathrm{Proj}(R),Ϙ_0)$ for the groupoid of such Poincaré objects. We will say that a Poincaré object $(P,q) \in \mathrm{Pn}(\mathrm{Proj}(R),Ϙ_0)$ is *strictly metabolic* if it admits a Lagrangian $L \to P$ such that L is concentrated in degree 0. The question of whether a given (P,q) as above is strictly metabolic again depends only on $\mathrm{Proj}(R)$ and $Ϙ_0$. We may then associate to the pair $(\mathrm{Proj}(R),Ϙ_0)$ its corresponding *Witt group* $\mathrm{W}(\mathrm{Proj}(R);Ϙ_0)$, defined as the quotient of the monoid $\pi_0\mathrm{Pn}(\mathrm{Proj}(R),Ϙ_0)$ by the submonoid of strictly metabolic objects. We note that there is an evident map $\mathrm{W}(\mathrm{Proj}(R);Ϙ_0) \to \mathrm{L}_0(R;Ϙ)$ induced by the inclusion $\mathrm{Proj}(R) \to \mathcal{D}^{\mathrm{p}}(R)$.

1.2.10. **Proposition.** *Let $Ϙ$ be a Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$ which is compatible with the invertible module with involution M, and such that $Ϙ_0 := Ϙ|_{\mathrm{Proj}(R)}$ takes values in $\mathcal{A}b \subseteq \mathcal{S}p$. Then the natural map*

$$\mathrm{W}(\mathrm{Proj}(R);Ϙ_0) \longrightarrow \mathrm{L}_0(R;Ϙ)$$

is an isomorphism.

Proof. As observed above, $\Qoppa$ is 0-quadratic. Applying Proposition 1.2.3 to $\Qoppa$, we conclude that the map $\mathrm{L}_0^{0,1}(R;\Qoppa) \to \mathrm{L}_0(R;\Qoppa)$ is an isomorphism. On the other hand, by construction, the map under consideration factors through a surjective map

$$\mathrm{W}(\mathrm{Proj}(R);\Qoppa_0) \longrightarrow \mathrm{L}_0^{0,1}(R;\Qoppa).$$

It will hence suffice to show that this map is also injective. By Remark 1.2.5 it will suffice to show that if $(P[0],q)$ is a Poincaré object concentrated in degree 0, and it admits a Lagrangian $(f\colon L \to P[0],q,\eta)$ such that L can be represented by a complex $[Q \xrightarrow{d} N]$ concentrated in degrees $[-1,0]$, then $(P[0],q)$ represents zero in $\mathrm{W}(\mathrm{Proj}(R);\Qoppa_0)$. To see this, we note that the object L sits in a fibre sequence of the form $N[-1] \to L \to Q[0]$. Consider the commutative square

$$\begin{array}{ccc} N[-1] & = & N[-1] \\ \downarrow & & \downarrow \\ L & \xrightarrow{\ f\ } & P[0] \end{array}$$

as a morphism in $\mathrm{Met}(\mathcal{D}^{\mathrm{p}}(R),\Qoppa)$ from the top row to the bottom row. In particular, the object corresponding to the bottom horizontal arrow carries a Poincaré structure in $\mathrm{Met}(\mathcal{D}^{\mathrm{p}}(R),\Qoppa)$ which exhibits $L \to P[0]$ as a Lagrangian. The top row on the other hand admits no non-trivial hermitian structure in $\mathrm{Met}(\mathcal{D}^{\mathrm{p}}(R),\Qoppa)$ since $\Qoppa_{\mathrm{met}}(\mathrm{id}_{N[-1]}) = 0$. In particular, the Poincaré structure on the bottom row becomes uniquely null-homotopic when restricted to the top row. We may thus consider it as providing a surgery datum on the bottom row (note that this does not constitute a Lagrangian surgery datum in the sense of Construction 1.1.15 since $N[-1] \neq 0$). Performing surgery, i.e. applying Proposition 1.1.13, we obtain a new Poincaré object $(L' \to P', q', \eta')$ in $\mathrm{Met}(\mathcal{D}^{\mathrm{p}}(R),\Qoppa)$ in which (P',q') is the Poincaré object obtained from performing surgery along $N[-1] \to P[0]$ and where $L' = Q[0]$ is concentrated in degree 0. Let us now identify (P',q'). The surgery datum on $P[0]$ is given by restricting the null homotopy of $q_{|L}$ to $N[-1]$ along the map $N[-1] \to L$. However, the map $N[-1] \to P[0]$ is null homotopic for degree reasons as N is projective, so that we may identify the restriction of q to $N[-1]$ with 0 using a null homotopy. The surgery datum on $P[0]$ is hence equivalently given by a loop in $\Qoppa(N[-1])$. It follows that (P',q') is the orthogonal sum of (P,q) and the output of surgery on $N[-1] \to 0$, with surgery datum given by said loop in $\Qoppa(N[-1])$. By construction, a surgery on 0 along $N[-1]$ is a metabolic form on $N[0]$: it is given by $(\mathrm{D}N[0] \oplus N[0], q'')$ for which $\mathrm{D}N[0]$ is a Lagrangian. Hence we obtain

$$(P',q') \simeq (P[0],q) \oplus (\mathrm{D}N[0] \oplus N[0], q'').$$

Thus, since (P',q') and $(\mathrm{D}N[0] \oplus N[0], q'')$ admit Lagrangians concentrated in degree 0, namely $Q[0]$ and $\mathrm{D}N[0]$ respectively, we find that $(P[0],q)$ vanishes in $\mathrm{W}(\mathrm{Proj}(R),\Qoppa_0)$ as claimed. □

1.2.11. **Remark.** Suppose that $\Qoppa$ is 1-quadratic and that (P,q) is a Poincaré object with P finitely generated projective concentrated in degree 0. Suppose (P,q) admits a Lagrangian L which is itself concentrated in degree 0. Since L is a Lagrangian, we find that the fibre of the map $L \to P$ is equivalent to $(\mathrm{D}L)[-1]$ and that $P \cong L \oplus \mathrm{D}L$. By the algebraic Thom isomorphism [I].2.4.6, the space of Poincaré structures on the object $L \to P$ of the metabolic category is equivalently described by the space of shifted forms $\Omega\Qoppa(\mathrm{D}L[-1])$, which is connected as $\Qoppa$ is 1-quadratic. It follows that (P,q) is equivalent to $\mathrm{hyp}(L)$, the hyperbolic form on L. This recovers the well-known classical fact that a strictly metabolic quadratic form on a finitely generated projective module is hyperbolic.

1.2.12. **Corollary.**

i) $\mathrm{L}_0^{\mathrm{gs}}(R;M)$ *is naturally isomorphic to the Witt group of M-valued symmetric forms over R.*
ii) $\mathrm{L}_0^{\mathrm{ge}}(R;M)$ *is naturally isomorphic to the Witt group of M-valued even forms over R.*
iii) $\mathrm{L}_0(R;\Qoppa_M^{\geq m})$ *is naturally isomorphic to the Witt group of M-valued quadratic forms over R for every $m \geq 2$.*

Proof. For $\mathrm{L}_0(R;\Qoppa_M^{\geq m})$ with $m = 0,1,2$ we simply apply Proposition 1.2.10 and invoke the explicit description of $\Qoppa_M^{\geq m}(P[0])$ for $m = 0,1,2$ in terms of symmetric, even and quadratic forms, respectively. The case of $\mathrm{L}_0(R;\Qoppa_M^{\geq m})$ for $m > 2$ reduces to that of $m = 2$ since the natural map

$$\mathrm{L}_0^{\mathrm{q}}(R;M) \longrightarrow \mathrm{L}_0(R;\Qoppa_M^{\geq m})$$

is an isomorphism for $m \geq 2$ by Corollary 1.2.8. □

1.2.13. **Remark.** Combining Corollary 1.2.12 and the equivalences $\mathrm{L}^{\mathrm{gs}}_{-2k}(R; M((-1)^k)) \cong \mathrm{L}_0\left(R; \Qoppa^{\geq k}_M\right)$ given by Corollary R.10 we obtain a description of all the *even non-positive* genuine symmetric L-groups of R in terms of Witt groups of symmetric, even or quadratic forms, see Theorem 1.2.18. A similar description can be obtained for the corresponding *odd* L-groups of degrees ≤ 1 in terms of symmetric (in degree 1), even (in degree -1) and quadratic (in odd degrees ≤ -3) *formations*. We leave the details to the motivated reader.

1.2.14. **Remark.** When M is free as an R-module, the results of this section apply equally well if we restrict attention to the full subcategory $\mathcal{D}^{\mathrm{f}}(R) \subseteq \mathcal{D}^{\mathrm{p}}(R)$ of finitely generated complexes. Indeed, in the proof of Proposition 1.2.3 we may simply choose P to be free, in which case the algebraic surgery procedure stays within $\mathcal{D}^{\mathrm{f}}(R)$. In addition, the connectivity bounds obtained by that proposition translate into the same type of representation by complexes concentrated in certain intervals, only that now these complexes consist of stably free modules, see Remark 1.1.8. For example, if $\Qoppa$ is a 0-quadratic Poincaré structure then any element of $\mathrm{L}_n(\mathcal{D}^{\mathrm{f}}(R), \Qoppa)$ with $n \geq 0$ can be represented by a Poincaré form on a complex of stable free R-modules concentrated in degrees $[-n, 0]$, and such a Poincaré complex represents zero if and only if it admits a Lagrangian represented by a complex of stable free modules concentrated in degrees $[-n-1, 0]$. Similarly, the proof of Proposition 1.2.10 can be run verbatim with stably free modules instead of projective modules, and so we get that in the situation of that proposition, $\mathrm{L}_0(\mathcal{D}^{\mathrm{f}}(R), \Qoppa)$ is isomorphic to the corresponding Witt groups of stably free Poincaré objects. More generally, one can take any intermediate subcategory $\mathcal{D}^{\mathrm{f}}(R) \subseteq \mathcal{C} \subseteq \mathcal{D}^{\mathrm{p}}(R)$ which is closed under the duality. A typical such $\mathcal{C}$ is the full subcategory of objects whose class in $\mathrm{K}_0(R)$ lies in a given involution-closed subgroup of $\mathrm{K}_0(R)$. We will consider this framework again in §2 when we will discuss *control* on GW and L spectra.

Genuine symmetric L-theory. In this subsection, we use the previous surgery results to identify the genuine symmetric L-groups

$$\mathrm{L}^{\mathrm{gs}}_n(R; M) := \mathrm{L}_n(R; \Qoppa^{\mathrm{gs}}_M)$$

with Ranicki's original definition of symmetric L-groups, which we recall now.

We let $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Proj}(R))$ be the category of bounded chain complexes of finitely generated projective R-modules. We will say that $C \in \mathrm{Ch}^{\mathrm{b}}(\mathrm{Proj}(R))$ is *n-dimensional* if it is concentrated in the range $[0, n]$, that is, if $C_i = 0$ whenever $i < 0$ or $i > n$. Recall that the ∞-category $\mathcal{D}^{\mathrm{p}}(R)$ of perfect left R-modules can be identified with the ∞-categorical localisation $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Proj}(R))[W^{-1}]$ of $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Proj}(R))$ with respect to the collection W of quasi-isomorphisms.

1.2.15. **Definition.** We let $n \geq 0$ be a non-negative integer. An *n-dimensional Poincaré complex* in the sense of [Ran80, §1] is a pair (C, q) where $C \in \mathrm{Ch}^{\mathrm{b}}(\mathrm{Proj}(R))$ is an n-dimensional complex and q is an element of $\mathrm{H}_n(\mathcal{H}om_R(\mathrm{D}C, C)^{\mathrm{hC}_2})$ whose image in $\mathrm{H}_n(\mathcal{H}om_R(\mathrm{D}C, C)) = [\mathrm{D}(C)[n], C]$ is an isomorphism $\mathrm{D}(C)[n] \to C$ in the homotopy category of $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Proj}(R))$. Here, $\mathcal{H}om_R(-,-)$ denotes the internal Hom complex and $(-)^{\mathrm{hC}_2}$ is the homotopy fixed point construction (described explicitly in [Ran80] using the standard projective resolution of $\mathbb{Z}$ as a trivial C_2-module). An $(n+1)$*-dimensional Poincaré pair* is a pair (f, η) where $f : C \to C'$ is a map in $\mathrm{Ch}^{\mathrm{b}}(\mathrm{Proj}(R))$ from an n-dimensional complex to an $(n+1)$-dimensional complex and η is an element of

$$\mathrm{H}_n(\mathrm{fib}[\mathcal{H}om_R(\mathrm{D}C, C) \to \mathcal{H}om_R(\mathrm{D}C', C')]^{\mathrm{hC}_2})$$

whose respective images in $[\mathrm{D}(C)[n], C]$ and $[\mathrm{cof}(\mathrm{D}C' \to \mathrm{D}C)[n], C']$ are isomorphisms in the homotopy category. Every Poincaré pair $(C \to C', \eta)$ determines, in particular, a Poincaré complex $(C, \eta|_C)$, and we say that a Poincaré complex is *null-cobordant* if it is obtained in this way. Similarly, two n-dimensional Poincaré complexes $(C, q), (C', q')$ are said to be *cobordant* if $(C \oplus C', q \oplus -q')$ is null-cobordant. The set of equivalence classes of n-dimensional Poincaré complexes modulo the cobordism relation above forms an abelian group $\mathrm{L}^{\mathrm{short}}_n(R; M)$ under direct sum, with the inverse of (C, q) given by $(C, -q)$. We will refer to these groups as the *short* symmetric L-groups of R.

1.2.16. **Remark.**

i) The groups $\mathrm{L}^{\mathrm{short}}_n(R; M)$ are formally defined in [Ran80] only when M is of the form $R(\epsilon)$ for some ϵ-involution on R. However, the definition only makes use of the induced duality on chain complexes and it therefore makes sense for any invertible module M with involution.

ii) In [Ran80] Ranicki extends the definition of the classical symmetric L-groups to negative integers as follows:

$$\mathrm{L}^{\mathrm{short}}_n(R;M) = \begin{cases} \mathrm{L}^{\mathrm{short,ev}}_{n+2}(R;-M) & \text{for } n=-2,-1 \\ \mathrm{L}^{\mathrm{q}}_n(R;M) & \text{for } n\leq -3 \end{cases}$$

Here $\mathrm{L}^{\mathrm{short,ev}}_n(R;M)$ are the even L-groups from [Ran80, §3]. We recall that an n-dimensional Poincaré complex (C,φ) is called even in [Ran80] if a certain Wu class $v_0(\phi)\colon \mathrm{H}^n(C)\to\hat{\mathrm{H}}^0(\mathrm{C}_2;M)$ vanishes. Likewise, an $(n+1)$-dimensional Poincaré pair $f\colon C\to C'$ is called even if its relative Wu class $\mathrm{H}^{n+1}(f)\to\hat{\mathrm{H}}^0(\mathrm{C}_2;M)$ vanishes. The short even L-groups $\mathrm{L}^{\mathrm{short,ev}}_n(R;M)$ are then the cobordism groups of n-dimensional even complexes. For $n\geq 0$ we will also show that they are equivalent to our genuine even L-groups, see the proof of Theorem 1.2.18.

1.2.17. **Remark.** Two Poincaré complexes $(C,q),(C',q')$ are said to be quasi-isomorphic if there exists a quasi-isomorphism $f\colon C\to C'$ such that $f_*q=q'$. This yields an equivalence relation which is finer than cobordism: Given a quasi-isomorphism $f\colon (C,q)\to(C',q')$ one can construct a Poincaré pair of the form $(\mathrm{id},f)\colon C\to C\oplus C'$ witnessing (C,q) and (C',q') as cobordant. In particular, if $\mathrm{Pn}^{\mathrm{short}}_n(R)$ denotes the monoid of quasi-isomorphism classes of n-dimensional Poincaré complexes and $\mathcal{M}^{\mathrm{short}}_n(R)$ the monoid of n-dimensional Poincaré pairs, then $\mathrm{L}^{\mathrm{short}}_n(R;M)$ is naturally isomorphic to the cokernel of $\mathcal{M}^{\mathrm{short}}_n(R)\to\mathrm{Pn}^{\mathrm{short}}_n(R)$ in the category of commutative monoids.

The remainder of this subsection is devoted to a proof of the following theorem.

1.2.18. **Theorem.** *Let R be a ring and M an invertible module with involution over R. Then for all integers n, there is a natural isomorphism*

$$\mathrm{L}^{\mathrm{short}}_n(R;M)\cong\mathrm{L}^{\mathrm{gs}}_n(R;M)$$

between Ranicki's classical symmetric L-groups and the genuine symmetric L-groups.

The proof will proceed in several steps. We first compare, for $n\geq 0$, Ranicki's classical L-group $\mathrm{L}^{\mathrm{short}}_n(R;M)$ to the group $\mathrm{L}^{n,n+1}_n(R;\mathrm{Ϙ}^{\mathrm{s}}_M)$ of Definition 1.2.1 as follows. Let $\mathcal{I}_n\subseteq\mathrm{Ch}^{\mathrm{b}}(\mathrm{Proj}(R))$ be the subcategory consisting of the n-dimensional complexes and quasi-isomorphisms between them, and let $\mathcal{J}_{[0,n]}\subseteq\mathcal{D}^{\mathrm{p}}(R)^{\simeq}$ be the full sub-∞-groupoid spanned by those perfect R-modules which can be represented by a complex in $\mathrm{Proj}(R)$ concentrated in degrees $[0,n]$. The canonical localisation map then restricts to a functor $\pi\colon\mathcal{I}_n\to\mathcal{J}_{[0,n]}$, and it induces isomorphisms

$$\rho\colon \mathrm{H}_n(\mathcal{H}om_R(\mathrm{D}(C),C)^{\mathrm{hC}_2})\xrightarrow{\cong}\pi_n(\hom_R(\mathrm{D}(\pi C),\pi C)^{\mathrm{hC}_2})\cong\pi_0\Omega^n\mathrm{Ϙ}^{\mathrm{s}}_M(\pi\mathrm{D}C)$$

natural in the object C of $\mathcal{I}_n$. We can then define a map of sets

$$\mathrm{Pn}^{\mathrm{short}}_n(R)\longrightarrow\pi_0\mathrm{Pn}^n_n(R;\mathrm{Ϙ}^{\mathrm{s}}_M)$$

by sending an n-dimensional Poincaré complex (C,q) to the component determined by $(\pi(\mathrm{D}C),\rho(q))$. This map is in fact an isomorphism, since the functor $\pi\colon\mathcal{I}_n\to\mathcal{J}_{[0,n]}$ is an equivalence on homotopy categories, and ρ is an isomorphism. Since the localisation functor $\mathrm{Ch}^{\mathrm{b}}(R)\to\mathcal{D}^{\mathrm{p}}(R)$ preserves direct sums this is moreover an isomorphism of monoids.

1.2.19. **Proposition.** *For every $n\geq 0$, the previously defined map induces a group isomorphism*

$$\mathrm{L}^{\mathrm{short}}_n(R;M)\cong\mathrm{L}^{n,n+1}_n(R;\mathrm{Ϙ}^{\mathrm{s}}_M).$$

Proof. By replacing $\mathrm{Ch}^{\mathrm{b}}(R)$ and $\mathcal{D}^{\mathrm{p}}(R)$ by their arrow categories a similar construction provides a morphism of monoids

$$\mathcal{M}^{\mathrm{short}}_n(R)\longrightarrow\pi_0\mathcal{M}^{n,n+1}_n(R;\mathrm{Ϙ}^{\mathrm{s}}_M)$$

which is compatible with the morphism $\mathrm{Pn}^{\mathrm{short}}_n(R)\to\pi_0\mathrm{Pn}^n_n(R;\mathrm{Ϙ}^{\mathrm{s}}_M)$. Thus we obtain a well-defined group homomorphism on L-groups. Since every arrow in $\mathcal{D}^{\mathrm{p}}(R)$ can be lifted to a map of chain complexes, an argument similar to the one above shows that this map is also surjective. This suffices to induce an isomorphism on quotients. □

The next step for the proof of Theorem 1.2.18 is to see that on n-dimensional complexes, the datum of a symmetric form is the same as the datum of a genuine symmetric form. We record here the corresponding statement for L-groups, For the following result, we keep in mind that $Ϙ^{\mathrm{gs}}_M$ is 2-symmetric:

1.2.20. **Lemma.** *Let $Ϙ$ be r-symmetric for $r \in \mathbb{Z}$. Let $a, b, n \in \mathbb{Z}$ be as in Definition 1.2.1, and suppose additionally that $a \leq n + 2r - 4$ and $b \leq n + 2r - 3$. Then the map $\mathrm{L}^{a,b}_n(R; Ϙ) \to \mathrm{L}^{a,b}_n(R; Ϙ^{\mathrm{s}}_M)$ is an isomorphism. In particular, the map*

$$\mathrm{L}^{n,n+1}_n(R; Ϙ^{\mathrm{gs}}_M) \longrightarrow \mathrm{L}^{n,n+1}_n(R; Ϙ^{\mathrm{s}}_M)$$

is an isomorphism for every $n \geq 0$.

Proof. It will suffice to show that the monoid homomorphisms

$$\pi_0\mathrm{Pn}^a_n(R; Ϙ) \to \pi_0\mathrm{Pn}^a_n(R; Ϙ^{\mathrm{s}}_M) \quad \text{and} \quad \pi_0\mathcal{M}^{a,b}_n(R; Ϙ) \to \pi_0\mathcal{M}^{a,b}_n(R; Ϙ^{\mathrm{s}}_M)$$

are isomorphisms. Now the left homomorphism is an isomorphism since $Ϙ$ is r-symmetric and so the map $\Omega^{\infty+n}Ϙ(X) \to \Omega^{\infty+n}Ϙ^{\mathrm{s}}_M(X)$ is an equivalence by Remark 1.1.3 whenever X is $\left(\frac{-n-a}{2}\right)$-connective, taking into account that $-r - \frac{-n-a}{2} = \frac{-2r+n+a}{2} \leq n-2$ by our assumption. Concerning the right map, it will suffice to show that whenever $L \to X$ is such that L is $\lceil\frac{-n-1-b}{2}\rceil$-connective and X is $\left(\frac{-n-a}{2}\right)$-connective the map

$$Ϙ_{\mathrm{Met}}(L \to X) = \mathrm{fib}[Ϙ(X) \to Ϙ(L)] \to \mathrm{fib}[Ϙ^{\mathrm{s}}_M(X) \to Ϙ^{\mathrm{s}}_M(L)] = Ϙ^{\mathrm{s}}_{\mathrm{Met}}(L \to X)$$

has an $(n-2)$-truncated fibre. Equivalently, this is the same as saying that the square

$$\begin{array}{ccc} Ϙ(X) & \longrightarrow & Ϙ^{\mathrm{s}}_M(X) \\ \downarrow & & \downarrow \\ Ϙ(L) & \longrightarrow & Ϙ^{\mathrm{s}}_M(L) \end{array}$$

has $(n-2)$-truncated total fibre. As in the first part of the proof, we find that the top horizontal map is $(n-2)$-truncated and the bottom horizontal map is $(n-1)$-truncated since L is $\lceil\frac{-n-1-b}{2}\rceil$-connective and $-r - \lceil\frac{-n-1-b}{2}\rceil \leq \frac{-2r+n+1+b}{2} \leq n-1$. □

Proof of Theorem 1.2.18. First, we consider the case $n \geq 0$ where we simply combine the isomorphisms

$$\mathrm{L}^{\mathrm{short}}_n(R; M) \cong \mathrm{L}^{n,n+1}_n(R; Ϙ^{\mathrm{s}}_M) \cong \mathrm{L}^{n,n+1}_n(R; Ϙ^{\mathrm{gs}}_M) \cong \mathrm{L}_n(R; Ϙ^{\mathrm{gs}}_M)$$

of Proposition 1.2.19, Lemma 1.2.20 and Corollary 1.2.7, respectively.

The case $n \leq -3$ is covered by Corollary 1.2.8. It then suffices to treat the case $n = -2, -1$. Here, we use the "periodicity"

$$\mathrm{L}^{\mathrm{gs}}_n(R; M) \cong \mathrm{L}^{\mathrm{ge}}_{n+2}(R; -M)$$

of Corollary R.10 and will now argue more generally that for $n \geq 0$, a canonical map $\mathrm{L}^{\mathrm{short,ev}}_n(R; M) \to \mathrm{L}^{\mathrm{ge}}_n(R; M)$ is an isomorphism. To construct the map, we consider an n-dimensional even complex (C, φ) and obtain from the construction preceding Proposition 1.2.19 a canonical element (X, q) of $\mathrm{L}^{n,n+1}_n(R; Ϙ^{\mathrm{gs}}_M)$, represented by $\mathrm{D}C$. We want to argue that (X, q) refines to an element of $\mathrm{L}^{n,n+1}_n(R; Ϙ^{\mathrm{ge}}_M)$. Let us consider the fibre sequence

$$\Omega^n Ϙ^{\mathrm{ge}}(X) \longrightarrow \Omega^n Ϙ^{\mathrm{gs}}(X) \longrightarrow \hom_R(X; \hat{\mathrm{H}}^0(\mathrm{C}_2; M)[-n]).$$

We note the equivalence $\Omega^\infty \hom_R(X; \hat{\mathrm{H}}^0(\mathrm{C}_2; M)[-n]) \simeq \mathrm{Hom}_R(\mathrm{H}_{-n}(X), \hat{\mathrm{H}}^0(\mathrm{C}_2; M))$. Tracing through the definitions, the symmetric structure q is sent to the Wu class $v_0(\varphi)$ which is zero by assumption. We deduce that (X, q) canonically refines to an element of $\mathrm{L}^{n,n+1}_n(R; Ϙ^{\mathrm{ge}}_M)$. Likewise, an $(n+1)$-dimensional even pair gives rise to a genuine even structure on the associated cobordism. Reversing the above argument, we deduce that the map

$$\mathrm{L}^{\mathrm{short,ev}}_n(R; M) \longrightarrow \mathrm{L}^{n,n+1}_n(R; Ϙ^{\mathrm{ge}}_M)$$

is an isomorphism. Combining this with the isomorphism

$$\mathrm{L}^{n,n+1}_n(R; Ϙ^{\mathrm{ge}}_M) \longrightarrow \mathrm{L}_n(R; Ϙ^{\mathrm{ge}}_M)$$

obtained from Proposition 1.2.3 just as Corollary 1.2.7, the theorem follows. □

1.2.21. **Remark.** In [Ran80] Ranicki also defines, for $n \geq 0$, the n'th *quadratic* L-*group*, using quadratic Poincaré complexes of dimension n. The same argument as above shows that this group coincides with $\mathrm{L}_n^{n,n+1}(R;\Omega^{\mathrm{q}}_M)$, and hence with $\mathrm{L}_n(R;\Omega^{\mathrm{q}}_M)$ by Proposition 1.2.3. As shown in [Ran80], these are also the same as the quadratic L-groups of Wall [Wal99], which arise in manifold theory as the natural recipient of surgery obstructions. We warn the reader that these groups do not agree with $\mathrm{L}_n(R;\Omega^{\mathrm{gq}}_M)$, which by periodicity are isomorphic to $\mathrm{L}_{n-4}(R;\Omega^{\mathrm{gs}}_M)$.

Surgery for connective ring spectra. In this section we apply the surgery arguments previously developed to the case where R is a connective ring spectrum (that is a connective $\mathcal{E}_1$-algebra in the monoidal ∞-category of spectra). The perfect derived category is then replaced with the ∞-category Mod^{ω}_R of compact R-module spectra, and finitely generated projective modules with the full subcategory $\mathrm{Proj}(R)$ of Mod^{ω}_R consisting of the retracts of finitely generated free modules, i.e. those equivalent to $R^{\oplus n}$ for some n. The duality underlying a Poincaré structure Ω on Mod^{ω}_R is then induced by an invertible module with involution M as defined in Definition [I].3.1.1. We refer to §[I].3 for a complete treatment of Poincaré structures on categories of module spectra.

As in Definition 1.1.2, we say that Ω is m-quadratic if the cofibre of the map

$$\Omega^{\mathrm{q}}_M(X) = \hom_{R\otimes R}(X \otimes X, M)_{\mathrm{hC}_2} \longrightarrow \Omega(X)$$

sends R, or equivalently the category $\mathrm{Proj}(R)$, to m-connective spectra. Here and in the rest of the section $\otimes$ denotes the tensor product over the sphere spectrum $\mathbb{S}$, or in other words the smash product of spectra. When R and M are discrete Ω^{q}_M on $\mathrm{Mod}^{\omega}_{\mathrm{H}R}$ corresponds to the quadratic Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$ under the equivalence $\mathcal{D}^{\mathrm{p}}(R) \simeq \mathrm{Mod}^{\omega}_{\mathrm{H}R}$.

We notice that the definition of the groups $\mathrm{L}_n^{a,b}(R;\Omega)$ carry over to this more general situation, as they only make use of the notion of connectivity for objects of Mod^{ω}_R.

1.2.22. **Remark.** Since a connective module over a connective ring spectrum is projective if and only if its dual is also connective, the proof of Proposition 1.2.3 applies equally well to the case where R is a connective ring spectrum. As a result, one obtains for instance that

$$\mathrm{L}_n^{a,b}(R;\Omega) \longrightarrow \mathrm{L}_n(R;\Omega)$$

is an isomorphism for $n \leq 2m$ if $a, b \in \{0, 1\}$ are such that $n + a$ and $n + 1 + b$ are even, and hence that for $n \leq 2m - 3$ the canonical map

$$\mathrm{L}_n(R;\Omega^{\mathrm{q}}_M) \longrightarrow \mathrm{L}_n(R;\Omega)$$

is an isomorphism, as in Corollary 1.2.8.

One may for instance apply this for the universal Poincaré structure Ω^{u} on $\mathrm{Mod}^{\omega}_{\mathbb{S}}$ from Example [I].1.2.15, to obtain that $\mathrm{L}^{\mathrm{q}}_n(\mathbb{S}) \to \mathrm{L}^{\mathrm{u}}_n(\mathbb{S})$ is an equivalence in degrees ≤ -3. Interestingly, since $\mathbb{S}^{\mathrm{tC}_2} \simeq \mathbb{S}^{\wedge}_2$ is connective by Lin's Theorem [Lin80], the symmetric Poincaré structure Ω^{s} on $\mathrm{Mod}^{\omega}_{\mathbb{S}}$ is also 0-quadratic. Thus, the map $\mathrm{L}^{\mathrm{q}}_n(\mathbb{S}) \to \mathrm{L}^{\mathrm{s}}_n(\mathbb{S})$ is also an isomorphism for $n \leq -3$. Both of these observations also follow from work of Weiss-Williams [WW14] who give an explicit formula for the cofibre of the maps in question.

Using such surgery methods, we obtain the following results, see also [Lur11, Lecture 14] for a proof of the algebraic π-π-theorem, Corollary 1.2.24i) below. We say that a map of spectra is k-connective for some $k \in \mathbb{Z}$ if its fibre is. We also write Λ_{Ω} for the linear approximation of a quadratic functor Ω.

1.2.23. **Proposition.** *Let $f\colon R \to S$ be a 1-connective map of connective ring spectra and Ω_R and Ω_S be m-quadratic Poincaré structures on Mod^{ω}_R and Mod^{ω}_S respectively. Suppose that the extension of scalars functor $f_!$ is enhanced to a Poincaré functor $(\mathrm{Mod}^{\omega}_R, \Omega_R) \to (\mathrm{Mod}^{\omega}_S, \Omega_S)$ such that the induced map $\Lambda_{\Omega_R}(R) \to \Lambda_{\Omega_S}(S)$ is $(m+1)$-connective. Then the induced map*

$$\mathrm{L}_n(R;\Omega_R) \longrightarrow \mathrm{L}_n(S;\Omega_S) \tag{5}$$

is an isomorphism for $n \leq 2m$ and surjective for $n = 2m + 1$.

Proof. First, we observe that for every $r \leq m$ and every projective R-module P, the map $Ϙ_R(P[-r]) \to Ϙ_S(f_!P[-r])$ is $(2r+1)$-connective: Consider the diagram of horizontal cofibre sequences

$$\begin{array}{ccccc} Ϙ^{\mathrm{q}}_R(P[-r]) & \longrightarrow & Ϙ_R(P[-r]) & \longrightarrow & \Lambda_{Ϙ_R}(P)[r] \\ \downarrow & & \downarrow & & \downarrow \\ Ϙ^{\mathrm{q}}_S(f_!P[-r]) & \longrightarrow & Ϙ_S(f_!P[-r]) & \longrightarrow & \Lambda_{Ϙ_S}(f_!P)[r] \end{array}$$

in which the left vertical map is $(2r+1)$-connective by inspection: By writing P as a retract of a free module $R^{\oplus n}$, we may assume that P is itself free, in which case it follows immediately from the assumption that f is 1-connective. Furthermore, the right vertical map is $(r+m+1)$-connective by assumption. Since $r \leq m$ we have $2r+1 \leq r+m+1$ and the claim follows.

Let us now discuss the bijectivity of the map (5). Suppose first that $n = 2k$ is even, and let (X, q) represent an element of $\mathrm{L}_{2n}(S; Ϙ_S)$. Since $n \leq 2m$ the surgery step above allows us to assume that $X = P'[-k]$ for some projective S-module P'. We can find a projective R-module P such that $f_!(P) = P'$ by the assumption that f is a π_0-isomorphism. Also, $n = 2k \leq 2m$, so that $k \leq m$ and we find that

$$Ϙ_R(P[-k]) \longrightarrow Ϙ_S(P'[-k])$$

is $(2k+1)$-connective, and hence surjective on π_{2k} (in fact an isomorphism) by our first observation. We may thus lift the form q to a form on P which is automatically Poincaré (as this may be tested after base change to $\pi_0 R \cong \pi_0 S$), and we deduce surjectivity. Next, we show that it is also injective. So let (X, q) be a Poincaré object for $Ϙ_R[-2k]$ which is sent to zero in $\mathrm{L}_{2k}(S; Ϙ_S)$. We may assume that $X = P[-k]$ for some projective R-module P, and set $P'[-k] := f_!X$. Using an argument similar to that of Corollary 1.2.10, we may assume that its image $(P'[-k], q')$ admits a Lagrangian $L'[-k] \to P'[-k]$ which is itself of the form $L' = f_!L$ for a projective S-module L. We can then lift the map $L'[-k] \to P'[-k]$ to a map $L[-k] \to P[-k]$ as f is a π_0-isomorphism. Since the map $Ϙ_R(L[-k]) \to Ϙ_S(L'[-k])$ is injective on π_{2k}, we deduce that the form q restricted to $L[-k]$ is 0. We obtain an induced null homotopy of the composite $L[-k] \to P[-k] \to (\mathrm{D}L)[-k]$ which becomes a fibre sequence after applying $f_!$. It follows that it is in fact a fibre sequence before applying $f_!$, so that $L[-k]$ is indeed a Lagrangian. We conclude that for $n = 2k \leq 2m$, the map (5) is an isomorphism as claimed.

We now turn to the case $n = 2k+1$. By the above surgery arguments, we find that for the maps

$$\mathrm{L}^{1,0}_{2k+1}(R; Ϙ_R) \longrightarrow \mathrm{L}^{1,2}_{2k+1}(R; Ϙ_R) \longrightarrow \mathrm{L}_{2k+1}(R; Ϙ_R)$$

the composite is an isomorphism provided $2k+1 \leq 2m$ and the latter map is an isomorphism for $2k+1 = 2m+1$, and likewise for S. In order to prove surjectivity of the map (5) we may thus represent an element of $\mathrm{L}_{2k+1}(S; Ϙ_S)$ by a Poincaré object (X', q') where X' is the cofibre of a map $P'[-k-1] \to Q'[-k-1]$, for some P', Q' in $\mathrm{Proj}(S)$. We can lift this map to a map $P[-k-1] \to Q[-k-1]$, and set X to be its cofibre. We wish to argue that the map $Ϙ_R(X) \to Ϙ_S(X')$ is surjective on π_{2k+1}. We consider the diagram

$$\begin{array}{ccccc} Ϙ^{\mathrm{q}}_R(X) & \longrightarrow & Ϙ_R(X) & \longrightarrow & \Lambda_{Ϙ_R}(X) \\ \downarrow & & \downarrow & & \downarrow \\ Ϙ^{\mathrm{q}}_S(X') & \longrightarrow & Ϙ_S(X') & \longrightarrow & \Lambda_{Ϙ_S}(X') \end{array}$$

where the left vertical map is $(2k+1)$-connective. The right vertical map is the fibre of two maps which are each $(m+k+2)$-connective, and is consequently $(m+k+1)$-connective. Again, $2k+1 \leq m+k+1$ as we have assumed $2k+1 \leq 2m+1$. Thus $Ϙ_R(X) \to Ϙ_S(X')$ is $(2k+1)$-connective, and we can again lift the form on X' to a form on X which is automatically Poincaré. The surjectivity claim is hence proven.

It remains to prove that the map (5) is also injective. So let (X, q) be a Poincaré object over R, without loss of generality assume that $X = \mathrm{cof}(P[-k-1] \to Q[-k-1])$ and that $X' = f_!(X)$ admits a Lagrangian $L'[-k-1] \to X'$; it is here where we use that $2k+1 \leq 2m$, in the case $2k+1 = 2m+1$ we can in general not push the Lagrangian into a single degree. Now we recall from the algebraic Thom construction that the Lagrangian $L'[-k-1] \to X'$ is determined by an induced (possibly degenerate) form on $\mathrm{fib}(L'[-k-1] \to X')$ with respect to the Poincaré structure $Ϙ_S[-2k-2]$. We find that this fibre is given by $N'[-k-1]$ for a projective S-module N' (namely $\mathrm{D}L'$). We may then lift N' to a projective R-module N shifted in degree $-k-1$. Moreover, since $k+1 \leq m$, we find that the map

$$\Qoppa_R(N[-k-1]) \longrightarrow \Qoppa_S(N'[-k-1])$$

is $(2(k+1)+1)$-connective and hence surjective on π_{2k+2} as needed. The proposition follows. □

We can now apply Proposition 1.2.23 to compare the L-spectra of some canonical Poincaré structures over sphere spectrum and over the integers. We compare the universal Poincaré structure $\Qoppa^{\mathrm{u}}_{\mathbb{S}}$ on $\mathrm{Mod}^{\omega}_{\mathbb{S}}$ from §[I].4.1, the Tate Poincaré structure $\Qoppa^{\mathrm{t}}_{\mathbb{Z}}$ on $\mathcal{D}^{\mathrm{p}}(\mathbb{Z})$ from Example [I].3.2.11, and the Burnside Poincaré structure $\Qoppa^{\mathrm{b}}_{\mathbb{Z}}$ on $\mathcal{D}^{\mathrm{p}}(\mathbb{Z})$. These are defined respectively by the pullback squares

$$\begin{array}{ccc} \Qoppa^{\mathrm{u}}_{\mathbb{S}}(X) & \longrightarrow & \hom_{\mathbb{S}}(X,\mathbb{S}) \\ \downarrow & & \downarrow \\ \Qoppa^{\mathrm{s}}_{\mathbb{S}}(X) & \longrightarrow & \hom_{\mathbb{S}}(X,\mathbb{S}^{\mathrm{tC}_2}) \end{array} \qquad \begin{array}{ccc} \Qoppa^{\mathrm{t}}_{\mathbb{Z}}(Y) & \longrightarrow & \hom_{\mathbb{Z}}(Y,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \Qoppa^{\mathrm{s}}_{\mathbb{Z}}(Y) & \longrightarrow & \hom_{\mathbb{Z}}(Y,\mathbb{Z}^{\mathrm{tC}_2}) \end{array} \qquad \begin{array}{ccc} \Qoppa^{\mathrm{b}}_{\mathbb{Z}}(Y) & \longrightarrow & \hom_{\mathbb{Z}}(Y,\tau_{\geq 1/2}\mathbb{Z}^{\mathrm{tC}_2}) \\ \downarrow & & \downarrow \\ \Qoppa^{\mathrm{s}}_{\mathbb{Z}}(Y) & \longrightarrow & \hom_{\mathbb{Z}}(Y,\mathbb{Z}^{\mathrm{tC}_2}) \end{array}$$

where the right-hand vertical maps in the first two diagrams are induced by the Tate-valued Frobenii of the $\mathcal{E}_\infty$-rings $\mathbb{S}$ and $H\mathbb{Z}$. In the third diagram we have denoted

$$\tau_{\geq 1/2}\mathbb{Z}^{\mathrm{tC}_2} := (\tau_{\geq 0}\mathbb{Z}^{\mathrm{tC}_2}) \times_{\mathrm{H}\mathbb{Z}/2} \mathrm{H}\mathbb{Z},$$

and the name Burnside indicates the fact that $\Qoppa^{\mathrm{b}}(\mathbb{Z})$ is the Burnside ring of C_2. There are canonical Poincaré functors

$$(\mathrm{Mod}^{\omega}_{\mathbb{S}}, \Qoppa^{\mathrm{u}}_{\mathbb{S}}) \longrightarrow (\mathcal{D}^{\mathrm{p}}(\mathbb{Z}), \Qoppa^{\mathrm{t}}_{\mathbb{Z}}) \longrightarrow (\mathcal{D}^{\mathrm{p}}(\mathbb{Z}), \Qoppa^{\mathrm{b}}_{\mathbb{Z}}) \longrightarrow (\mathcal{D}^{\mathrm{p}}(\mathbb{Z}), \Qoppa^{\mathrm{gs}}_{\mathbb{Z}})$$

where the first functor is the base change along $\mathbb{S} \to \mathrm{H}\mathbb{Z}$, and the second Poincaré functor is induced by the Tate valued Frobenius of $\mathrm{H}\mathbb{Z}$. We denote the corresponding L-spectra by $\mathrm{L}^{\mathrm{u}}(\mathbb{S})$, $\mathrm{L}^{\mathrm{t}}(\mathbb{Z})$ and $\mathrm{L}^{\mathrm{b}}(\mathbb{Z})$, respectively. The connective cover of $\mathrm{L}^{\mathrm{b}}(\mathbb{Z})$ is equivalent to the spectrum $\mathrm{L}^{\mathrm{g}}(\mathbb{A})$ considered in [DO19].

1.2.24. **Corollary.** *For any connective ring spectrum R we have that:*

i) *The map $\mathrm{L}^{\mathrm{q}}(R) \to \mathrm{L}^{\mathrm{q}}(\pi_0 R)$ is an equivalence.*
ii) *The map $\mathrm{L}^{\mathrm{u}}_n(\mathbb{S}) \to \mathrm{L}^{\mathrm{t}}_n(\mathbb{Z})$ is an isomorphism for $n \leq 0$ and surjective for $n = 1$.*
iii) *The map $\mathrm{L}^{\mathrm{u}}_n(\mathbb{S}) \to \mathrm{L}^{\mathrm{b}}_n(\mathbb{Z})$ is an isomorphism for $n \leq 0$ and surjective for $n = 1$.*
iv) *The map $\mathrm{L}^{\mathrm{u}}_n(\mathbb{S}) \to \mathrm{L}^{\mathrm{gs}}_n(\mathbb{Z})$ is an isomorphism for $n \leq -1$ and surjective for $n = 0$.*

Proof. All claims follow from Proposition 1.2.23. For i) we observe that $\Qoppa^{\mathrm{q}}_R$ and $\Qoppa^{\mathrm{q}}_{\pi_0 R}$ are m-quadratic for every m. Furthermore the linear terms vanish, and therefore they are equivalent. For ii) notice that both $\Qoppa^{\mathrm{u}}_{\mathbb{S}}$ and $\Qoppa^{\mathrm{t}}_{\mathbb{Z}}$ are 0-quadratic, and that the map on linear terms evaluated at the sphere spectrum is the map $\mathbb{S} \to \mathrm{H}\mathbb{Z}$ which is 1-connective. For iii) we observe that $\Qoppa^{\mathrm{t}}_{\mathbb{Z}}$ is also 0-quadratic, and that the map on linear terms is $\mathbb{S} \to (\tau_{\geq 0}\mathbb{Z}^{\mathrm{tC}_2}) \times_{\mathrm{H}\mathbb{Z}/2} \mathrm{H}\mathbb{Z}$, which is 1-connective (notice that the target has trivial π_1). For iv), recall that also $\Qoppa^{\mathrm{gs}}$ is 0-quadratic and that the map on linear terms on the sphere spectrum is the map $\mathbb{S} \to \tau_{\geq 0}\mathbb{Z}^{\mathrm{tC}_2}$, which is 0-connective. We can then apply Proposition 1.2.23 for $m = -1$ which shows the claim for $n \leq -2$. We conclude that $\mathrm{L}^{\mathrm{u}}_{-2}(\mathbb{S}) = 0$ since $\mathrm{L}^{\mathrm{gs}}_{-2}(\mathbb{Z})$ is well-known to vanish, but see also Example 2.2.12 for a proof. The vanishing of $\mathrm{L}^{\mathrm{u}}_{-2}(\mathbb{S})$ was also shown in Proposition [II].4.6.4 by explicit means. In loc. cit. it is also argued how one can use the fibre sequence of Weiss and Williams [WW14, Theorem 4.5]

$$\mathrm{L}^{\mathrm{q}}(\mathbb{S}) \longrightarrow \mathrm{L}^{\mathrm{u}}(\mathbb{S}) \longrightarrow \mathbb{S} \oplus \mathrm{MTO}(1)$$

and i) to deduce that also $\mathrm{L}^{\mathrm{u}}_{-1}(\mathbb{S}) = 0$, so the map $\mathrm{L}^{\mathrm{u}}_n(\mathbb{S}) \to \mathrm{L}^{\mathrm{gs}}_n(\mathbb{Z})$ is a bijection up to degree -1. Here, $\mathrm{MTO}(1)$ denotes the Thom spectrum of $-\gamma$, where γ is the tautological bundle over $\mathrm{BO}(1)$. Namely, one uses that $\mathrm{MTO}(1)$ is (-1)-connective and that $\pi_{-1}(\mathrm{MTO}(1)) = \mathbb{Z}/2$. As $\pi_0(\mathrm{MTO}(1)) = 0$, we also deduce from the fibre sequence that $\mathrm{L}^{\mathrm{u}}_0(\mathbb{S}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and that the map in π_0 has target $\mathbb{Z}$ and hits 1. The low dimensional homotopy groups of $\mathrm{MTO}(1)$ are calculated by describing $\mathrm{MTO}(1)$ as the fibre of the transfer for the universal cover of $\mathrm{BO}(1)$, see again Proposition [II].4.6.4 for the details. □

1.2.25. **Remark.** We end this section with the observation that $\mathrm{L}^{\mathrm{u}}_2(\mathbb{S}) \to \mathrm{L}^{\mathrm{t}}_2(\mathbb{Z})$ is not an isomorphism. Using algebraic surgery, one can show that the map $\mathrm{L}^{\mathrm{q}}_2(\mathbb{S}) \to \mathrm{L}^{\mathrm{t}}_2(\mathbb{Z})$ is surjective. However, using the exact sequence

$$\mathrm{L}^{\mathrm{q}}_2(\mathbb{S}) \longrightarrow \mathrm{L}^{\mathrm{u}}_2(\mathbb{S}) \longrightarrow \pi_2(\mathbb{S} \oplus \mathrm{MTO}(1)) \longrightarrow 0$$

we deduce that the map $\mathrm{L}^{\mathrm{q}}_2(\mathbb{S}) \to \mathrm{L}^{\mathrm{u}}_2(\mathbb{S})$ is not surjective.

1.3. **Surgery for *r*-symmetric structures.** In this section, we prove a comparison result between genuine symmetric and symmetric L-theory. Algebraic surgery for symmetric Poincaré structures is not as straightforward as for the quadratic ones, and we will need to further assume that the base ring is left-coherent of finite left-global dimension. We recall that a ring is called left-coherent, if its finitely presented left modules form an abelian category. From what follows we omit the word *left* from the notation, and stress here that in the present section, commutativity of R is not needed. We note that Noetherian rings are coherent, and that a standard example of coherent but not necessarily Noetherian rings are valuation rings.

Let R be a ring and M an invertible module with involution over R. For an integer $d \geq 0$, we recall that R has global dimension $\leq d$ if every R-module N has a projective resolution of length at most d. When R is in addition coherent, one can find such a resolution where the modules are moreover finitely generated, provided N is finitely presented. In this case the connective cover functor $\tau_{\geq 0}$ and the truncation functor $\tau_{\leq 0}$ preserve perfect R-modules, and so $\mathcal{D}^{\mathrm{p}}(R)$ inherits from $\mathcal{D}(R)$ its Postnikov t-structure, so that $\mathcal{D}^{\mathrm{p}}(R)_{\geq 0}$ consists of the 0-connective perfect R-modules and $\mathcal{D}^{\mathrm{p}}(R)_{\leq 0}$ of the 0-truncated perfect R-modules. This uses that the lowest non-trivial homotopy group of a perfect complex is finitely presented. The duality $\mathrm{D}\colon \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \to \mathcal{D}^{\mathrm{p}}(R)$ induced by M interacts with the t-structure as follows:

$$\mathrm{D}(\mathcal{D}^{\mathrm{p}}(R)_{\geq 0}) \subseteq \mathcal{D}^{\mathrm{p}}(R)_{\leq 0} \quad \text{and} \quad \mathrm{D}(\mathcal{D}^{\mathrm{p}}(R)_{\leq 0}) \subseteq \mathcal{D}^{\mathrm{p}}(R)_{\geq -d}.$$

The first inclusion is immediate from Remark 1.1.7, and the second one follows from the Universal Coefficient spectral sequence computing $\mathrm{H}_* \hom_R(X, M)$, since $\mathrm{Ext}^i_R = 0$ for every $i \geq d+1$ as R has global dimension $\leq d$.

We will cast the algebraic surgery argument for symmetric Poincaré structures in the setting of a general Poincaré ∞-category $(\mathcal{C}, \Qoppa)$ equipped with a t-structure which interacts with the underlying duality $\mathrm{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ in the way described above. Similarly to Definition 1.1.2 (see also Remark 1.1.3), we will say that the Poincaré structure $\Qoppa$ is *r-symmetric* if the fibre of $\Qoppa(X) \to \Qoppa^{\mathrm{s}}_{\mathrm{D}}(X)$ is $(-r)$-truncated for every $X \in \mathcal{C}_{\geq 0}$, where

$$\Qoppa^{\mathrm{s}}_{\mathrm{D}}(X) = \hom_{\mathcal{C}}(X, \mathrm{D}X)^{\mathrm{hC}_2}$$

is the symmetric Poincaré structure associated to the duality D. Given integers $a, b \geq -1$ we define $\mathrm{Pn}^a_n(\mathcal{C}, \Qoppa)$, $\mathcal{M}^{a,b}_n(\mathcal{C}, \Qoppa)$ and $\mathrm{L}^{a,b}_n(\mathcal{C}, \Qoppa)$ as in Definition 1.2.1, where we interpret the connectivity requirement on Poincaré objects and Lagrangians as pertaining to the given t-structure on $\mathcal{C}$.

1.3.1. **Proposition** (Surgery for r-symmetric Poincaré structures)**.** *Let $\mathcal{C}$ be a stable ∞-category with a bounded t-structure $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$. Let $\Qoppa$ an r-symmetric Poincaré structure on $\mathcal{C}$ with duality $\mathrm{D}\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ such that $\mathrm{D}(\mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\geq -d}$ for some integer $d \geq 0$. Fix an $n \in \mathbb{Z}$ and let $a \geq d-1$, $b \geq d$ be integers with $b \geq a$, and such that*

- *$n + a$ is even, and*
- *$a \geq -n + 2d - 2r$.*

Then the canonical map $\mathrm{L}^{a,b}_n(\mathcal{C}, \Qoppa) \to \mathrm{L}_n(\mathcal{C}, \Qoppa)$ is an isomorphism.

Proof. We start with the surjectivity of the map in question. For this, it suffices to show that every Poincaré object $(X, q) \in \mathrm{Pn}(\mathcal{C}, \Qoppa^{[-n]})$ is cobordant to one whose underlying object is $(\frac{-n-a}{2})$-connective. Let $k = \frac{-n-a-2}{2}$, define $W := \Omega^n \mathrm{D}\tau_{\leq k} X$ and let

$$f\colon W \longrightarrow \Omega^n \mathrm{D}X \overset{q}{\simeq} X$$

be the map dual to the truncation map $X \to \tau_{\leq k} X$. Since $\mathrm{D}(\mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\geq -d}$ we have that W is $(-n-k-d)$-connective. Since $\mathrm{D}W \simeq \Sigma^n \tau_{\leq k} X$ is $(n+k)$-truncated we conclude that $\Omega^n \hom_{\mathcal{C}}(W, \mathrm{D}W)^{\mathrm{hC}_2}$ is $(n+2k+d)$-truncated. On the other hand, since $\Qoppa$ is r-symmetric, the fibre of the map

$$\Omega^n \Qoppa(W) \longrightarrow \Omega^n \Qoppa^{\mathrm{s}}_{\mathrm{D}}(W) = \Omega^n \hom_{\mathcal{C}}(W, \mathrm{D}W)^{\mathrm{hC}_2}$$

is $(k+d-r)$-truncated, so that $\Omega^n Ϙ(W)$ is $\max(n+2k+d, k+d-r)$-truncated. Spelling out the definition of k and using the estimates in the assumptions, we find that

$$\max(n+2k+d, k+d-r) < 0.$$

We hence get that $\Omega^n Ϙ(W)$ is (-1)-truncated and so $\Omega^{\infty+n}Ϙ(W) \simeq *$. The restriction of q to W is consequently null-homotopic, and we may therefore perform surgery along $f : W \to X$ to obtain a new Poincaré object (X', q'), given by the cofibre of the resulting map $W \to \tau_{\geq k+1}X$ (see diagram (4)). Since W is $(-n-k-d)$-connective it is in particular $(k+1)$-connective (since $-2k \geq n+a+2 \geq n+d+1$), and so X' is $(k+1)$-connective. Since $k+1 = \frac{-n-a}{2}$, surjectivity is shown.

To prove injectivity, we may represent an element of $\mathrm{L}_n^{a,b}(\mathcal{C}, Ϙ)$ by a Poincaré complex (X, q) such that X is $(\frac{-n-a}{2})$-connective. Such an element maps to zero in $\mathrm{L}(\mathcal{C}, Ϙ)$ if and only if it admits a Lagrangian $(L \to X, q, \eta)$. We need to verify that in this case (X, q) already represents the zero element in $\mathrm{L}_n^{a,b}(\mathcal{C}, Ϙ)$. Let $N := \mathrm{fib}(L \to X)$, so that $L \simeq \Omega^{n+1}\mathrm{D}N$. If L is $\lceil\frac{-n-1-b}{2}\rceil$-connective then N is $\lfloor\frac{-n-1-b}{2}\rfloor$-connective (since X is $(\frac{-n-a}{2})$-connective and $b \geq a$) and we are done. Otherwise, let $l = \lceil\frac{-n-1-b}{2}\rceil - 1$, define $N' := \Omega^{n+1}\mathrm{D}\tau_{\leq l}L$ and let

$$f : N' \longrightarrow N$$

be the map dual to the truncation map $L \to \tau_{\leq l}L$. We may view this map as a map $(N' \to 0) \to (L \to X)$ in the metabolic category, and we claim that it extends to a Lagrangian surgery datum for which it suffices to show that $Ϙ_{\mathrm{met}}(N' \to 0) \simeq \Omega Ϙ(N')$ is $(n-1)$-truncated. Since $\mathrm{D}(\mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\geq -d}$ we have that N' is $(-n-1-l-d)$- connective. Since $\mathrm{D}N' \simeq \Sigma^{n+1}\tau_{\leq l}L$ is $(n+1+l)$-truncated we have that $\Omega^{n+1}\hom_{\mathcal{C}}(N', \mathrm{D}N')^{\mathrm{hC}_2}$ is $(n+1+2l+d)$-truncated. On the other hand since $Ϙ$ is r-symmetric the fibre of the map

$$\Omega^{n+1}Ϙ(N') \longrightarrow \Omega^{n+1}\hom_{\mathcal{C}}(N', \mathrm{D}N')^{\mathrm{hC}_2}$$

is $(l+d-r)$-truncated, so that $\Omega^{n+1}Ϙ(N')$ is $\max(n+1+2l+d, l+d-r)$-truncated. Now by definition of l and the estimates in the assumptions we have that

$$\max(n+1+2l+d, l+d-r) < 0$$

We hence get that $\Omega^{n+1}Ϙ(N')$ is (-1)-truncated as needed. We may therefore perform Lagrangian surgery along $N' \to N$ to obtain a new Lagrangian $L' \to X$, such that L' is given by the cofibre of the resulting map $N' \to \tau_{\geq l+1}L$. Since N' is $(-n-1-l-d)$-connective it is in particular $(l+1)$-connective (since $-2l \geq n+b+2 \geq n+d+2$), and so L' is $(l+1)$-connective. Since $l+1 = \lceil\frac{-n-1-b}{2}\rceil$, it follows that the class (X, q) already represents zero in $\mathrm{L}_n^{a,b}(\mathcal{C}, Ϙ)$, and so we have established injectivity. The proposition is shown. □

1.3.2. **Remark.** Similarly to Remark 1.2.5, the surgery argument in the proof of Proposition 1.3.1 allows us to conclude that in the situation of that proposition, the sequence

$$\pi_0\mathcal{M}_n^{a,b}(\mathcal{C}, Ϙ) \longrightarrow \pi_0\mathrm{Pn}_n^a(\mathcal{C}, Ϙ) \longrightarrow \mathrm{L}_n^{a,b}(\mathcal{C}, Ϙ)$$

is exact in the middle, when a, b and n satisfy the assumptions of Proposition 1.3.1.

Unwinding the definitions, the case $d = 0$ of Proposition 1.3.1 gives the following result. Let $\mathcal{C}$ be a stable ∞-category with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ and a duality $\mathrm{D} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ such that D sends $\mathcal{C}_{\leq 0}$ to $\mathcal{C}_{\geq 0}$ and vice versa. In particular, D induces a duality $\mathrm{D}^\heartsuit : \mathcal{C}^\heartsuit \to \mathcal{C}^\heartsuit$ on the heart of $\mathcal{C}$, and we may consider $\mathcal{C}^\heartsuit$ an abelian category with duality. As in the previous section, we let $Ϙ^{\mathrm{gs}}_{\mathrm{D}^\heartsuit} : \mathcal{C}^\heartsuit \to \mathcal{A}b$ be the quadratic functor that takes $A \in \mathcal{C}^\heartsuit$ to the abelian subgroup of strict invariants

$$Ϙ^{\mathrm{gs}}_{\mathrm{D}^\heartsuit}(A) := \hom_{\mathcal{C}^\heartsuit}(A, \mathrm{D}^\heartsuit A)^{\mathrm{C}_2}.$$

We let $\mathrm{W}(\mathcal{C}^\heartsuit, Ϙ^{\mathrm{gs}}_{\mathrm{D}^\heartsuit})$ be the corresponding symmetric Witt group, defined as the quotient of the monoid $\pi_0\mathrm{Pn}(\mathcal{C}^\heartsuit, Ϙ^{\mathrm{gs}}_{\mathrm{D}^\heartsuit})$ by the submonoid of strictly metabolic objects. Thus two elements of $\pi_0\mathrm{Pn}(\mathcal{C}^\heartsuit, Ϙ^{\mathrm{gs}}_{\mathrm{D}^\heartsuit})$ are identified in the Witt group if they are isomorphic after adding strictly metabolic objects. We also write $-\mathrm{D}^\heartsuit$ for the duality on $\mathcal{C}^\heartsuit$ defined by the functor $\mathrm{D}^\heartsuit$ but where we replace the isomorphism $\eta : \mathrm{id} \to (\mathrm{D}^\heartsuit)^{\mathrm{op}}\mathrm{D}^\heartsuit$ with $-\eta$.

1.3.3. **Corollary.** *In the above situation, let* $\Omega^{\mathrm{s}}_{\mathrm{D}} \colon \mathcal{C} \to \mathcal{S}p$ *be the symmetric Poincaré structure associated to* D*. Then there are canonical isomorphisms*

$$\mathrm{L}_n(\mathcal{C}; \Omega^{\mathrm{s}}_{\mathrm{D}}) \cong \begin{cases} \mathrm{W}(\mathcal{C}^{\heartsuit}, \Omega^{\mathrm{gs}}_{\mathrm{D}^{\heartsuit}}) & \text{for } n \equiv 0 \bmod 4, \\ \mathrm{W}(\mathcal{C}^{\heartsuit}, \Omega^{\mathrm{gs}}_{-\mathrm{D}^{\heartsuit}}) & \text{for } n \equiv 2 \bmod 4, \\ 0 & \text{else.} \end{cases}$$

In particular, every element of $\pi_0\mathrm{Pn}(\mathcal{C}^{\heartsuit}, \Omega^{\mathrm{gs}}_{\mathrm{D}^{\heartsuit}})$ *which is zero in* $\mathrm{W}(\mathcal{C}^{\heartsuit}, \Omega^{\mathrm{gs}}_{\mathrm{D}^{\heartsuit}})$ *is metabolic.*

Proof. Apply Proposition 1.3.1 in the case of $r = \infty, d = 0$ and take (a, b) to be $(0, 0)$ when n is even and $(-1, 0)$ when n is odd. □

We now specialize Proposition 1.3.1 to the main case of interest, where $\mathcal{C} = \mathcal{D}^{\mathrm{p}}(R)$:

1.3.4. **Corollary.** *Let M be an invertible module with involution over R and suppose that R is coherent of finite global dimension d. Let Ω be an r-symmetric compatible Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$, for $r \in \mathbb{Z}$, e.g. $\Omega = \Omega^{\geq 2-r}_M$. Then for $n \geq d - 2r$ the following holds:*

i) *If $n + d$ is even, the canonical map*
$$\mathrm{L}^{d,d}_n(R; \Omega) \longrightarrow \mathrm{L}_n(R; \Omega)$$
is an isomorphism.

ii) *If $n + d$ is odd, the canonical map*
$$\mathrm{L}^{d-1,d}_n(R; \Omega) \longrightarrow \mathrm{L}_n(R; \Omega)$$
is an isomorphism.

1.3.5. **Corollary.** *Let M be an invertible module with involution over R and suppose that R is coherent of finite global dimension* 0*. Then the following holds:*

i) $\mathrm{L}_{2k-1}(R; \Omega^{\geq m}_M) = 0$ *whenever* $k \geq m - 1$*;*
ii) $\mathrm{L}_{2k}(R; \Omega^{\geq m}_M) \cong \mathrm{W}(\mathrm{Proj}(R), \pi_0\Omega^{\geq m+k}_{(-1)^k M})$ *whenever* $k \geq m - 2$*;*
iii) *every symmetric, even or quadratic M-valued Poincaré object in* $\mathrm{Proj}(R)$ *which is zero in the Witt group is strictly metabolic.*

Proof. We apply Corollary 1.3.4, noting that $\Omega^{\geq m}_M$ is $(2 - m)$-symmetric, with $d = 0$. To see part ii), we then need to recall that there is an equivalence $\Omega^{2k}\Omega^{\geq m}_M(P[-k]) \simeq \Omega^{\geq m+k}_{(-1)^k M}(P)$. □

1.3.6. **Remark.** We notice that a ring R is of global dimension 0 if and only if it is semisimple, and that semisimple rings are Noetherian and thus coherent. Part i) above hence recovers Ranicki's result that the odd-dimensional symmetric and quadratic L-groups of semisimple rings vanish [Ran92, Proposition 22.7]: Indeed, the symmetric case follows from the above with $m = -\infty$. For the quadratic case, by Corollary R.10 applied to Ω^{q}_M, it suffices to show that $\mathrm{L}^{\mathrm{q}}_{-3}(R; M) = 0$. But by Corollary 1.2.8 we have that $\mathrm{L}^{\mathrm{q}}_{-3}(R; M) \cong \mathrm{L}^{\mathrm{gs}}_{-3}(R; M) = \mathrm{L}_{-3}(R; \Omega^{\mathrm{gs}}_M)$, so i) applies for $k = -1$.

For completeness, we note that if K is a field of characteristic different from 2, also $\mathrm{L}^{\mathrm{q}}_{4k+2}(K) \cong \mathrm{L}^{\mathrm{s}}_{4k+2}(K)$ vanishes: By Corollary 1.3.5 it is given by the Witt group of anti-symmetric forms over K, but any such form admits a symplectic basis and hence a Lagrangian.

Our next goal is to use the surgery results above in order to identify the L-groups of an r-symmetric structure with the corresponding symmetric L-groups in a suitable range. The following corollary should be compared with Corollary 1.2.8 above:

1.3.7. **Corollary.** *Let M be an invertible module with involution over R and suppose that R is coherent of finite global dimension d. Let Ω be an r-symmetric Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$ compatible with M, for $r \in \mathbb{Z}$. Then the canonical map*

$$\mathrm{L}_n(R; \Omega) \longrightarrow \mathrm{L}_n(R; \Omega^{\mathrm{s}}_M) = \mathrm{L}^{\mathrm{s}}_n(R; M)$$

is injective for $n \geq d - 2r + 2$ and bijective for $n \geq d - 2r + 3$.

Proof. We consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{L}^{d,d}_n(R;\Qoppa) & \longrightarrow & \mathrm{L}_n(R;\Qoppa) & \longleftarrow & \mathrm{L}^{d-1,d}_n(R;\Qoppa) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{L}^{d,d}_n(R;\Qoppa^{\mathrm{s}}_M) & \longrightarrow & \mathrm{L}_n(R;\Qoppa^{\mathrm{s}}_M) & \longleftarrow & \mathrm{L}^{d-1,d}_n(R;\Qoppa^{\mathrm{s}}_M) \end{array}$$

and use Corollary 1.3.4 and Lemma 1.2.20 to conclude the bijectivity claim of the corollary. To see injectivity for $n = d - 2r + 2$, again using Corollary 1.3.4, it will suffice to show that the left vertical map in the above diagram is injective. In light of Remark 1.3.2 it will suffice to show that if (X, q) is a $(-d + r - 1)$-connective Poincaré object in $(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^{[-n]})$ whose associated Poincaré object in $(\mathcal{D}^{\mathrm{p}}(R), (\Qoppa^{\mathrm{s}}_M)^{[-n]})$ admits a Lagrangian $L \to X$ such that L is $(-d+r-1)$-connective then L can be refined to a Lagrangian of (X, q) with respect to $\Qoppa$. For this, it will suffice to show that for an L with this connectivity bound, the map

$$\Omega^n\Qoppa(L) \longrightarrow \Omega^n\Qoppa^{\mathrm{s}}_M(L)$$

is surjective on π_1 and injective on π_0. Indeed, this map is (-1)-truncated by Remark 1.1.3 since $\Qoppa$ is r-symmetric. □

As a consequence, we obtain the following result, which proves Theorem 6 and the first part of Theorem 3 from the introduction.

1.3.8. **Corollary.** *Let M be an invertible module with involution over R and suppose that R is coherent of finite global dimension d. Then the canonical maps*

$$\mathrm{L}^{\mathrm{gs}}_n(R;M) \longrightarrow \mathrm{L}^{\mathrm{s}}_n(R;M) \quad \text{and} \quad \mathrm{GW}^{\mathrm{gs}}_n(R;M) \longrightarrow \mathrm{GW}^{\mathrm{s}}_n(R;M)$$

are injective for $n \geq d-2$ and bijective for $n \geq d-1$. Likewise, the maps

$$\mathrm{L}^{\mathrm{gq}}_n(R;M) \longrightarrow \mathrm{L}^{\mathrm{s}}_n(R;M) \quad \text{and} \quad \mathrm{GW}^{\mathrm{gq}}_n(R;M) \longrightarrow \mathrm{GW}^{\mathrm{s}}_n(R;M)$$

are injective for $n \geq d+2$ and bijective for $n \geq d+3$.

Proof. By Theorem 1, the canonical squares

$$\begin{array}{ccccc} \mathrm{GW}^{\mathrm{gq}}(R;M) & \longrightarrow & \mathrm{GW}^{\mathrm{gs}}(R;M) & \longrightarrow & \mathrm{GW}^{\mathrm{s}}(R;M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{L}^{\mathrm{gq}}(R;M) & \longrightarrow & \mathrm{L}^{\mathrm{gs}}(R;M) & \longrightarrow & \mathrm{L}^{\mathrm{s}}(R;M) \end{array}$$

are pullbacks. Hence the Grothendieck–Witt part of the corollary follows from the L-theory part. Moreover, the L-theory parts then follow from Corollary 1.3.7 using that $\Qoppa^{\mathrm{gs}}$ is 2-symmetric and $\Qoppa^{\mathrm{gq}}$ is 0-symmetric. □

1.3.9. **Example.** In general, the bounds obtained in Corollary 1.3.8 are sharp, as the following example shows. Consider the d-dimensional ring $\mathbb{F}_2[\mathbb{Z}^d]$ as a ring with anti-involution induced by the inversion of the group $\mathbb{Z}^d$. We claim that the map $\mathrm{L}^{\mathrm{gs}}(\mathbb{F}_2[\mathbb{Z}^d) \to \mathrm{L}^{\mathrm{s}}(\mathbb{F}_2[\mathbb{Z}^d])$ is not surjective on π_{d-2}. To see this we use the Shaneson splitting proved by Ranicki for L^{s} and Milgram–Ranicki for L^{gs}. In Paper [IV] we give a proof of this result which works simultaneously for both variants, but for our purposes the following version is sufficient. Let R be a ring with anti-involution and consider the ring $R[\mathbb{Z}]$ with anti-involution induced by the group inversion of $\mathbb{Z}$. Suppose that $\mathrm{K}_0(R) \cong \mathrm{K}_0(R[\mathbb{Z}]) \cong \mathbb{Z}$, for instance R could be a field or the integers. Then, for $\Qoppa = \Qoppa^{\mathrm{gs}}, \Qoppa^{\mathrm{s}}$, there is a natural equivalence

$$\mathrm{L}(R[\mathbb{Z}];\Qoppa) \simeq \mathrm{L}(R;\Qoppa) \oplus \Sigma\,\mathrm{L}(R;\Qoppa).$$

By induction, we deduce that the map $\Sigma^d\,\mathrm{L}^{\mathrm{gs}}(\mathbb{F}_2) \to \Sigma^d\,\mathrm{L}^{\mathrm{s}}(\mathbb{F}_2)$ is a retract of the map $\mathrm{L}^{\mathrm{gs}}(\mathbb{F}_2[\mathbb{Z}^d]) \to \mathrm{L}^{\mathrm{s}}(\mathbb{F}_2[\mathbb{Z}^d])$. Therefore, in order to see that the latter map is not surjective on π_{d-2} it suffices to argue that the map $\mathrm{L}^{\mathrm{gs}}(\mathbb{F}_2) \to \mathrm{L}^{\mathrm{s}}(\mathbb{F}_2)$ is not surjective on π_{-2}. Since $\mathrm{L}^{\mathrm{s}}_2(\mathbb{F}_2) \cong \mathbb{Z}/2$ but $\mathrm{L}^{\mathrm{gs}}(\mathbb{F}_2) = 0$, this is indeed the case. A similar argument shows that for $\epsilon = -1$, the map $\mathrm{L}^{\mathrm{gs}}_0(\mathbb{Z}[\mathbb{Z}];\epsilon) \to \mathrm{L}^{\mathrm{s}}_0(\mathbb{Z}[\mathbb{Z}];\epsilon)$ is not surjective on π_0.

We note that for $d = 0$, this shows the the obtained bounds are sharp also for commutative rings viewed as rings with trivial anti-involution. However, at the time of writing we do not have a specific example of a commutative ring where the map $\mathrm{L}^{\mathrm{gs}}(R) \to \mathrm{L}^{\mathrm{s}}(R)$ is not an isomorphism on some non-negative homotopy group, though we believe that they must exist in abundance.

1.3.10. **Remark.** If R is a right-coherent ring of finite right-global dimension, we may apply the results of Corollaries 1.3.5, 1.3.7, and 1.3.8 and Remark 1.3.6 to the ring R^{op} with Poincaré structure $Ϙ^\vee$ as described in Remark R.11: By Example 1.1.5 $Ϙ^\vee$ is r-symmetric if $Ϙ$ is. Using then the equivalence of Poincaré ∞-categories $(\mathcal{D}^{\mathrm{p}}(R^{\mathrm{op}}), Ϙ^\vee) \simeq (\mathcal{D}^{\mathrm{p}}(R), Ϙ)$, we obtain the conclusions of Corollaries 1.3.5, 1.3.7, and 1.3.8 and Remark 1.3.6 also for right-coherent rings of finite right-global dimension.

1.3.11. **Remark.** As described in Paper [I] Definition [I].4.2.23, there is a canonical non-abelian derived Poincaré structure $Ϙ^{\mathrm{g}\lambda}_M$ associated to a generalised form parameter λ in the sense of Schlichting [Sch19a] (extending the classical notion of Bak) on an invertible module with involution M. It then follows from the fact that $Ϙ^{\mathrm{g}\lambda}_M(P[0])$ is discrete (by definition) that $Ϙ^{\mathrm{g}\lambda}_M$ is 0-quadratic and 0-symmetric. In particular, the comparison results of Corollary 1.2.8 and Corollary 1.3.7 apply to $Ϙ^{\mathrm{g}\lambda}_M$. Depending on λ, the Poincaré structure $Ϙ^{\mathrm{g}\lambda}_M$ might in fact be 1-symmetric (as is the case for even forms) or 2-symmetric (as is the case for symmetric forms), or likewise 1-quadratic (as in the case of even forms) or 2-quadratic (as in the case of quadratic forms).

1.3.12. **Remark.** In Theorem 1.2.18 we have shown that the non-negative genuine symmetric L-groups coincide with Ranicki's L-groups of short complexes. The comparison range above then improves on Ranicki's classical theorem that established injectivity of the map $\mathrm{L}^{\mathrm{gs}}_n(R) \to \mathrm{L}^{\mathrm{s}}(R)$ for non-negative $n \geq 2d-3$ and bijectivity for non-negative $n \geq 2d-2$ for Noetherian rings of global dimension d.

Since Dedekind rings have global dimension ≤ 1, and by applying the fibre sequence of Theorem 1 we immediately find:

1.3.13. **Corollary.** *Let R be a Dedekind ring, e.g. the ring of integers in an algebraic number field. Then the canonical maps*

$$\mathrm{L}^{\mathrm{gs}}_n(R; M) \longrightarrow \mathrm{L}^{\mathrm{s}}_n(R; M) \quad \text{and} \quad \mathrm{GW}^{\mathrm{gs}}_n(R; M) \longrightarrow \mathrm{GW}^{\mathrm{s}}_n(R; M)$$

are injective for $n = -1$ and bijective for $n \geq 0$. In particular, the non-negative homotopy groups of $\mathrm{L}^{\mathrm{gs}}_n(R; M)$ are 4-periodic. Similarly, the maps

$$\mathrm{L}^{\mathrm{gq}}_n(R; M) \longrightarrow \mathrm{L}^{\mathrm{s}}_n(R; M) \quad \text{and} \quad \mathrm{GW}^{\mathrm{gq}}_n(R; M) \longrightarrow \mathrm{GW}^{\mathrm{s}}_n(R; M)$$

are injective for $n = 3$ and bijective for $n \geq 4$.

We recall that by the main Theorem of [HS21] the connective covers of $\mathrm{GW}^{\mathrm{gs}}_n(R; M)$ and $\mathrm{GW}^{\mathrm{gq}}_n(R; M)$ are equivalent to the classical Grothendieck-Witt groups respectively of symmetric and quadratic forms, and we further find that:

1.3.14. **Corollary.** *Let R be a Dedekind ring and M an invertible module with involution over R. Then the canonical maps*

$$\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R; M) \longrightarrow \tau_{\geq 0}\, \mathrm{GW}^{\mathrm{s}}(R; M) \quad \text{and} \quad \tau_{\geq 4}\, \mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(R; M) \longrightarrow \tau_{\geq 4}\, \mathrm{GW}^{\mathrm{s}}(R; M)$$

are equivalences.

1.3.15. **Remark.** We now prove the second part of Theorem 3 from the introduction. So let R be a coherent ring of finite global dimension d. We have equivalences $\Sigma^2\, \mathrm{L}^{\mathrm{gs}}(R) \simeq \mathrm{L}^{-\mathrm{ge}}(R)$, and $\Sigma^2\, \mathrm{L}^{\mathrm{ge}}(R) \simeq \mathrm{L}^{-\mathrm{gq}}(R)$. If R is in addition 2-torsion free, for instance a Dedekind domain whose fraction field is of characteristic different from 2, then the canonical maps $Ϙ^{-\mathrm{ge}} \to Ϙ^{-\mathrm{gs}}$ and $Ϙ^{\mathrm{gq}} \to Ϙ^{\mathrm{ge}}$ are equivalences by Remark R.4. We deduce that for such rings, there are in fact canonical equivalences

$$\Sigma^2\, \mathrm{L}^{\mathrm{gs}}(R) \simeq \mathrm{L}^{-\mathrm{gs}}(R) \quad \text{and} \quad \Sigma^2\, \mathrm{L}^{\mathrm{gq}}(R) \simeq \mathrm{L}^{-\mathrm{gq}}(R).$$

The comparison map is compatible with these equivalences, so we deduce that the map $\mathrm{L}^{\mathrm{gq}}(R) \to \mathrm{L}^{\mathrm{gs}}(R)$ is a 2-fold loop of the map $\mathrm{L}^{-\mathrm{gq}}(R) \to \mathrm{L}^{-\mathrm{gs}}(R)$. Corollary 1.3.8 then implies that the maps $\mathrm{L}^{\mathrm{gq}}_n(R) \to \mathrm{L}^{\mathrm{gs}}_n(R)$ and $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl},n}(R) \to \mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},n}(R)$ are injective for $n = d$ and an isomorphism for $n \geq d+1$.

1.3.16. **Remark.** The surgery results of this section can now be used to determine the L-groups of $\mathrm{L}^{\mathrm{b}}(\mathbb{Z})$ appearing in Corollary 1.2.24, or equivalently $\mathrm{L}^{\mathrm{g}}(\mathbb{A})$ from [DO19]. Recall that in Corollary 1.2.24 we have determined the non-positive homotopy groups of $\mathrm{L}^{\mathrm{b}}(\mathbb{Z})$. Making use of the fact that $\Omega^{\mathrm{b}}_{\mathbb{Z}}$ is also 0-symmetric, we obtain from Corollary 1.3.7 that the map $\mathrm{L}^{\mathrm{b}}(\mathbb{Z}) \to \mathrm{L}^{\mathrm{s}}(\mathbb{Z})$ is an isomorphism on homotopy groups of degree at least 3 (here we have in addition used that $\mathrm{L}^{\mathrm{s}}_3(\mathbb{Z}) = 0$). It remains to determine $\mathrm{L}^{\mathrm{b}}_n(\mathbb{Z})$ for $n = 1, 2$, and we claim that both groups vanish. We treat the case $n = 1$ first. From Corollary 1.2.24, we also deduce that the map $\mathrm{L}^{\mathrm{u}}_1(\mathbb{S}) \to \mathrm{L}^{\mathrm{b}}_1(\mathbb{Z})$ is surjective. By the results of Weiss-Williams [WW14, Theorem 4.5] mentioned in the proof of Corollary 1.2.24, we find that $\mathrm{L}^{\mathrm{u}}_1(\mathbb{S}) \cong (\mathbb{Z}/2)^2$, with generators having underlying object $\mathbb{S} \oplus \Omega\mathbb{S}$. We find that both generators are mapped to an $\Omega\Omega^{\mathrm{b}}$-form on $\mathbb{Z} \oplus \mathbb{Z}[-1]$. But since $\Omega\Omega^{\mathrm{b}}(\mathbb{Z})$ is connected, we find that both these forms admit a Lagrangian. Hence the map $\mathrm{L}^{\mathrm{u}}_1(\mathbb{S}) \to \mathrm{L}^{\mathrm{b}}_1(\mathbb{Z})$ is both zero and surjective from which we conclude that $\mathrm{L}^{\mathrm{b}}_1(\mathbb{Z}) = 0$. To see that $\mathrm{L}^{\mathrm{b}}_2(\mathbb{Z})$ vanishes we argue as follows: By Corollary 1.3.4, we may assume that a representative for $\mathrm{L}^{\mathrm{b}}_2(\mathbb{Z})$ is concentrated in degrees $[-2, 0]$, and by shifted self-duality we find that π_0 of the underlying complex is torsion free. One can then perform more surgeries until π_0 vanishes (this uses that π_0 is torsion-free). It is then not hard to see that the map $\mathrm{L}^{\mathrm{q}}_2(\mathbb{Z}) \to \mathrm{L}^{\mathrm{b}}_2(\mathbb{Z})$ is surjective. Similarly as above, we know a concrete generator of $\mathrm{L}^{\mathrm{q}}_2(\mathbb{Z})$, which is a quadratic form on $(\mathbb{F}_2 \oplus \mathbb{F}_2)[-1]$. One of the $\mathbb{F}_2[-1]$ summands is then a Lagrangian for the image of this generator in $\Omega^2\Omega^{\mathrm{b}}((\mathbb{F}_2 \oplus \mathbb{F}_2)[-1])$. Hence the map $\mathrm{L}^{\mathrm{q}}_2(\mathbb{Z}) \to \mathrm{L}^{\mathrm{b}}_2(\mathbb{Z})$ is both surjective and zero, so that $\mathrm{L}^{\mathrm{b}}_2(\mathbb{Z}) = 0$ as claimed.

2. L-theory of Dedekind rings

The goal of this section is to extend Quillen's localisation-dévissage sequence [Qui73, Corollary of Theorem 5] for Dedekind rings to hermitian K-theory, thereby proving Theorem 7 of the introduction. We recall that Dedekind rings are commutative regular Noetherian domains of global dimension 1, and the main example of interest to us are those whose fraction field is a number field. Other notable examples are discrete valuation rings and rings of functions of smooth affine curves over fields.

To prove the theorem we first construct a Poincaré-Verdier sequence induced by the map from R to its localisation *away from a set S of non-zero prime ideals*. The functor GW takes such sequences to cofibre sequences of spectra. Using a dévissage result for symmetric GW-theory we then identify the fibre term as the sum of GW-spectra of the residue fields $R/\mathfrak{p}$, where $\mathfrak{p}$ ranges through the set S.

Results of this type have appeared in the literature from early on. Three- or four-term localisation sequences for Witt groups appear in the work of Knebusch [Kne70], Milnor-Husemoller [MH73] and these are extended to long exact sequences of L-groups by Ranicki [Ran81, §4.2] and Balmer-Witt groups in [Bal05, §1.5.2]. For Grothendieck-Witt groups there are exact sequences due to Karoubi [Kar74, Kar75] as well as Hornbostel-Schlichting [Hor02, HS04], both under the assumption that 2 is invertible in the ring.

Our results hold with no assumption on invertibility of 2, for the homotopy theoretic symmetric Grothendieck-Witt spectrum GW^{s}. By the results of the previous section (see Corollary 1.3.13 and 1.3.14) the non-negative homotopy groups of $\mathrm{GW}^{\mathrm{s}}(R)$ agree with the classical higher Grothendieck-Witt groups, but it is only $\mathrm{GW}^{\mathrm{s}}(R)$ which is well-behaved in all degrees, see Remark 2.1.11.

2.1. The localisation-dévissage sequence.

Let R be a Dedekind ring and S a set of non-zero prime ideals of R. We let $R_S = \mathcal{O}(U)$ where $\mathcal{O}$ is the structure sheaf of $\mathrm{spec}(R)$ and $U = \mathrm{spec}(R) \setminus S$ is the complement of the set S (in case S is infinite U is not open in $\mathrm{spec}(R)$, and $\mathcal{O}(U)$ is defined as the colimit of $\mathcal{O}$ on the open subsets of $\mathrm{spec}(R)$ containing U). Concretely, one can describe the ring R_S as follows: For any non-zero prime ideal $\mathfrak{p}$, the localisation $R_{(\mathfrak{p})}$ at $\mathfrak{p}$ is a discrete valuation ring, and the fraction field K of R hence acquires a $\mathfrak{p}$-adic valuation $v_{\mathfrak{p}}$. Then R_S identifies with the subring of K given by all elements $x \in K$ such that $v_{\mathfrak{p}}(x) \geq 0$ for all $\mathfrak{p}$ not contained in S. The ring R_S can be thought of as the localisation of R *away* from the set of primes S.

2.1.1. **Example.** Let R be a Dedekind ring.

i) If S consists of all the non-zero prime ideals of R, then R_S is the fraction field K,
ii) Given a multiplicative subset $T \subset R$ we may consider the set of primes ideals $S = \{\mathfrak{p} \mid \mathfrak{p} \cap T \neq \emptyset\}$. Then $R_S = R[T^{-1}]$ is obtained from R by inverting the elements of T. In particular:
iii) If $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is such that the ideal $\mathfrak{p}_1^{r_1} \cdot \cdots \cdot \mathfrak{p}_n^{r_n} = (x)$ is a principal ideal, then $R_S = R[\frac{1}{x}]$.

These are the cases that we will use most. We warn the reader that, in general, R_S is not obtained from R by inverting a multiplicative subset.

2.1.2. **Lemma.** *Let R be a Dedekind ring and S a set of non-zero prime ideals of R. Then the map $R \to R_S$ is a derived localisation, R_S is a flat R-module, and the extension of scalars functor $\mathcal{D}(R) \to \mathcal{D}(R_S)$ has perfectly generated fibre.*

Proof. We recall from Definition [II].A.4.2 that a map of rings $A \to B$ is called a derived localisation if the map $\mathrm{H}B \otimes_{\mathrm{H}A} \mathrm{H}B \to \mathrm{H}B$ of *ring spectra* is an equivalence. We note that R_S is described as the filtered colimit

$$R_S \cong \operatorname*{colim}_{S' \subseteq S} \mathcal{O}(U')$$

where $U' = \operatorname{spec}(R) \setminus S'$ and S' ranges through the finite subsets of S. As R is a Dedekind ring, a finite set of primes S' is a closed subset of $\operatorname{spec}(R)$, so the map $R \to R_S$ arises as a filtered colimit of ring maps, each of which is given by restricting the structure sheaf to an affine open subset. Any such map is a derived localisation and has perfectly generated fibre by [Rou10, Theorem 3.6]. By passing to filtered colimits, we deduce that also the map $\mathcal{D}(R) \to \mathcal{D}(R_S)$ is a derived localisation and has perfectly generated fibre. Finally R_S, as a submodule of the fraction field K, is a torsion free and hence flat R-module [SP18, Lemma 0AUW]. □

2.1.3. **Remark.** From the description of R_S as a subring of the fraction field, it also follows immediately that $R \to R_S$ is a derived localisation: Since R_S is a flat module, it suffices to note that the underived tensor product $R_S \otimes_R R_S$ is, via the multiplication map, isomorphic to R_S. Likewise, we remark that the kernel $\mathcal{D}(R)_S$ of $\mathcal{D}(R) \to \mathcal{D}(R_S)$ is in fact generated by the perfect R-modules $R/\mathfrak{p}$ where $\mathfrak{p}$ ranges through the elements of S. Note that $\mathfrak{p}$ is a finitely generated projective R-module (again, because it is a torsion free module, hence flat, and thus also projective by finite generation), so that $R/\mathfrak{p}$ is equivalent to the perfect complex $(\mathfrak{p} \to R)$.

First, we observe that for each non-zero prime $\mathfrak{p}$, the module $R/\mathfrak{p}$ is indeed in $\mathcal{D}(R)_S$. To see that the objects $R/\mathfrak{p}$ indeed generate the kernel it suffices to consider the case where $S = \{\mathfrak{p}\}$ for a prime ideal $\mathfrak{p}$ of R. Let us write $R[\frac{1}{\mathfrak{p}}]$ for the ring R_S. The fibre sequence,

$$R \longrightarrow R[\tfrac{1}{\mathfrak{p}}] \longrightarrow R[\tfrac{1}{\mathfrak{p}}]/R$$

together with the fact that R generates $\mathcal{D}(R)$, implies that it suffices to prove that $R[\frac{1}{\mathfrak{p}}]/R$ is in the subcategory generated by $R/\mathfrak{p}$. For this, one constructs a filtration on $R[\frac{1}{\mathfrak{p}}]$ using a lower bound on the $\mathfrak{p}$-adic valuation of elements of $R_S \subseteq K$. This induces a filtration on the quotient $R[\frac{1}{\mathfrak{p}}]/R$ and the successive filtration quotients are then equivalent to $R/\mathfrak{p}$ as an R-module.

We denote by $\mathcal{D}^{\mathrm{p}}(R)_S \subseteq \mathcal{D}^{\mathrm{p}}(R)$ the fibre of the map $\mathcal{D}^{\mathrm{p}}(R) \to \mathcal{D}^{\mathrm{p}}(R_S)$. By the above, it coincides with the full subcategory spanned by those perfect R-modules whose homotopy groups are S-primary torsion modules. Now we let M be a line bundle over R with an R-linear involution (which in fact can only be multiplication with ± 1), which we regard as a module with involution over R as in Definition R.1. We recall that a line bundle over a commutative ring is a finitely generated projective module of rank 1.

We then endow $\mathcal{D}^{\mathrm{p}}(R)$ with the symmetric Poincaré structure $Ϙ^{\mathrm{s}}_M$ and $\mathcal{D}^{\mathrm{p}}(R_S)$ with the symmetric Poincaré structure $Ϙ^{\mathrm{s}}_{M_S}$ associated to the localised line bundle

$$M_S := R_S \otimes_R M.$$

The extension of scalars is then a Poincaré functor, see Lemma [I].3.3.3, so that $\mathcal{D}^{\mathrm{p}}(R)_S$ is closed under the duality of $\mathcal{D}^{\mathrm{p}}(R)$ induced by M and becomes a Poincaré subcategory, with the restricted Poincaré structure. We will denote this restricted Poincaré structure again by $Ϙ^{\mathrm{s}}_M : \mathcal{D}^{\mathrm{p}}(R)_S{}^{\mathrm{op}} \to \mathcal{S}p$.

2.1.4. **Proposition.** *Let R be a Dedekind ring and M a line-bundle over R with R-linear involution. Then the sequence of Poincaré ∞-categories*

$$(\mathcal{D}^{\mathrm{p}}(R)_S, Ϙ^{\mathrm{s}}_M) \longrightarrow (\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{s}}_M) \longrightarrow (\mathcal{D}^{\mathrm{p}}(R_S), Ϙ^{\mathrm{s}}_{M_S})$$

is a Poincaré-Verdier sequence. In particular, it induces a fibre sequence of GW *and* L*-spectra.*

Proof. The last statement says that GW and L are Verdier-localising functors, which was proven in Corollary [II].4.4.15 and Corollary [II].4.4.6. We now wish to apply Proposition [II].1.4.8. For this we need to check that M is compatible with the localisation $R \to R_S$ in the sense of Definition [II].1.4.3, which follows from the fact that R and R_S are commutative and M is an $R \otimes R$-module through the multiplication map of R, see Example [II].1.4.4. Furthermore, we have observed earlier that R_S is a flat R-module. In addition, the map $\mathrm{K}_0(R) \to \mathrm{K}_0(R_S)$ is surjective: As filtered colimits along surjections are surjections, it suffices to argue this in the case where S is finite. In this case, R_S is itself a Dedekind ring, so it suffices to argue that the map $\mathrm{Pic}(R) \to \mathrm{Pic}(R_S)$ is surjective. This follows from the observation that $\mathrm{Pic}(R)$ and $\mathrm{Pic}(R_S)$ are respectively the quotients of the free abelian groups generated by the prime ideals of R and R_S. The proposition then follows from Lemma 2.1.2. □

The objective of dévissage is then to identify $\mathrm{GW}(\mathcal{D}^{\mathrm{p}}(R)_S, Ϙ^{\mathrm{s}}_M)$ in terms of the GW-spectra of the residue fields $\mathbb{F}_{\mathfrak{p}} := R/\mathfrak{p}$ for $\mathfrak{p} \in S$. To establish this, we begin by refining the restriction of scalars functor to a Poincaré functor.

We recall that for a ring homomorphism $f : A \to B$, the extension of scalars functor $f_! : \mathcal{D}(A) \to \mathcal{D}(B)$ is left adjoint to the restriction of scalars functor $f^* : \mathcal{D}(B) \to \mathcal{D}(A)$, which admits a further right adjoint $f_* : \mathcal{D}(A) \to \mathcal{D}(B)$. If B is moreover perfect as an A-module (that is, it admits a finite resolution by finitely generated projective A-modules), then f^* restricts to a functor $f^* : \mathcal{D}^{\mathrm{p}}(B) \to \mathcal{D}^{\mathrm{p}}(A)$ on perfect objects.

2.1.5. **Lemma.** *Let $f : A \to B$ be a ring homomorphism such that B is a perfect A-module, and let M and N be invertible modules with involution, respectively over A and B. Then any map $\Psi : (f \otimes f)^*(N) \to M$ induces a hermitian structure on the restriction functor $f^* : (\mathcal{D}^{\mathrm{p}}(B), Ϙ^{\mathrm{s}}_N) \to (\mathcal{D}^{\mathrm{p}}(A), Ϙ^{\mathrm{s}}_M)$. This hermitian structure is Poincaré if and only if the map $N \to f_*(M)$ induced by Ψ is an equivalence in $\mathcal{D}(B)$.*

Proof. For such a Ψ, the hermitian structure on $f^* : \mathcal{D}^{\mathrm{p}}(B) \to \mathcal{D}^{\mathrm{p}}(A)$ is given by the natural transformation

$$Ϙ^{\mathrm{s}}_N(X) = \hom_{B\otimes B}(X \otimes X, N)^{\mathrm{hC}_2} \longrightarrow \hom_{A\otimes A}((f \otimes f)^*(X \otimes X), (f \otimes f)^* N)^{\mathrm{hC}_2} \longrightarrow$$

$$\hom_{A\otimes A}(f^*(X) \otimes f^*(X), (f \otimes f)^* N)^{\mathrm{hC}_2} \xrightarrow{\Psi_*} \hom_{A\otimes A}(f^*(X) \otimes f^*(X), M)^{\mathrm{hC}_2} = Ϙ^{\mathrm{s}}_M(f^*(X)).$$

This hermitian structure is Poincaré if and only if the map $\gamma : N \to f_* M$, adjoint to the map $f^* M \to N$ obtained from Ψ by restricting the $B \otimes B$-module structure to one factor, is an equivalence: To see this, observe that since $\mathcal{D}^{\mathrm{p}}(B)$ is generated by B via finite colimits and retracts it will suffice to show that the associated natural transformation

$$f^*\mathrm{D}(X) \longrightarrow \mathrm{D} f^*(X)$$

evaluates to an equivalence on $X = B$. Unwinding the definitions, the above map for $X = B$ identifies with the map $f^* N \to \hom_A(f^* B, M) = f^* f_* M$ which is the image under f^* of $\gamma : N \to f_* M$. Since f^* is conservative this image is an equivalence if and only if γ is an equivalence. □

Now suppose that $\mathfrak{p} \subseteq R$ is a non-zero prime ideal, and let $p : R \to \mathbb{F}_{\mathfrak{p}} := R/\mathfrak{p}$ be the quotient map. We note that $\mathfrak{p}$ is a rank 1 projective module and hence is $\otimes$-invertible, where the inverse is given by the dual $\mathfrak{p}^{-1} \simeq \hom_R(\mathfrak{p}, R)$, see [SP18, Tag 0AUW]. We find that $\mathbb{F}_{\mathfrak{p}}$ is perfect as an R-module, as it is represented by the chain complex $(\mathfrak{p} \to R)$ with $\mathfrak{p}$ in degree 1 and R in degree 0. Let us then consider the adjunction

$$p^* : \mathcal{D}(\mathbb{F}_{\mathfrak{p}}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{D}(R) : p_*$$

and note that p^* preserves compact objects since $p^*(\mathbb{F}_{\mathfrak{p}})$ is compact. It follows that p_* preserves filtered colimits, and since it is exact it must in fact preserve all colimits. We recall that p_* is given by the formula

$$p_*(X) = \hom_R(p^* \mathbb{F}_{\mathfrak{p}}, X)$$

regarded as an $\mathbb{F}_{\mathfrak{p}}$-module via the functoriality in the first variable. Moreover $p_*(R)$ is an $\mathbb{F}_{\mathfrak{p}} \otimes R$-module via the functoriality of p_*, and there is an equivalence of functors

$$p_*(X) \simeq X \otimes_R p_*(R)$$

as both preserve colimits and agree on $X = R$. To determine the functor p_* it hence suffices to calculate $p_*(R)$ which we do in the following lemma, see also [SP18, Tag 0BZH].

2.1.6. **Lemma.** *Let R be a Dedekind ring and $\mathfrak{p} \subseteq R$ a non-zero prime ideal. Then there is a canonical equivalence $p_*(R)[1] \simeq p_!(\mathfrak{p}^{-1})$. A choice of uniformiser π for $\mathfrak{p}$ thus induces an equivalence $p_*(R)[1] \simeq \mathbb{F}_{\mathfrak{p}}$.*

Proof. By applying the functor $\hom_R(-, R)$ to the fibre sequence $\mathfrak{p} \to R \to \mathbb{F}_\mathfrak{p}$, and using that $\hom_R(\mathfrak{p}, R) = \mathfrak{p}^{-1}$, we find a fibre sequence of R-modules

$$p^* p_*(R) \longrightarrow R \longrightarrow \mathfrak{p}^{-1}.$$

From this, we find a canonical equivalence

$$p^* p_*(R)[1] \simeq \mathbb{F}_\mathfrak{p} \otimes_R \mathfrak{p}^{-1} \simeq p^* p_!(\mathfrak{p}^{-1}).$$

In fact, one obtains a canonical equivalence $p_*(R)[1] \simeq p_!(\mathfrak{p}^{-1})$ since on discrete modules p^* is fully faithful.

Finally, we note that the map $p\colon R \to \mathbb{F}_\mathfrak{p}$ factors through the localisation $R_{(\mathfrak{p})}$ of R at $\mathfrak{p}$. The base change $\mathfrak{p} \otimes_R R_{(\mathfrak{p})}$ is the maximal ideal in the local Dedekind ring $R_{(\mathfrak{p})}$. As any local Dedekind ring is a principal ideal domain, one can choose a generator $\pi \in R_{(\mathfrak{p})}$ for this maximal ideal, called a uniformiser. Hence, the choice of a uniformiser determines an equivalence $p_!(\mathfrak{p}) \simeq \mathbb{F}_\mathfrak{p}$ and hence an equivalence $p_!(\mathfrak{p}^{-1}) \simeq p_!(\mathfrak{p})^{-1} \simeq \mathbb{F}_\mathfrak{p}$. □

2.1.7. **Remark.** For a choice of uniformiser π for $\mathfrak{p}$, the induced equivalence $p_*(R)[1] \simeq \mathbb{F}_\mathfrak{p}$ is concretely given as follows. As before, it suffices to describe the equivalence after applying p^*. In this case we find that

$$p^* p_*(R)[1] = \hom_R(\mathbb{F}_\mathfrak{p}, R)[1] = \mathrm{cof}(R \xrightarrow{\pi} R) = R/\pi$$

since $\mathbb{F}_\mathfrak{p}$ is the cofibre of the map $\pi\colon R \to R$. In addition, the counit map $p^* p_*(R) \to R$ is the R-linear dual of the map $R \to R/\pi$, which is the Bockstein map $R/\pi[-1] \to R$. We will use this observation in Lemma 2.2.1.

For what follows, we fix a uniformiser π for $\mathfrak{p}$, and remark that the corresponding isomorphism $p_*(R)[1] \simeq \mathbb{F}_\mathfrak{p}$ only depends on the class of π in $\mathfrak{p}/\mathfrak{p}^2$. Suppose as earlier that M is a line-bundle over R with R-linear involution. From the discussion preceding Lemma 2.1.6, we obtain an induced equivalence

$$p_! M := \mathbb{F}_\mathfrak{p} \otimes_R M \simeq (p_* R[1]) \otimes_R M = p_* M[1]$$

where the involution on $p_! M$ has the same sign as the one on M. The adjoint of the equivalence above defines a map of modules with involution $p^* p_! M \to M[1]$, and we may apply Lemma 2.1.5 to promote the restriction functor $p^*\colon \mathcal{D}^\mathrm{p}(\mathbb{F}_\mathfrak{p}) \to \mathcal{D}^\mathrm{p}(R)$ to a Poincaré functor

$$(\mathcal{D}^\mathrm{p}(\mathbb{F}_\mathfrak{p}), Ϙ^\mathrm{s}_{p_! M}) \longrightarrow (\mathcal{D}^\mathrm{p}(R), Ϙ^\mathrm{s}_{M[1]}) \simeq (\mathcal{D}^\mathrm{p}(R), (Ϙ^\mathrm{s}_M)^{[1]}).$$

The image of this functor lands in $\mathcal{D}^\mathrm{p}(R)_S$, yielding in particular a Poincaré functor

$$\psi_\mathfrak{p}\colon (\mathcal{D}^\mathrm{p}(\mathbb{F}_\mathfrak{p}), Ϙ^\mathrm{s}_{p_! M}) \longrightarrow (\mathcal{D}^\mathrm{p}(R)_S, (Ϙ^\mathrm{s}_M)^{[1]}).$$

2.1.8. **Theorem** (Dévissage)**.** *Let R be a Dedekind ring, S a set of non-zero prime ideals of R with chosen uniformisers, and M a line bundle over R with R-linear involution. Then for every $m \in \mathbb{Z}$ the direct sum Poincaré functor*

$$\psi_S\colon \bigoplus_{\mathfrak{p}\in S} (\mathcal{D}^\mathrm{p}(\mathbb{F}_\mathfrak{p}), (Ϙ^\mathrm{s}_{p_! M})^{[m]}) \longrightarrow (\mathcal{D}^\mathrm{p}(R)_S, (Ϙ^\mathrm{s}_M)^{[m+1]}) \tag{6}$$

induces equivalences on algebraic K*-theory,* GW*-theory and* L*-theory spectra.*

Proof. First, note that by the fibre sequence of Corollary [II].4.4.14, it will be enough to prove the theorem for algebraic K-theory and L-theory. Second, both sides of ψ_S depend on S in a manner that preserves filtered colimits. More specifically, if we write S as a filtered colimit $S = \mathrm{colim}_{S' \subseteq S, |S| < \infty} S'$ of its finite subsets then the direct sum on the left hand side of (6) is the colimit of the corresponding finite direct sums, while on the right hand side the full subcategory $\mathcal{D}^\mathrm{p}(R)_S \subseteq \mathcal{D}^\mathrm{p}(R)$ is the union of all the full subcategories $\mathcal{D}^\mathrm{p}(R)_{S'}$ for finite $S' \subseteq S$. Since both algebraic K-theory and L-theory commute with filtered colimits we may reduce to the case where S is finite.

In this case, the left hand side of (6) can also be written as the product of $\mathcal{D}^\mathrm{p}(\mathbb{F}_\mathfrak{p})$ for varying $\mathfrak{p} \in S$ (recall that $\mathrm{Cat}^\mathrm{p}_\infty$ is semi-additive, see Proposition [I].6.1.7). In addition, each $\mathcal{D}^\mathrm{p}(\mathbb{F}_\mathfrak{p})$, being the perfect derived category of a field, supports a t-structure inherited from $\mathcal{D}(\mathbb{F}_\mathfrak{p})$, and so we can endow the left hand side of (6) with the product of the corresponding t-structures. The heart of this product t-structure is then the direct sum $\bigoplus_{\mathfrak{p}\in S} \mathrm{Vect}(\mathbb{F}_\mathfrak{p})$ where $\mathrm{Vect}(\mathbb{F}_\mathfrak{p})$ is the abelian category of finite dimensional $\mathbb{F}_\mathfrak{p}$-vector spaces. We can also identify it with the category of finitely generated modules over the product ring $\prod_{\mathfrak{p}\in S} \mathbb{F}_\mathfrak{p}$.

Concerning the right hand side, since R and R_S have global dimension ≤ 1, the perfect derived categories $\mathcal{D}^{\mathrm{p}}(R)$ and $\mathcal{D}^{\mathrm{p}}(R_S)$ inherit t-structures from the respective unbounded derived categories. In addition, since R_S is flat over R the localisation functor $\mathcal{D}^{\mathrm{p}}(R) \to \mathcal{D}^{\mathrm{p}}(R_S)$ preserves both connective and truncated objects, and hence commutes with truncations and connective covers. As a result, its kernel $\mathcal{D}^{\mathrm{p}}(R)_S \subseteq \mathcal{D}^{\mathrm{p}}(R)$ is closed under truncation and connective covers, and so inherits a t-structure from $\mathcal{D}^{\mathrm{p}}(R)$, so that the inclusion $\mathcal{D}^{\mathrm{p}}(R)_S \subseteq \mathcal{D}^{\mathrm{p}}(R)$ commutes with truncations and connective covers. The heart of this t-structure is then the abelian category $\mathrm{Mod}^{\mathrm{fin}}(R)_S$ of finitely generated S-primary torsion R-modules. Now since (6) is induced by the various restriction functors $p^* : \mathcal{D}^{\mathrm{p}}(\mathbb{F}_{\mathfrak{p}}) \to \mathcal{D}^{\mathrm{p}}(R)$ it preserves connective and truncated objects with respect to the t-structures just discussed. The functor

$$\bigoplus_{\mathfrak{p}\in S} \mathrm{Vect}(\mathbb{F}_{\mathfrak{p}}) \longrightarrow \mathrm{Mod}^{\mathrm{fin}}(R)_S \tag{7}$$

induced by (6) on the respective hearts is then fully-faithful (even though (6) itself is not fully-faithful) and can be identified with the inclusion of the full subcategory of finitely generated S-torsion modules inside all finitely generated S-primary torsion modules (i.e. the full subcategory of *semi-simple* objects inside the abelian category $\mathrm{Mod}^{\mathrm{fin}}(R)_S$). By the main result of Barwick [Bar15] the inclusion of hearts induces an equivalence on algebraic K-theory spectra on both the domain and codomain of (6). The desired claim for algebraic K-theory is hence equivalent to saying that the inclusion of abelian categories (7) induces an equivalence on algebraic K-theory, which in turn follows from Quillen's classical dévissage theorem for algebraic K-theory [Qui73, Theorem §5.4].

We will now show that (6) induces an equivalence on L-theory. Recall that by Corollary R.10, L-theory supports natural equivalences $\mathrm{L}(\mathcal{C}, \Omega^{[1]}) \simeq \Sigma\,\mathrm{L}(\mathcal{C}, \Omega)$. It will thus suffice to prove the claim for $m = 0$. Now since each $\mathbb{F}_{\mathfrak{p}}$ has global dimension 0 the duality $\mathrm{D}_{p_! M}$ on $\mathcal{D}^{\mathrm{p}}(\mathbb{F}_{\mathfrak{p}})$ maps 0-connective objects to 0-truncated objects and *vice versa*. The same hence holds for the product duality on $\prod_{\mathfrak{p}\in S} \mathcal{D}^{\mathrm{p}}(\mathbb{F}_{\mathfrak{p}})$ with respect to the product t-structure. We now claim that this also holds for the Poincaré ∞-category $(\mathcal{D}^{\mathrm{p}}(R)_S, (\Omega^{\mathrm{s}}_M)^{[1]})$. To see this, note first that since R has global dimension 1 the shifted duality $\Sigma\mathrm{D}_M$ on $\mathcal{D}^{\mathrm{p}}(R)$ sends $\mathcal{D}^{\mathrm{p}}(R)_{\geq 0}$ to $\mathcal{D}^{\mathrm{p}}(R)_{\leq 1}$ and $\mathcal{D}^{\mathrm{p}}(R)_{\leq 0}$ to $\mathcal{D}^{\mathrm{p}}(R)_{\geq 0}$. On the other hand, if $X \in \mathcal{D}^{\mathrm{p}}(R)_{\geq 0}$ is S^∞-torsion then

$$\pi_1 \Sigma \mathrm{D}_M(X) \cong \pi_0 \hom_R(X, M[0]) \cong \pi_0 \hom_R(\tau_{\leq 0}X, M[0]) = \mathrm{Hom}_R(\pi_0(X), M) = 0$$

since $\pi_0(X)$ is S-primary torsion and M is torsion free. We then get that the duality on $\mathcal{D}^{\mathrm{p}}(R)_S$ restricted from $\mathrm{D}_M[1]$ sends $(\mathcal{D}^{\mathrm{p}}(R)_S)_{\geq 0}$ to $(\mathcal{D}^{\mathrm{p}}(R)_S)_{\leq 0}$ and vice versa. Hence for both sides of (6) we are in the situation of Corollary 1.3.3, and so to finish the proof it will suffice to show that (7) induces an isomorphism on symmetric and anti-symmetric Witt groups. But this follows from the dévissage result of [QSS79, Corollary 6.9, Theorem 6.10]. ☐

The combination of the classical dévissage and localisation theorems of Quillen give rise to the fibre sequence of K-theory spectra

$$\bigoplus_{\mathfrak{p}\in S} \mathrm{K}(\mathbb{F}_{\mathfrak{p}}) \longrightarrow \mathrm{K}(R) \longrightarrow \mathrm{K}(R_S).$$

From Theorem 2.1.8 we obtain the corresponding sequences for the symmetric L and GW-spectra.

2.1.9. **Corollary** (Localisation-dévissage)**.** *Under the assumptions of Theorem 2.1.8, the restriction and localisation functors yield fibre sequences of spectra*

$$\bigoplus_{\mathfrak{p}\in S} \mathrm{GW}(\mathbb{F}_{\mathfrak{p}}; (\Omega^{\mathrm{s}}_{p_! M})^{[m-1]}) \longrightarrow \mathrm{GW}(R; (\Omega^{\mathrm{s}}_M)^{[m]}) \longrightarrow \mathrm{GW}(R_S; (\Omega^{\mathrm{s}}_{M_S})^{[m]})$$

$$\bigoplus_{\mathfrak{p}\in S} \mathrm{L}(\mathbb{F}_{\mathfrak{p}}; (\Omega^{\mathrm{s}}_{p_! M})^{[m-1]}) \longrightarrow \mathrm{L}(R; (\Omega^{\mathrm{s}}_M)^{[m]}) \longrightarrow \mathrm{L}(R_S; (\Omega^{\mathrm{s}}_{M_S})^{[m]})$$

for every $m \in \mathbb{Z}$.

2.1.10. **Remark.** A variant of Corollary 2.1.9 for L-theory of short complexes in non-negative degrees (which, for Dedekind rings, coincides with symmetric L-theory in non-negative degrees by Theorem 1.2.18 and Corollary 1.3.13), was proven by Ranicki in [Ran81, §4.2]. For Grothendieck-Witt theory, Hornbostel and Schlichting prove a dévissage statement and obtain a localisation sequence of the type of Corollary 2.1.9 under the assumption that 2 is a unit in R, see [Hor02], [HS04]. Apart from the announcement [Sch19b, Theorem 3.2] which provides the above fibre sequence for GW after passing to connective covers, we are not aware of any previous results in the literature for Grothendieck-Witt spaces, along the lines of Corollary 2.1.9, for rings in which 2 is not invertible.

2.1.11. **Remark.** The dévissage result above is a special feature of the symmetric Poincaré structure: It is the only among the Poincaré structures $Ϙ^{\geq m}_M$ for which this result holds at the spectrum level (not just in a range of degrees). Indeed, to see this it suffices, by Corollary 1.2.8, to argue that dévissage fails for quadratic L-theory. For an explicit example, one can note that the maps

$$\Omega\,\mathrm{L}^{\mathrm{q}}(\mathbb{F}_2) \longrightarrow \mathrm{L}^{\mathrm{q}}(\mathbb{Z}) \longrightarrow \mathrm{L}^{\mathrm{q}}(\mathbb{Z}[\tfrac{1}{2}])$$

cannot be part of a fibre sequence, for instance because it would imply that $\mathrm{L}^{\mathrm{q}}_2(\mathbb{Z}) = 0$, which is not the case. Here, we use that $\mathrm{L}^{\mathrm{q}}_2(\mathbb{Z}[\frac{1}{2}]) \cong \mathrm{L}^{\mathrm{s}}_2(\mathbb{Z}[\frac{1}{2}]) = 0$, which is of course well-known, but see also Corollary 2.2.4 below, as well as the vanishing of $\mathrm{L}^{\mathrm{q}}_3(\mathbb{F}_2)$, see Remark 1.3.6.

However, dévissage in quadratic L-theory fails only at dyadic primes. More precisely, if no prime in S is dyadic, then the localisation-dévissage sequence exists also in quadratic L-theory; see Remark 2.2.14 for details. For instance, it can be used to calculate $\mathrm{L}^{\mathrm{q}}(\mathbb{Z}[\frac{1}{p}])$ for odd primes p.

In the next subsection, we will use the localisation-dévissage sequence to calculate the symmetric L-theory of Dedekind rings. By Remark 2.1.11, this strategy, however, will not allow us to calculate the quadratic L-groups of Dedekind rings in general. Instead we will make use of a general localisation-completion property, Proposition 2.1.12 below, and a rigidity property of quadratic L-theory, Proposition 2.1.13. Following §[II].A.4, for a subgroup $\mathrm{c} \subset \mathrm{K}_0(R)$ fixed by the involution, we let $\mathcal{D}^{\mathrm{c}}(R)$ denote the full subcategory of $\mathcal{D}^{\mathrm{p}}(R)$ spanned by the complexes whose K_0-class lies in c.

2.1.12. **Proposition.** *Let R be a ring, M an invertible module with involution over R, and S the multiplicatively closed subset generated by an integer $\ell \in R$. Assume that the ℓ^∞-torsion in R is bounded, for instance that ℓ is a non-zero divisor. Then the square*

$$\begin{array}{ccc} (\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\geq m}_M) & \longrightarrow & (\mathcal{D}^{\mathrm{p}}(R^\wedge_\ell), Ϙ^{\geq m}_{M^\wedge_\ell}) \\ \downarrow & & \downarrow \\ (\mathcal{D}^{\mathrm{c}}(R[\frac{1}{\ell}]), Ϙ^{\geq m}_{S^{-1}M}) & \longrightarrow & (\mathcal{D}^{\mathrm{c}'}(R^\wedge_\ell[\frac{1}{\ell}]), Ϙ^{\geq m}_{S^{-1}(M^\wedge_\ell)}) \end{array}$$

is a Poincaré-Verdier square for all $m \in \mathbb{Z} \cup \{\pm\infty\}$, where $\mathrm{c} = \mathrm{im}(\mathrm{K}_0(R) \to \mathrm{K}_0(R[\frac{1}{\ell}]))$, and $\mathrm{c}' = \mathrm{im}(\mathrm{K}_0(R^\wedge_\ell) \to \mathrm{K}_0(R^\wedge_\ell[\frac{1}{\ell}]))$. In particular it becomes a pullback after applying GW *or* L.

Proof. We show that the canonical maps $f\colon R \to R^\wedge_\ell$ and $\alpha\colon M \to (f\otimes f)^*(M^\wedge_\ell)$ satisfy the conditions of Proposition [II].4.4.21. The morphism

$$R^\wedge_\ell \otimes_R M \to (R^\wedge_\ell \otimes R^\wedge_\ell) \otimes_{R\otimes R} M \to M^\wedge_\ell$$

is indeed an equivalence: This is clear for $M = R$, which implies the general case since M is a finitely generated projective R-module. Moreover the square

$$\begin{array}{ccc} R & \longrightarrow & R^\wedge_\ell \\ \downarrow & & \downarrow \\ R[\frac{1}{\ell}] & \longrightarrow & R^\wedge_\ell[\frac{1}{\ell}] \end{array}$$

is a derived pullback, see for instance [DG02, §4], as the assumption on ℓ^∞-torsion implies that $R^\wedge_\ell$ is also a derived completion. The final thing to check is that the map $M^\wedge_\ell[\frac{1}{\ell}] \to M[1]$ induces the zero map in C_2-Tate cohomology. This follows from the fact that the domain is a $\mathbb{Q}$-vector space, and so has trivial C_2-Tate cohomology. □

To make efficient use of the localisation-completion square, we shall also need the following result due to Wall [Wal73, Lemma 5]. We include a guide through the proof merely for convenience of the reader, as to avoid confusion about different definitions (and versions) of L-theory. We warn the reader that what is denoted by $L_i^K(R)$ in [Wal73] is what we would denote $\mathrm{L}(\mathcal{D}^{\mathrm{f}}(R), \Qoppa^{\mathrm{q}})$, i.e. quadratic L-theory based on complexes of (stably) free modules.

2.1.13. **Proposition.** *Let R be ring, complete in the I-adic topology for an ideal I of R. Then the canonical map $\mathrm{L}^{\mathrm{q}}(R) \to \mathrm{L}^{\mathrm{q}}(R/I)$ is an equivalence.*

Proof. First, we claim that the functor $\mathrm{Unimod}^{\mathrm{q}}(R;\epsilon) \to \mathrm{Unimod}^{\mathrm{q}}(R/I;\epsilon)$ induces a bijection on isomorphism classes, for $\epsilon = \pm 1$. To see this, we first observe that the functor $\mathrm{Proj}(R) \to \mathrm{Proj}(R/I)$ is full and essentially surjective. Moreover, for any finitely generated projective R module P, the map

$$\Qoppa^{\mathrm{q}}_{\epsilon}(P) \to \Qoppa^{\mathrm{q}}_{\epsilon}(P \otimes_R R/I)$$

is surjective on π_0, and furthermore an ϵ-quadratic form is unimodular if and only if its image over R/I is; this is as a consequence of Nakayama's lemma: I is contained in the Jacobsen radical as we have assumed that R is I-complete. We deduce that the above map is surjective. To see injectivity, we apply [Wal70, Theorem 2]: amongst other things, it says that given forms (P,q) and (P',q') over R, then any isometry between their induced forms over R/I can be lifted to an isometry over R. In particular, the map $\mathrm{Unimod}^{\mathrm{q}}(R;\epsilon) \to \mathrm{Unimod}^{\mathrm{q}}(R/I;\epsilon)$ is also injective on isomorphism classes. We deduce that the map $\mathrm{GW}^{\mathrm{q}}_0(R;\epsilon) \to \mathrm{GW}^{\mathrm{q}}_0(R/I;\epsilon)$ is an isomorphism. Since likewise the map $\mathrm{K}_0(R) \to \mathrm{K}_0(R/I)$ is an isomorphism, we deduce that $\mathrm{L}^{\mathrm{q}}_0(R;\epsilon) \to \mathrm{L}^{\mathrm{q}}_0(R/I;\epsilon)$ is an isomorphism as well. We then consider the diagram

$$\begin{array}{ccccccccc}
\pi_1(\mathrm{K}(R;\epsilon)_{\mathrm{hC}_2}) & \longrightarrow & \mathrm{GW}^{\mathrm{q}}_1(R;\epsilon) & \longrightarrow & \mathrm{L}^{\mathrm{q}}_1(R;\epsilon) & \longrightarrow & \mathrm{K}_0(R;\epsilon)_{\mathrm{C}_2} & \longrightarrow & \mathrm{GW}^{\mathrm{q}}_0(R;\epsilon) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow & & \downarrow\cong & & \downarrow\cong \\
\pi_1(\mathrm{K}(R/I;\epsilon)_{\mathrm{hC}_2}) & \longrightarrow & \mathrm{GW}^{\mathrm{q}}_1(R/I;\epsilon) & \longrightarrow & \mathrm{L}^{\mathrm{q}}_1(R/I;\epsilon) & \longrightarrow & \mathrm{K}_0(R/I;\epsilon)_{\mathrm{C}_2} & \longrightarrow & \mathrm{GW}^{\mathrm{q}}_0(R/I;\epsilon)
\end{array}$$

where [Wal73, Corollary 1 & Lemma 1] give that the two left most vertical maps are surjective, and [Wal73, Proposition 4] that the induced map on vertical kernels is surjective. This implies that the map $\mathrm{L}^{\mathrm{q}}_1(R;\epsilon) \to \mathrm{L}^{\mathrm{q}}_1(R/I;\epsilon)$ is an isomorphism. From the general periodicity $\mathrm{L}^{\mathrm{q}}_n(R;\epsilon) \cong L^{\mathrm{q}}_{n+2}(R;-\epsilon)$ we deduce the proposition. ☐

2.2. **Symmetric and quadratic L-groups of Dedekind rings.** In this section we show that the classical symmetric and quadratic Grothendieck-Witt groups of certain Dedekind rings are finitely generated. By Theorem 1 it will suffice to prove the finite generation of the corresponding L-groups, provided the finite generation of the K-groups is known. On the L-theory side we in fact do much more: We give a full calculation of the quadratic and symmetric L-groups of Dedekind rings whose field of fractions is not of characteristic 2. We first treat the symmetric case, where we need to make the boundary map of the localisation-dévissage sequence explicit: Shifting the L-theory fibre sequence of Corollary 2.1.9 once to the right, we obtain a fibre sequence

$$\mathrm{L}^{\mathrm{s}}(R) \longrightarrow \mathrm{L}^{\mathrm{s}}(K) \xrightarrow{\partial} \bigoplus_{\mathfrak{p}} \mathrm{L}^{\mathrm{s}}(\mathbb{F}_{\mathfrak{p}})$$

where we recall that R is a Dedekind ring with field of fractions K, and $\mathbb{F}_{\mathfrak{p}}$ is the residue field at a prime ideal $\mathfrak{p}$ of R. As we shall use it momentarily, let us make the effect of the map ∂ on π_0 explicit. Clearly, it suffices to describe the composite of ∂ with the projection to $\mathrm{L}^{\mathrm{s}}(\mathbb{F}_{\mathfrak{p}})$ for each prime $\mathfrak{p}$ of R. By naturality of the dévissage theorem and the localisation sequence, to describe this composition we may replace R by its localisation $R_{(\mathfrak{p})}$ which is a local Dedekind ring and hence a discretely valued ring, as the choice of uniformiser for $\mathfrak{p}$ is (by definition) also a uniformiser for $\mathfrak{p}$, viewed as prime ideal in $R_{(\mathfrak{p})}$. Without loss of generality, we may hence assume that R was a discretely valued ring to begin with. Let π be a uniformiser of the maximal ideal of R, so that every non-zero element in K is uniquely of the form $\pi^i u$ for some unit u in R. Clearly, it suffices to describe the map $\partial_0\colon \mathrm{L}^{\mathrm{s}}_0(K) \to \mathrm{L}^{\mathrm{s}}_0(\mathbb{F}_{\mathfrak{p}})$ on generators of the L-group, which are given by the forms $\langle x \rangle = (K, x)$ for units x of K, where we have identified canonically $\pi_0(\Qoppa^{\mathrm{s}}(K))$ with K. Indeed, if the characteristic of K is not 2, then every form itself is isomorphic to a diagonal form. If the characteristic is 2, then any unimodular form is the sum of a diagonalisable form and one which admits a Lagrangian, see [MH73, I §3]. By a change of basis, one finds the relation $\langle x \rangle = \langle xy^2 \rangle$ for any other unit y. We may thus suppose without loss of generality that x is either of the form πu or of the form u, again for u a unit in R. By exactness of the localisation-dévissage sequence, we have $\partial_0\langle u\rangle = 0$, so it remains to describe $\partial_0\langle \pi u\rangle$.

2.2.1. **Lemma.** *In the notation just established, we have $\partial_0(\langle \pi u\rangle) = \langle u\rangle$ in $\mathrm{L}_0^{\mathrm{s}}(\mathbb{F}_{\mathfrak{p}})$. Here we view u also as a unit of $\mathbb{F}_{\mathfrak{p}}$ via the projection $R \to \mathbb{F}_{\mathfrak{p}}$. In particular, the map $\mathrm{L}_0^{\mathrm{s}}(K) \to \mathrm{L}_0^{\mathrm{s}}(\mathbb{F}_{\mathfrak{p}})$ is surjective.*

Proof. We note that, by construction, the composite

$$\mathrm{L}_0^{\mathrm{s}}(K) \xrightarrow{\partial_0} \mathrm{L}_0^{\mathrm{s}}(\mathbb{F}_p) \xrightarrow{\cong} \mathrm{L}_0(\mathcal{D}^{\mathrm{p}}(R)_{(\pi)}, (Ϙ^{\mathrm{s}})^{[1]}),$$

where the second map is the dévissage isomorphism of Theorem 2.1.8, is the boundary map associated to the Poincaré-Verdier sequence

$$(\mathcal{D}^{\mathrm{p}}(R)_{(\pi)}, Ϙ^{\mathrm{s}}) \longrightarrow (\mathcal{D}^{\mathrm{p}}(R), Ϙ^{\mathrm{s}}) \longrightarrow (\mathcal{D}^{\mathrm{p}}(K), Ϙ^{\mathrm{s}}).$$

It hence suffices to prove that the image of $\langle u\rangle$ under the dévissage isomorphism is mapped to the image of $\langle \pi u\rangle$ under this boundary map. The boundary map for this localisation sequence is explicitly described as follows, see Proposition [II].4.4.8. Starting with the Poincaré object $(K, \pi u)$, we may view the hermitian object $(R, \pi u)$ as a surgery datum on 0 for $(Ϙ^{\mathrm{s}})^{[1]}$. The output of surgery is a Poincaré object for $(\mathcal{D}^{\mathrm{p}}(R)_{(\pi)}, (Ϙ^{\mathrm{s}})^{[1]})$ which is the value of the boundary map at $(K, \pi u)$. It is immediate from the diagram of Remark 1.1.14 that the underlying object of the surgery output is the complex $R/\pi u$. In order to describe its Poincaré form we interpret the surgery as an instance of the algebraic Thom isomorphism, which induces a map from (possibly degenerate) forms for $Ϙ$ to Poincaré objects for $Ϙ^{[1]}$. In general, this map is implemented by the following construction. Let α be a form in $\pi_0(Ϙ(L))$ regarded as a surgery datum (L, α) on 0 for $Ϙ^{[1]}$. The output of surgery is a Poincaré form on the cofibre C of the map $\alpha_\sharp\colon L \to \mathrm{D}L$. The form is constructed from the cofibre sequence

$$Ϙ(\mathrm{D}L) \longrightarrow Ϙ(L) \times_{\hom(L,\mathrm{D}L)} \hom(\mathrm{D}L, \mathrm{D}L) \xrightarrow{\partial} \SigmaϘ(C), \tag{8}$$

from Example [I].1.1.21, by noticing that the pair $(\alpha, \mathrm{id}_{\mathrm{D}L})$ canonically refines to a point in the pullback above. Its image under ∂ is the form on C we seek. We wish to make the boundary map ∂ explicit, as the form on C is the image of an explicit point in the pullback. We again take a step back and consider a general cofibre sequence $X \to Y \to C$ instead of $L \to \mathrm{D}L \to C$. The above cofibre sequence (8) is a reformulation of the fact that $Ϙ(C)$ is the total fibre of the left square in the diagram of horizontal cofibre sequences

$$\begin{array}{ccccc} Ϙ(Y) & \longrightarrow & Ϙ(X) & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{B}(Y,X) & \longrightarrow & \mathrm{B}(X,X) & \longrightarrow & M' \end{array} \tag{9}$$

where the left vertical map is the composite $Ϙ(Y) \to \mathrm{B}(Y,Y) \to \mathrm{B}(Y,X)$ where the latter map is induced by the map $X \to Y$. We note that $\SigmaϘ(C)$, being the suspension of the total fibre, is equivalently given by the fibre of the map of horizontal cofibres, i.e. the fibre of the map $M \to M'$ displayed above. The boundary map above is then given as follows: a map to the fibre of $M \to M'$ consists of a map to M, together with a null homotopy of the composite to M'. Now the pullback of the left upper square canonically maps to $Ϙ(A)$ which in turn maps to M. The composite of this map to M' factors through the lower horizontal cofibre sequence and is thus canonically trivialised. These two pieces of data together determine the map from the pullback to $\SigmaϘ(C)$.

Spelling this diagram out for the fibre sequence $R \xrightarrow{x} R \to R/x$, for an element x in R, we obtain the diagram

$$\begin{array}{ccccc} Ϙ^{\mathrm{s}}(R) & \xrightarrow{(\cdot x)^*} & Ϙ^{\mathrm{s}}(R) & \longrightarrow & (R/x^2)^{\mathrm{hC}_2} \\ \downarrow & & \downarrow & & \downarrow \\ \hom_R(R,R) & \xrightarrow{(\cdot x)^*} & \hom_R(R,R) & \longrightarrow & R/x \end{array}$$

where we have used the identification $Ϙ^{\mathrm{s}}(R) = R^{\mathrm{hC}_2}$ and that the map induced by multiplication with x on R becomes the map induced by multiplication with x^2. We notice that the left vertical map identifies with the composite of the forgetful map $R^{\mathrm{hC}_2} \to R$ and the multiplication by x on R. Now the point in the pullback is given by the point 1 in R, the point x in R^{hC_2} and the canonical identification of their images in R along the two maps. Since R has no nonzero zero divisors, R/x is discrete, and hence the first map in the sequence

$$\pi_0(\Sigma\Qoppa^{\mathrm{s}}_R(R/x)) \longrightarrow \pi_0((R/x^2)^{\mathrm{hC}_2}) \longrightarrow \pi_0(R/x)$$

is injective. We hence obtain an isomorphism $\alpha_x\colon (x)/(x)^2 \to \pi_0(\Sigma\Qoppa^{\mathrm{s}}_R(R/x))$, as the latter of the above two maps canonically identifies with the map $R/x^2 \to R/x$. Now, the image of the point $(x,1)$ in $\pi_0((R/x^2)^{\mathrm{hC}_2}) \cong R/x^2$ is the image of x in the quotient R/x^2. Summarising, we find that for the element πu of R, we have that the output of surgery is given by the pair

$$\big(R/(\pi u), \alpha_{\pi u}(\pi u) \in \pi_0(\Sigma\Qoppa^{\mathrm{s}}_R(R/(\pi u)))\big).$$

Next, we claim that the diagram

$$\begin{array}{ccc}
\Qoppa^{\mathrm{s}}_{\mathbb{F}_{\mathfrak{p}}}(R/\pi) \simeq (R/\pi)^{\mathrm{hC}_2} & \xrightarrow{p^*} & \hom_R(R/\pi \otimes_R R/\pi, R/\pi)^{\mathrm{hC}_2} \\
& \searrow^{\mathrm{dev}_\pi} & \downarrow{\scriptstyle \beta_\pi} \\
& & \hom_R(R/\pi \otimes_R R/\pi, R[1])^{\mathrm{hC}_2}
\end{array}$$

commutes, where dev_π denotes the map induced from the Poincaré functor p^* as constructed after Lemma 2.1.6. This follows from the observation that the counit $p^*p_*(R)[1] \to R[1]$ identifies with the Bockstein map $R/\pi \to R[1]$, see Remark 2.1.7. It is easy to see that both upper terms have π_0 canonically isomorphic to R/π, and that the horizontal map identifies with the identity. To identify the effect of the Bockstein map on π_0 of the above diagram, we note that the functor

$$\Phi(X) = \hom_R(X \otimes_R X, R/\pi)^{\mathrm{hC}_2}$$

is a quadratic functor (though not non-degenerate), so we may apply the same method as for $\Sigma\Qoppa^{\mathrm{s}}_R(R/x)$, indicated in diagram (9), to calculate $\Phi(R/x)$. The Bockstein map induces a comparison map of the diagrams. Passing to π_0 in the above triangle gives the diagram

$$\begin{array}{ccccc}
R/\pi & \longrightarrow & \pi_0(\hom_R(R/\pi \otimes_R R/\pi, R/\pi)^{\mathrm{hC}_2}) & \longleftarrow & R/\pi \\
& \searrow & \downarrow & & \vdots \\
& & \pi_0(\hom_R(R/\pi \otimes_R R/\pi, R[1])^{\mathrm{hC}_2}) & \xleftarrow[\alpha_\pi]{} & (\pi)/(\pi)^2
\end{array}$$

and from this it is an explicit calculation to see that the dashed vertical map in the above diagram is the map given by multiplication by π. We deduce that the map dev_π takes the form u on R/π to the form $\alpha_\pi(\pi u)$ on R/π. We are thus left to compare the elements

$$(R/\pi, \alpha_\pi(\pi u)) \quad \text{and} \quad (R/(\pi u), \alpha_{\pi u}(\pi u))$$

of $\mathrm{L}_0(\mathcal{D}^{\mathrm{p}}(R)_{\mathfrak{p}}, \Sigma\Qoppa^{\mathrm{s}})$. For this, we need to work out a formula for the isomorphism $\alpha_{\pi u}$ in terms of α_π. This we do as follows. We consider the diagram

$$\begin{array}{ccccc}
R & \xrightarrow{\pi} & R & \longrightarrow & R/\pi \\
\| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u} \\
R & \xrightarrow{\pi u} & R & \longrightarrow & R/\pi u
\end{array}$$

which induces a map $u^*\colon \Sigma\Qoppa^{\mathrm{s}}(R/\pi u) \to \Sigma\Qoppa^{\mathrm{s}}(R/\pi)$. One checks that the diagram

$$\begin{array}{ccc}
(\pi u)/(\pi u)^2 & \xrightarrow{\alpha_{\pi u}} & \pi_0(\Sigma\Qoppa^{\mathrm{s}}(R/\pi u)) \\
\downarrow{\scriptstyle u^{-2}} & & \downarrow{\scriptstyle u^*} \\
(\pi)/(\pi)^2 & \xrightarrow{\alpha_\pi} & \pi_0(\Sigma\Qoppa^{\mathrm{s}}(R/\pi))
\end{array}$$

commutes. In formulas, we find that

$$u^*(\alpha_{\pi u}(\pi u)) = \alpha_\pi(\pi u^{-1})$$

so that we obtain that the output of surgery on $(R, \pi u)$ is given by the pair $(R/\pi, \alpha_\pi(\pi u^{-1}))$ which equals the image of $(R/\pi, u^{-1})$ under the dévissage map. Since the two forms u^{-1} and u differ by a square, they represent the same class in L-theory, so we finally deduce the lemma. □

2.2.2. **Remark.** Lemma 2.2.1 identifies the map $\partial_0 \colon \mathrm{L}^{\mathrm{s}}_0(K) \to \bigoplus_{\mathfrak{p}} \mathrm{L}^{\mathrm{s}}_0(\mathbb{F}_{\mathfrak{p}})$ with the map induced by the maps $\psi^1 \colon \mathrm{W}^{\mathrm{s}}(K) \to \mathrm{W}^{\mathrm{s}}(\mathbb{F}_{\mathfrak{p}})$ constructed in [MH73, Chapter IV §1].

We obtain the following calculation. Recall that a prime is called dyadic if it contains the ideal (2).

2.2.3. **Corollary.** *Let R be a Dedekind ring whose field of fractions K is not of characteristic 2, and let $\mathfrak{I}$ be the (finite) set of dyadic primes of R. Then we have*

$$\mathrm{L}^{\mathrm{s}}_n(R) \cong \begin{cases} \mathrm{W}^{\mathrm{s}}(R) & \text{for } n \equiv 0(4) \\ \bigoplus_{\mathfrak{p}\in\mathfrak{I}} \mathrm{W}^{\mathrm{s}}(\mathbb{F}_{\mathfrak{p}}) & \text{for } n \equiv 1(4) \\ 0 & \text{for } n \equiv 2(4) \\ \mathrm{coker}(\partial_0) & \text{for } n \equiv 3(4) \end{cases}$$

Proof. The case $n \equiv 0(4)$ follows from combining Corollary 1.2.12 and Corollary 1.3.13. Since the symmetric L-groups of K and each residue field $\mathbb{F}_{\mathfrak{p}}$ vanish in odd degrees by Corollary 1.3.5, the long exact sequence in L-groups furnished by Corollary 2.1.9 yields for every k an exact sequence

$$0 \longrightarrow \mathrm{L}^{\mathrm{s}}_{2k}(R) \longrightarrow \mathrm{L}^{\mathrm{s}}_{2k}(K) \xrightarrow{\partial_{2k}} \bigoplus\nolimits_{\mathfrak{p}} \mathrm{L}^{\mathrm{s}}_{2k}(\mathbb{F}_{\mathfrak{p}}) \longrightarrow \mathrm{L}^{\mathrm{s}}_{2k-1}(R) \longrightarrow 0.$$

This shows the case $n \equiv 2(4)$, as for odd numbers k, the group $\mathrm{L}^{\mathrm{s}}_{2k}(K)$ is isomorphic to the anti-symmetric Witt group of K by Corollary 1.3.5, which vanishes as the characteristic of K is not 2. The remaining cases are obvious from the above exact sequence, making use of the fact that the symmetric L-theory of fields of characteristic 2, like $\mathbb{F}_{\mathfrak{p}}$ for dyadic primes, is 2-periodic, whereas the symmetric L-theory of fields of odd characteristic vanishes in degrees different from $0 \equiv 4$, see Remark 1.3.6. □

2.2.4. **Corollary.** *Under the assumptions of Corollary 2.2.3, assume in addition that K is a global field and let $d = |\mathfrak{I}|$ be the (finite) number of dyadic primes of R. Then we have*

$$\mathrm{L}^{\mathrm{s}}_n(R) = \begin{cases} \mathrm{W}^{\mathrm{s}}(R) & \text{for } n \equiv 0(4) \\ (\mathbb{Z}/2)^d & \text{for } n \equiv 1(4) \\ 0 & \text{for } n \equiv 2(4) \\ \mathrm{Pic}(R)/2 & \text{for } n \equiv 3(4) \end{cases}$$

Proof. The case $n \equiv 1(4)$ follows since the assumption that K is global says that the residue fields $\mathbb{F}_{\mathfrak{p}}$ at non-zero primes are finite fields. The claim then follows from the fact that the symmetric Witt group of a finite field of characteristic 2 is given by $\mathbb{Z}/2$. For the other non-trivial case, Lemma 2.2.1 gives the following commutative diagram

$$\begin{array}{ccc} \mathrm{W}^{\mathrm{s}}(K) & \xrightarrow{\psi^1} & \bigoplus_{\mathfrak{p}} \mathrm{W}^{\mathrm{s}}(\mathbb{F}_{\mathfrak{p}}) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{L}^{\mathrm{s}}_0(K) & \xrightarrow{\partial_0} & \bigoplus_{\mathfrak{p}} \mathrm{L}^{\mathrm{s}}_0(\mathbb{F}_{\mathfrak{p}}) \end{array}$$

It is then shown in [MH73, Chapter IV §4] that the cokernel of the upper horizontal map is given by $\mathrm{Pic}(R)/2$, provided K is a number field. In [Sch12, Chapter 6, §6, Theorem 6.11] this is extended to hold for a general global field K. □

2.2.5. **Remark.** We recall that there is a canonical equivalence $\mathrm{L}^{-\mathrm{s}}(R) \simeq \Sigma^2 \mathrm{L}^{\mathrm{s}}(R)$, so that Corollaries 2.2.3 and 2.2.4 also determine the (-1)-symmetric L-groups.

When the fraction field of R is a global field of characteristic 2 we have a similar result:

2.2.6. **Corollary.** *Let R be a Dedekind ring whose field of fractions K is a global field of characteristic* 2*. Then*

$$\mathrm{L}^{\mathrm{s}}_n(R) = \begin{cases} \mathrm{W}^{\mathrm{s}}(R) & \text{for } n \equiv 0(2) \\ \mathrm{Pic}(R)/2 & \text{for } n \equiv 1(2) \end{cases}$$

Proof. As in the proof of Corollary 2.2.3, we have an exact sequence

$$0 \longrightarrow \mathrm{L}^{\mathrm{s}}_{2k}(R) \longrightarrow \mathrm{L}^{\mathrm{s}}_{2k}(K) \xrightarrow{\partial_{2k}} \bigoplus_{\mathfrak{p}} \mathrm{L}^{\mathrm{s}}_{2k}(\mathbb{F}_{\mathfrak{p}}) \longrightarrow \mathrm{L}^{\mathrm{s}}_{2k-1}(R) \longrightarrow 0.$$

Since R is an $\mathbb{F}_2$-algebra, the L-groups $\mathrm{L}^{\mathrm{s}}_n(R)$ are 2-periodic and since K is a global field, all residue fields $\mathbb{F}_{\mathfrak{p}}$ are (finite) fields of characteristic 2. We recall that there is an exact sequence of abelian groups

$$K^{\times} \xrightarrow{\mathrm{div}} \mathrm{Div}(R) \longrightarrow \mathrm{Pic}(R) \longrightarrow 0$$

where $\mathrm{Div}(R)$ is the free abelian group generated by the prime ideals of R, and the group homomorphism div is determined by the following: For a non-zero element x of R, write $(x) = \mathfrak{p}_1^{r_1} \cdot \cdots \cdot \mathfrak{p}_n^{r_n}$ with natural numbers r_i. Then $\mathrm{div}(x) = \sum_{i=1}^n r_i \cdot \mathfrak{p}_i$. Now consider the diagram

$$\begin{array}{ccccccc} \mathbb{Z}/2[K^{\times}] & \longrightarrow & \mathrm{Div}(R)/2 & \longrightarrow & \mathrm{Pic}(R)/2 & \longrightarrow & 0 \\ \downarrow{\scriptstyle\langle - \rangle} & & \downarrow{\scriptstyle\cong} & & \downarrow & & \\ \mathrm{W}^{\mathrm{s}}(K) & \xrightarrow{\partial_0} & \bigoplus_{\mathfrak{p}} \mathrm{W}^{\mathrm{s}}(\mathbb{F}_{\mathfrak{p}}) & \longrightarrow & \mathrm{L}^{\mathrm{s}}_1(R) & \longrightarrow & 0 \end{array}$$

consisting of exact horizontal sequences and the left most top vertical map is induced by the map div above. Here, the middle vertical isomorphism is induced from the isomorphism $\mathrm{Div}(R)/2 \cong \bigoplus_{\mathfrak{p}} \mathbb{Z}/2$ and the isomorphisms $\mathrm{W}^{\mathrm{s}}(\mathbb{F}_{\mathfrak{p}}) \cong \mathbb{Z}/2$. The left square commutes by an explicit check, so that there exists a dashed arrow as indicated. By construction, the dashed map is a surjection, and an injection by the observation that the left most vertical map is surjective, as the Witt group $\mathrm{W}^{\mathrm{s}}(K)$ is generated by the forms $\langle x \rangle$ for $x \in K^{\times}$ by [MH73, I §3]. □

2.2.7. **Remark.** Let R be a local Dedekind ring with fraction field K and residue field k. Then the map $\partial_0 \colon \mathrm{L}^{\mathrm{s}}_0(K) \to \mathrm{L}^{\mathrm{s}}_0(k)$ is surjective by Lemma 2.2.1, and we have seen earlier that its kernel is $\mathrm{L}^{\mathrm{s}}_0(R)$. Assuming that the characteristic of K is not 2, we deduce from Corollary 2.2.3 that $\mathrm{L}^{\mathrm{s}}_n(R)$ vanishes for $n \equiv 2, 3 \mod (4)$. Furthermore, for $n \equiv 1 \mod (4)$, we find that $\mathrm{L}^{\mathrm{s}}_n(R)$ is either isomorphic to $\mathrm{W}^{\mathrm{s}}(k)$, if the characteristic of k is 2, or is trivial otherwise. If the characteristic of K is 2, we deduce from the proof of Corollary 2.2.6 that $\mathrm{L}^{\mathrm{s}}_1(R) = 0$.

We now want to give a formula for the quadratic L-groups of Dedekind rings, similar to Corollary 2.2.3 and Corollary 2.2.4. We set out to prove the following result:

2.2.8. **Proposition.** *Let R be a Dedekind ring whose field of fractions K is not of characteristic 2, and let $\mathcal{I}$ be the (finite) set of dyadic primes of R. Then we have*

$$\mathrm{L}^{\mathrm{q}}_n(R) \cong \begin{cases} \mathrm{W}^{\mathrm{q}}(R) & \text{for } n \equiv 0(4) \\ 0 & \text{for } n \equiv 1(4) \\ \bigoplus_{\mathfrak{p} \in \mathcal{I}} \mathrm{W}^{\mathrm{q}}(\mathbb{F}_{\mathfrak{p}}) & \text{for } n \equiv 2(4) \end{cases}$$

The isomorphism in degrees $n \equiv 2(4)$ is induced by the canonical maps $R \to \mathbb{F}_{\mathfrak{p}}$ for each dyadic prime $\mathfrak{p}$. For $n \equiv 3(4)$ there is a short exact sequence

$$0 \longrightarrow A \longrightarrow \mathrm{L}^{\mathrm{q}}_n(R) \longrightarrow \mathrm{L}^{\mathrm{s}}_n(R) \longrightarrow 0$$

where A is the total cokernel, that is the cokernel of the map induced on cokernels, of the commutative square

$$\begin{array}{ccc} \mathrm{L}^{\mathrm{q}}_0(R) & \longrightarrow & \mathrm{L}^{\mathrm{s}}_0(R) \\ \downarrow & & \downarrow \\ \mathrm{L}^{\mathrm{q}}_0(R^{\wedge}_2) & \longrightarrow & \mathrm{L}^{\mathrm{s}}_0(R^{\wedge}_2) \end{array}$$

Proof. The canonical map $\mathrm{W}^{\mathrm{q}}(R) \to \mathrm{L}_0^{\mathrm{q}}(R)$ is an isomorphism by Corollary 1.2.12. To see the other cases, we consider the cube

$$\begin{array}{ccccccc}
 & & \mathrm{L}^{\mathrm{s}}(R) & & \longrightarrow & & \mathrm{L}^{\mathrm{s}}(R_2^\wedge) \\
 & \nearrow & & & & \nearrow & \\
\mathrm{L}^{\mathrm{q}}(R) & & \longrightarrow & & \mathrm{L}^{\mathrm{q}}(R_2^\wedge) & & \downarrow \\
 & & \downarrow & & & & \\
\downarrow & & \mathrm{L}^{\mathrm{s}}(R[\frac{1}{2}]) & & \longrightarrow & & \mathrm{L}^{\mathrm{s}}(R_2^\wedge[\frac{1}{2}]) \\
 & \nearrow & & & \downarrow & \nearrow & \\
\mathrm{L}^{\mathrm{q}}(R[\frac{1}{2}]) & & \longrightarrow & & \mathrm{L}^{\mathrm{q}}(R_2^\wedge[\frac{1}{2}]) & &
\end{array}$$

which is obtained by mapping the quadratic localisation-completion square appearing in Proposition 2.1.12 to the symmetric one. We note that no control terms are needed since localisations of Dedekind rings induce surjections on K_0. In this cube, the front and back squares are pullbacks by Proposition 2.1.12, and the bottom square is a pullback since in all rings that appear 2 is invertible. We deduce that the diagram

$$\begin{array}{ccc}
\mathrm{L}^{\mathrm{q}}(R) & \longrightarrow & \mathrm{L}^{\mathrm{q}}(R_2^\wedge) \\
\downarrow & & \downarrow \\
\mathrm{L}^{\mathrm{s}}(R) & \longrightarrow & \mathrm{L}^{\mathrm{s}}(R_2^\wedge)
\end{array} \tag{10}$$

is also a pullback.

Now all remaining statements to be proven follow from the long exact Mayer-Vietoris sequence associated to this pullback, using the following:

i) $\mathrm{L}_n^{\mathrm{q}}(R_2^\wedge) = 0$ for odd n, by Proposition 2.1.13,
ii) $\mathrm{L}_n^{\mathrm{s}}(R_2^\wedge) = 0$ for $n \equiv 3(4)$, because $R_2^\wedge$ is a product of local Dedekind rings; see Remark 2.2.7,
iii) $\mathrm{L}_n^{\mathrm{s}}(R) = \mathrm{L}_n^{\mathrm{s}}(R_2^\wedge) = 0$ for $n \equiv 2(4)$ by Corollary 2.2.3, and
iv) the map $\mathrm{L}_1^{\mathrm{s}}(R) \to \mathrm{L}_1^{\mathrm{s}}(R_2^\wedge)$ is an isomorphism. This can be seen from the localisation-completion square for symmetric L-theory and iii).

□

2.2.9. **Corollary.** *Under the assumptions of Proposition 2.2.8, assume in addition that K is a number field and let $d = |\mathfrak{I}|$ be the (finite) number of dyadic primes of R. Then we have*

$$\mathrm{L}_n^{\mathrm{q}}(R) \cong \begin{cases} \mathrm{W}^{\mathrm{q}}(R) & \text{for } n \equiv 0(4) \\ 0 & \text{for } n \equiv 1(4) \\ (\mathbb{Z}/2)^d & \text{for } n \equiv 2(4) \end{cases}$$

The invariants in the case $n \equiv 2(4)$ are given by the Arf invariants of the images in the L-theory of $\mathbb{F}_{\mathfrak{p}}$ for each dyadic prime $\mathfrak{p}$. Moreover, there is an exact sequence

$$0 \longrightarrow A \longrightarrow \mathrm{L}_{-1}^{\mathrm{q}}(R) \longrightarrow \mathrm{Pic}(R)/2 \longrightarrow 0$$

where A is as in Proposition 2.2.8 and is a finite 2-group.

Proof. First we recall from Corollary 2.2.4 that for $n \equiv 3(4)$, we have $\mathrm{L}^{\mathrm{s}}(R) \cong \mathrm{Pic}(R)/2$, and that A is a quotient of $\mathrm{L}_0^{\mathrm{s}}(R_2^\wedge)$. We have $(2) = (\mathfrak{p}_1^{e_1} \cdot \dots \cdot \mathfrak{p}_k^{e_k})$ for some numbers e_i, where the $\mathfrak{p}_i$ are the dyadic primes. It follows that there is an isomorphism

$$\mathrm{L}_0^{\mathrm{s}}(R_2^\wedge) \cong \prod_{i=1}^{k} \mathrm{L}_0^{\mathrm{s}}(R_{\mathfrak{p}_i}^\wedge).$$

It thus suffices to recall that

i) the map $\mathrm{L}_0^{\mathrm{s}}(R_{\mathfrak{p}_i}^\wedge) \to \mathrm{L}_0^{\mathrm{s}}(R_{\mathfrak{p}_i}^\wedge[\frac{1}{2}])$ is injective; see the proof of Corollary 2.2.3, and that
ii) $\mathrm{L}_0^{\mathrm{s}}(R_{\mathfrak{p}_i}^\wedge[\frac{1}{2}])$ is a finite 2-group: The fraction field $R_{\mathfrak{p}_i}^\wedge[\frac{1}{2}]$ of $R_{\mathfrak{p}_i}^\wedge$ is a finite extension of $\mathbb{Q}_2$, so we may appeal to [Lam05, Theorem VI 2.29].

Finally, we note that the residue fields $\mathbb{F}_{\mathfrak{p}}$ are finite fields of characteristic 2, so that the Arf invariant provides an isomorphism $\mathrm{W}^{\mathrm{q}}(\mathbb{F}_{\mathfrak{p}}) \cong \mathbb{Z}/2$. □

2.2.10. **Remark.** As in the symmetric case, we recall that there is a canonical equivalence $\mathrm{L}^{-\mathrm{q}}(R) \simeq \Sigma^2 \mathrm{L}^{\mathrm{q}}(R)$, so that Proposition 2.2.8 and Corollary 2.2.9 also determine the (-1)-quadratic L-groups.

2.2.11. **Remark.** If the number d of dyadic primes of R is at least 2, then A is not trivial: Taking the rank mod 2 induces the right horizontal surjections in the following diagram.

$$\begin{array}{ccccc} \mathrm{L}^{\mathrm{q}}_0(R) & \longrightarrow & \mathrm{L}^{\mathrm{s}}_0(R) & \twoheadrightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{L}^{\mathrm{q}}_0(R^\wedge_2) & \longrightarrow & \mathrm{L}^{\mathrm{s}}_0(R^\wedge_2) & \twoheadrightarrow & (\mathbb{Z}/2)^d \end{array}$$

Both horizontal composites are zero, therefore we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{coker}(\mathrm{L}^{\mathrm{q}}_0(R) \to \mathrm{L}^{\mathrm{s}}_0(R)) & \longrightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow \\ \mathrm{coker}(\mathrm{L}^{\mathrm{q}}_0(R^\wedge_2) \to \mathrm{L}^{s}_0(R^\wedge_2)) & \longrightarrow & (\mathbb{Z}/2)^d \end{array}$$

whose horizontal arrows are surjective. The induced map on vertical cokernels is a map $A \to (\mathbb{Z}/2)^{d-1}$ which is therefore again surjective.

2.2.12. **Example.** Let us consider the case $R = \mathbb{Z}$. From the pullback diagram (10), we obtain an exact sequence

$$0 \longrightarrow \mathrm{L}^{\mathrm{q}}_0(\mathbb{Z}) \xrightarrow{(0,8)} \mathrm{L}^{\mathrm{q}}_0(\mathbb{Z}^\wedge_2) \oplus \mathrm{L}^{\mathrm{s}}_0(\mathbb{Z}) \longrightarrow \mathrm{L}^{\mathrm{s}}_0(\mathbb{Z}^\wedge_2) \longrightarrow \mathrm{L}^{\mathrm{q}}_{-1}(\mathbb{Z}) \longrightarrow 0,$$

where the map $\mathrm{L}^{\mathrm{q}}_0(\mathbb{Z}) \to \mathrm{L}^{\mathrm{q}}_0(\mathbb{Z}^\wedge_2)$ is the zero map: By Proposition 2.1.13, it suffices to know that the map $\mathrm{L}^{\mathrm{q}}_0(\mathbb{Z}) \to \mathrm{L}^{\mathrm{q}}_0(\mathbb{F}_2)$ is the zero map. For this, one calculates that the Arf invariant of the E_8-form (viewed as a form over $\mathbb{F}_2$) is zero. Furthermore, the map $\mathrm{L}^{\mathrm{q}}_0(\mathbb{Z}) \to \mathrm{L}^{\mathrm{s}}_0(\mathbb{Z})$ is isomorphic to multiplication by 8, as the E_8 form generates $\mathrm{L}^{\mathrm{q}}_0(\mathbb{Z})$. We therefore obtain a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/8 \longrightarrow \mathrm{L}^{\mathrm{s}}_0(\mathbb{Z}^\wedge_2) \longrightarrow \mathrm{L}^{\mathrm{q}}_{-1}(\mathbb{Z}) \longrightarrow 0. \tag{11}$$

Furthermore, by localisation-dévissage, there is a short exact sequence

$$0 \longrightarrow \mathrm{L}^{\mathrm{s}}_0(\mathbb{Z}^\wedge_2) \longrightarrow \mathrm{L}^{\mathrm{s}}_0(\mathbb{Q}_2) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

and from [Lam05, Theorem 2.29 & Corollary 2.23], we know that $\mathrm{L}^{\mathrm{s}}_0(\mathbb{Q}_2)$ has 32 elements. We deduce that $\mathrm{L}^{\mathrm{s}}_0(\mathbb{Z}^\wedge_2)$ has 16 elements, and hence the above injection $\mathbb{Z}/2 \oplus \mathbb{Z}/8 \subseteq \mathrm{L}^{\mathrm{s}}_0(\mathbb{Z}^\wedge_2)$ is an isomorphism. For completeness, we observe that the exact sequence involving $\mathrm{L}^{\mathrm{s}}_0(\mathbb{Q}_2)$ splits, so one obtains the well known isomorphism $\mathrm{L}^{\mathrm{s}}_0(\mathbb{Q}_2) \cong (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/8$ [Lam05, Theorem 2.29]. A concrete splitting is given by the element $\langle -1, 2\rangle$. The only thing that needs checking is that this element has order 2.

From the above and the exact sequence (11), we find that $\mathrm{L}^{\mathrm{q}}_{-1}(\mathbb{Z}) = 0$. In particular, we obtain the well known calculations of the symmetric and quadratic L-groups of $\mathbb{Z}$:

$$\mathrm{L}^{\mathrm{s}}_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n \equiv 0(4) \\ \mathbb{Z}/2 & \text{for } n \equiv 1(4) \\ 0 & \text{for } n \equiv 2(4) \\ 0 & \text{for } n \equiv 3(4) \end{cases} \qquad \mathrm{L}^{\mathrm{q}}_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n \equiv 0(4) \\ 0 & \text{for } n \equiv 1(4) \\ \mathbb{Z}/2 & \text{for } n \equiv 2(4) \\ 0 & \text{for } n \equiv 3(4) \end{cases}$$

Together with Theorem 1.2.18, Corollary 1.3.13, and Remark 1.3.15 this determines $\mathrm{L}^{\mathrm{gs}}_n(\mathbb{Z})$. In addition, we find that the map $\mathrm{L}^{\mathrm{gs}}(\mathbb{Z})[\frac{1}{2}] \to \mathrm{L}^{\mathrm{s}}(\mathbb{Z})[\frac{1}{2}]$ is an equivalence. We will make use of this fact in Proposition 3.1.13.

2.2.13. **Example.** Consider the quadratic extension $K = \mathbb{Q}[\sqrt{-3}]$ of $\mathbb{Q}$ and let R be its ring of integers. Concretely, R is the ring of Eisenstein integers $R = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, which is a euclidean domain and hence a principal ideal domain. The discriminant of K is (3), and as (2) does not divide (3), we deduce that (2) is a prime ideal in R [Neu99, Corollary III.2.12], and hence is the single dyadic prime. We deduce that

$\mathrm{L}^{\mathrm{s}}_2(R) = \mathrm{L}^{\mathrm{s}}_3(R) = 0$, as the Picard group of a principal ideal domain vanishes. Furthermore $\mathrm{L}^{\mathrm{s}}_1(R) \cong \mathbb{Z}/2$ and $\mathrm{L}^{\mathrm{s}}_0(R) \cong \mathrm{W}^{\mathrm{s}}_0(R) \cong \mathbb{Z}/4$ [MH73, Corollary 4.2]. To calculate the quadratic L-groups we consider the diagram of exact sequences

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{L}^{\mathrm{q}}_0(\mathbb{Z}) & \xrightarrow{(8,0)} & \mathbb{Z}\oplus\mathbb{Z}/2 & \longrightarrow & \mathrm{L}^{\mathrm{s}}_0(\mathbb{Z}^\wedge_2) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (\mathrm{pr},\mathrm{id})} & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{L}^{\mathrm{q}}_0(R) & \longrightarrow & \mathbb{Z}/4\oplus\mathbb{Z}/2 & \longrightarrow & \mathrm{L}^{\mathrm{s}}_0(R^\wedge_2) & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

and deduce that $A \cong \mathrm{coker}(\theta)$ and that there is an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \ker(\theta) \longrightarrow \mathrm{L}^{\mathrm{q}}_0(R) \longrightarrow 0.$$

Now, from the commutative diagram of localisation-dévissage sequences (note that 2 is a uniformiser in both cases)

$$\begin{array}{ccccc} \mathrm{L}^{\mathrm{s}}(\mathbb{Z}^\wedge_2) & \longrightarrow & \mathrm{L}^{\mathrm{s}}(\mathbb{Q}_2) & \longrightarrow & \mathrm{L}^{\mathrm{s}}(\mathbb{Z}/(2)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{L}^{\mathrm{s}}(R^\wedge_2) & \longrightarrow & \mathrm{L}^{\mathrm{s}}(K^\wedge_2) & \longrightarrow & \mathrm{L}^{\mathrm{s}}(R/(2)) \end{array}$$

we deduce that the kernel and the cokernel of θ are respectively isomorphic to the kernel and the cokernel of the map

$$\theta' : \ \mathrm{L}^{\mathrm{s}}_0(\mathbb{Q}_2) \longrightarrow \mathrm{L}^{\mathrm{s}}_0(K^\wedge_2).$$

It is a general theorem about quadratic extensions of fields that the kernel of θ' is, as an ideal, generated by the element $\langle 1, 3\rangle$, [Lam05, VII Theorem 3.5]. Since $-5/3$ is a square in $\mathbb{Q}_2$, we deduce that $\langle 1, 3\rangle = \langle 1, -5\rangle$. From [Lam05, VI Remark 2.31], we then deduce that the kernel of θ' is spanned by $4\langle 1\rangle$ and $\langle 1, 3\rangle$ and thus isomorphic to $(\mathbb{Z}/2)^2$. We deduce that $\mathrm{L}^{\mathrm{q}}_0(R) \cong \mathbb{Z}/2$. From [Lam05, VII Theorem 3.5], we also find that

$$\mathrm{coker}(\theta') \cong \ker\left(\mathrm{L}^{\mathrm{s}}_0(\mathbb{Q}_2) \xrightarrow{\cdot\langle 1,3\rangle} \mathrm{L}^{\mathrm{s}}_0(\mathbb{Q}_2)\right)$$

and again from [Lam05, Remark 2.31], we find that the kernel of $\cdot\langle 1, 3\rangle$ is additively generated by $2\langle 1\rangle$ and $\langle 1, -2\rangle$, and deduce an isomorphism

$$\ker\left(\mathrm{L}^{\mathrm{s}}_0(\mathbb{Q}_2) \xrightarrow{\cdot\langle 1,3\rangle} \mathrm{L}^{\mathrm{s}}_0(\mathbb{Q}_2)\right) \cong \mathbb{Z}/4\oplus\mathbb{Z}/2.$$

In summary, we obtain the following L-groups for R:

$$\mathrm{L}^{\mathrm{s}}_n(R) \cong \begin{cases} \mathbb{Z}/4 & \text{for } n \equiv 0(4) \\ \mathbb{Z}/2 & \text{for } n \equiv 1(4) \\ 0 & \text{for } n \equiv 2(4) \\ 0 & \text{for } n \equiv 3(4) \end{cases} \qquad \mathrm{L}^{\mathrm{q}}_n(R) \cong \begin{cases} \mathbb{Z}/2 & \text{for } n \equiv 0(4) \\ 0 & \text{for } n \equiv 1(4) \\ \mathbb{Z}/2 & \text{for } n \equiv 2(4) \\ \mathbb{Z}/4\oplus\mathbb{Z}/2 & \text{for } n \equiv 3(4) \end{cases}$$

2.2.14. **Remark.** Let R be a Dedekind ring and S a set of primes not containing a dyadic one. We note that in this case, R and R_S have the same 2-adic completions, i.e. the canonical map $R^\wedge_2 \to (R_S)^\wedge_2$ is an isomorphism. We then consider the following diagram

$$\begin{array}{ccccc} \mathrm{L}^{\mathrm{q}}(R) & \longrightarrow & \mathrm{L}^{\mathrm{q}}(R_S) & \longrightarrow & \mathrm{L}^{\mathrm{q}}(R^\wedge_2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{L}^{\mathrm{s}}(R) & \longrightarrow & \mathrm{L}^{\mathrm{s}}(R_S) & \longrightarrow & \mathrm{L}^{\mathrm{s}}(R^\wedge_2). \end{array}$$

We have seen in the proof of Proposition 2.2.8 that the big and the right squares are pullbacks. Therefore, so is the left square. Using that for any non-dyadic prime $\mathfrak{p}$ of R, the residue field $\mathbb{F}_{\mathfrak{p}}$ is a field of odd characteristic, so that its quadratic and symmetric L-theories agree, we deduce from Corollary 2.1.9 that there is a fibre sequence

$$\mathrm{L}^{\mathrm{q}}(R) \longrightarrow \mathrm{L}^{\mathrm{q}}(R_S) \longrightarrow \bigoplus_{\mathfrak{p}\in S} \mathrm{L}^{\mathrm{q}}(\mathbb{F}_{\mathfrak{p}})$$

so that the failure of dévissage in quadratic L-theory is related only to dyadic primes, as indicated in Remark 2.1.11.

2.2.15. **Corollary.** *Let $\mathcal{O}$ be a number ring, that is, a localisation of the rings of integers in a number field away from finitely many primes, and $\epsilon = \pm 1$. Then the ϵ-symmetric L-groups $\mathrm{L}^{\mathrm{s}}_n(\mathcal{O};\epsilon)$ and the ϵ-quadratic L-groups $\mathrm{L}^{\mathrm{q}}_n(\mathcal{O};\epsilon)$ are finitely generated.*

Proof. It follows from Corollaries 2.2.3 and 2.2.9 and Remarks 2.2.5 and 2.2.10 that it suffices to show that the symmetric Witt group $\mathrm{W}^{\mathrm{s}}(\mathcal{O})$, the quadratic Witt group $\mathrm{W}^{\mathrm{q}}(\mathcal{O})$, and the Picard group $\mathrm{Pic}(\mathcal{O})$ are finitely generated. The statement for the symmetric Witt group is proven in [MH73, §4, Theorem 4.1], and in fact $\mathrm{W}^{\mathrm{s}}(\mathcal{O})$ is an extension of a finite group by a free abelian group of rank given by the number of real embeddings of the number field F. Now we claim that generally for a Dedekind ring R whose fraction field K is of characteristic different from 2, the canonical map $\mathrm{W}^{\mathrm{q}}(R) \to \mathrm{W}^{\mathrm{s}}(R)$ is injective, so that $\mathrm{W}^{\mathrm{q}}(R)$ is finitely generated if $\mathrm{W}^{\mathrm{s}}(R)$ is. This follows from the fact that the map $\mathrm{W}^{\mathrm{q}}(R) \to \mathrm{W}^{\mathrm{q}}(K)$ is injective, see [KS71]. As the argument in loc. cit. is not explicitly written out, let us sketch a direct argument that the map $\mathrm{W}^{\mathrm{q}}(R) \to \mathrm{W}^{\mathrm{s}}(R)$ is injective: First, assume that a symmetric form (P,φ) vanishes in $\mathrm{W}^{\mathrm{s}}(R)$. Then the same is true for its image in $\mathrm{W}^{\mathrm{s}}(K)$. By Corollary 1.3.5 we deduce that $(P\otimes_R K, \varphi\otimes_R K)$ admits a strict Lagrangian. The argument written in the proof of [KS71, Lemma 1.4] then shows that (P,φ) indeed itself admits a strict Lagrangian. Now, let (P,q) be a quadratic form whose image in $\mathrm{W}^{\mathrm{s}}(R)$ vanishes. We deduce that the underlying symmetric bilinear form of (P,q) admits a strict Lagrangian L. We then observe that for each x in L, we have $2q(x) = b(x,x) = 0$, so that $q_{|L} = 0$ as R is 2-torsion free. It follows that L is a Lagrangian for the quadratic form (P,q) as needed. Finally, the Picard group of the ring of integers in a number field is finite (in other words, the class number of a ring of integers is finite), and hence the Picard group of a localisation of such a ring receives a surjection from a finite group and is thus itself finite, compare to the proof of Proposition 2.1.4. □

2.2.16. **Remark.** In the above proof, we have again restricted our attention to Dedekind rings whose field of fractions K has characteristic different from 2. If the characteristic of K is 2 we find that:

i) The map $\mathrm{W}^{\mathrm{s}}(R) \to \mathrm{W}^{\mathrm{s}}(K)$ is injective, but
ii) the map $\mathrm{W}^{\mathrm{q}}(R) \to \mathrm{W}^{\mathrm{s}}(R)$ is zero.

Indeed i) follows from the same argument given above, since also for fields K of characteristic 2 a form (P,q) is zero in $\mathrm{W}^{\mathrm{s}}(K)$ if and only if it admits a strict Lagrangian, see Corollary 1.3.5. To see ii), it suffices to show that the composite $\mathrm{W}^{\mathrm{q}}(R) \to \mathrm{W}^{\mathrm{s}}(R) \to \mathrm{W}^{\mathrm{s}}(K)$ vanishes, as the latter map is injective, see the proof of Corollary 2.2.6. This composite factors through the map $\mathrm{W}^{\mathrm{q}}(K) \to \mathrm{W}^{\mathrm{s}}(K)$ which is zero as the underlying bilinear form of any quadratic form over a field of characteristic 2 has a symplectic basis and hence admits a Lagrangian.

2.2.17. **Corollary.** *Let $\mathcal{O}$ be a number ring and $\epsilon = \pm 1$. Then for all $m,n \in \mathbb{Z}$, the groups $\mathrm{L}_n(\mathcal{O};\Omega^{\geq m}_{\epsilon})$, and consequently the groups $\mathrm{GW}_n(\mathcal{O};\Omega^{\geq m}_{\epsilon})$, are finitely generated.*

Proof. We saw in Example 1.1.4 that the functor $\Omega^{\geq m}_{\epsilon}$ is m-quadratic and $(2-m)$-symmetric. Hence, on the one hand, it follows from Corollary 1.2.8 that for $n \leq 2m-2$ the map $\mathrm{L}^{\mathrm{q}}_n(\mathcal{O};\epsilon) \to \mathrm{L}_n(\mathcal{O};\Omega^{\geq m}_{\epsilon})$ is surjective. By Corollary 2.2.15 the left hand group is finitely generated, so the same is true for $\mathrm{L}_n(\mathcal{O};\Omega^{\geq m}_{\epsilon})$.

On the other hand, Corollary 1.3.7 implies that the map $\mathrm{L}_n(\mathcal{O};\Omega^{\geq m}_{\epsilon}) \to \mathrm{L}^{\mathrm{s}}_n(\mathcal{O};\epsilon)$ is injective for $n \geq 2m-1$. Again, by Corollary 2.2.15 the target group is finitely generated, it follows that $\mathrm{L}_n(\mathcal{O};\Omega^{\geq m}_{\epsilon})$ is so as well. To obtain the consequences for Grothendieck-Witt groups, we recall from Quillen's results that the algebraic K-groups of number rings are finitely generated. From the homotopy orbits spectral sequence, it follows that also the homotopy groups of $\mathrm{K}(\mathcal{O};\epsilon)_{\mathrm{hC}_2}$ are finitely generated, so that the desired result follows from the fibre sequence

$$\mathrm{K}(\mathcal{O};\epsilon)_{\mathrm{hC}_2} \longrightarrow \mathrm{GW}(\mathcal{O};\Omega^{\geq m}_{\epsilon}) \longrightarrow \mathrm{L}(\mathcal{O};\Omega^{\geq m}_{\epsilon}).$$

□

Combining the above with the comparison theorem of [HS21] gives the following corollary.

2.2.18. **Corollary.** *Let $\mathcal{O}$ be a number ring, and $\epsilon = \pm 1$. Then the classical ϵ-symmetric and ϵ-quadratic Grothendieck-Witt groups $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},n}(\mathcal{O};\epsilon)$ and $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl},n}(\mathcal{O};\epsilon)$ are finitely generated for all $n \geq 0$.*

2.2.19. **Remark.** The finite generation of the groups $\mathrm{GW}^{\mathrm{q}}(\mathcal{O};\epsilon)$ can also be deduced by a homological stability argument similar to the one of Quillen for algebraic K-theory of number rings: By Serre class theory, it suffices to show that the ordinary homology groups of the components of $\Omega^\infty \mathrm{GW}^{\mathrm{q}}(\mathcal{O};\epsilon)$ are finitely generated. Since every ϵ-quadratic form is a direct summand in an ϵ-hyperbolic form, the group completion theorem identifies any such component with the space

$$\begin{cases} \mathrm{BO}_{\infty,\infty}(\mathcal{O})^+ & \text{for } \epsilon = 1, \\ \mathrm{BSp}^{\mathrm{q}}_\infty(\mathcal{O})^+ & \text{for } \epsilon = -1 \end{cases}$$

where $\mathrm{O}_{\infty,\infty}(\mathcal{O})$ and $\mathrm{Sp}^{\mathrm{q}}_\infty(\mathcal{O})$ denote the colimit of the automorphism group of an n-fold sum of the (1)- and (−1)-quadratic hyperbolic form, respectively. Charney [Cha87] has proved a homological stability result for those groups, so that it suffices to show that the groups $\mathrm{O}_{n,n}(\mathcal{O})$ and $\mathrm{Sp}^{\mathrm{q}}_n(\mathcal{O})$ have finitely generated homology. Let us briefly explain why that is: First we note that both groups are arithmetic. Second, every arithmetic group has a torsion free finite index subgroup [Ser79, 1.3 (4)], and hence also a *normal* torsion free finite index subgroup. By the Serre spectral sequence for the quotient by this normal subgroup, we find that it suffices to know that torsion free arithmetic groups have finitely generated homology, which follows from the fact they they admit a finite classifying space [Ser79, 1.3 (5)]. We wish to thank Manuel Krannich for a helpful discussion about this and for making us aware of Serre's survey.

Finally, we note that in the symmetric case, it is not generally true that every form embeds into a hyperbolic form (as any such form admits a quadratic refinement), so in order to run a similar argument one first needs to find a symmetric bilinear form b such that every other form embeds into a suitable number of orthogonal copies of b, and one needs to prove homological stability for the family of automorphism groups of such orthogonal copies of b. To our knowledge, this is not known to hold in the generality of number rings, though it does hold for the integers.

3. Grothendieck-Witt groups of Dedekind rings

In this final section we consider the homotopy limit problem for Dedekind domains and finite fields of characteristic 2. In the latter case, we extend the solution of the homotopy limit problem from the Grothendieck-Witt space (where it is known to hold by the work of Friedlander) to its Grothendieck-Witt spectrum. We then combine this with the dévissage results of §3.1 to solve the homotopy limit problem for Dedekind rings whose fraction field is a global field of characteristic 0, i.e. a number field, proving Theorem 2 from the introduction. Finally, we apply these ideas to the particular case of $\mathbb{Z}$ and calculate its ± 1-symmetric and genuine ± 1-quadratic Grothendieck-Witt groups conditionally on Vandiver's conjecture, and in the range $n \leq 20000$ unconditionally.

3.1. The homotopy limit problem.

A prominent question in the hermitian K-theory of rings and schemes is when the map from the Grothendieck-Witt space/spectrum to the homotopy fixed points of the associated algebraic K-theory space/spectrum is an equivalence. This question, first raised by Thomason in [Tho83], is commonly known as the *homotopy limit problem*. In the case of fields, the following theorem represents the current state of the art; see [HKO11, BKSØ15, BH20]. We recall that the virtual mod 2 cohomological dimension vcd_2 of a field k can be defined as the ordinary mod 2 cohomological dimension cd_2 of (the absolute Galois group of) $k[\sqrt{-1}]$. In particular, we have $\mathrm{vcd}_2(k) \leq \mathrm{cd}_2(k)$. Given $\epsilon = \pm 1$ we let $\mathrm{K}(k;\epsilon)$ denote the K-theory spectrum of k with C_2-action induced by the duality $\mathrm{D} = \hom_k(-, k(\epsilon))$.

3.1.1. **Theorem.** *Let k be a field of characteristic different from 2 and such that $\mathrm{vcd}_2(k) < \infty$. Then the map of spectra*

$$\mathrm{GW}^{\mathrm{s}}(k;\epsilon) \longrightarrow \mathrm{K}(k;\epsilon)^{\mathrm{hC}_2}$$

is an equivalence after 2-completion.

3.1.2. **Remark.** The cited Theorem 3.1.1 was stated in [BKSØ15] using Schlichting's model for Grothedieck-Witt spectra. Since k is assumed to have characteristic $\neq 2$ we may invoke the comparison statement of Proposition [II].B.2.2 and identify Schlichting's construction with ours.

The characteristic 0 case of Theorem 3.1.1 was proven in [HKO11], while the positive odd characteristic case is established in [BKSØ15]. An alternative proof of this theorem is also provided in recent work of Bachmann and Hopkins [BH20]. Special cases of the above theorem were already known before: the case of the field $\mathbb{C}$ of complex numbers, for example, can be reduced to the classical equivalence $\mathrm{BO} \simeq \mathrm{BU}^{\mathrm{hC}_2}$ see, e.g., [BK05, Lemma 7.3]. In fact, in loc. cit. the authors prove this also for the (-1)-symmetric variant. The equivalence for $\mathbb{C}$ can in turn be used to deduce the same for finite fields $\mathbb{F}_q$. One can express the Grothendieck-Witt spaces of $\mathbb{F}_q$ in terms of the Adams operations on BO and BSp in a way analogous to the main results of Quillen's famous paper [Qui72] on the algebraic K-theory of $\mathbb{F}_q$. These results were first established by Friedlander in [Fri76], and later expanded and refined in see [FP78] (where also a small mistake was corrected in the case of q even and quadratic forms: Friedlander computed $\pi_1(\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(\mathcal{F}_q))$ to be trivial, but in fact it is isomorphic to $\mathbb{Z}/2$). Combined with the positive solution of the homotopy limit problem for $\mathbb{C}$ they imply the following.

3.1.3. **Theorem.** *For $\epsilon = \pm 1$ and every prime power q the natural map*

$$\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(\mathbb{F}_q; \epsilon) \longrightarrow \mathrm{K}(\mathbb{F}_q; \epsilon)^{\mathrm{hC}_2}$$

is an equivalence on connective covers.

The results of [Fri76] and [FP78] on which this approach relies use lengthy computations in the cohomology of various finite matrix groups. We shall now present an alternative and significantly shorter proof of the Theorem 3.1.3 in the case of q even, using Theorem 1 from the introduction. We recall that $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(\mathbb{F}_q; \epsilon) \to \mathrm{GW}^{s}(\mathbb{F}_q; \epsilon)$ is an equivalence on connective covers, Corollary 1.3.14, so it suffices to prove the following proposition, covering not only the Grothendieck-Witt space, but also the corresponding spectrum, and which applies to arbitrary shifts of the symmetric Poincaré structure:

3.1.4. **Proposition.** *Let $q = 2^r$ for some positive integer r. Then the map of spectra*

$$\mathrm{GW}(\mathbb{F}_q; (\mathrm{Ϙ}^{\mathrm{s}})^{[m]}) \longrightarrow \mathrm{K}(\mathbb{F}_q; (\mathrm{Ϙ}^{\mathrm{s}})^{[m]})^{\mathrm{hC}_2}$$

is an equivalence for every $m \in \mathbb{Z}$.

Proof. Corollary [II].4.4.14 provides, for every ring R and Poincaré structure Ϙ on $\mathcal{D}^{\mathrm{p}}(R)$, a pullback square

$$\begin{array}{ccc} \mathrm{GW}(R; Ϙ) & \longrightarrow & \mathrm{L}(R; Ϙ) \\ \downarrow & & \downarrow \\ \mathrm{K}(R; Ϙ)^{\mathrm{hC}_2} & \longrightarrow & \mathrm{K}(R; Ϙ)^{\mathrm{tC}_2}, \end{array}$$

It therefore suffices to show that the canonical map

$$\mathrm{L}(\mathbb{F}_q; (Ϙ^{\mathrm{s}})^{[m]}) \longrightarrow \mathrm{K}(\mathbb{F}_q; (Ϙ^{\mathrm{s}})^{[m]})^{\mathrm{tC}_2} \tag{12}$$

is an equivalence for every m. Applying the transformation $\mathrm{L}(-) \to \mathrm{K}(-)^{\mathrm{tC}_2}$ to the Bott-Genauer sequence of Example [II].1.2.5 gives a commutative diagram

$$\begin{array}{ccc} \mathrm{L}(\mathbb{F}_q; (Ϙ^{\mathrm{s}})^{[m]}) & \xrightarrow{\simeq} & \Sigma^m \mathrm{L}(\mathbb{F}_q; Ϙ^{\mathrm{s}}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathbb{F}_q; (Ϙ^{\mathrm{s}})^{[m]})^{\mathrm{tC}_2} & \xrightarrow{\simeq} & \Sigma^m \mathrm{K}(\mathbb{F}_q; Ϙ^{\mathrm{s}})^{\mathrm{tC}_2} \end{array}$$

whose horizontal arrows are equivalences; see the discussion before Corollary R.10. It will thus suffice to treat the case $m = 0$. Furthermore, L and Tate of K-theory are 2-periodic, see Corollary R.10, and it suffices to check that (12) induces an isomorphism on π_0 and π_1. By Corollary 1.3.4 we have that $\mathrm{L}_0(\mathbb{F}_q; Ϙ^{\mathrm{s}}) \cong \mathrm{W}^{\mathrm{s}}(\mathbb{F}_q) \cong \mathbb{Z}/2$ is the Witt group of symmetric bilinear forms over $\mathbb{F}_q$, which is isomorphic to $\mathbb{Z}/2$ generated by the class of the symmetric bilinear form $(\mathbb{F}_q, b)$ with $b(1, 1) = 1$. On the other hand, the same corollary also gives that $\mathrm{L}_1(\mathbb{F}_q; Ϙ^{\mathrm{s}}) = 0$. To finish the proof it will hence suffice to show that $\pi_1 \mathrm{K}(\mathbb{F}_q; Ϙ^{\mathrm{s}})^{\mathrm{tC}_2} = 0$, that $\pi_0 \mathrm{K}(\mathbb{F}_q; Ϙ^{\mathrm{s}})^{\mathrm{tC}_2} = \mathbb{Z}/2$, and that the map $\mathrm{L}_0(\mathbb{F}_q; Ϙ^{\mathrm{s}}) \to \pi_0 \mathrm{K}(\mathbb{F}_q; Ϙ^{\mathrm{s}})^{\mathrm{tC}_2}$ is non-zero.

Now, by Quillen's calculation of the K-theory of finite fields [Qui72], the K-groups $\mathrm{K}_*(\mathbb{F}_q)$ are odd torsion groups in positive degrees, so that the map $\mathrm{K}(\mathbb{F}_q) \to \mathrm{H}\mathbb{Z}$ is a 2-adic equivalence. It follows that the induced map $\mathrm{K}(\mathbb{F}_q; Ϙ^{\mathrm{s}})^{\mathrm{tC}_2} \to \mathbb{Z}^{\mathrm{tC}_2}$ is an equivalence as well, which shows that the Tate-K-groups are as claimed.

To finish the proof it will hence suffice to show that the map $L_0(\mathbb{F}_q;Ϙ^s) \to \pi_0(K(\mathbb{F}_q;Ϙ^s)^{tC_2})$ sends the generator to the generator. Indeed, in light of the commutative diagram

$$\begin{array}{ccccc} \pi_0\mathrm{Pn}(\mathcal{D}^p(\mathbb{F}_q),Ϙ^s) & \twoheadrightarrow & GW_0(\mathbb{F}_q;Ϙ^s) & \twoheadrightarrow & L_0(\mathbb{F}_q;Ϙ^s) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_0\mathrm{Cr}(\mathcal{D}^p(\mathbb{F}_q),Ϙ^s)^{C_2} & \twoheadrightarrow & K_0(\mathbb{F}_q;Ϙ^s)^{C_2} & \twoheadrightarrow & \widehat{H}^0(C_2,K_0(\mathbb{F}_q;Ϙ^s)) \end{array}$$

this simply follows from the fact that the composed forgetful functor

$$\pi_0\mathrm{Pn}(\mathcal{D}^p(\mathbb{F}_q),Ϙ^s) \longrightarrow \pi_0\mathrm{Cr}(\mathcal{D}^p(\mathbb{F}_q),Ϙ^s)^{C_2} \longrightarrow K_0(\mathbb{F}_q;Ϙ^s)^{C_2} \longrightarrow K_0(\mathbb{F}_q) \cong \mathbb{Z}$$

sends $(\mathbb{F}_q,b)$ to the generator $1 \in K_0(\mathbb{F}_q)$. □

3.1.5. **Remark.** An alternative argument can be given making use of multiplicative structures: In Paper [IV], we prove that the map $L(R;Ϙ^s) \to K(R;Ϙ^s)^{tC_2}$ is a map of E_∞-rings if R is a commutative ring. For $R = \mathbb{F}_q$ with q even, we then know that both homotopy rings are isomorphic to $\mathbb{F}_2[x^{\pm 1}]$, for $|x| = 2$. As any ring endomorphism of this ring is an isomorphism, the map we investigate is an equivalence.

3.1.6. **Remark.** Suppose k is a perfect field of characteristic 2. In this case, the map $\mathbb{F}_2 \to k$ induces an equivalence on 2-complete K-theory and on L-theory. For K-theory, this follows from an analysis of Adams operations on $K(k)$, see [Hil81, Theorem 5.4], and for L-theory it follows from Remark 1.3.6 that the odd L-groups vanish in both cases. By 2-periodicity of L-theory, Corollary R.10, and Corollary 1.3.13 together with Corollary 1.2.12, it therefore suffices to note that the map $\mathbb{F}_2 \to k$ induces an isomorphism on symmetric Witt groups. This in turn follows as every element in the symmetric Witt group of a field is a sum of one-dimensional forms $\langle x\rangle$ for $x \in k^\times$. Since k is perfect, the Frobenius is surjective and hence $\langle x\rangle = \langle y^2\rangle = \langle 1\rangle$ showing that the rank mod 2 map $W^s(k) \to \mathbb{Z}/2$ is an isomorphism for any perfect field of characteristic k, including $\mathbb{F}_2$. Considering the commutative diagram

$$\begin{array}{ccc} L^s(\mathbb{F}_2) & \xrightarrow{\simeq} & L^s(k) \\ \downarrow{\scriptstyle\simeq} & & \downarrow \\ K(\mathbb{F}_2)^{tC_2} & \xrightarrow{\simeq} & K(k)^{tC_2} \end{array}$$

we deduce that the homotopy limit problem has an affirmative answer for every perfect field of characteristic 2.

The results of Berrick et al. [BKSØ15] on the homotopy limit problem extend significantly beyond the realm of fields. It is shown, for example, that for any Noetherian scheme X of finite Krull dimension over $\mathbb{Z}[\frac{1}{2}]$, if $\mathrm{vcd}_2(k(x)[\sqrt{-1}])$ is uniformally bounded across all points $x \in X$, then the map

$$GW(X) \longrightarrow K(X)^{hC_2}$$

is an equivalence after 2-completion. Using the results of the previous sections we can now relax the assumption that 2 is invertible from the above result. Recall that for a Dedekind ring R with line bundle M with involution ± 1, the canonical map

$$GW^s_{cl}(R;M) \longrightarrow GW(R;Ϙ^s_M)$$

is an equivalence in non-negative degrees, by Corollary 1.3.14. Combining this with the following result gives Theorem 2 from the introduction.

3.1.7. **Theorem** (The homotopy limit problem)**.** *Let R be a Dedekind ring whose fraction field is a number field. Then for every $m \in \mathbb{Z}$ and every line bundle M over R with involution ± 1, the map*

$$GW(R;(Ϙ^s_M)^{[m]}) \longrightarrow K(R;(Ϙ^s_M)^{[m]})^{hC_2}$$

is a 2-adic equivalence.

Proof. Let S be the (finite) set of all prime ideals in R lying over 2. We observe that then $R_S = R[\frac{1}{2}]$ and similarly that $M_S = M[\frac{1}{2}]$ and consider the commutative diagram

$$\begin{array}{ccccc} \bigoplus_{\mathfrak{p}\in S} \mathrm{GW}(\mathbb{F}_\mathfrak{p}; (\Omega^{\mathrm{s}}_{M_\mathfrak{p}})^{[m-1]}) & \longrightarrow & \mathrm{GW}(R; (\Omega^{\mathrm{s}}_M)^{[m]}) & \longrightarrow & \mathrm{GW}(R[\frac{1}{2}]; (\Omega^{\mathrm{s}}_{M_S})^{[m]}) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{\mathfrak{p}\in S} \mathrm{K}(\mathbb{F}_\mathfrak{p}; (\Omega^{\mathrm{s}}_{M_\mathfrak{p}})^{[m-1]})^{\mathrm{hC}_2} & \longrightarrow & \mathrm{K}(R; (\Omega^{\mathrm{s}}_M)^{[m]})^{\mathrm{hC}_2} & \longrightarrow & \mathrm{K}(R[\frac{1}{2}]; (\Omega^{\mathrm{s}}_{M_S})^{[m]})^{\mathrm{hC}_2} \end{array}$$

obtained via the localisation-dévissage sequences of Corollary 2.1.9. Note that we have commuted the homotopy fixed points with the *finite* direct sum in the lower left corner. The left most vertical map is an equivalence by Proposition 3.1.4, and the right most vertical map is a 2-adic equivalence by [BKSØ15, Theorem 2.2]: We need to argue that all residue fields of $R[\frac{1}{2}]$ have finite mod 2 virtual cohomological dimension. Indeed, the residue fields at non-zero prime ideals are finite fields and hence have cohomological dimension one (the Galois group is $\widehat{\mathbb{Z}}$), and the residue field at 0 is the fraction field which is number field and hence also has finite vcd_2; [Ser02, §II.4.4]. It then follows that the middle vertical map is a 2-adic equivalence, as desired. □

3.1.8. **Remark.** The conclusion of Theorem 3.1.7 thus holds for all Dedekind rings whose field of fractions is a global field of characteristic different from 2: In the odd characteristic case [BKSØ15] applies, and the case of characteristic zero is the content of Theorem 3.1.7.

3.1.9. **Remark.** Suppose again that R is a Dedekind ring with global fraction field K. Suppose that K has characteristic different from 2 and is not formally real, that is, that -1 is a sum of squares. In other words, suppose that K has positive odd characteristic or is a totally imaginary number field. Then the Witt group $\mathrm{W}^{\mathrm{s}}(K)$ is a 2-primary torsion group of bounded exponent by [Sch12, Theorem 2.7.9]. As $\mathrm{W}^{\mathrm{s}}(R)$ is a subgroup of $\mathrm{W}^{\mathrm{s}}(K)$, see the proof of Corollary 2.2.3, Corollary 2.2.4 implies that $\mathrm{L}^{\mathrm{s}}(R)$ is (derived) 2-complete. As $\mathrm{K}(R)^{\mathrm{tC}_2}$ is also 2-complete, the pullback

$$\begin{array}{ccc} \mathrm{GW}(R; \Omega^{\mathrm{s}}) & \longrightarrow & \mathrm{L}(R; \Omega^{\mathrm{s}}) \\ \downarrow & & \downarrow \\ \mathrm{K}(R; \Omega^{\mathrm{s}})^{\mathrm{hC}_2} & \longrightarrow & \mathrm{K}(R; \Omega^{\mathrm{s}})^{\mathrm{tC}_2} \end{array}$$

together with Theorem 3.1.7 implies that the map of Theorem 3.1.7 is in fact an equivalence before 2-completion. Conversely, if K admits a real embedding, then $\mathrm{L}^{\mathrm{s}}(R)$ is not 2-complete: We have seen in Corollary 2.2.15 that all homotopy groups are finitely generated, so $\mathrm{L}^{\mathrm{s}}(R)$ is 2-complete if and only if all symmetric L-groups of R are 2-complete. However, as observed in the proof of Corollary 2.2.15, $\mathrm{W}^{\mathrm{s}}(R)$ has rank equal to the number of real embeddings of K, and is thus not 2-complete. It hence follows that the map under investigation in Theorem 3.1.7 is not an integral equivalence if K admits a real embedding. See also [BKSØ15, Theorem 2.4 & Proposition 4.7]. In fact, in our situation, the same result is true for $\mathrm{W}^{\mathrm{s}}(R; M)$ for any line bundle M on R: The map $\mathrm{W}^{\mathrm{s}}(R; M) \to \mathrm{W}^{\mathrm{s}}(K)$ is an isomorphism after inverting 2, and the map $\mathrm{W}^{\mathrm{s}}(K) \to \mathrm{W}^{\mathrm{s}}(\mathbb{R})$ induced from a real embedding of K is surjective. Hence the composite is non-zero and consequently $\mathbb{Z}$ is a direct summand inside $\mathrm{W}^{\mathrm{s}}(R; M)$. Hence $\mathrm{L}^{\mathrm{s}}(R; M)$ is not 2-complete.

As a side remark, we note that in the case where K admits a real embedding, $\mathrm{L}^{\mathrm{s}}(R)$ contains $\mathrm{L}^{s}(\mathbb{R})$ as a retract, and $\mathrm{L}^{\mathrm{s}}(\mathbb{R})$ is not 2-complete. To see that $\mathrm{L}^{\mathrm{s}}(\mathbb{R})$ is indeed a retract, consider the following composite

$$\mathrm{L}^{\mathrm{s}}(\mathbb{Z}) \longrightarrow \mathrm{L}^{\mathrm{s}}(R) \longrightarrow \mathrm{L}^{\mathrm{s}}(\mathbb{R})$$

where the two maps are induced by the canonical map $\mathbb{Z} \to R$ and the map $R \to K \subseteq \mathbb{R}$ induced by a real embedding of K. This composite admits a splitting, as was observed in [HLN20, Theorem A].

3.1.10. **Remark.** The proof of Theorem 3.1.7 reveals that the assumptions are not optimal. Assume for instance that R is the ring of integers in a non-archimedean local field K of mixed characteristic $(0, 2)$ and let k be the residue field of the local ring R. For instance, assume that R is a dyadic completion of the ring of integers in a number field. Since R is local the line bundle M is trivial. We again consider the diagram consisting of horizontal fibre sequences

$$\begin{array}{ccccc} \mathrm{GW}(k; (\Omega^{\mathrm{s}})^{[m-1]}) & \longrightarrow & \mathrm{GW}(R; (\Omega^{\mathrm{s}})^{[m]}) & \longrightarrow & \mathrm{GW}(K; (\Omega^{\mathrm{s}})^{[m]}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}(k; (\Omega^{\mathrm{s}})^{[m-1]})^{\mathrm{hC}_2} & \longrightarrow & \mathrm{K}(R; (\Omega^{\mathrm{s}})^{[m]})^{\mathrm{hC}_2} & \longrightarrow & \mathrm{K}(K; (\Omega^{\mathrm{s}})^{[m]})^{\mathrm{hC}_2} \end{array}$$

First, we note that $\mathrm{cd}_2(K) = 2$ [Ser02, §4.3], so the right vertical map is a 2-adic equivalence. We deduce that the middle vertical map is a 2-adic equivalence if and only if the left vertical map is a 2-adic equivalence. Thus if we assume that k is a finite field, the middle vertical map is a 2-adic equivalence. In fact, in this case, K is a finite extension of $\mathbb{Q}_2^\wedge$, and as observed earlier, $\mathrm{L}^{\mathrm{s}}(K)$ is 2-complete, in fact 2-power torsion [Lam05, Theorem 2.29]. It follows that the middle vertical map is in fact an equivalence.

The proof of Theorem 3.1.7 allows us to also deduce the following result, which, in case of the integers was conjectured by Berrick and Karoubi [BK05].

3.1.11. **Proposition.** *Let R be a Dedekind ring whose fraction field is a global field of characteristic zero. Then the map*

$$\mathrm{GW}^{\mathrm{s}}(R; \epsilon) \longrightarrow \mathrm{GW}^{\mathrm{s}}(R[\tfrac{1}{2}]; \epsilon)$$

is a 2-local equivalence on connective covers.

Proof. The fibre of the map in question is given by a sum of terms of the kind $\mathrm{GW}(\mathbb{F}_{\mathfrak{p}}; (\Omega^{\mathrm{s}}_{\epsilon})^{[-1]})$, with $\mathbb{F}_{\mathfrak{p}}$ a finite field of characteristic 2 by Corollary 2.1.9. It therefore suffices to show that each of these term, is 2-locally (-1)-truncated. To see this, we note that the map

$$\mathrm{GW}(\mathbb{F}_{\mathfrak{p}}; (\Omega^{\mathrm{s}}_{\epsilon})^{[-1]}) \longrightarrow \mathrm{K}(\mathbb{F}_{\mathfrak{p}}; (\Omega^{\mathrm{s}}_{\epsilon})^{[-1]})^{\mathrm{hC}_2}$$

is an equivalence by Proposition 3.1.4. Let us denote by $\mathbb{Z}(-1)$ the complex $\mathbb{Z}$ in degree 0 with the sign action of C_2. The map $\mathrm{K}(\mathbb{F}_{\mathfrak{p}}; (\Omega^{\mathrm{s}}_{\epsilon})^{[-1]}) \to \mathbb{Z}(-1)$ is a C_2-equivariant map whose fibre has finite and odd torsion homotopy groups. It follows that this map induces a 2-local equivalence after applying $(-)^{\mathrm{hC}_2}$. The proposition follows. □

We finish this subsection by noting the following obstruction to a positive solution of the homotopy limit problem for classical Grothendieck-Witt-theory of a discrete ring R, see also [BKSØ15, Remark 4.9]. The result implies that the map $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R)/2 \to \mathrm{K}(R)^{\mathrm{hC}_2}/2$ cannot be an equivalence in non-negative degrees unless the comparison map $\mathrm{L}^{\mathrm{gs}}(R) \to \mathrm{L}^{\mathrm{s}}(R)$ is so as well. Recall from Example 1.3.9 that there rings for which this is not the case.

3.1.12. **Proposition.** *Suppose that the fibre of the map $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R)/2 \to \mathrm{K}(R)^{\mathrm{hC}_2}/2$ is n-truncated for some integer n, where we view $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R)/2$ as a (connective) spectrum. Then the map $\mathrm{L}^{\mathrm{gs}}(R) \to \mathrm{L}^{\mathrm{s}}(R)$ is an equivalence on $(n+2)$-connective covers.*

Proof. Let F be the fibre of the map $\tau_{\geq n+2}\,\mathrm{L}^{\mathrm{gs}}(R) \to \tau_{\geq n+2}\,\mathrm{L}^{\mathrm{s}}(R)$. By Proposition 3.1.13 below, the map is an equivalence after inverting 2, so it follows that $F[\tfrac{1}{2}]$ vanishes. We will show that also $F/2$ vanishes, which implies that F is trivial. We recall that there are canonical shift maps

$$\cdots \longrightarrow \Sigma^4\,\mathrm{L}^{\mathrm{gs}}(R) \xrightarrow{\sigma} \mathrm{L}^{\mathrm{gs}}(R) \xrightarrow{\sigma} \Sigma^{-4}\,\mathrm{L}^{\mathrm{gs}}(R) \longrightarrow \cdots$$

whose filtered colimit is given by $\mathrm{L}^{\mathrm{s}}(R)$. Now, we need to show that the map $\pi_k(\mathrm{L}^{\mathrm{gs}}(R)/2) \to \pi_k(\mathrm{L}^{\mathrm{s}}(R)/2)$ is an isomorphism for $k \geq n+2$. It hence suffices to argue that the top horizontal map in the diagram

$$\begin{array}{ccc} \pi_k(\mathrm{L}^{\mathrm{gs}}(R)/2) & \xrightarrow{\sigma} & \pi_{k+4}(\mathrm{L}^{\mathrm{gs}}(R)/2) \\ \downarrow & & \downarrow \\ \pi_k(\mathrm{K}(R)^{\mathrm{tC}_2}/2) & \xrightarrow{\sigma'} & \pi_{k+4}(\mathrm{K}(R)^{\mathrm{tC}_2}/2) \end{array}$$

is an isomorphism for all $k \geq n+2$. Here, σ' denotes the corresponding shift map on the Tate construction of algebraic K-theory, which is an equivalence by Corollary R.10. We claim that the vertical maps are isomorphisms, concluding the proof of the proposition. To see the claim, we first observe that the map

$\mathrm{GW}^{\mathrm{gs}}(R)/2 \to \mathrm{K}(R)^{\mathrm{hC}_2}/2$ is also n-truncated, because the map $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(R)/2 \to \mathrm{GW}^{\mathrm{gs}}(R)/2$ is 0-truncated. We then consider the pullback diagram

$$\begin{array}{ccc} \mathrm{GW}^{\mathrm{gs}}(R)/2 & \longrightarrow & \mathrm{L}^{\mathrm{gs}}(R)/2 \\ \downarrow & & \downarrow \\ \mathrm{K}(R)^{\mathrm{hC}_2}/2 & \longrightarrow & \mathrm{K}(R)^{\mathrm{tC}_2}/2 \end{array}$$

and conclude that the map $\mathrm{L}^{\mathrm{gs}}(R)/2 \to \mathrm{K}(R)^{\mathrm{tC}_2}/2$ is n-truncated as well. Thus the vertical maps induce isomorphisms for $k > n+1$ as needed. □

We finish this section with the promised calculation of 2-inverted genuine L-theory. At this point, we will invoke multiplicative structures on L-theory which we develop in detail in Paper [IV].

3.1.13. **Proposition.** *Let R be a ring with invertible module with involution M, and let $m \in \mathbb{Z} \cup \{\pm\infty\}$. Then the natural map*

$$\mathrm{L}(R; \Omega^{\geq m}_M)[\tfrac{1}{2}] \longrightarrow \mathrm{L}(R; \Omega^{\mathrm{s}}_M)[\tfrac{1}{2}]$$

is an equivalence.

Proof. We first observe that the canonical map $\mathrm{L}^{\mathrm{gs}}(\mathbb{Z}) \to \mathrm{L}^{\mathrm{s}}(\mathbb{Z})$ is an equivalence after inverting 2, see Example 2.2.12. Moreover, the shift maps appearing in the proof of Proposition 3.1.12 are in fact given by multiplication with an element $x \in \mathrm{L}^{\mathrm{gs}}_4(\mathbb{Z})$, namely the Poincaré object $\mathbb{Z}[-2]$ with its standard genuine symmetric Poincaré structure of signature 1. Thus we find

$$\mathrm{L}(R; \Omega^{\mathrm{s}}_M) \simeq \mathrm{L}(R; \Omega^{\geq m}_M)[x^{-1}] \simeq \mathrm{L}(R; \Omega^{\geq m}_M) \otimes_{\mathrm{L}^{\mathrm{gs}}(\mathbb{Z})} \mathrm{L}^{\mathrm{s}}(\mathbb{Z}),$$

and the result follows. □

3.2. **Grothendieck-Witt groups of the integers.** In this section, we will specialise the results established earlier in the paper to the ring of integers $\mathbb{Z}$, and calculate its classical ϵ-symmetric and ϵ-quadratic Grothendieck-Witt groups. We will exploit Corollary 1.3.13 and instead calculate the non-negative Grothendieck-Witt groups $\mathrm{GW}^{\mathrm{s}}(\mathbb{Z};\epsilon) = \mathrm{GW}(\mathbb{Z}; \Omega^{\mathrm{s}}_\epsilon)$ for the non-genuine symmetric Poincaré structure. Our calculation crucially relies on the knowledge of the algebraic K-groups [Wei13] of $\mathbb{Z}$, and on the calculations of Berrick-Karoubi [BK05] of the Grothendieck-Witt groups of $\mathbb{Z}[\frac{1}{2}]$. Before we start the computation, we give a brief account of the 4 types of classical Grothendieck-Witt groups that we are considering.

(1)-**Symmetric**: We recall that $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(\mathbb{Z})$ denotes the homotopy theoretic group completion of the maximal subgroupoid of the category of non-degenerate symmetric bilinear forms over $\mathbb{Z}$. By [Ser61, Théorème 1], there is an isomorphism $\pi_0\,\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ where the summands are generated by the classes of the forms $\langle 1\rangle$ and $\langle -1\rangle$ on a free module of rank 1 $\mathbb{Z}$; they send (x,y) to xy and $-xy$, respectively. We write $\mathrm{O}_{\langle n,n\rangle}(\mathbb{Z}) = \mathrm{Aut}((\langle 1\rangle \perp \langle -1\rangle)^{\perp n}) \subseteq \mathrm{GL}_{2n}(\mathbb{Z})$ and $\mathrm{O}_{\langle\infty,\infty\rangle}(\mathbb{Z}) = \mathrm{colim}_n\,\mathrm{O}_{\langle n,n\rangle}(\mathbb{Z})$. Then the commutator subgroup of $\mathrm{O}_{\langle\infty,\infty\rangle}(\mathbb{Z})$ is perfect by e.g. [RW13, Proposition 3.1]. Moreover, it is a direct consequence of [Ser61, Théorème 4] that any non-degenerate symmetric bilinear form over $\mathbb{Z}$ is an orthogonal summand in $(\langle 1\rangle \perp \langle -1\rangle)^{\perp n}$ for some $n \geq 0$. Hence, the group completion theorem yields a homotopy equivalence of spaces

$$\tau_{>0}\,\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(\mathbb{Z}) \simeq \mathrm{BO}_{\langle\infty,\infty\rangle}(\mathbb{Z})^+,$$

see [MS76], or [RW13, Corollary 1.2].

(-1)-**Symmetric:** Similarly $\mathrm{GW}^{-\mathrm{s}}_{\mathrm{cl}}(\mathbb{Z})$ is the homotopy theoretic group completion of the maximal subgroupoid of the category of non-degenerate symplectic bilinear forms over $\mathbb{Z}$. We let $\mathrm{H}_{-\mathrm{s}}$ be the standard symplectic bilinear form on $\mathbb{Z}^2$. As every symplectic form over $\mathbb{Z}$ is isomorphic to a finite orthogonal sum of copies of $\mathrm{H}_{-\mathrm{s}}$, we find $\pi_0\,\mathrm{GW}^{-\mathrm{s}}_{\mathrm{cl}}(\mathbb{Z}) \cong \mathbb{Z}$, generated by $\mathrm{H}_{-\mathrm{s}}$. We write $\mathrm{Sp}_{2n}(\mathbb{Z}) = \mathrm{Aut}((\mathrm{H}_{-\mathrm{s}})^{\perp n}) \subseteq \mathrm{GL}_{2n}(\mathbb{Z})$ and $\mathrm{Sp}_\infty(\mathbb{Z}) = \mathrm{colim}_n\,\mathrm{Sp}_{2n}(\mathbb{Z})$. The group $\mathrm{Sp}_\infty(\mathbb{Z})$ is again perfect, see e.g. [RW13, Proposition 3.1], and the group completion theorem yields a homotopy equivalence of spaces

$$\tau_{>0}\,\mathrm{GW}^{-\mathrm{s}}_{\mathrm{cl}}(\mathbb{Z}) \simeq \mathrm{BSp}_\infty(\mathbb{Z})^+.$$

(1)**-Quadratic:** Now $\mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(\mathbb{Z})$ is the homotopy theoretic group completion of the maximal subgroupoid of the category of non-degenerate quadratic forms over $\mathbb{Z}$. Let H_{q} be the standard hyperbolic quadratic form and E_8 the classical 8-dimensional quadratic form associated to the Dynkin diagram of the same name. By [Ser61, Théorème 5], every quadratic form (P, q) satisfies $P \perp \mathrm{H}_{\mathrm{q}} \cong \mathrm{H}^n_{\mathrm{q}} \oplus E^m_8$ for some n and m and $\pi_0 \mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ with generators H_{q} and E_8. We write $\mathrm{O}_{n,n}(\mathbb{Z}) = \mathrm{Aut}((\mathrm{H}_{\mathrm{q}})^{\perp n}) \subseteq \mathrm{GL}_{2n}(\mathbb{Z})$ and $\mathrm{O}_{\infty,\infty}(\mathbb{Z}) = \mathrm{colim}_n\, \mathrm{O}_{n,n}(\mathbb{Z})$. As above the group $\mathrm{O}_{\infty,\infty}(\mathbb{Z})$ has perfect commutator subgroup and since any quadratic form over $\mathbb{Z}$ is a direct summand of $(\mathrm{H}_{\mathrm{q}})^{\perp n}$ for some $n \geq 0$ there is a homotopy equivalence of spaces

$$\tau_{>0} \mathrm{GW}^{\mathrm{q}}_{\mathrm{cl}}(\mathbb{Z}) \simeq \mathrm{BO}_{\infty,\infty}(\mathbb{Z})^+.$$

(-1)**-Quadratic**: Finally, $\mathrm{GW}^{-\mathrm{q}}_{\mathrm{cl}}(\mathbb{Z})$ is similarly built from (-1)-quadratic forms over $\mathbb{Z}$. Such a form is determined by its rank (which is an even number) and its Arf invariant, see [Bro12, §III.1]. Let

$$\mathrm{H}^0_{-\mathrm{q}} = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, xy\right) \quad \text{and} \quad \mathrm{H}^1_{-\mathrm{q}} = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, x^2 + xy + y^2\right),$$

be the standard hyperbolic (-1)-quadratic forms with Arf invariant 0 and 1, respectively. Then every (-1)-quadratic form with Arf invariant 0 is isomorphic to a direct sum of copies of $\mathrm{H}^0_{-\mathrm{q}}$, and every (-1)-quadratic form with Arf invariant 1 is isomorphic to a direct sum of copies of $\mathrm{H}^0_{-\mathrm{q}}$ plus one copy of $\mathrm{H}^1_{-\mathrm{q}}$. Thus, $\pi_0 \mathrm{GW}^{-\mathrm{q}}_{\mathrm{cl}} \cong \mathbb{Z} \oplus \mathbb{Z}/2$. We define $\mathrm{Sp}^{\mathrm{q}}_{2n}(\mathbb{Z}) = \mathrm{Aut}((\mathrm{H}^0_{-\mathrm{q}})^{\perp n}) \subseteq \mathrm{Sp}_{2n}(\mathbb{Z})$ to be the group of matrices preserving both the bilinear form and its quadratic refinement and set $\mathrm{Sp}^{\mathrm{q}}_{\infty}(\mathbb{Z}) = \mathrm{colim}_n\, \mathrm{Sp}^{\mathrm{q}}_{2n}(\mathbb{Z})$. As above, the group completion theorem yields a homotopy equivalence of spaces

$$\tau_{>0} \mathrm{GW}^{-\mathrm{q}}_{\mathrm{cl}}(\mathbb{Z}) \simeq \mathrm{BSp}^{\mathrm{q}}_{\infty}(\mathbb{Z})^+.$$

The Grothendieck-Witt groups of $\mathbb{Z}$. We now proceed to calculate the ϵ-symmetric Grothendieck-Witt groups of $\mathbb{Z}$. Recall that the Bernoulli numbers $\{B_n\}_{n \geq 0}$ are rational numbers determined by the equation

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

We write c_n for the numerator of $|\frac{B_{2n}}{4n}|$ which is an odd number. For each $k \geq 0$ we have the equations

$$|\,\mathrm{K}_{8k+2}(\mathbb{Z})| = 2 \cdot c_{2k+1} \quad \text{and} \quad |\,\mathrm{K}_{8k+6}(\mathbb{Z})| = c_{2k+2},$$

by [Wei13, Theorem 10.1], and if the Vandiver conjecture holds the groups in question are cyclic. The denominator of $|\frac{B_{2n}}{4n}|$ will be denoted by w_{2n}. There are isomorphisms

$$\mathrm{K}_{8k+3}(\mathbb{Z}) \cong \mathbb{Z}/2w_{4k+2} \quad \text{and} \quad \mathrm{K}_{8k+7}(\mathbb{Z}) \cong \mathbb{Z}/w_{4k+4},$$

for all $k \geq 0$ by [Wei13, Theorem 10.1].

We now arrive at the main computation of the ϵ-symmetric Grothendieck-Witt groups of $\mathbb{Z}$ in degree $n \geq 1$. We determine these groups completely up to the precise group structure in degrees which are 2 mod 4, which depend on Vandiver's conjecture. Thanks to the work of Weibel [Wei13] on the algebraic K-theory of $\mathbb{Z}$ the last uncertainty can be removed in the range $n \leq 20000$, see Remark 3.2.2 below. For an abelian group A, we write A_{odd} for the odd torsion subgroup of A.

3.2.1. **Theorem.** *The classical ϵ-symmetric Grothendieck-Witt groups $\mathbb{Z}$ are given in degrees $n \geq 1$ by the following table:*

3.2.2. **Remark.** For $m \leq 5000$ the group $(\mathrm{K}_{4m-2})_{\mathrm{odd}}$ is known to be cyclic of order c_m, see [Wei13, Example 10.3.2]. This holds for all m if Vandiver's conjecture is true [Wei13, Theorem 10.2].

3.2.3. **Remark.** The number w_{2n} is equal to the cardinality of the image of the J-homomorphism $\pi_{4n-1}(O) \to \pi_{4n-1}(\mathbb{S})$ in the stable stem. By [Qui76, pg. 186] the unit map $\pi_{4n-1}(\mathbb{S}) \to \mathrm{K}_{4n-1}(\mathbb{Z})$ is injective on this image. Since the unit map for $\mathrm{K}(\mathbb{Z})$ factors through the unit map for $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(\mathbb{Z})$, it follows that the groups $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},8k+3}(\mathbb{Z})$ and $\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},8k+7}(\mathbb{Z})$ consist precisely of image of J-classes.

$n =$	$\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},n}(\mathbb{Z})$	$\mathrm{GW}^{-\mathrm{s}}_{\mathrm{cl},n}(\mathbb{Z})$
$8k$	$\mathbb{Z} \oplus \mathbb{Z}/2$	0
$8k+1$	$(\mathbb{Z}/2)^3$	0
$8k+2$	$(\mathbb{Z}/2)^2 \oplus \mathrm{K}_{8k+2}(\mathbb{Z})_{odd}$	$\mathbb{Z} \oplus \mathrm{K}_{8k+2}(\mathbb{Z})_{\mathrm{odd}}$
$8k+3$	$\mathbb{Z}/w_{4k+2}$	$\mathbb{Z}/2w_{4k+2}$
$8k+4$	$\mathbb{Z}$	$\mathbb{Z}/2$
$8k+5$	0	$\mathbb{Z}/2$
$8k+6$	$\mathrm{K}_{8k+6}(\mathbb{Z})_{\mathrm{odd}}$	$\mathbb{Z} \oplus \mathrm{K}_{8k+6}(\mathbb{Z})_{\mathrm{odd}}$
$8k+7$	$\mathbb{Z}/w_{4k+4}$	$\mathbb{Z}/w_{4k+4}$

Proof of Theorem 3.2.1. Since the groups in question are finitely generated, it suffices to prove that the theorem holds after localisation at 2 and after inverting 2. First, we argue 2-locally. Proposition 3.1.11 and Corollary 1.3.14 imply that for $\epsilon = \pm 1$, the canonical map

$$\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(\mathbb{Z};\epsilon) \longrightarrow \mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(\mathbb{Z}[\tfrac{1}{2}];\epsilon)$$

is a 2-local equivalence in degrees ≥ 1. One can then compare with [BK05, Theorem B][1], where the 2-local GW-groups of $\mathbb{Z}[\frac{1}{2}]$ are determined as displayed. In order to compare their values for $8k+3$ and $8k+7$ with ours, note that by work of von Staudt the largest power of 2 which divides w_{2n} is the same as the largest power of 2 which divides $8n$.

For the 2-inverted case, we observe that the fibre sequence

$$\mathrm{K}(\mathbb{Z};\epsilon)_{\mathrm{hC}_2} \longrightarrow \mathrm{GW}^{\mathrm{s}}(\mathbb{Z};\epsilon) \longrightarrow \mathrm{L}^{\mathrm{s}}(\mathbb{Z};\epsilon)$$

from Theorem 1 splits after inverting 2, see e.g. Corollary [II].4.4.17, so that there is an equivalence of spectra

$$\mathrm{GW}^{\mathrm{s}}(\mathbb{Z};\epsilon)[\tfrac{1}{2}] \simeq (\mathrm{K}(\mathbb{Z};\epsilon)_{\mathrm{hC}_2})[\tfrac{1}{2}] \oplus \mathrm{L}^{\mathrm{s}}(\mathbb{Z};\epsilon)[\tfrac{1}{2}]$$

Furthermore, we note that there is an isomorphism $\pi_n(\mathrm{K}(\mathbb{Z};\epsilon)_{\mathrm{hC}_2}[\frac{1}{2}]) \cong (\mathrm{K}_n(\mathbb{Z};\epsilon)[\frac{1}{2}])_{\mathrm{C}_2}$. It then follows from Lemma 3.2.4 below that

$$\mathrm{GW}^{\mathrm{s}}_n(\mathbb{Z};\epsilon)[\tfrac{1}{2}] \cong \begin{cases} \mathrm{L}^{\mathrm{s}}_n(\mathbb{Z};\epsilon)[\tfrac{1}{2}] & \text{for } n \equiv 0,1 \bmod 4 \\ \mathrm{K}_n(\mathbb{Z};\epsilon)[\tfrac{1}{2}] \oplus \mathrm{L}^{\mathrm{s}}_n(\mathbb{Z};\epsilon)[\tfrac{1}{2}] & \text{for } n \equiv 2,3 \bmod 4 \end{cases}$$

This matches with the values in the above table after tensoring with $\mathbb{Z}[\frac{1}{2}]$ and so the desired result follows. □

3.2.4. **Lemma.** *The* C_2*-actions induced by the Poincaré structures* Ω^s *and* Ω^s_- *on* $\mathcal{D}^{\mathrm{p}}(\mathbb{Z})$ *induce multiplication by* $(-1)^n$ *on the groups* $\mathrm{K}_{2n-1}(\mathbb{Z})[\frac{1}{2}]$ *and* $\mathrm{K}_{2n-2}(\mathbb{Z})[\frac{1}{2}]$ *for each* $n \geq 2$.

Proof. This follows from [FGV20, §2], and we briefly collect the arguments. We first note that the dualities associated to Ω^{s} and Ω^{s}_- have the same underlying equivalences $\mathcal{D}^{\mathrm{p}}(\mathbb{Z}) \to \mathcal{D}^{\mathrm{p}}(\mathbb{Z})^{\mathrm{op}}$ so that the induced C_2-action on homotopy groups is the same in both cases. Hence it will suffice to prove the claim for Ω^{s}. Since the K-groups of $\mathbb{Z}$ are finitely generated, it suffices to prove the claim on the ℓ-completed K-groups $\mathrm{K}_n(\mathbb{Z})^\wedge_\ell$ for all odd primes ℓ. We then have the following:

i) The map $\mathrm{K}(\mathbb{Z})^\wedge_\ell \to L_{K(1)}\mathrm{K}(\mathbb{Z})$ induces an isomorphism on π_i for $i \geq 2$; see [FGV20, Proposition 2.9] and use that $L_{K(1)}\mathrm{K}(\mathbb{Z})/\ell \simeq (\mathrm{K}(\mathbb{Z})/\ell)[\beta^{-1}]$ where β is the mod ℓ Bott element; compare [FGV20, Remark 2.8].
ii) The map $L_{K(1)}\mathrm{K}(\mathbb{Z}) \to L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}])$ is an equivalence; this follows from the fibre sequence $\mathrm{K}(\mathbb{F}_\ell) \to \mathrm{K}(\mathbb{Z}) \to \mathrm{K}(\mathbb{Z}[\frac{1}{\ell}])$; see also [BCM20, LMT20] for a generalisation of this equivalence.
iii) The resulting map $\mathrm{K}(\mathbb{Z})^\wedge_\ell \to L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}])$ is equivariant with respect to the duality action on both sides; this action is usually denoted by ψ^{-1}, see also [FGV20, 2.3.1].

[1] Note that what we denote GW is denoted $\mathcal{L}$ in loc. cit. and that the homotopy groups of $\mathcal{L}(R)$ are denoted by $L_i(R)$.

iv) On the odd homotopy groups $\pi_{2n-1}(L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}])/\ell^k)$ the action of ψ^{-1} is multiplication by $(-1)^n$ (independently of k) [FGV20, Lemma 2.14], hence the same is true for the inverse limit over k tending to ∞. This inverse limit is $\pi_{2n-1}(L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}]))$, as this group is finitely generated.

v) For the action on the even homotopy groups, one shows that the (-1)-eigenspace $\pi_{2n}(L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}]))^{(-)}$ of ψ^{-1} is trivial for n odd, and that the $(+1)$-eigenspace $\pi_{2n}(L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}]))^{(+)}$ of ψ^{-1} is trivial for even n: Indeed, one has

$$\begin{cases} \pi_{4n-2}(L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}])/\ell^k)^{(-)} \cong \mathrm{H}^0_{\text{ét}}(\mathrm{spec}(\mathbb{Z}[\frac{1}{\ell}]);\mu_{\ell^k}^{\otimes(2n-1)}) \\ \pi_{4n}(L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}])/\ell^k)^{(+)} \cong \mathrm{H}^0_{\text{ét}}(\mathrm{spec}(\mathbb{Z}[\frac{1}{\ell}]);\mu_{\ell^k}^{\otimes(2n)}) \end{cases}$$

Since $\mathbb{Z}[\frac{1}{\ell}]$ does not have non-trivial ℓ-power roots of unity, one finds that the étale cohomology terms vanish upon passing to the inverse limit over k tending to ∞.

vi) We deduce that $\pi_{4n-2}(L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}]))^{(+)} = \pi_{4n-2}(L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}]))$ and that $\pi_{4n}(L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}]))^{(-)} = \pi_{4n}(L_{K(1)}\mathrm{K}(\mathbb{Z}[\frac{1}{\ell}]))$.

□

3.2.5. **Remark.** A calculation of the Grothendieck-Witt groups of the integers has also been announced in [Sch19b], but with a different odd torsion: there it is claimed that the C_2-action on $\mathrm{K}_*(\mathbb{Z})[\frac{1}{2}]$ is multiplication by $(-1)^{n+1}$ on $\mathrm{K}_{2n-2}(\mathbb{Z})[\frac{1}{2}]$ and $\mathrm{K}_{2n-1}(\mathbb{Z})[\frac{1}{2}]$, but we believe this comes from an error in equation (3.3) of [Sch19b, Proof of Lemma 3.1].

In low degrees the groups can be worked out explicitly.

3.2.6. **Proposition.** *The first 24 non-negative Grothendieck-Witt groups of $\mathbb{Z}$ are given by the table 3.2.6 below.*

TABLE 1. The first 24 Grothendieck-Witt groups of $\mathbb{Z}$

k	$\mathrm{GW}^{\mathrm{s}}_k(\mathbb{Z})$	k	$\mathrm{GW}^{s}_k(\mathbb{Z})$	k	$\mathrm{GW}^{s}_k(\mathbb{Z})$
0	$\mathbb{Z}\oplus\mathbb{Z}$	8	$\mathbb{Z}\oplus\mathbb{Z}/2$	16	$\mathbb{Z}\oplus\mathbb{Z}/2$
1	$(\mathbb{Z}/2)^3$	9	$(\mathbb{Z}/2)^3$	17	$(\mathbb{Z}/2)^3$
2	$(\mathbb{Z}/2)^2$	10	$(\mathbb{Z}/2)^2$	18	$(\mathbb{Z}/2)^2$
3	$\mathbb{Z}/24$	11	$\mathbb{Z}/504$	19	$\mathbb{Z}/264$
4	$\mathbb{Z}$	12	$\mathbb{Z}$	20	$\mathbb{Z}$
5	0	13	0	21	0
6	0	14	0	22	$\mathbb{Z}/691$
7	$\mathbb{Z}/240$	15	$\mathbb{Z}/480$	23	$\mathbb{Z}/65520$

k	$\mathrm{GW}^{-\mathrm{s}}_k(\mathbb{Z})$	k	$\mathrm{GW}^{-\mathrm{s}}_k(\mathbb{Z})$	k	$\mathrm{GW}^{-\mathrm{s}}_k(\mathbb{Z})$
0	$\mathbb{Z}$	8	0	16	0
1	0	9	0	17	0
2	$\mathbb{Z}$	10	$\mathbb{Z}$	18	$\mathbb{Z}$
3	$\mathbb{Z}/48$	11	$\mathbb{Z}/1008$	19	$\mathbb{Z}/528$
4	$\mathbb{Z}/2$	12	$\mathbb{Z}/2$	20	$\mathbb{Z}/2$
5	$\mathbb{Z}/2$	13	$\mathbb{Z}/2$	21	$\mathbb{Z}/2$
6	$\mathbb{Z}$	14	$\mathbb{Z}$	22	$\mathbb{Z}\oplus\mathbb{Z}/691$
7	$\mathbb{Z}/240$	15	$\mathbb{Z}/480$	23	$\mathbb{Z}/65520$

Proof. The only information not already present in the table of Theorem 3.2.1 is the structure of the odd torsion in $\mathrm{K}_n(\mathbb{Z})$ for $n = 2, 6 \bmod 8$. This can be read off from the list of K-groups [Wei13, Example 10.3] - the only non-trivial one in this range is $\mathrm{K}_{22}(\mathbb{Z}) = \mathbb{Z}/691$. □

We now turn to the computation of the classical ϵ-quadratic Grothendieck-Witt groups of $\mathbb{Z}$. Recall that for $\epsilon = \pm 1$ there is a Poincaré functor $(\mathcal{D}^{\mathrm{p}}(\mathbb{Z}), \Omega^{\mathrm{gq}}_{\epsilon}) \to (\mathcal{D}^{\mathrm{p}}(\mathbb{Z}), \Omega^{\mathrm{gs}}_{\epsilon})$, which by the fibre sequence of Corollary [II].4.4.14 induces a cartesian square of spectra

$$\begin{array}{ccc} \mathrm{GW}^{\mathrm{gq}}(\mathbb{Z};\epsilon) & \longrightarrow & \mathrm{L}^{\mathrm{gq}}(\mathbb{Z};\epsilon) \\ \downarrow & & \downarrow \\ \mathrm{GW}^{\mathrm{gs}}(\mathbb{Z};\epsilon) & \longrightarrow & \mathrm{L}^{\mathrm{gs}}(\mathbb{Z};\epsilon). \end{array}$$

The non-negative homotopy groups of the bottom left hand spectrum were computed in Theorem 3.2.1 above. To understand the spectrum $\mathrm{GW}^{\mathrm{gq}}(\mathbb{Z};\epsilon)$ we will calculate the homotopy groups of the cofibre of the right hand vertical map, which is equivalent to the cofibre of the left hand vertical map. We begin with the case $\epsilon = 1$. Write C for the cofibre of the map $\mathrm{L}^{\mathrm{gq}}(\mathbb{Z}) \to \mathrm{L}^{\mathrm{gs}}(\mathbb{Z})$ and C_i for the homotopy group $\pi_i(C)$.

3.2.7. **Lemma.** *The groups C_i are given by*

i) $C_1 \cong \mathbb{Z}/2$,
ii) $C_0 \cong \mathbb{Z}/8$
iii) $C_{-1} \cong \mathbb{Z}/2$,
iv) $C_i = 0$ *for all other values of* i.

Proof. Let us consider the commutative diagram

$$\begin{array}{ccccc} & & \mathrm{L}^{\mathrm{gq}}(\mathbb{Z}) & & \\ & \nearrow^{\simeq_{\leq 1}} & & \searrow^{\simeq_{\geq 2}} & \\ \mathrm{L}^{\mathrm{q}}(\mathbb{Z}) & & \downarrow & & \mathrm{L}^{\mathrm{s}}(\mathbb{Z}) \\ & \searrow_{\simeq_{\leq -3}} & & \nearrow_{\simeq_{\geq -2}} & \\ & & \mathrm{L}^{\mathrm{gs}}(\mathbb{Z}), & & \end{array}$$

where the subscript on the symbol $\simeq$ indicates the range of dimensions i in which the map induces an isomorphism on π_i. These ranges are obtained from Corollaries 1.2.8 and 1.3.7, using that by Example 1.1.4 the Poincaré structures $\Omega^{\mathrm{gq}} = \Omega^{\mathrm{ge}}$ and $\Omega^{\mathrm{gs}} = \Omega^{\geq(-1)}$ are respectively 1-symmetric and (-1)-quadratic. Using in addition that $\mathrm{L}^{\mathrm{gs}}_{-2}(\mathbb{Z}) = 0$, it follows that C_i is at most non-trivial in the range $-1 \leq i \leq 1$ as claimed. We then find that $C_{-1} \cong \mathrm{L}^{\mathrm{gq}}_{-2}(\mathbb{Z}) \cong \mathrm{L}^{\mathrm{q}}_{-2}(\mathbb{Z}) \cong \mathbb{Z}/2$. The remaining two groups sit in the exact sequence

$$0 \longrightarrow \mathrm{L}^{\mathrm{gs}}_1(\mathbb{Z}) \longrightarrow C_1 \longrightarrow \mathrm{L}^{\mathrm{gq}}_0(\mathbb{Z}) \longrightarrow \mathrm{L}^{\mathrm{gs}}_0(\mathbb{Z}) \longrightarrow C_0 \to 0.$$

Since the map $\mathrm{L}^{\mathrm{gq}}_0(\mathbb{Z}) \to \mathrm{L}^{\mathrm{gs}}_0(\mathbb{Z})$ identifies with the multiplication by 8 map on $\mathbb{Z}$ it follows that $C_0 \cong \mathbb{Z}/8$ and $C_1 \cong \mathrm{L}^{\mathrm{gs}}_1(\mathbb{Z}) \cong \mathbb{Z}/2$; see [Ran81, Prop 4.3.1]. ☐

3.2.8. **Remark.** Let us denote by $\mathrm{L}^n(R)$ the cofibre of the symmetrisation map $\mathrm{L}^{\mathrm{q}}(R) \to \mathrm{L}^{\mathrm{s}}(R)$, called *normal* or *hyperquadratic* L-theory in Ranicki's work [Ran79, Ran92]. We then have $C \simeq \tau_{[-1,1]}\, \mathrm{L}^n(\mathbb{Z})$.

3.2.9. **Theorem.** *The classical quadratic Grothendieck-Witt groups of $\mathbb{Z}$ are given by*

i) $\mathrm{GW}^{\mathrm{gq}}_0(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$,
ii) $\mathrm{GW}^{\mathrm{gq}}_1(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$,
iii) $\mathrm{GW}^{\mathrm{gq}}_n(\mathbb{Z}) \cong \mathrm{GW}^{\mathrm{gs}}_n(\mathbb{Z})$ *for* $n \geq 2$.

Proof. The group $\mathrm{GW}^{\mathrm{gq}}_0(\mathbb{Z})$ is well known to be freely generated by the standard hyperbolic form and the positive definite even form E_8 (see the discussion at the beginning of the section). For ii) consider the exact sequence

$$C_2 \to \mathrm{GW}^{\mathrm{gq}}_1(\mathbb{Z}) \to \mathrm{GW}^{\mathrm{gs}}_1(\mathbb{Z}) \to C_1 \to \mathrm{GW}^{\mathrm{gq}}_0(\mathbb{Z}) \to \mathrm{GW}^{\mathrm{gs}}_0(\mathbb{Z}).$$

The map $\mathrm{GW}^{\mathrm{gq}}_0(\mathbb{Z}) \to \mathrm{GW}^{\mathrm{gs}}_0(\mathbb{Z})$ is injective and the image has index 8. It follows that $\mathrm{GW}^{\mathrm{gs}}_1(\mathbb{Z}) \cong (\mathbb{Z}/2)^3$ maps surjectively onto $C_1 \cong \mathbb{Z}/2$ and since $C_2 = 0$ by Lemma 3.2.7 we get that $\mathrm{GW}^{\mathrm{gq}}_1(\mathbb{Z}) \cong (\mathbb{Z}/2)^2$. Finally, iii) is implied by Lemma 3.2.7iii). ☐

We now turn to the case $\epsilon = -1$.

3.2.10. **Lemma.** *Let D be the cofibre of the map* $\mathrm{L}^{-\mathrm{gq}}(\mathbb{Z}) \to \mathrm{L}^{-\mathrm{gs}}(\mathbb{Z})$. *Then* $D \simeq \Sigma^2 C$.

Proof. By Theorem R.6 and Remark R.4, we have canonical equivalences $\mathrm{L}^{-\mathrm{gq}}(\mathbb{Z}) \simeq \Sigma^2 \mathrm{L}^{\mathrm{gq}}(\mathbb{Z})$ and $\mathrm{L}^{-\mathrm{gs}}(\mathbb{Z}) \simeq \Sigma^2 \mathrm{L}^{\mathrm{gs}}(\mathbb{Z})$. Under these equivalences, the symmetrisation map in the definition of D corresponds to the one of the definition of C. □

3.2.11. **Lemma.** *There are group isomorphisms*

i) $\pi_1 \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{gq}}_-)_{\mathrm{hC}_2} \cong \mathbb{Z}/4$,
ii) $\pi_2 \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{gq}}_-)_{\mathrm{hC}_2} = 0$.

Proof. Since the involution on $\mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{gq}}_-)$ only depends on the underlying duality the canonical map

$$\pi_n \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{gq}}_-)_{\mathrm{hC}_2} \to \pi_n \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2}$$

is an isomorphism, and we shall henceforth replace $Ϙ^{\mathrm{gq}}_-$ with $Ϙ^{\mathrm{s}}_-$.

We first compute $\pi_1 \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2}$. Consider the homotopy orbit spectral sequence

$$E^2_{s,t} = \mathrm{H}_s(\mathrm{C}_2; \pi_t \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-)) \implies \pi_{s+t} \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2}.$$

Since $\mathrm{H}_2(\mathrm{C}_2; \pi_0 \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-)) = 0$ the generator of $\mathrm{H}_0(\mathrm{C}_2; \pi_1 \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-)) \cong \mathbb{Z}/2$ is a permanent cycle. The group $\pi_1 \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2}$ also gets a contribution from $\mathrm{H}_1(\mathrm{C}_2; \pi_0 \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-))$ which has order 2, so in total it must have order 4. The former group sits in an exact sequence

$$\mathrm{L}^{-\mathrm{s}}_2(\mathbb{Z}) \to \pi_1 \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2} \to \mathrm{GW}^{-\mathrm{s}}_1(\mathbb{Z}),$$

where the left hand group is isomorphic to $\mathrm{L}^{\mathrm{s}}_0(\mathbb{Z}) \cong \mathbb{Z}$ and the right hand group is trivial by Table 3.2.6. It follows that the middle group is cyclic and is hence isomorphic to $\mathbb{Z}/4$.

We will now compute $\pi_2 \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2}$. For this, it will be useful to embed $\mathbb{Z}$ in the field $\mathbb{R}$ of real numbers, and consider the *topological* variants of K-theory and GW-theory for $\mathbb{R}$, equipped with its usual topology. For this we follow the approach of [Sch17, §10] and define these in terms of the simplicial ring $\mathbb{R}^{\Delta^\bullet} \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ring})$, whose n-simplices are the set $\mathbb{R}^{\Delta^n}$ of continuous maps of topological spaces $|\Delta^n| \to \mathbb{R}$, considered as a ring via pointwise operations. One then defines the topological variants of K-theory, GW-theory and L-theory by

$$\mathrm{K}^{\mathrm{top}}(\mathbb{R}) := |\,\mathrm{K}(\mathbb{R}^{\Delta^\bullet})| = \operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{K}(\mathbb{R}^{\Delta^n}) \in \mathcal{S}p$$

$$\mathrm{GW}^{\mathrm{top}}(\mathbb{R}; Ϙ^{\mathrm{s}}_\epsilon) := |\,\mathrm{GW}(\mathbb{R}^{\Delta^\bullet}; Ϙ^{\mathrm{s}}_\epsilon)| = \operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{GW}(\mathbb{R}^{\Delta^n}; Ϙ^{\mathrm{s}}_\epsilon) \in \mathcal{S}p$$

and

$$\mathrm{L}^{\mathrm{top}}(\mathbb{R}; Ϙ^{\mathrm{s}}_\epsilon) := |\,\mathrm{L}(\mathbb{R}^{\Delta^\bullet}; Ϙ^{\mathrm{s}}_\epsilon)| = \operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{L}(\mathbb{R}^{\Delta^n}; Ϙ^{\mathrm{s}}_\epsilon) \in \mathcal{S}p.$$

The construction above furnishes a natural map of spectra $\mathrm{K}^{\mathrm{top}}(\mathbb{R}) \to \mathrm{ko} = \mathbb{Z} \times \mathrm{BGL}^{\mathrm{top}}_\infty(\mathbb{R})$ which is an equivalence by [Sch17, Proposition 10.2]. Similarly, by the same proposition $\mathrm{GW}^{\mathrm{top}}_0(\mathbb{R}; Ϙ^{\mathrm{s}}_\epsilon) \cong \mathrm{GW}_0(\mathbb{R}; Ϙ^{\mathrm{s}}_\epsilon)$ and $\tau_{\geq 1} \mathrm{GW}^{\mathrm{top}}(\mathbb{R}; Ϙ^{\mathrm{s}}_\epsilon)$ is naturally equivalent to $\mathrm{BO}^{\mathrm{top}}_{\infty,\infty}(\mathbb{R})$ when $\epsilon = 1$ and to $\mathrm{BSp}^{\mathrm{top}}_\infty(\mathbb{R})$ when $\epsilon = -1$. The superscript top indicates that we topologise the groups as sequential colimits of Lie groups. In addition, by [Sch17, Remark 10.4] the natural map $\mathrm{L}(\mathbb{R}; Ϙ^{\mathrm{s}}_\epsilon) \to \mathrm{L}^{\mathrm{top}}(\mathbb{R}; Ϙ^{\mathrm{s}}_\epsilon)$ is an equivalence.

We now claim that the map $\mathrm{K}(\mathbb{Z}) \to \mathrm{K}^{\mathrm{top}}(\mathbb{R}) \simeq \mathrm{ko}$ induces isomorphisms on π_i for $i \leq 2$. The groups in question are in fact isomorphic, furthermore the composite $\mathbb{S} \to \mathrm{K}(\mathbb{Z}) \to \mathrm{ko}$ is an isomorphism in the claimed range, so the result follows.

Let us write $\mathrm{K}^{\mathrm{top}}(\mathbb{R}; Ϙ^{\mathrm{s}}_-)$ for the spectrum $\mathrm{K}^{\mathrm{top}}(\mathbb{R})$ considered together with the C_2-action induced by the duality associated to $Ϙ^{\mathrm{s}}_-$. Since taking homotopy orbits preserves connectivity we get from the above that the map $\pi_i \mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2} \to \pi_i \mathrm{K}^{\mathrm{top}}(\mathbb{R}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2}$ is an isomorphism for $i \leq 2$. To finish the proof it will hence suffice to show that $\pi_2 \mathrm{K}^{\mathrm{top}}(\mathbb{R}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2}$ vanishes. Since geometric realisations preserve fibre sequences of spectra, the latter group sits in an exact sequence

$$\mathrm{L}^{\mathrm{top}}_3(\mathbb{R}; Ϙ^{\mathrm{s}}_-) \longrightarrow \pi_2 \mathrm{K}^{\mathrm{top}}(\mathbb{R}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2} \longrightarrow \mathrm{GW}^{\mathrm{top}}_2(\mathbb{R}; Ϙ^{\mathrm{s}}_-).$$

Since $\mathbb{R}$ is a field we have that $\mathrm{L}^{\mathrm{top}}_3(\mathbb{R}; Ϙ^{\mathrm{s}}_-) \cong \mathrm{L}_3(\mathbb{R}; Ϙ^{\mathrm{s}}_-) \cong 0$ and since $\mathrm{GW}^{\mathrm{top}}_2(\mathbb{R}; Ϙ^{\mathrm{s}}_-) \cong \pi_1 \mathrm{Sp}^{\mathrm{top}}_\infty(\mathbb{R}) \cong \mathbb{Z}$ it follows that the group $\pi_2 \mathrm{K}^{\mathrm{top}}(\mathbb{R}; Ϙ^{\mathrm{s}}_-)_{\mathrm{hC}_2}$ is free. But from the homotopy orbit spectral sequence we see that it has order at most 4, and so we conclude that it is trivial. □

3.2.12. **Remark.** By Karoubi periodicity (as formulated e.g. in Corollary [II].4.5.4), we know that $\mathrm{K}(\mathbb{Z}; Ϙ^s_-) \simeq \mathbb{S}^{2-2\sigma} \otimes \mathrm{K}(\mathbb{Z}; Ϙ^\mathrm{s})$ as spectrum with C_2-action. Furthermore, the map $\mathrm{K}(\mathbb{Z}) \to \mathrm{ko}$ is C_2-equivariant with respect to the C_2-action induced by $Ϙ^\mathrm{s}$ on $\mathrm{K}(\mathbb{Z})$ and the trivial action on ko. The above lemma is then a statement about low dimensional homotopy groups of $(\mathbb{S}^{2-2\sigma} \otimes \mathrm{ko})_{\mathrm{hC}_2}$. These can also be computed using the cofibre sequence $\mathrm{C}_{2+} \to S^0 \to S^\sigma$ and some elaborations thereof.

3.2.13. **Theorem.** *There are isomorphisms*

i) $\mathrm{GW}_0^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$,
ii) $\mathrm{GW}_1^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathbb{Z}/4$,
iii) $\mathrm{GW}_2^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathbb{Z}$
iv) $\mathrm{GW}_3^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathbb{Z}/24$
v) $\mathrm{GW}_i^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathrm{GW}_i^{-\mathrm{gs}}(\mathbb{Z})$ *for* $i \geq 4$.

We remark that statements ii) and iii) have been shown previously by Krannich and Kupers using geometric methods, see [KK20].

Proof. Part v) follows immediately from Lemma 3.2.10 and Lemma 3.2.7iii). Part i) is well known, see the discussion at the beginning of the section. For Part iii) it suffices to note that by Lemma 3.2.11 the map $\mathrm{GW}_2^{-\mathrm{gq}}(\mathbb{Z}) \to \mathrm{L}_2^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathrm{L}_0^{\mathrm{gq}}(\mathbb{Z}) \cong \mathbb{Z}$ is injective with finite cokernel.

Now to show ii) consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{L}_2^{-\mathrm{gq}}(\mathbb{Z}) & \longrightarrow & \pi_1\,\mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{gq}}_-)_{\mathrm{hC}_2} & \longrightarrow & \mathrm{GW}_1^{-\mathrm{gq}}(\mathbb{Z}) & \longrightarrow & \mathrm{L}_1^{-\mathrm{gq}}(\mathbb{Z}) \\
\downarrow & & \downarrow{\cong} & & \downarrow & & \downarrow \\
\mathrm{L}_2^{-\mathrm{gs}}(\mathbb{Z}) & \longrightarrow & \pi_1\,\mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{gs}}_-)_{\mathrm{hC}_2} & \longrightarrow & \mathrm{GW}_1^{-\mathrm{gs}}(\mathbb{Z}) & \longrightarrow & \mathrm{L}_1^{-\mathrm{gs}}(\mathbb{Z})
\end{array}$$

Since $\mathrm{GW}_1^{-\mathrm{gs}}(\mathbb{Z}) = 0$ by Table (3.2.6) the bottom left hand map must be surjective. As in the proof of Lemma 3.2.10, the map $\mathrm{L}_2^{-\mathrm{gq}}(\mathbb{Z}) \to \mathrm{L}_2^{-\mathrm{gs}}(\mathbb{Z})$ identifies with the map $\mathrm{L}_0^{\mathrm{gq}}(\mathbb{Z}) \to \mathrm{L}_0^{\mathrm{gs}}(\mathbb{Z})$ and hence with the inclusion $8\mathbb{Z} \hookrightarrow \mathbb{Z}$. Since $\pi_1\,\mathrm{K}(\mathbb{Z}, Ϙ^{\mathrm{gq}}_-)_{\mathrm{hC}_2} \cong \mathbb{Z}/4$ the upper left hand map must be 0. In addition $\mathrm{L}_1^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathrm{L}_1^{-\mathrm{q}}(\mathbb{Z}) \cong 0$ and so the upper middle map gives an isomorphism $\mathrm{GW}_1^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathbb{Z}/4$ by Lemma 3.2.11i).

Finally, to prove iv) consider the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{L}_4^{-\mathrm{gq}}(\mathbb{Z}) & \longrightarrow & \pi_3\,\mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{gq}}_-)_{\mathrm{hC}_2} & \longrightarrow & \mathrm{GW}_3^{-\mathrm{gq}}(\mathbb{Z}) & \longrightarrow & \mathrm{L}_3^{-\mathrm{gq}}(\mathbb{Z}) \\
\downarrow & & \downarrow{\cong} & & \downarrow & & \downarrow \\
\mathrm{L}_4^{-\mathrm{gs}}(\mathbb{Z}) & \longrightarrow & \pi_3\,\mathrm{K}(\mathbb{Z}; Ϙ^{\mathrm{gs}}_-)_{\mathrm{hC}_2} & \longrightarrow & \mathrm{GW}_3^{-\mathrm{gs}}(\mathbb{Z}) & \twoheadrightarrow & \mathrm{L}_3^{-\mathrm{gs}}(\mathbb{Z})
\end{array}$$

where the bottom right map is surjective by Lemma 3.2.11ii). Then $\mathrm{L}_3^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathrm{L}_1^{\mathrm{gq}}(\mathbb{Z}) = 0$ and $\mathrm{L}_4^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathrm{L}_4^{-\mathrm{gs}}(\mathbb{Z}) \cong \mathrm{L}_2^{\mathrm{gs}}(\mathbb{Z}) \cong 0$, which implies that the top middle horizontal map in the above diagram is an isomorphism and the bottom middle horizontal map is injective with cokernel $\mathrm{L}_3^{-\mathrm{gs}}(\mathbb{Z}) \cong \mathrm{L}_3^{-\mathrm{s}}(\mathbb{Z}) \cong \mathbb{Z}/2$; see Corollary 2.2.3. Since $\mathrm{GW}_3^{-\mathrm{gs}}(\mathbb{Z}) \cong \mathbb{Z}/48$ by Table (3.2.6) this implies that $\mathrm{GW}_3^{-\mathrm{gq}}(\mathbb{Z}) \cong \mathbb{Z}/24$, as claimed. □

3.2.14. **Remark.** By Proposition 3.1.11 and Corollary 1.3.14, we know that for the ring of integers $\mathcal{O}$ in a number field, the canonical map

$$\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(\mathcal{O}; \epsilon) \longrightarrow \mathrm{GW}^{\mathrm{s}}_{\mathrm{cl}}(\mathcal{O}[\tfrac{1}{2}]; \epsilon)$$

is a 2-local equivalence in degrees ≥ 1. In principle, one can then use the results of [KRØ20] to calculate the 2-local Grothendieck-Witt groups of $\mathcal{O}$. As before, the odd torsion is controlled by the isomorphisms

$$\mathrm{GW}^{\mathrm{s}}_{\mathrm{cl},n}(\mathcal{O}; \epsilon)[\tfrac{1}{2}] \cong \mathrm{K}_n(\mathcal{O}; \epsilon)[\tfrac{1}{2}]_{\mathrm{C}_2} \oplus \mathrm{L}^{\mathrm{s}}_n(\mathcal{O}; \epsilon)[\tfrac{1}{2}].$$

To make efficient use of this, one needs a version of Lemma 3.2.4, determining the C_2-action on the 2-inverted K-groups of $\mathcal{O}$, which can again be described in terms of étale cohomology of $\mathcal{O}[\tfrac{1}{2}]$.

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