In this talk I will describe recent work with Olivier Wittenberg concerning rational points and 0-cycles on fibered varieties. I will focus on the case of rational points. The case of 0-cycles will be explained in the next lecture. Our starting point is a conjecture of Colliot-Thélène and Sansuc, first formulated in 1979 for the case of surfaces:

**Conjecture 1 ([CT93, p. 174]).** For any smooth, proper, geometrically irreducible, rationally connected variety $X$ over a number field $k$, the set $X(k)$ is dense in $X(\mathbb{A}_k)^{Br(X)}$.

By “rationally connected”, we mean that for any algebraically closed field $\bar{k}$ containing $k$, the base change of $X$ to $\bar{k}$ is rationally connected in the usual sense. Let us adopt the abbreviation RC to denote “rationally connected".

We shall now fix our base number field $k$. It is well-known that the class of proper, smooth, RC varieties over $k$ is “closed under fibrations”. In other words, if $X$ and $B$ are smooth proper RC varieties and $\pi : X \rightarrow B$ is a surjective map whose generic fiber is RC then $X$ is RC. It is hence natural to ask the following question:

**Question 2.** Let $X$ be smooth proper RC varieties and $\pi : X \rightarrow B$ a surjective map whose generic fiber is GRC. Assume that $B$ satisfies conjecture 1 and that all but finitely many fibers of $\pi$ satisfies conjecture 1. Can we deduce directly that $X$ satisfies conjecture 1?

The theory developed around giving a positive answer to Question 2 is often referred to as ”The fibration method”. In the typical application the base is the projective line $\mathbb{P}^1_k$. Hasse was the first to use such a method in his theorem on quadratic hypersurfaces in order to reduce the general case to the 1-dimensional case. In 1982, Colliot-Thélène and Sansuc [CTS82] noticed that a variant of Hasse’s proof yields Conjecture 1 for a large family of conic bundle surfaces over $\mathbb{P}^1_Q$ if one assumes Schinzel’s hypothesis, a far reaching conjecture regarding polynomials taking simultaneous prime values. Further work of Serre [Ser92] and of Swinnerton-Dyer [SD94] led to the systematic study of fibrations over $\mathbb{P}^1$ into varieties which satisfy weak approximation (see [CT94], [CTSSD98a], [HSW]). All the above-cited papers rely on the same reciprocity
argument as Hasse’s original proof, and as a result they are all required to assume that every singular fiber of \( \pi \) contains an irreducible component of multiplicity 1 split by an abelian extension of its base field. Other approaches were able to dispense with the abelianness condition under strong assumptions on \( \pi \).

By using the theory of descent, Colliot-Thélène and Skorobogatov were able to remove this abelianness condition (but not the weak approximation condition) when the subscheme \( S \subseteq \mathbb{P}^1_k \) of non-split fibers has degree \( \leq 2 \) ([CTS00]). Harari ([Har97]) was able to further allow for a non-trivial Brauer-Manin obstruction on the fibers when this degree is \( \leq 1 \).

Our goal in this talk is to outline a proof of the following theorem

**Theorem 3 ([HW14]).** Let \( X \) be a smooth, proper, geometrically irreducible variety \( X \) over \( \mathbb{Q} \) and let \( \pi : X \to \mathbb{P}^n \) be a dominant morphism whose generic fiber is RC and such that all non-split fibers are all defined over \( \mathbb{Q} \). If there exists a Hilbert subset \( H \subseteq \mathbb{P}^n_\mathbb{Q} \) such that Conjecture 1 holds for \( X_c \) whenever \( c \in H \) then conjecture 1 holds for \( X \).

Even though we present this result for \( k = \mathbb{Q} \) it will be useful to keep the notation general. A first step in trying to give a positive answer to Question 2 is to be able to approximate a given adelic point \( (x_v) \in X(\mathbb{A}_k)^{Br} \) by another adelic point \( (x'_v) \) such that \( \pi(x'_v) \) is rational, i.e., comes from a rational point \( t_0 \in \mathbb{P}^1(k) \). This particular question turns out to fit more naturally in a setting where \( X \) is not necessarily proper, but \( \pi : X \to Y \) is smooth and surjective. If one starts from a proper \( \pi \), a smooth one can always be obtained by restricting to the smooth locus \( X' \subseteq X \) with respect to \( \pi \). For \( \pi \) to remain surjective one needs that each fiber of \( \pi \) contains an irreducible component of multiplicity 1. This indeed turns out to be an essential condition if one wants to approximate an adelic point by one whose image under \( \pi \) is rational.

The notions of adelic points and adelic topology become a bit more subtle when \( X \) is not assumed to be proper. Let us recall the definition.

**Definition 4.** Let \( X \) be a variety over \( k \), \( S \) a finite set of places and \( X \) an \( S \)-integral model for \( X \). A point \( (x_v) \in \prod_{v \in O_k} X(k_v) \) is adelic with respect to \( X \) if for all but finitely many \( v \notin S \) the point \( x_v \) comes from a point in \( X(O_v) \). Let

\[
X(\mathbb{A}_k) \subseteq \prod_{v \in O_k} X(k_v)
\]

denote the set of adelic points with respect to \( X \). We endow \( X(\mathbb{A}_k) \) with the coarsest topology such that for every finite \( S' \supseteq S \) the inclusion

\[
\prod_{v \in S'} X(k_v) \times \prod_{v \notin S'} X(O_v) \hookrightarrow X(\mathbb{A}_k)
\]

is open (where the left hand side is endowed with the product topology). Then a typical neighbourhood of a point \( (x_v) \in X(\mathbb{A}_k) \) looks like

\[
U = \prod_{v \in S'} W_v \times \prod_{v \notin S'} X(O_v)
\]
where \( S' \supseteq S \) is a finite set of places containing all the places where \( x_v \) is not integral and \( x_v \in W_v \) is an open neighbourhood in \( X(k_v) \).

**Remark 5.** Given a variety \( X \), a pair \((S, \mathcal{X})\) as above always exists. Furthermore, neither the set \( X(\mathbb{A}_k) \) nor its topology as defined above depend on \((S, \mathcal{X})\). In particular, given an adelic point \((x_v)\), when we talk about approximating it by an adelic point \((x'_v)\), we mean that for some \( S \) and some \( S\)-integral model \( \mathcal{X} \), the point \((x'_v)\) is arbitrarily close to \( x_v \) for every place in \( S \) and \( x'_v \) is integral for every \( v \not\in S \).

At this point it will be useful to describe a particular type of fibration, whose behaviour is in some sense universal with respect to “fibration problems”. Let \( m \geq 2, n \geq 1 \) be integers. For each \( i = 1, \ldots, n \), let \( k_i/k \) and \( L_i/k_i \) be finite field extensions. For each \( i = 1, \ldots, n, j = 1, \ldots, m \) let \( a_{i,j} \in k_i \) be an element. We assume the following:

**Assumption 6.** The \( k \)-linear map \( k^m \longrightarrow \bigoplus_{i=1}^k k_i \) given by \((x_1, \ldots, x_m) \mapsto (\sum_j a_{1,j}x_j, \ldots, \sum_j a_{n,j}x_j)\) is of full rank.

To this data we associate an irreducible quasi-affine variety \( W \) over \( k \) endowed with a smooth morphism \( p : W \longrightarrow \mathbb{P}_k^{m-1} \) with a geometrically irreducible generic fiber as follows.

For \( i = 1, \ldots, n \), us denote by \( T_i = R_{L_i/k}(\mathbb{G}_{m, L_i}) \) the extension of scalars torus and by \( D_i = R_{L_i/k}(\mathbb{A}_{L_i}^1) \setminus T \) its complement in the corresponding affine space. We then observe that \( D_i \) is a divisor with normal crossings (geometrically isomorphic to the union of all coordinate hyperplanes). Let \( F_i \subseteq D_i \) denote the singular locus of \( D_i \) (corresponding to the locus where two different hyperplanes meet). Then \( F_i \) has codimension 2 insude \( R_{L_i/k}(\mathbb{A}_{L_i}^1) \). Let

\[
W \subseteq (\mathbb{A}_k^m \setminus \{(0, 0)\}) \times \prod_{i=1}^n (R_{L_i/k}(\mathbb{A}_{L_i}^1) \setminus F_i)
\]  

(0.1)

be the subvariety given by the equations

\[
\sum_{j=1}^m a_{i,j}x_j = N_{L_i/k_i}(y_i)
\]

for \( i = 1, \ldots, n \), where \( x_1, \ldots, x_m \) are the coordinates of \( \mathbb{A}_k^m \) and \( y_i \) is a coordinate on \( R_{L_i/k}(\mathbb{A}_{L_i}^1) \). Finally, denote by \( p : W \longrightarrow \mathbb{P}_k^{m-1} \) the composition of the projection to the first factor with the natural map \( \mathbb{A}_k^m \setminus \{(0, 0)\} \to \mathbb{P}_k^{m-1} \). We note that since the \( F_i \)'s were removed the map \( p \) is always smooth.

Let \( S \subseteq \Omega_k \) be a finite set of places such that each \( k_i \) and each \( L_i \) are unramified outside \( S \) and such that for each \( v \not\in S \) and each place \( w \in \Omega_k \) lying above \( v \) we have that all the \( a_{i,j} \) are \( w \)-integral and at least one of the \( a_{i,j} \) is a \( w \)-unit. Let \( S_i \) denote the set of places of \( k_i \) which lie above \( S \) and let \( T_i \) denote the set of places of \( L_i \) which lie above \( S \). Then there is a natural \( S \)-integral model \( \mathcal{W} \) for \( W \). Define \( \mathcal{T}_i = R_{\mathcal{O}_{\mathcal{W}}/\mathcal{O}_{\mathcal{T}_i}}(\mathbb{G}_{m, \mathcal{O}_{\mathcal{T}_i}}) \) and let \( \mathcal{T}_i \) denote
the singular locus of \(D_i\). Since the norm of a \(T_i\)-integral element is \(S_i\)-integral and since \(L_i/k_i\) is unramified over \(S_i\) we get that the norm map \(N_{L_i/k_i}\) can be refined to a map of \(O_S\)-schemes

\[
N_{\mathcal{O}_{T_i}/\mathcal{O}_{S_i}}: \mathcal{O}_{\mathcal{O}_{T_i}/\mathcal{O}_{S_i}}^{1} \rightarrow \mathcal{O}_{\mathcal{O}_{S_i}/\mathcal{O}_{S_i}}^{1}.
\]

We then define the \(O_S\)-subscheme

\[
W \subseteq (\mathbb{A}^m_{\mathcal{O}_S} \setminus \{(0,0)\}) \times \prod_{i=1}^n \left( \mathcal{O}_{\mathcal{O}_{T_i}/\mathcal{O}_{S_i}}^{1} \setminus \mathcal{F}_i \right)
\]

by the equations

\[
\sum_{j=1}^m a_{i,j} x_j = N_{\mathcal{O}_{T_i}/\mathcal{O}_{S_i}}(y_i)
\]

where we interpret \(x_1, \ldots, x_m\) as coordinates of \(\mathbb{A}^m_{\mathcal{O}_S}\) and \(y_1, \ldots, y_n\) as coordinates on \(\mathcal{O}_{\mathcal{O}_{T_i}/\mathcal{O}_{S_i}}^{1}\). The corresponding map of \(O_S\)-schemes \(p: W \rightarrow \mathbb{P}^{m-1}\) is smooth.

It is worthwhile to have an explicit description of local integral points of \(W\) for \(v \notin S\).

**Lemma 7.** Let \(S \subseteq \Omega_k\) and \(W\) be as above. If \(v \notin S\) is a place and \(x_j \in k_v, y_i \in L_i\) are elements considered as coordinates for a local point \(P = (x_1, \ldots, x_m, y_1, \ldots, y_n) \in W(k_v)\), then \(P\) comes from \(W(O_v)\) if and only the following holds:

1. Each \(x_i\) belongs to \(O_v\) and at least one of the \(x_i\)'s belongs to \(O_v^*\).
2. Each \(y_i\) belongs to \(O_{L_i,v} = \mathcal{O}_{T_i} \otimes_{\mathcal{O}_S} O_v\).
3. For each \(i = 1, \ldots, n\) there is at most one place \(u \in \Omega_{L_i}\) lying over \(v\) such that \(\text{val}_u(y_i) > 0\). Furthermore, if such a place \(u\) exists then it has degree 1 over \(v\).

**Proof.** We first observe that each \(x_i\) must be \(v\)-integral in order for \((x_1, \ldots, x_m)\) to come from a \(v\)-integral point of \(\mathbb{A}^m_{\mathcal{O}_S}\). If we further want it to come from \(\mathbb{A}^m_{\mathcal{O}_S} \setminus \{(0, \ldots, 0)\}\) one needs to assume that the reduction of the vector \((x_1, \ldots, x_m)\) mod \(v\) is not the 0-vector. This is equivalent to saying that at least one of the \(x_i\)'s is in \(O_v^*\). We hence see that condition (1) is necessary. Similarly, we see that condition 2 is necessary. Now let \(y_i \in O_{L_i,v}\) be an element. By definition we have that \(y_i\) comes from a \(v\)-integral point of

\[
\left[ \mathcal{O}_{L_i,v} \otimes_{\mathcal{O}_v} \mathcal{A}_{\mathcal{O}_{L_i,v}}^{1} \right] \setminus \mathcal{F}_i
\]

if and only if the reduction \(\overline{y_i}\) of \(y_i\) mod \(v\) does not belong to \(\mathcal{F}_i \otimes_{\mathcal{O}_v} F_v\). In particular, \(\overline{y_i}\) either belongs to \(\mathcal{F}_i \otimes_{\mathcal{O}_v} F_v\) or is a smooth point of \(D_i \otimes_{\mathcal{O}_v} F_v\). The first case is equivalent to \(y_i \in O_{L_i,c}\), i.e., no place of \(L\) above \(v\) divides \(y_i\). Since the geometric irreducible components of \(D_i \otimes_{\mathcal{O}_v} F_v\) are smooth, the second case is equivalent to \(\overline{y_i}\) lying on exactly one irreducible component of
$\mathcal{D}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$, and that this component is geometrically irreducible. Let $E_{i,v} = \mathcal{O}_{L_i,v} \otimes_{\mathcal{O}_v} \mathbb{F}_v$. Then the irreducible components of $\mathcal{D}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$ correspond to the coordinate hyperplanes of $R_{E_{i,v}/\mathbb{F}_v} \mathbb{A}^1_{E_{i,v}}$, and are hence classified by $\text{spec}(E_{i,v})$. In particular, geometrically irreducible components of $\mathcal{D}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$ (defined over $\mathbb{F}_v$) correspond to $\mathbb{F}_v$-points of $\text{spec}(E_{i,v})$, i.e., to places $u \in \Omega_{L_i}$ over $v$. Now $\overline{y}_i$ sits on the component corresponding to $u$ if and only if the $u$-coordinate of $\overline{y}_i$ vanishes, i.e., if and only if $\text{val}_u(y_i) > 0$. We hence see that $\overline{y}_i$ sits on exactly one irreducible component $\mathcal{D}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$ if and only if there is exactly one place $u \in \Omega_{L_i}$ over $v$ such that $\text{val}_u(y_i) > 0$, and that the component corresponding to $u$ is geometrically irreducible if and only if $u$ is of degree 1. The desired result now follows.

**Conjecture 8 (Conjecture (W)).** Assume the data defining the variety $W$ satisfies assumption $\mathcal{A}$. Then the inclusion

$$\bigcup_{t_0 \in \mathbb{P}^{m-1}(k)} W_{t_0}(\mathbb{A}_k) \subseteq W(\mathbb{A}_k)$$

is dense in the adelic topology.

**Theorem 9** ([Mat14], building on earlier work with Tim Browning, building on a deep results of Green, Tao and Ziegler). Conjecture (W) holds whenever $k_1 = k_2, ..., k_n = \mathbb{Q}$.

**Remark 10.** The result of [Mat14] is in fact stronger. Given an adelic point $(x_v) \in W(\mathbb{A}_\mathbb{Q})$ and a large enough finite set of places $S$, one may find an $S$-integral point $x_0 \in W(\mathbb{Z}_S)$ such that $x_0$ is arbitrarily close to $(x_v)$ for every non-archimedean place $v \in S$ and such that the $\mathbb{A}^m$-coordinates of $x_0$ belongs to any convex cone containing the $\mathbb{A}^m$-coordinates of $x_\infty$.

**Theorem 11** ([HW14]). If $m \geq \sum_{i=1}^n \lceil k_i : k \rceil$ then conjecture (W) holds.

**Proof.** Will be given in Olivier’s talk.

The following can be proven using the standard fibration method techniques.

**Theorem 12.** Assume Schinzel’s hypothesis holds. Then Conjecture (W) holds whenever the extensions $L_i/k_i$ are abelian (or semi-abelian, see [HW14]).

In order to state our main result we will need to recall the following standard notion.

**Definition 13.** Let $\pi : X \rightarrow Y$ be a map. We will denote by

$$\text{Br}_{\text{vert}}(X) = \text{Br}(X) \cap \text{Br}(k(Y)) \subseteq \text{Br}(X)$$

the subgroup of $\text{Br}(X)$ consisting of classes which come from the Brauer group of the generic point of $Y$. The subgroup $\text{Br}_{\text{vert}}(X)$ is known as the **vertical Brauer group** of $X$ with respect to $\pi$. 

5
Remark 14. If $X,Y$ are smooth and geometrically irreducible and $\pi$ is smooth and surjective with a geometrically irreducible fiber then the quotient $\text{Br}(X)_{\text{vert}}/\text{Br}(k)$ is finite (see [CTS00] Lemma 3.1).

Remark 15. It is not hard to show that $\text{Br}_v(W) = \text{Br}(k)$. See [HSW] for a proof in a particular case.

Theorem 16 ([HW14]). Assume conjecture (W) holds for $m = 2$. Let $X$ be a smooth geometrically integral variety over $k$ endowed with a smooth morphism $\pi : X \rightarrow \mathbb{P}^1$ with a geometrically integral generic fiber. Let $V \subseteq \mathbb{P}^1_k$ be an open set such that the fiber of $\pi$ above every point of $U$ is geometrically integral. Let $U = \pi^{-1}(V) \subseteq X$ and let $B \subseteq \text{Br}(U)/\text{Br}(k)$ be a finite subgroup containing $\text{Br}(U)_{\text{vert}}/\text{Br}(k)$. Let $B' = B \cap [\text{Br}(X)/\text{Br}(k)]$. Then the inclusion

$$\bigcup_{c \in V(k)} U_c(A_k)^B \subseteq X(A_k)^{B'}$$

is dense in the adelic topology.

Remark 17. In light of Remark 15 we see that the conclusion of Theorem 16 includes Conjecture (W) is a particular case. This can be interpreted as saying that the fibrations $\pi : W \rightarrow \mathbb{P}^1$ are, in some sense, “complete fibration problems”. According to Theorem 16, if one can solve all the fibration problems of type (W) (with $m = 2$) then one can solve any fibration problem.

Before we give the proof let us recall Harari’s “formal lemma”:

Lemma 18 ([Har94]). Let $X$ be a smooth geometrically integral variety and $U \subseteq X$ an open subset. Let $B \subseteq \text{Br}(U)$ be a finite subgroup. Let $(x_v) \in X(A_k)$ be an adelic point which is orthogonal to $B \cap \text{Br}(X)$. Then there exists an adelic point $(x_v') \in U(A_k)$, arbitrarily close to $(x_v)$ in the adelic topology on $X(A_k)$, and such that $(x_v')$ is orthogonal to $B$.

We are now ready to prove Theorem 16.

Proof of Theorem 16. For simplicity we will prove the claim for $B = \text{Br}_v(U)/\text{Br}(k)$. We note that in this case $B \cap X = \text{Br}_v(X)/\text{Br}(k)$. The proof in the general case is similar.

Let $m_1,\ldots,m_n \in \mathbb{P}^1$ denote the closed points in the complement of $V$. For each $i = 1,\ldots,n$ we will denote by $Y_i$ the fiber of $\pi$ over $m_i$ and fix an irreducible component $Y_i \subseteq X_i$. Let $k_i = k[m_i]$ be the function field of $m_i$ and $L_i/k_i$ the algebraic closure of $k$ in the field of functions of $Y_i$. Without loss of generality we can assume that each $m_i$ lies in $\mathbb{A}^1$. Let $t$ be a coordinate on $\mathbb{A}^1$ and let $e_i \in k_i$ denote the value $t$ takes on $m_i$.

For each $i$, let $\chi_{i,1},\ldots,\chi_{i,n_i} : \Gamma_{k_i} \rightarrow \mathbb{Q}/\mathbb{Z}$ be a finite set of generators for the group $\text{Ker}[\text{Hom}(\Gamma_{k_i},\mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(\Gamma_{L_i},\mathbb{Q}/\mathbb{Z})]$.

We denote by $K_{i,j}$ the fixed field of $\chi_{i,j}$. Consider the Brauer element $A_{i,j} = \text{cores}_{K_{i}(t)/k(t)}(t - e_i, K_{i,j}/k_i) \in \text{Br}(k(t))$.
Then $A_{i,j}$ is ramified on $\mathbb{P}^1$ only at $m_i$ and $\infty$ with residues $\chi_{i,j}$ and $-\text{cores}_{k_i/k}(\chi_{i,j})$, respectively. In particular, $A_{i,j}$ is unramified on $V$ and $\pi^* A_{i,j}$ is unramified on $U$. Furthermore, $\pi^* A_{i,j} \in \text{Br}(U)_{\text{vert}}$.

Now let $(x_v) \in X(\mathbb{A}_k)^{\text{Br}_{\text{vert}}}$ be a point that we wish to approximate. Let $S$ be a finite set of places which is big enough so that the following holds:

1. $S$ contains all the archimedean places.
2. All the the $k_i$'s and all the $L_i$'s are unramified outside $S$.
3. Each $e_i$ is $S$-integral and the norm of each $e_i - e_j \in k_i k_j$ is an $S$-unit.
4. $X$ admits a smooth $S$-integral model $\mathcal{X}$ such that the induced map $\pi : \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_S}$ is smooth.

Our goal is to construct to show that for every large enough $S' \supset S$ there exists an adelic point $(x_v)^' \in U(\mathbb{A}_k)$ such that $(x_v)^'$ is arbitrarily close to $(x_v)$ for every $v \in S'$ and $x_v$ is $v$-integral with respect to $\mathcal{X}$ for every $v \notin S'$.

In light of our conditions on $S$ one may consider the natural $S$-integral model $\mathcal{V}$ for the open subset $V$. If $v \notin S$ and $t_v \in k_v$ is a coordinate of a point in $p_v \in V(k_v)$ then $p_v$ belongs to $\mathcal{V}(\mathcal{O}_v)$ if and only if $\text{val}(t_v) \geq 0$ and $\text{val}(t_v - e_i) = 0$ for every $i$. Furthermore, by condition (3) above, if we denote by $m_i \subseteq \mathbb{P}^1_{\mathcal{O}_S}$ the Zariski closure of $m_i$ then $m_i$ and $m_j$ do not intersect in $\mathcal{V}$. We will denote by $\mathcal{X}_i \subseteq \mathcal{X}$ the fiber of $\pi$ over $m_i$ and by $\mathcal{Y}_i \subseteq \mathcal{X}_i$ the Zariski closure of the component $\mathcal{Y}_i$. We then observe that the smooth $\mathcal{O}_S$-scheme $\text{spec}(\mathcal{O}_{L_i,S})$ classifies the irreducible components of $\mathcal{Y}_i$. In particular, if $w \in \Omega_k$ is a place of $k_i$ corresponding to a closed point $x \in m_i$, then the irreducible components of $\mathcal{Y}_i \times_M \{x\}$ are in bijection with places $u \in \Omega_L$ lying above $w$. Finally, let us add three more assumptions which can be met by enlarging $S$ if necessary:

5 Each $A_{i,j}$ extends to an element $A_{i,j} \in \text{Br}(V)$.
6 For every $v \notin S$ and every $x \in \mathcal{V}(\mathcal{O}_v)$ there exists a $y \in \mathcal{X}(\mathcal{O}_v)$ lying above $t$.
7 For every $v \notin S$, every place $w \in \Omega_k$, lying above $v$ (corresponding to a closed point $x \in m_i$) and every place $u \in \Omega_L$ of degree 1 over $w$, the component of $\mathcal{Y}_i \times_M \{x\}$ classified by $u$ has an $\mathcal{F}_w$-point.

By Harari’s formal lemma [Har94] there exists an adelic point $(x_v^') \in U(\mathbb{A}_k)$ such that

1. $(x_v^')$ is arbitrarily close to $(x_v)$ for every $v \in S$.
2. For every $A_{i,j}$ as above we have

$$\sum_{v \in \Omega_k} A_{i,j}(\pi(x_v^')) = 0. \quad (0.3)$$
Let \( U = V \times_p V \) and let \( S' \) be a finite set of places containing \( S \), such that \((x_v)\) belongs to \( \mathcal{U}(\mathcal{O}_v) \) for every \( v \notin S' \). Let \( t'_v \) be the coordinate of \( \pi(x'_v) \) on \( \mathbb{A}^1 \), and let \( S'_t \) be the set of places of \( k_t \) lying above \( S' \). Given a place \( v \in \Omega_{k_t} \) we will denote by \( \overline{v} \) the place of \( k \) lying below it. We then have for each \( i, j \) as above

\[
\sum_{w \in \Omega_{k_t}} \text{inv}_w(t'_w - e_i, \chi_{i,j}) = 0. \tag{0.4}
\]

Let us now fix an \( i = 1, \ldots, n \) and let \( T_i = R_{L_i/k_i} \mathbb{G}_m \) be the restriction-of-scalars torus of \( L_i/k_i \) and \( T^1_i \) the associated norm one torus. Since \((x_v)\) lies in \( \mathcal{U}(\mathcal{O}_v) \) for all but finitely many \( v \)'s we see that \( t'_v - e_i \) lies in \( \mathbb{G}_m(\mathcal{O}_w) = \mathcal{O}^*_w \) for all but finitely many places \( w \in \Omega_{k_i} \). Let us denote by

\[
(\sigma^i_w) = (t'_w - e_i) \in \prod_{w \in \Omega_{k_i}} \mathbb{G}_m(A_{k_i})
\]

the resulting adelic point. Consider the following short exact sequence \( k_i \)-groups schemes

\[
1 \longrightarrow T^1_i \longrightarrow T_i \longrightarrow \mathbb{G}_m \longrightarrow 1
\]

and the associated boundary map

\[
\partial : \mathbb{G}_m(A_{k_i}) \longrightarrow H^1(A_{k_i}, T^1_i).
\]

We claim that this element is in fact \( k_i \)-rational, i.e., comes from an element \( \rho_i \in H^1(k_i, T^1_i) \). In light of Tate-Poitou’s sequence for tori what we need to prove is that \((\partial \sigma^i_w)\) is orthogonal to all the elements in \( H^1(k_i, T^1_i) \). To compute the latter consider the short exact sequence

\[
1 \longrightarrow \mathbb{Z} \longrightarrow \hat{T}_i \longrightarrow \hat{T}^1_i \longrightarrow 1
\]

The boundary map

\[
\partial : H^1(k_i, \hat{T}^1_i) \longrightarrow H^2(k_i, \mathbb{Z}) \cong H^1(k_i, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\Gamma_{k_i}, \mathbb{Q}/\mathbb{Z})
\]

maps \( H^1(k_i, \hat{T}^1_i) \) into the kernel of the map

\[
\text{Hom}(\Gamma_{k_i}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \text{Hom}(\Gamma_{L_i}, \mathbb{Q}/\mathbb{Z}).
\]

In particular, for each \( \alpha \in H^1(k_i, \hat{T}^1_i) \) we have that \( \partial \alpha \) is some linear combination of the \( \chi_{i,j} \)'s. From the naturality of the pairing and equation \( 0.4 \) we then get

\[
\langle \partial \sigma^i, \alpha \rangle = \langle \sigma^i, \partial \alpha \rangle = \sum_{w \in \Omega_{k_i}} \text{inv}_w(t'_w - e_i, \partial \alpha) = 0.
\]

This proves that \( \partial \sigma^i \) comes from some element \( \rho_i \in H^1(k_i, T^1_i) \). Now since \( H^1(k_i, T) = 0 \) it follows that \( \rho_i = \partial b_i \) for some \( b_i \in \mathbb{G}_m(k_i) \). We may hence conclude that for every \( w \in \Omega_{k_i} \), there exists a \( y_{w,i} \in (L_i)_w \) such that

\[
b_i(t'_w - e_i) = N_{(L_i)_w/(k_i)_w}(y_{w,i}).
\]
Let $S'$ be a finite set of places containing $S$ such that $b_i$ (and hence $b_i(t_{w,i} - e_i)$) is in $\mathcal{O}_v^\ast$ whenever $w \in \Omega_{k_i}$, is such that $\mathfrak{m} \not\in S''$. We may then assume that if $w \in \Omega_{k_i}$ is such that $\mathfrak{m} \not\in S''$ then $y_{w,i} \in \mathcal{O}_v^\ast$.

Let $W$ be the quasi-affine variety associated as above to the fields $k_1, \ldots, k_n$, $L_1, \ldots, L_n$, and to the values $a_{i,0} = b_i, a_{i,1} = b_i e_i$ (note that here $m = 2$). Let $W$ be the smooth $S''$-integral model established in Lemma 7. According to Lemma 7 the values $(t_{w,i}^\prime, 1, y_{w,1}, \ldots, y_{w,n})$ determine an adelic point $(p_v)$ on $W$ such that $p_v$ belongs to $W(\mathcal{O}_v)$ for every $v \not\in S''$. By Conjecture (W) there exists an adelic point $(q_v) = (\lambda_v, \mu_v, y_{w,1}^\prime, \ldots, y_{w,n}^\prime)$ such that

1. $q_v$ is arbitrarily close to $p_v$ for every $v \in S''$.
2. $q_v$ belongs to $W(\mathcal{O}_v)$ for every $v \not\in S''$.
3. $q_v$ lies above a rational point $t_0 \in \mathbb{P}^1(k)$.

We shall now construct an adelic point $(x_v^\prime)$ on the fiber $X_1$ which is arbitrarily close to $(x_v^\prime)$ for $v \in S''$ and is $X$-integral outside $S''$. For $v \in S''$, we have that $t_0$ is arbitrarily close to $t_v$ and so we may construct $x_v^\prime$ by means of the inverse function theorem. Let us hence consider the case $v \not\in S''$. Then at least one of $\lambda_v, \mu_v$ is in $\mathcal{O}_v^\ast$ and $(\lambda_v : \mu_v) = (t_0 : 1)$. Furthermore, if there exists a place $w \in \Omega_{k_i}$ lying over $v$ such that $\text{val}_w(b_i(\lambda_v - e_i \mu_v)) > 0$ then by Lemma 7 $u$ would have to be of degree 1 over $v$. In particular, in this case $w$ has of degree 1 over $v$ and $u$ has of degree 1 over $w$.

Let us now construct an $\mathcal{O}_v$-point on $X$. First suppose that $\text{val}_w(t_0 - e_i) \leq 0$ for every $i = 1, \ldots, n$ and every $w \in \Omega_{k_i}$ lying above $v$. In this case the reduction of $t_0$ modulo $v$ lies in $\mathcal{V}$ and hence by our choice of $S$ there is a smooth $\mathbb{F}_v$-point on $X_{t_0}(\mathbb{F}_v)$ and so an $\mathcal{O}_v$-point on $X_{t_0}$ by Hensel’s lemma.

Now suppose that $\text{val}_w(t_0 - e_i) > 0$ for some $e_i$ and some $w \in \Omega_{k_i}$ lying above $v$ (in which case the pair $(i, w)$ is unique by our choice of $S$). By the above considerations we know that $w$ must have degree 1 over $v$ and that there must exist a place $u \in \Omega_{k_i}$ of degree 1 over $w$. Let $x \in \mathcal{M}_i$ be the closed point corresponding to $w$. By our choice of $S$ the irreducible component of $\mathcal{M}_i \times_X \{x\}$ classified by $u$ has a smooth $\mathbb{F}_w$-point and hence a smooth $\mathbb{F}_v$-point. This point can then be lifted to an $\mathcal{O}_v$-point of $X_{t_0}$ by Hensel’s lemma.

\textbf{Corollary 19 (Main theorem). Assume that conjecture (W) holds for $m = 2$. Let $X$ be a smooth, proper, geometrically integral variety $X$ over $k$ and let $\pi : X \to \mathbb{P}^n$ be a dominant morphism whose generic fiber is RC. If there exists a Hilbert subset $H \subseteq \mathbb{P}^n$ such that Conjecture 7 holds for $X_c$ whenever $c \in H$ then conjecture 7 holds for $X$.}

\textbf{Proof.} Let $X' \subseteq X$ be the smooth locus of $X$. Since the generic fiber of $\pi$ is RC it follows that every fiber of $\pi$ has an irreducible component of multiplicity one. This implies that the restricted map $\pi : X' \to \mathbb{P}^1_k$ is surjective and that the complement of $X'$ in $X$ has codimension 2, and so $\text{Br}(X') = \text{Br}(X)$.}
Let \( \eta \in \mathbb{P}^1 \) be the generic point. Since the generic fiber \( X_\eta \) is RC, it follows from [CT13, Lemma 1.3] that \( Br(X_\eta)/Br(\eta) \) is finite. Hence there exists an open subset \( V \subseteq \mathbb{P}^1 \) and a finite subgroup \( B \subseteq Br(\pi^{-1}(V))/Br(k) \), containing the vertical Brauer group of \( U = \pi^{-1}(V) \), and such that the map \( B \to Br(X_\eta)/Br(\eta) \) is surjective. By [Har77, Théorème 2.3.1] there exists a Hilbert subset \( H' \subseteq H \) such that the restriction \( B \to Br(X_c)/Br(k) \) is surjective for every \( c \in \mathbb{P}^1 \). Let \( h \in H'(k) \) be a point. Let \( (x_v) \in X(A_k) \) be an adelic point which is orthogonal to \( Br(X) \) and \( S \) a finite set of places. By a small deformation we may assume that \( (x_v) \) lies on \( X' \) and since \( Br(X') = Br(X) \) we know that \( (x_v) \) is orthogonal to \( Br(X') \).

Let \( h \in H'(k) \) be a point. By enlarging \( S \) we may assume that \( X'_h(\mathbb{O}_v) \neq \emptyset \) for every \( v \notin S \) and that all the elements of \( B \cap [Br(X)/Br(k)] \) extend to \( S \)-integral elements on \( X' \). We may hence replace \( x_v \) with some \( x'_v \in X_h(\mathbb{O}_v) \) for every \( v \notin S \), and set \( x'_v = x_v \) for \( v \in S \). Then \( (x'_v) \) is orthogonal to \( B \cap [Br(X')/Br(k)] \), and we may then apply Theorem 16 to it. This results in an adelic point \( (x''_v) \) approximating \( (x_v) \) arbitrarily well (and hence approximating \( (x_v) \) over \( S \) and integral outside \( S \)), orthogonal to \( B \), and such that \( \pi(x''_v) \) is rational. By [Sme13, Proposition 6.1] it follows that the \( t_0 \) satisfying Conjecture 1 holds for \( X_t \) whenever \( c \in H \). By the above we get that \( (x''_v) \) is orthogonal to \( Br(X_{t_0})/Br(k) \) and since \( X'_{t_0} = X_{t_0} \) satisfies Conjecture 1 it follows that \( (x''_v) \) can be approximated by a rational point of \( X_{t_0} \).

**Theorem 20** (Main theorem - optimized version). Let \( X \) be a smooth, proper, geometrically integral variety \( X \) over \( k \) and let \( \pi : X \to \mathbb{P}^n \) be a dominant morphism whose generic fiber is RC. Assume that there exists a Hilbert subset \( H \subseteq \mathbb{P}^n \) such that Conjecture 1 holds for \( X_c \) whenever \( c \in H \). Let \( M \subseteq \mathbb{P}^1 \) be the scheme classifying the non-split fibers of \( \pi \). Assume that the following two conditions hold:

1. Either \( k \) is totally imaginary or \( M \) contains a rational point.
2. Conjecture (W) holds for \( m = 2 \) in those cases where the singular locus of \( W \to \mathbb{P}^1 \) coincides with \( M \).

Then conjecture 1 holds for \( X \).

**Corollary 21.** Let \( X \) be a smooth, proper, geometrically integral variety \( X \) over \( \mathbb{Q} \) and let \( \pi : X \to \mathbb{P}^n \) be a dominant morphism whose generic fiber is RC and such that all non-split fibers are all defined over \( \mathbb{Q} \). If there exists a Hilbert subset \( H \subseteq \mathbb{P}^n \) such that Conjecture 1 holds for \( X_c \) whenever \( c \in H \) then conjecture 1 holds for \( X \).

**Corollary 22.** Let \( X \) be a smooth, proper, geometrically integral variety \( X \) over a totally imaginary number field \( k \) and let \( \pi : X \to \mathbb{P}^n \) be a dominant morphism whose generic fiber is RC. Assume that the scheme \( M \) classifying the non-split fibers of \( \pi \) is of degree \( \leq 2 \). If there exists a Hilbert subset \( H \subseteq \mathbb{P}^n \) such that Conjecture 1 holds for \( X_c \) whenever \( c \in H \) then conjecture 1 holds for \( X \).
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