The Fibration Method

Yonatan Harpaz

May 4, 2016

In this talk I will describe recent work with Olivier Wittenberg concerning rational points and 0-cycles on fibered varieties. I will focus on the case of rational points. The case of 0-cycles will be explained in the next lecture. Our starting point is a conjecture of Colliot-Thélène and Sansuc, first formulated in 1979 for the case of surfaces:

Conjecture 1 ([CT03, p. 174]). *For any smooth, proper, geometrically irreducible, rationally connected variety X over a number field k, the set $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$.*

By "rationally connected', we mean that for any algebraically closed field $\overline{k}$ containing k, the base change of X to $\overline{k}$ is rationally connected in the usual sense. Let us adopt the abbreviation RC to denote "rationally connected".

We shall now fix our base number field k. It is well-known that the class of proper, smooth, RC varieties over k is "closed under fibrations". In other words, if X and B are smooth proper RC varieties and $\pi : X \longrightarrow B$ is a surjective map whose generic fiber is RC then X is RC. It is hence natural to ask the following question:

Question 2. *Let X be smooth proper RC varieties and $\pi : X \longrightarrow B$ a surjective map whose generic fiber is GRC. Assume that B satisfies conjecture 1 and that all but finitely many fibers of π satisfies conjecture 1. Can we deduce directly that X satisfies conjecture 1?*

The theory developed around giving a positive answer to Question 2 is often referred to as "The fibration method". In the typical application the base is the projective line $\mathbb{P}^1_k$. Hasse was the first to use such a method in his theorem on quadratic hypersurfaces in order to reduce the general case to the 1-dimensional case. In 1982, Colliot-Thélène and Sansuc [CTS82] noticed that a variant of Hasse's proof yields Conjecture 1 for a large family of conic bundle surfaces over $\mathbb{P}^1_{\mathbb{Q}}$ if one assumes **Schinzel's hypothesis**, a far reaching conjecture regarding polynomials taking simultaneous prime values. Further work of Serre [Ser92] and of Swinnerton-Dyer [SD94] led to the systematic study of fibrations over $\mathbb{P}^1_k$ into varieties which satisfy weak approximation (see [CT94], [CTSSD98a], [HSW]). All the above-cited papers rely on the same reciprocity argument as Hasse's original proof, and as a result they are all required to assume that every singular fiber of π contains an irreducible component of multiplicity 1 split by an **abelian extension** of its base field. Other approaches were able to dispense with the abelianness condition under strong assumptions on π. By using the theory of descent, Colliot-Theélène and Skorobogatov were able to remove this abelianness condition (but not the weak approximation condition) when the subscheme $S \subseteq \mathbb{P}^1_k$ of non-split fibers has degree ≤ 2 ([CTS00]). Harari ([Har97]) was able to further allow for a non-trivial Brauer manin obsrtuction on the fibers when this degree is ≤ 1.

Our goal in this talk is to outline a proof of the following theorem

Theorem 3 ([HW14])**.** *Let X be a smooth, proper, geometrically irreducible variety X over $\mathbb{Q}$ and let $\pi : X \longrightarrow \mathbb{P}^n$ be a dominant morphism whose generic fiber is RC and such that all non-split fibers are all defined over $\mathbb{Q}$. If there exists a Hilbert subset $H \subseteq \mathbb{P}^n_{\mathbb{Q}}$ such that Conjecture 1 holds for X_c whenever $c \in H$ then conjecture 1 holds for X.*

Even though we present this result for $k = \mathbb{Q}$ it will be useful to keep the notation general. A first step in trying to give a positive answer to Question 2 is to be able to approximate a given adelic point $(x_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$ by another adelic point (x'_v) such that $\pi(x'_v)$ is **rational**, i.e. comes from a rational point $t_0 \in \mathbb{P}^1(k)$. This particular question turns out to fit more naturally in a setting where X is not necessarily proper, but $\pi : X \longrightarrow Y$ is **smooth and surjective**. If one starts from a proper π, a smooth one can always be obtained by restricting to the smooth locus $X' \subseteq X$ with respect to π. For π to remain surjective one needs that each fiber of π contains an irreducible component of multiplicity 1. This indeed turns out to be an essential condition if one wants to approximate an adelic point by one whose image under π is rational.

The notions of adelic points and adelic topology become a bit more subtle when X is not assumed to be proper. Let us recall the definition.

Definition 4. Let X be a variety over k, S a finite set of places and $\mathcal{X}$ an S-integral model for X. A point $(x_v) \in \prod_{v \in \Omega_k} X(k_v)$ is **adelic** with respect to $\mathcal{X}$ if for all but finitely many $v \notin S$ the point x_v comes from a point in $\mathcal{X}(\mathcal{O}_v)$. Let

$$X(\mathbf{A}_k) \subseteq \prod_{v \in \Omega_k} X(k_v)$$

denote the set of adelic points with respect to $\mathcal{X}$. We endow $X(\mathbb{A}_k)$ with the coarsest topology such that for every finite $S' \supset S$ the inclusion

$$\prod_{v \in S'} X(k_v) \times \prod_{v \notin S'} \mathcal{X}(\mathcal{O}_v) \hookrightarrow X(\mathbb{A}_k)$$

is open (where the left hand side is endowed with the product topology). Then a typical neighbourhood of a point $(x_v) \in X(\mathbf{A}_k)$ looks like

$$U = \prod_{v \in S'} W_v \times \prod_{v \notin S'} \mathcal{X}(\mathcal{O}_v)$$

where $S' \supseteq S$ is a finite set of places containing all the places where x_v is not integral and $x_v \in W_v$ is an open neighbourhood in $X(k_v)$.

Remark 5. Given a variety X, a pair $(S, \mathcal{X})$ as above always exists. Furthermore, neither the set $X(\mathbf{A}_k)$ nor its topology as defined above depend on $(S, \mathcal{X})$. In particular, given an adelic point (x_v), when we talk about approximating it by an adelic point (x'_v), we mean that for some S and some S-integral model $\mathcal{X}$, the point (x'_v) is arbitrarily close to x_v for every place in S and x'_v is integral for every $v \notin S$.

At this point it will be useful to describe a particular type of fibration, whose behaviour is in some sense universal with respect to "fibration problems". Let $m \geq 2, n \geq 1$ be integers. For each $i = 1, ..., n$, let k_i/k and L_i/k_i be finite field extensions. For each $i = 1, ..., n, j = 1, ..., m$ let $a_{i,j} \in k_i$ be an element. We assume the following:

Assumption 6. *The k-linear map $k^m \longrightarrow \oplus_{i=1}^n k_i$ given by $(x_1, ..., x_m) \mapsto (\sum_j a_{1,j} x_j, ..., \sum_j a_{n,j} x_j)$ is of full rank.*

To this data we associate an irreducible quasi-affine variety W over k endowed with a smooth morphism $p : W \longrightarrow \mathbb{P}^{m-1}_k$ with a geometrically irreducible generic fiber as follows.

For $i = 1, ..., n$, us denote by $T_i = \mathrm{R}_{L_i/k}(\mathbb{G}_{\mathrm{m},L_i})$ the extension of scalars torus and by $D_i = \mathrm{R}_{L_i/k}(\mathbb{A}^1_{L_i}) \setminus T$ its complement in the corresponding affine space. We then observe that D_i is a divisor with normal crossings (geometrically isomorphic to the union of all coordinate hyperplanes). Let $F_i \subseteq D_i$ denote the singular locus of D_i (corresponding to the locus where two different hyperplanes meet). Then F_i has codimension 2 insude $\mathrm{R}_{L_i/k}(\mathbb{A}^1_{L_i})$. Let

$$W \subseteq (\mathbb{A}^m_k \setminus \{(0,0)\}) \times \prod_{i=1}^{n} \left(\mathrm{R}_{L_i/k}(\mathbb{A}^1_{L_i}) \setminus F_i\right) \tag{0.1}$$

be the subvariety given by the equations

$$\sum_{j=1}^{m} a_{i,j} x_j = \mathrm{N}_{L_i/k_i}(y_i)$$

for $i = 1, ..., n$, where $x_1, ..., x_m$ are the coordinates of $\mathbb{A}^m$ and y_i is a coordinate on $\mathrm{R}_{L_i/k}(\mathbb{A}^1_{L_i})$. Finally, denote by $p : W \longrightarrow \mathbb{P}^{m-1}_k$ the composition of the projection to the first factor with the natural map $\mathbb{A}^m_k \setminus \{(0,0)\} \to \mathbb{P}^{m-1}_k$. We note that since the F_i's were removed the map p is always **smooth**.

Let $S \subseteq \Omega_k$ be a finite set of places such that each k_i and each L_i are unramified outside S, and such that for each $v \notin S$ and each place $w \in \Omega_{k_i}$ lying above v we have that all the $a_{i,j}$ are w-integral and at least one of the $a_{i,j}$ is a w-unit. Let S_i denote the set of places of k_i which lie above S and let T_i denote the set of places of L_i which lie above S. Then there is a natural **S-integral model** $\mathcal{W}$ for W. Define $\mathcal{T}_i = \mathrm{R}_{\mathcal{O}_{T_i}/\mathcal{O}_S}(\mathbb{G}_{\mathrm{m},\mathcal{O}_{T_i}})$ and let $\mathcal{F}_i$ denote the singular locus of $\mathcal{D}_i$. Since the norm of a T_i-integral element is S_i-integral and since L_i/k_i is unramified over S_i we get that the norm map N_{L_i/k_i} can be refined to a map of $\mathcal{O}_S$-schemes

$$\mathrm{N}_{\mathcal{O}_{T_i}/\mathcal{O}_{S_i}} : \mathrm{R}_{\mathcal{O}_{T_i}/\mathcal{O}_S} \mathbb{A}^1_{\mathcal{O}_{T_i}} \longrightarrow \mathrm{R}_{\mathcal{O}_{S_i}/\mathcal{O}_S} \mathbb{A}^1_{\mathcal{O}_{S_i}}.$$

We then define the $\mathcal{O}_S$-subscheme

$$\mathcal{W} \subseteq \left(\mathbb{A}^m_{\mathcal{O}_S} \setminus \{(0,0)\}\right) \times \prod_{i=1}^{n} \left(\mathrm{R}_{\mathcal{O}_{T_i}/\mathcal{O}_S}(\mathbb{A}^1_{\mathcal{O}_{T_i}}) \setminus \mathcal{F}_i\right) \tag{0.2}$$

by the equations

$$\sum_{j=1}^{m} a_{i,j} x_j = \mathrm{N}_{\mathcal{O}_{T_i}/\mathcal{O}_{S_i}}(y_i)$$

where we interpret $x_1, .., x_j$ as coordinates of $\mathbb{A}^m_{\mathcal{O}_S}$ and $y_1, ..., y_n$ as coordinates on $\mathrm{R}_{\mathcal{O}_{T_i}/\mathcal{O}_S} \mathbb{A}^1_{\mathcal{O}_{T_i}}$. The corresponding map of $\mathcal{O}_S$-schemes $p : \mathcal{W} \longrightarrow \mathbb{P}^{m-1}_{\mathcal{O}_S}$ is smooth.

It is worthwhile to have an explicit description of local integral points of $\mathcal{W}$ for $v \notin S$.

Lemma 7. *Let $S \subseteq \Omega_k$ and $\mathcal{W}$ be as above. If $v \notin S$ is a place and $x_j \in k_v, y_i \in L_i$ are elements considered as coordinates for a local point $P = (x_1, ..., x_m, y_1, ..., y_n) \in W(k_v)$, then P comes from $\mathcal{W}(\mathcal{O}_v)$ if and only the following holds:*

1. *Each x_i belongs to $\mathcal{O}_v$ and at least one of the x_i's belongs to $\mathcal{O}_v^*$.*
2. *Each y_i belongs to $\mathcal{O}_{L_i,v} = \mathcal{O}_{T_i} \otimes_{\mathcal{O}_S} \mathcal{O}_v$.*
3. *For each $i = 1, ..., n$ there is at most one place $u \in \Omega_{L_i}$ lying over v such that $\mathrm{val}_u(y_i) > 0$. Furthermore, if such a place u exists then it has degree 1 over v.*

Proof. We first observe that each x_i must be v-integral in order for $(x_1, ..., x_m)$ to come from a v-integral point of $\mathbb{A}^m_{\mathcal{O}_v}$. If we further want it to come from $\mathbb{A}^m_{\mathcal{O}_v} \setminus \{(0, ..., 0)\}$ one needs to assume that the reduction of the vector $(x_1, ..., x_m)$ mod v is not the 0-vector. This is equivalent to saying that at least one of the x_i's is in $\mathcal{O}_v^*$. We hence see that condition (1) is necessary. Similarly, we see that condition 2 is necessary. Now let $y_i \in \mathcal{O}_{L_i,v}$ be an element. By definition we have that y_i comes from a v-integral point of

$$\left[\mathrm{R}_{\mathcal{O}_{L_i,v}/\mathcal{O}_v} \mathbb{A}^1_{\mathcal{O}_{L_i,v}}\right] \setminus \mathcal{F}_i$$

if and only if the reduction $\overline{y_i}$ of y_i mod v does not belong to $\mathcal{F}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$. In particular, $\overline{y_i}$ either belongs to $\mathcal{T}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$ or is a smooth point of $\mathcal{D}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$. The first case is equivalent to $y_i \in \mathcal{O}^*_{L_i,c}$, i.e., no place of L above v divides y_i. Since the geometric irreducible components of $\mathcal{D}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$ are smooth, the second case is equivalent to $\overline{y_i}$ lying on exactly one irreducible component of

$\mathcal{D}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$, and that this component is geometrically irreducible. Let $\mathbb{E}_{i,v} = \mathcal{O}_{L_i,v} \otimes_{\mathcal{O}_v} \mathbb{F}_v$. Then the irreducible components of $\mathcal{D}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$ correspond to the coordinate hyperplanes of $\mathrm{R}_{\mathbb{E}_{i,v}/\mathbb{F}_v} \mathbb{A}^1_{\mathbb{E}_{i,v}}$ and are hence classified by $\mathrm{spec}(\mathbb{E}_{i,v})$. In particular, geometrically irreducible components of $\mathcal{D}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$ (defined over $\mathbb{F}_v$) correspond to $\mathbb{F}_v$-points of $\mathrm{spec}(\mathbb{E}_{i,v})$, i.e., to places $u \in \Omega_{L_i}$ of degree 1 over v. Now $\overline{y_i}$ sits on the component corresponding to u if and only if the u-coordinate of $\overline{y_i}$ vanishes, i.e, if and only if $\mathrm{val}_u(y_i) > 0$. We hence see that $\overline{y_i}$ sits on exactly one irreducible component $\mathcal{D}_i \otimes_{\mathcal{O}_v} \mathbb{F}_v$ if and only if there is exactly one place $u \in \Omega_{L_i}$ over v such that $\mathrm{val}_u(y_i) > 0$, and that the component corresponding to u is geometrically irreducible if and only if u is of degree 1. The desired result now follows.

□

Conjecture 8 (Conjecture (W))**.** *Assume the data defining the variety W satisfies assumption 6. Then the inclusion*

$$\bigcup_{t_0 \in \mathbb{P}^{m-1}(k)} W_{t_0}(\mathbf{A}_k) \subseteq W(\mathbf{A}_k)$$

is dense in the adelic topology.

Theorem 9 ([Mat14], building on earlier work with Tim Browning, building on a deep results of Green, Tao and Ziegler)**.** *Conjecture (W) holds whenever $k_1 = k_2, ..., k_n = \mathbb{Q}$.*

Remark 10. The result of [Mat14] is in fact stronger. Given an adelic point $(x_v) \in W(\mathbf{A}_{\mathbb{Q}})$ and a large enough finite set of places S, one may find an S-integral point $x_0 \in \mathcal{W}(\mathbb{Z}_S)$ such that x_0 is arbitrarily close to (x_v) for every non-archimedean place $v \in S$ and such that the $\mathbb{A}^m$ coordinates of x_0 belongs to any convex cone containing the $\mathbb{A}^m$-coordinates of x_∞.

Theorem 11 ([HW14])**.** *If $m \geq \sum_{i=1}^{n} [k_i : k]$ then conjecture (W) holds.*

Proof. Will be given in Olivier's talk. □

The following can be proven using the standard fibration method techniques.

Theorem 12. *Assume Schinzel's hypothesis holds. Then Conjecture (W) holds whenever the extensions L_i/k_i are abelian (or semi-abeian, see [HW14]).*

In order to state our main result we will need to recall the following standard notion.

Definition 13. Let $\pi : X \longrightarrow Y$ be a map. We will denote by

$$\mathrm{Br}_{\mathrm{vert}}(X) = \mathrm{Br}(X) \cap \mathrm{Br}(k(Y)) \subseteq \mathrm{Br}(X)$$

the subgroup of $\mathrm{Br}(X)$ consisting of classes which come from the Brauer group of the generic point of Y. The subgroup $\mathrm{Br}_{\mathrm{vert}}(X)$ is known as the **vertical Brauer group** of X with respect to π.

Remark 14. If X, Y are smooth and geometrically irreducible and π is smooth and surjective with a geometrically irreducible fiber then the quotient $\mathrm{Br}(X)_{\mathrm{vert}}/\mathrm{Br}(k)$ is finite (see [CTS00, Lemma 3.1]).

Remark 15. It is not hard to show that $\mathrm{Br}_{\mathrm{vert}}(W) = \mathrm{Br}(k)$. See [HSW] for a proof in a particular case.

Theorem 16 ([HW14])**.** *Assume conjecture (W) holds for $m = 2$. Let X be a smooth geometrically integral variety over k endowed with a smooth morphism $\pi : X \longrightarrow \mathbb{P}^1$ with a geometrically integral generic fiber. Let $V \subseteq \mathbb{P}^1_k$ be an open set such that the fiber of π above every point of U is geometrically integral. Let $U = \pi^{-1}(V) \subseteq X$ and let $B \subseteq \mathrm{Br}(U)/\mathrm{Br}(k)$ a finite subgroup containing $\mathrm{Br}(U)_{\mathrm{vert}}/\mathrm{Br}(k)$. Let $B' = B \cap [\mathrm{Br}(X)/\mathrm{Br}(k)]$. Then the inclusion*

$$\bigcup_{c \in V(k)} U_c(\mathbf{A}_k)^B \subseteq X(\mathbf{A}_k)^{B'}$$

is dense in the adelic topology.

Remark 17. In light of Remark 15 we see that the conclusion of Theorem 16 includes Conjecture (W) is a particular case. This can be interpreted as saying that the fibrations $\pi : W \longrightarrow \mathbb{P}^1$ are, in some sense, "complete fibration problems". According to Theorem 16, if one can solve all the fibration problems of type (W) (with $m = 2$) then one can solve any fibration problem.

Before we give the proof let us recall Harari's "formal lemma":

Lemma 18 ([Har94])**.** *Let X be a smooth geometrically integral variety and $U \subseteq X$ an open subset. Let $B \subseteq \mathrm{Br}(U)$ be a finite subgroup. Let $(x_v) \in X(\mathbf{A}_k)$ be an adelic point which is orthogonal to $B \cap \mathrm{Br}(X)$. Then there exists an adelic point $(x'_v) \in U(\mathbf{A}_k)$, arbitrarily close to x_v in the adelic topology on $X(\mathbf{A}_k)$, and such that (x'_v) is orthogonal to B.*

We are now ready to prove Theorem 16.

Proof of Theorem 16. For simplicity we will prove the claim for $B = \mathrm{Br}_{\mathrm{vert}}(U)/\mathrm{Br}(k)$. We note that in this case $B \cap X = \mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k)$. The proof in the general case is similar.

Let $m_1, ..., m_n \in \mathbb{P}^1$ denote the closed points in the complement of V. For each $i = 1, \ldots, n$ we will denote by X_i the fiber of π over m_i and fix an irreducible component $Y_i \subseteq X_i$. Let $k_i = k[m_i]$ be the function field of m_i and L_i/k_i the algebraic closure of k in the field of functions of Y_i. Without loss of generality we can assume that each m_i lies in $\mathbb{A}^1$. Let t be a coordinate on $\mathbb{A}^1$ and let $e_i \in k_i$ denote the value t takes on m_i.

For each i, let $\chi_{i,1}, ..., \chi_{i,n_i} : \Gamma_{k_i} \to \mathbb{Q}/\mathbb{Z}$ be a finite set of generators for the group

$$\mathrm{Ker}[\mathrm{Hom}(\Gamma_{k_i}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(\Gamma_{L_i}, \mathbb{Q}/\mathbb{Z})].$$

We will denote by $K_{i,j}$ the fixed field of $\chi_{i,j}$. Consider the Brauer element

$$A_{i,j} = \mathrm{cores}_{k_i(t)/k(t)}(t - e_i, K_{ij}/k_i) \in \mathrm{Br}(k(t))$$

Then $A_{i,j}$ is ramified on $\mathbb{P}^1$ only at m_i and ∞ with residues $\chi_{i,j}$ and $-\mathrm{cores}_{k_i/k}(\chi_{i,j})$, respectively. In particular, $A_{i,j}$ is unramified on V and $\pi^* A_{i,j}$ is unramified on U. Furthermore, $\pi^* A_{i,j} \in \mathrm{Br}(U)_{\mathrm{vert}}$.

Now let $(x_v) \in X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$ be a point that we wish to approximate. Let S be a finite set of places which is big enough so that the following holds:

1. S contains all the archimedean places.
2. All the the k_i's and all the L_i's are unramified outside S.
3. Each e_i is S-integral and the norm of each $e_i - e_j \in k_i k_j$ is an S-unit.
4. X admits a smooth S-integral model $\mathcal{X}$ such that the induced map $\pi : \mathcal{X} \longrightarrow \mathbb{P}^1_{\mathcal{O}_S}$ is smooth.

Our goal is to construct to show that for every large enough $S' \supset S$ there exists an adelic point $(x_v)' \in U(\mathbf{A}_k)$ such that (x'_v) is arbitrarily close to (x_v) for every $v \in S'$ and x_v is v-integral with respect to $\mathcal{X}$ for every $v \notin S'$.

In light of our conditions on S one may consider the natural S-integral model $\mathcal{V}$ for the open subset V. If $v \notin S$ and $t_v \in k_v$ is a coordinate of a point in $p_v \in V(k_v)$ then p_v belongs to $\mathcal{V}(\mathcal{O}_v)$ if and only if $\mathrm{val}_v(t_v) \geq 0$ and $\mathrm{val}_v(t_v - e_i) = 0$ for every i. Furthermore, by condition (3) above, if we denote by $\mathcal{m}_i \subseteq \mathbb{P}^1_{\mathcal{O}_S}$ the Zariski closure of m_i then $\mathcal{m}_i$ and $\mathcal{m}_j$ do not intersect in $\mathcal{V}$. We will denote by $\mathcal{X}_i \subseteq \mathcal{X}$ the fiber of π over $\mathcal{m}_i$ and by $\mathcal{Y}_i \subseteq \mathcal{X}_i$ the Zariski closure of the component Y_i. We then observe that the smooth $\mathcal{O}_S$-scheme $\mathrm{spec}(\mathcal{O}_{L_i,S})$ **classifies** the irreducible components of $\mathcal{Y}_i$. In particular, if $w \in \Omega_k$ is a place of k_i corresponding to a closed point $x \in \mathcal{m}_i$, then the irreducible components of $\mathcal{Y}_i \times_{\mathcal{M}} \{x\}$ are in bijection with places $u \in \Omega_L$ lying above w. Finally, let us add three more assumptions which can be met by enlarging S if necessary:

5 Each $A_{i,j}$ extends to an element $\mathcal{A}_{i,j} \in \mathrm{Br}(V)$.

6 For every $v \notin S$ and every $x \in \mathcal{V}(\mathbb{F}_v)$ there exists a $y \in \mathcal{X}(\mathbb{F}_v)$ lying above t.

7 For every $v \notin S$, every place $w \in \Omega_{k_i}$ lying above v (corresponding to a closed point $x \in \mathcal{m}_i$) and every place $u \in \Omega_{L_i}$ of degree 1 over w, the component of $\mathcal{Y}_i \times_{\mathcal{M}} \{x\}$ classified by u has an $\mathbb{F}_w$-point.

By Harari's formal lemma [Har94] there exists an adelic point $(x'_v) \in U(\mathbf{A}_k)$ such that

1. (x'_v) is arbitrarily close to (x_v) for every $v \in S$.
2. For every $A_{i,j}$ as above we have

$$\sum_{v \in \Omega_k} A_{i,j}(\pi(x'_v)) = 0. \tag{0.3}$$

Let $\mathcal{U} = \mathcal{V} \times_{\mathbb{P}^1} \mathcal{V}$ and let S' be a finite set of places containing S, such that (x_v) belongs to $\mathcal{U}(\mathcal{O}_v)$ for every $v \notin S'$. Let t'_v be the coordinate of $\pi(x'_v)$ on $\mathbb{A}^1$, and let S'_i be the set of places of k_i lying above S'. Given a place $w \in \Omega_{k_i}$ we will denote by $\overline{w}$ the place of k lying below it. We then have for each i, j as above

$$\sum_{w \in \Omega_{k_i}} \mathrm{inv}_w(t'_{\overline{w}} - e_i, \chi_{i,j}) = 0. \tag{0.4}$$

Let us now fix an $i = 1, ..., n$ and let $T_i = \mathrm{R}_{L_i/k_i} \mathbb{G}_m$ be the restriction-of-scalars torus of L_i/k_i and $T_i^1 \subseteq T_i$ the associated norm one torus. Since (x_v) lies in $\mathcal{U}(\mathcal{O}_v)$ for all but finitely many v's we see that $t_{\overline{w}} - e_i$ lies in $\mathbb{G}_m(\mathcal{O}_w) = \mathcal{O}_w^*$ for all but finitely many places $w \in \Omega_{k_i}$. Let us denote by

$$(\sigma_w^i) \stackrel{\mathrm{def}}{=} (t'_{\overline{w}} - e_i) \in \prod_{w \in \Omega_{k_i}} \mathbb{G}_m(\mathbf{A}_{k_i})$$

the resulting adelic point. Consider the following short exact sequence k_i-groups schemes

$$1 \longrightarrow T_i^1 \longrightarrow T_i \xrightarrow{\mathrm{N}} \mathbb{G}_m \longrightarrow 1$$

and the associated boundary map

$$\partial : \mathbb{G}_m(\mathbf{A}_{k_i}) \longrightarrow H^1\left(\mathbf{A}_k, T_i^1\right).$$

We claim that this element is in fact k_i-rational, i.e., comes from an element $\rho_i \in H^1\left(k_i, T_i^1\right)$. In light of Tate-Poitou's sequence for tori what we need to prove is that $(\partial \sigma_w^i)$ is orthogonal to all the elements in $H^1\left(k_i, \widehat{T}_i^1\right)$. To compute the latter consider the short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widehat{T}_i \longrightarrow \widehat{T}_i^1 \longrightarrow 1$$

The boundary map

$$\partial : H^1\left(k_i, \widehat{T}_i^1\right) \longrightarrow H^2(k_i, \mathbb{Z}) \cong H^1(k_i, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\Gamma_{k_i}, \mathbb{Q}/\mathbb{Z})$$

maps $H^1\left(k_i, \widehat{T}_i^1\right)$ into the kernel of the map

$$\mathrm{Hom}(\Gamma_{k_i}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}(\Gamma_{L_i}, \mathbb{Q}/\mathbb{Z}).$$

In particular, for each $\alpha \in H^1\left(k_i, \widehat{T}_i^1\right)$ we have that $\partial\alpha$ is some linear combination of the $\chi_{i,j}$'s. From the naturality of the pairing and equation 0.4 we then get

$$\left\langle \partial\sigma^i, \alpha \right\rangle = \left\langle \sigma^i, \partial\alpha \right\rangle = \sum_{w \in \Omega_{k_i}} \mathrm{inv}_w(t'_{\overline{w}} - e_i, \partial\alpha) = 0.$$

This proves that $\partial\sigma^i$ comes from some element $\rho_i \in H^1\left(k_i, T_i^1\right)$. Now since $H^1(k_i, T) = 0$ it follows that $\rho_i = \partial b_i$ for some $b_i \in \mathbb{G}_m(k_i)$. We may hence conclude that for every $w \in \Omega_{k_i}$ there exists a $y_{w,i} \in (L_i)_w$ such that

$$b_i(t'_{\overline{w}} - e_i) = \mathrm{N}_{(L_i)_w/(k_i)_w}(y_{w,i})$$

Let S'' be a finite set of places containing S' such that b_i (and hence $b_i(t'_{\overline{w}} - e_i)$) is in $\mathcal{O}_w^*$ whenever $w \in \Omega_{k_i}$ is such that $\overline{w} \notin S''$. We may then assume that if $w \in \Omega_{k_i}$ is such that $\overline{w} \notin S''$ then $y_{w,i} \in \mathcal{O}_{L_i,w}^*$.

Let W be the quasi-affine variety associated as above to the fields $k_1, ..., k_n$, $L_1, ..., L_n$ and to the values $a_{i,0} = b_i, a_{i,1} = b_i e_i$ (note that here $m = 2$). Let $\mathcal{W}$ be the smooth S''-integral model established in Lemma 7. According to Lemma 7, the values $(t'_{\overline{w}}, 1, y_{w,1}, ..., y_{w,n})$ determine an adelic point (p_v) on W such that p_v belongs to $\mathcal{W}(\mathcal{O}_v)$ for every $v \notin S''$. By Conjecture (W) there exists an adelic point $(q_v) = (\lambda_v, \mu_v, y'_{w,1}, ..., y'_{w,n})$ such that

1. q_v is arbitrarily close to p_v for every $v \in S''$.
2. q_v belongs to $\mathcal{W}(\mathcal{O}_v)$ for every $v \notin S''$.
3. q_v lies above a rational point $t_0 \in \mathbb{P}^1(k)$.

We shall now construct an adelic point (x''_v) on the fiber X_t which is arbitrarily close to (x'_v) for $v \in S''$ and is $\mathcal{X}$-integral outside S''. For $v \in S''$, we have that t_0 is arbitrarily close to t_v and so we may construct x''_v by means of the inverse function theorem. Let us hence consider the case $v \notin S''$. Then at least one of λ_v, μ_v is in $\mathcal{O}_v^*$ and $(\lambda_v : \mu_v) = (t_0 : 1)$. Furthermore, if there exists a place $w \in \Omega_{k_i}$ lying over v such that $\mathrm{val}_w(b_i(\lambda_v - e_i\mu_v)) > 0$ then by 0.2 there exists a place $u \in \Omega_{L_i}$ lying over w such that $\mathrm{val}_u(y_{v,i}) > 0$. By Lemma **??** u would have to be of degree 1 over v. In particular, in this case w has of degree 1 over v and u has of degree 1 over w.

Let us now construct an $\mathcal{O}_v$-point on $\mathcal{X}$. First suppose that $\mathrm{val}_w(t_0 - e_i) \leq 0$ for every $i = 1, ...n$ and every $w \in \Omega_{k_i}$ lying above v. In this case the reduction of t_0 modulu v lies in $\mathcal{V}$ and hence by our choice of S there is a smooth $\mathbb{F}_v$-point on $X_{t_0}(\mathbb{F}_v)$ and so an $\mathcal{O}_v$-point on X_{t_0} by Hensel's lemma.

Now suppose that $\mathrm{val}_w(t_0 - e_i) > 0$ for some e_i and some $w \in \Omega_{k_i}$ lying above v (in which case the pair (i, w) is unique by our choice of S). By the above considerations we know that w must have degree 1 over v and that there must exist a place $u \in \Omega_{L_i}$ of degree 1 over w. Let $x \in m_i$ be the closed point corresponding to w. By our choice of S the irreducible component of $\mathcal{Y}_i \times_{\mathcal{M}} \{x\}$ classified by u has a smooth $\mathbb{F}_w$-point and hence a smooth $\mathbb{F}_v$-point. This point can then be lifted to an $\mathcal{O}_v$-point of $\mathcal{X}_{t_0}$ by Hensel's lemma.

□

Corollary 19 (Main theorem)**.** *Assume that conjecture (W) holds for $m = 2$. Let X be a smooth, proper, geometrically integral variety X over k and let $\pi : X \longrightarrow \mathbb{P}^n$ be a dominant morphism whose generic fiber is RC. If there exists a Hilbert subset $H \subseteq \mathbb{P}^n$ such that Conjecture 1 holds for X_c whenever $c \in H$ then conjecture 1 holds for X.*

Proof. Let $X' \subseteq X$ be the smooth locus of X. Since the generic fiber of π is RC it follows that every fiber of π has an irreducible component of multiplicity one. This implies that the restricted map $\pi : X' \longrightarrow \mathbb{P}^1_k$ is surjective and that the complement of X' in X has codimension 2, and so $\mathrm{Br}(X') = \mathrm{Br}(X)$.

Let $\eta \in \mathbb{P}^1_k$ be the generic point. Since the generic fiber X_η is RC, it follows from [CTS13, Lemma 1.3] that $\mathrm{Br}(X_\eta)/\mathrm{Br}(\eta)$ is finite. Hence there exists an open subset $V \subseteq \mathbb{P}^1$ and a finite subgroup $B \subseteq \mathrm{Br}(\pi^{-1}(V))/\mathrm{Br}(k)$, containing the vertical Brauer group of $U = \pi^{-1}(V)$, and such that the map $B \longrightarrow \mathrm{Br}(X_\eta)/\mathrm{Br}(\eta)$ is surjective. By [Har97, Théorème 2.3.1] there exists a Hilbert subset $H' \subseteq H$ such that the restriction $B \longrightarrow \mathrm{Br}(X_c)/\mathrm{Br}(k)$ is surjective for every $c \in \mathbb{P}^1$. Let $h \in H'(k)$ be a point. Let $(x_v) \in X(\mathbf{A}_k)$ be an adelic point which is orthogonal to $\mathrm{Br}(X)$ and S a finite set of places. By a small deformation we may assume that (x_v) lies on X' and since $\mathrm{Br}(X') = \mathrm{Br}(X)$ we know that (x_v) is orthogonal to $\mathrm{Br}(X')$.

Let $h \in H'(k)$ be a point. By enlarging S we may assume that $\mathcal{X}'_h(\mathcal{O}_v) \neq \varnothing$ for every $v \notin S$ and that all the elements of $B \cap [\mathrm{Br}(X)/\mathrm{Br}(k)]$ extend to S-integral elements on $\mathcal{X}'$. We may hence replace x_v with some $x'_v \in \mathcal{X}_h(\mathcal{O}_v)$ for every $v \notin S$, and set $x'_v = x_v$ for $v \in S$. Then (x'_v) is orthogonal to $B \cap [\mathrm{Br}(X')/\mathrm{Br}(k)]$, and we may then apply Theorem 16 to it. This results in an adelic point (x''_v) approximating (x_v) arbitrarily well (and hence approximating (x_v) over S and integral outside S), orthogonal to B, and such that $\pi(x''_v)$ is rational. By [Sme13, Proposition 6.1] it follows that the $t_0 = \pi(x''_v)$ is in H'. By the above we get that (x''_v) is orthogonal to $\mathrm{Br}(X_{t_0})/\mathrm{Br}(k)$ and since $X'_{t_0} = X_{t_0}$ satisfies Conjecture 1 it follows that (x''_v) can be approximated by a rational point of X_{t_0}. □

Theorem 20 (Main theorem - optimized version)**.** *Let X be a smooth, proper, geometrically integral variety X over k and let $\pi : X \longrightarrow \mathbb{P}^n$ be a dominant morphism whose generic fiber is RC. Assume that there exists a Hilbert subset $H \subseteq \mathbb{P}^n$ such that Conjecture 1 holds for X_c whenever $c \in H$. Let $M \subseteq \mathbb{P}^1$ be the scheme classifying the non-split fibers of π. Assume that the following two conditions hold:*

1. *Either k is totally imaginary or M contains a rational point.*

2. *Conjecture (W) holds for $m = 2$ in those cases where the singular locus of $W \longrightarrow \mathbb{P}^1$ coincides with M.*

Then conjecture 1 holds for X.

Corollary 21. *Let X be a smooth, proper, geometrically integral variety X over $\mathbb{Q}$ and let $\pi : X \longrightarrow \mathbb{P}^n$ be a dominant morphism whose generic fiber is RC and such that all non-split fibers are all defined over $\mathbb{Q}$. If there exists a Hilbert subset $H \subseteq \mathbb{P}^n$ such that Conjecture 1 holds for X_c whenever $c \in H$ then conjecture 1 holds for X.*

Corollary 22. *Let X be a smooth, proper, geometrically integral variety X over a totally imaginary number field k and let $\pi : X \longrightarrow \mathbb{P}^n$ be a dominant morphism whose generic fiber is RC. Assume that the scheme M classifying the non-split fibers of π is of degree ≤ 2. If there exists a Hilbert subset $H \subseteq \mathbb{P}^n$ such that Conjecture 1 holds for X_c wheneverc $\in H$ then conjecture 1 holds for X.*

References

[CT94] Colliot-Thélène, J.-L., Hasse principle and weak approximation for pencils of Severi-Brauer and similar varieties, *Journal fr die reine und angewandte Mathematik*, 453, 1994, p. 49–112.

[CT98] Colliot-Thélène, J.-L., The Hasse principle in a pencil of algebraic varieties, *Number Theory: Ramanujan Mathematical Society*, Jan. 3-6, 1996, Tiruchirapalli, India 210, 1998, 19.

[CT03] Colliot-Thélène, J.-L., Points rationnels sur les fibrations, *Higher dimensional varieties and rational points (Budapest 2001)*, 12, Springer, Berlin 2003, p. 181–194.

[CTS82] Colliot-Thélène, J.-L., Sansuc, J.-J., Sur le principe de Hasse et l'approximation faible, et sur une hypothèse de Schinzel, *Acta Arithmetica*, 41, 1982, p. 33–53.

[CTS00] Colliot-Thélène, J-L., Skorobogatov, A. N, Descent on fibrations over $\mathbb{P}^1_k$ revisited, *Mathematical Proceedings of the Cambridge Philosophical Society*, 128.3, Cambridge University Press, 2000.

[CTS13] Colliot-Thélène, J-L., Skorobogatov, A. N, Good reduction of the BrauerManin obstruction, *Transactions of the American Mathematical Society*, 365.2, 2013, p. 579–590.

[CTSSD98a] Colliot-Thélène, J.-L., Skorobogatov, A. N., and Swinnerton-Dyer, P., Rational points and zero-cycles on fibred varieties: Schinzel's hypothesis and Salberger's device, *Journal fr die reine und angewandte Mathematik*, 495, 1998, p. 1–28.

[Har94] Harari. D., Méthode des fibrations et obstruction de Manin, *Duke Mathematical Journal*, 75, 1994, p.221–260.

[Har97] Harari, D. Flches de spcialisations en cohomologie tale et applications arithmtiques, *Bulletin de la Société Mathématique de France*, 125.2, 1997, p. 143–166.

[HSW] Harpaz, Y., Skorobogatov A. N. and Wittenberg, O., The Hardy-Littlewood Conjecture and Rational Points, *Compositio Mathematica*, 150, 2014, p. 2095-2111.

[HW14] Harpaz, Y., Wittenberg, O., On the fibration method for zero-cycles and rational points, arXiv preprint `http://arxiv.org/abs/1409.0993`.

[Mat14] Matthiesen, L., On the square-free representation function of a norm form and nilsequences, arXiv preprint `http://arxiv.org/abs/1409.5028`.

[Ser92] Serre, J-.P., Résumé des cours au Collège de France, 1991–1992

[Sme13] Smeets, A., Principes locaux-globaux pour certaines fibrations en torseurs sous un tore, to appear, Math. Proc. Cambridge Philos. Soc., 2013.

[SD94] Swinnerton-Dyer, P., Rational points on pencils of conics and on pencils of quadrics, *Journal of the London Mathematical Society*, 50.2, 1994, p. 231–242.

[Wit12] Wittenberg, O., Zro-cycles sur les fibrations au-dessus dune courbe de genre quelconque, *Duke Mathematical Journal*, 161.11, 2012, p. 2113–2166.