Simplicial Methods

Yonatan Harpaz   Matan Prasma

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1 Lecture 1

Algebraic topology is the study of **nice** topological spaces and continuous maps between them. What are nice topological spaces? Well we all agree that the one-pointed space, let us denote it by $*$, is so basic that it has to be considered nice. Furthermore, given any space X, we are often considering various **points** of X. A point on a topological space X can be thought of as a map $* \longrightarrow X$ from the one-pointed space. Hence in order to talk about points we have to include $*$.

Now given a space X we want to know more than just the set of points of X as a discrete set. We want to do **topology**, so we want to know in what ways we can move continuously from one point to another. For example, we want to know when two points $x, y \in X$ can be connected by a **continuous path**. Such a path corresponds to a continuous map $\gamma : I \longrightarrow X$ (where $I = [0,1]$ is the unit segment such that $\gamma(0) = x$ and $\gamma(1) = y$. Hence we see that we should definitely include I in our collection of nice spaces.

What we said so far basically gives us the set of (path)-connected components of X: we know the points and we know when two points are in the same component. But a topological space contains more information. For example, some times two points can be connected by a path in **two different ways**. Given two points $x, y \in X$ and two paths $\alpha, \beta : I \longrightarrow X$ such that $\alpha(0) = \beta(0) = x$ and $\beta(0) = \beta(1) = y$ we can ask whether there exists a **homotopy** from α to β which fixes the end points x and y, i.e., when can we deform α continuously until it becomes β (which staying all the time in the space of paths from x to y). To describe such a homotopy formally one can consider the 2-disc D^2 and a partition of its boundary $\partial D^2 = S^1$ into a union of two hemispheres $S^1 = I_+ \cup I_-$ (where the intersection $I_+ \cap I_-$ is the north and south poles of S^1). A homotopy as above then corresponds to a map $h : D^2 \longrightarrow X$ such that $h|_{I_+} = \alpha$ and $h|_{I_-} = \beta$. Hence we see that if want to really understand the various ways in which two points can be connected we must include the spaces D^2 and S^1 in our collection of nice spaces.

Remark 1.1. When $x = y$ the questions becomes "in what ways I can connect x to itself inside X". Put formally, we obtain the set of homotopy classes of closed paths in X which start and end in x. This set of homotopy classes has an additional structure given by concatenation of loops, leading to the notion of the **fundamental group** of X based at x. In one wants to consider all paths (as opposed to just closed paths) then this can be done by considering the **fundamental groupoid** of X. This is the groupoid whose objects are the points of x and whose morphisms are homotopy classes of paths as above.

We can now take the considerations above one step further. Say we have two points $x, y \in X$ which are connected by two paths $\alpha, \beta : I \longrightarrow X$ and suppose that we know that this two paths are homotopic in the above sense. Then it is natural to ask what are the different ways in which α, β are homotopic, i.e., what are the various homotopies between them? As before, two such homotopies h, h' can them selves be homotopic to each other (in a boundary fixeing way), and such a homotopy can be encoded via a map $D^3 \longrightarrow X$ whose restriction to the two hemispheres of $\partial D^2 = S^2$ gives h and h'. We hence deduce that we should include D^3 and S^2 in our collection of nice spaces. But this can be continued on and on, and we see that all the spaces of the form D^n or S^n should be included in our collection of nice spaces.

Remark 1.2. When considering homotopies as above from the constant path at $x \in X$ to itself we obtain the notion of the **second homotopy group** $\pi_2(X, x)$ of X. Similarly, for higher homotopies we will obtain the notion of higher homotopy groups $\pi_n(X, x)$ of X.

We are now in a position to ask the big question: when is a topological space X nice? Here are are two possible heuristic answers to this question:

1. When X can be obtained by some procedure of **gluing** from balls and spheres of arbitrary dimension.
2. When all the homotopical information of X is determined by points, paths, homotopies, higher homotopies (including the concatenation structure) etc. as above. In other words, when all the homotopical information of X is seen by **maps** from spheres and balls.

A fundamental insight of classical algebraic topology, due to Whitehead, is that these two answers essentially **coincide**. The first answer leads to the notion of a **CW-complex** as a direct characterization of nice spaces. The second answer can be formulated by saying that the class of nice spaces should satisfy the property that if a map $f : X \longrightarrow Y$ induces an isomorphism on all homotopy groups (in which case all questions about paths, homotopies, higher homotopies etc. will have identical answers for X and Y) then f should be a homotopy equivalence. Such maps are called **weak homotopy equivalences**. Whitehead proved that this property is true for CW-complexes. Furthermore, the largest class of spaces for which this assertion holds is the class of spaces which are homotopy equivalent to a CW-complex.

At this point, classical algebraic topology suggests to simply study CW-complexes. This approach has some disadvantages. For example many reasonable constructions which start from CW-complexes don't stay inside the world of CW complex. For example, given two CW-complexes X, Y, the mapping space map(X, Y) with the compact open topology is not a CW-complex. In some cases (but not always) it will be homotopy equivalent to a CW-complex, but even then there is no obvious way to make it into an actual CW-complex. What one would want is some canonical way to take a general topological space, and functorially produce a CW-complex which approximates it as best as possible. In particular, this CW-complex should keep all the homotopical information of the type we described above, and this information alone.

This can technically be done in the world of CW-complexes, but is quite complicated and cumbersome to work with. One is then motivated to look for other, cleaner ways to make a general space into a CW-complex. This will lead us to our desired notion of **simplicial set**.

Let X be a topological space. We want to record all information regarding points, homotopies, higher homotopies etc. We basically need to recode the information of maps from discs and spheres into X. However, we observe that spheres can themselves be constructed from gluing two discs along a lower dimensional sphere, which can itself be decomposed further, leading to the observation that everything can be build out of discs if we have the right gluing maps. To streamline these gluings it is convenient to replace discs with **simplices**.

Remark 1.3. The passage from balls to simplices is not just a matter of convenience. As we shall see later, it enables one to record the concatenation structure of homotopies, which is not seen directly by just considering maps from balls and their restrictions along hemispheres inclusions.

Definition 1.4. The n-dimensional simplex is the subspace

$$|\Delta^n| = \left\{(x_0, ..., x_n) \in \mathbb{R}^{n+1} | x_n \geq 0, \sum_n x_n = 1\right\}$$

It is straightforward to verify that $|\Delta^n|$ is **homeomorphic** to the n-disc D^n. The point

$$v_i = (0, ..., 0, 1, 0, ..., 0) \in |\Delta^n|$$

is called the i'th vertex of $|\Delta^n|$. We will denote by $[n]$ the set $\{0, ..., n\}$ considered as an **ordered set** and will think of it as parametrizing the vertices of $|\Delta^n|$, where $i \in [n]$ corresponds to the vertex v_i above. In particular, we think of the vertices of $|\Delta^n|$ as **ordered**.

Definition 1.5. Let k, n be two natural numbers. We will say that a continuous map

$$f : |\Delta^k| \longrightarrow |\Delta^m|$$

is **simplicial** if it is **convex** and maps the vertices of $|\Delta^k|$ to the vertices of $|\Delta^n|$ in a (weakly) order preserving way.

We will denote by $\Delta \subseteq \texttt{Top}$ the category whose objects are the standard simplices and whose morphisms are the simplicial maps. Since each convex map is determined by its image on vertices this category admits a very combinatorial description: by associating with each simplex $|\Delta^n|$ its set of vertices $[n]$ we can identify Δ with the category whose objects are the ordered sets $[n]$ and whose morphisms are the order preserving maps. We will not distinguish between these two descriptions of Δ.

Now let X be a topological space. Then for each n we can consider the set

$$X_n = \mathrm{Hom}_{\mathtt{Top}}(|\Delta^n|, X)$$

of continuous maps from $|\Delta^n|$ to X. This collection of sets admits a bit of extra structure: the set X_n is **contraviantly functorial** in $[n]$. To see this, observe that any simplicial map $f : |\Delta^k| \longrightarrow |\Delta^n|$ induces a restriction map

$$f^* : \mathrm{Hom}_{\mathtt{Top}}(|\Delta^n|, X) \longrightarrow \mathrm{Hom}_{\mathtt{Top}}\left(|\Delta^k|, X\right)$$

In other words, the collection of sets X_n can be naturally organized into a functor

$$\Delta^{\mathrm{op}} \longrightarrow \mathtt{Set}$$

where Δ^{op} is the **opposite** category of Δ and $\mathtt{Set}$ is the category of sets. This is our formal way of dealing with contravariant functors from Δ to $\mathtt{Set}$.

We now claim that the resulting functor $X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathtt{Set}$ encodes all the homotopical information of the kind we described above. This can be seen as follows. First we started with the points of X. This, as a set, is just given by X_0. Next, we wanted to know which pairs $x, y \in X_0$ of points could be connected via a path. The set of all paths in X is just given by the set X_1. However, we have more information. Consider the morphisms

$$\sigma_{\{0\}}, \sigma_{\{1\}} : [0] \longrightarrow [1]$$

in Δ where $\sigma_{\{i\}}(0) = i$. Then the two maps $\sigma^*_{\{0\}}, \sigma^*_{\{1\}} : X_1 \longrightarrow X_0$ tell us for each path in X what are the starting point and ending point of that path. Now given two points $x, y \in X_0$ we see that there is a path from x to y in X if and only if there exists an element $\gamma \in X_1$ such that

$$\sigma^*_{\{0\}}(\gamma) = x$$

and

$$\sigma^*_{\{1\}}(\gamma) = y$$

Note that we also have a map

$$s : [1] \longrightarrow [0]$$

which is not injective - the constant map that sends both elements of $[1]$ to the single element of $[0]$. This map corresponds to the constant map from $|\Delta^1|$ to $|\Delta^0| = *$. Hence we also have a map

$$s^* : X_0 \longrightarrow X_1$$

This map gives the following information: it tells us for each point $x \in X$ who is the constant path at x.

Let us now see that we can obtain the homotopies between paths as well. Suppose we have two elements $\alpha, \beta \in X_1$ such that

$$\sigma^*_{\{0\}}(\alpha) = \sigma^*_{\{0\}}(\beta)$$

and

$$\sigma^*_{\{1\}}(\alpha) = \sigma^*_{\{1\}}(\beta)$$

Such data corresponds to a pair of paths in X which have the same starting point (call it x) and the same ending point (call it y). Now we want to know if there is an end-point preserving homotopy from α to β. Now for each $0 \le i < j \le 2$ let $\sigma_{\{i,j\}} : [1] \longrightarrow [2]$ denote the map which sends 0 to i and 1 to j. Then we see that α will be homotopic to β if and only if there exists an element $\tau \in X_2$, corresponding to a map from the triangle $|\Delta^2|$ to X, such that

$$\sigma^*_{\{0,1\}}(\tau) = \alpha$$

$$\sigma^*_{\{0,2\}}(\tau) = \beta$$

$$\sigma^*_{\{1,2\}}(\tau) = s^*(y)$$

where we recall that $s^*(y)$ is the constant path at y.

Remark 1.6. The above argument shows that one can recover from $X_\bullet$ the set of elements of the fundamental group $\pi_1(X, x)$ at any given base point. It is not hard to see that one can recover the **group structure** as well: any triangle $\tau \in X_2$ such that

$$\sigma^*_{\{0,1\}}(\tau) = \alpha$$

$$\sigma^*_{\{0,2\}}(\tau) = \beta$$

$$\sigma^*_{\{1,2\}}(\tau) = \gamma$$

is a witness to the fact that β is homotopic to the concatenation of α and γ. A similar statement holds for the higher homotopy groups as well.

We are now ready to define the main object of study in this course:

Definition 1.7. A **simplicial set** is a functor

$$\Delta^{\text{op}} : \Delta \longrightarrow \texttt{Set}$$

Maps of simplicial sets are given by natural transformations. We will denote by $\texttt{Set}_\Delta$ the resulting category and refer to it as the category of simplicial sets.

The operation $X \mapsto X_\bullet$ is then easily seen to give a functor from topological spaces to simplicial sets, which we denote by

$$\text{Sing} : \texttt{Top} \longrightarrow \texttt{Set}_\Delta$$

The simplicial set $\mathrm{Sing}(X) = X_\bullet$ is sometimes called the **singular simplicial set** of X. The purpose of this course is to explain in what way category of simplicial sets can serve as a **replacement** for the category of topological spaces, which captures exactly the kind of homotopy theoretic information we described above. Furthermore, the category of simplicial sets will have many desired formal properties, including a suitable notion of mapping space, making it into a far more convenient model then CW compelxes.

The equivalence between `Top` and $\mathtt{Set}_\Delta$ will be achieved by working in the setting of Quillen's **model categories**. In particular, we will endow the categories $\mathtt{Set}_\Delta$ and `Top` with suitable model structured, and show that the functor Sing fits into a suitable notion of **Quillen equivalence** between `Top` and $\mathtt{Set}_\Delta$.

2 lecture 2

2.1 Categorical Preliminaries

In the last lecture we introduced the notion of simplicial set and explain how you can associate with each topological space X its singular simplicial set $\mathrm{Sing}(X)$. Before we dive any deeper into the theory of simplicial sets let us recall a few categorical preliminaries.

2.2 Limits and Colimits

Definition 2.1. Let $\mathcal{C}$ be a category and $\mathcal{I}$ a small category. An **$\mathcal{I}$-shaped diagram** in $\mathcal{C}$ is simply a functor $f : \mathcal{I} \longrightarrow \mathcal{C}$. Given such an f we will define the category $\mathcal{C}_{f/}$ of **objects in $\mathcal{C}$ under** f as follows:

1. The objects of $\mathcal{C}_{f/}$ are pais $(X, \{\alpha_i\}_{i\in\mathcal{I}})$ where X is an object in $\mathcal{C}$ and $\alpha_i : f(i) \longrightarrow X$ is a compatible choice of morphisms in $\mathcal{C}$. The compatibility here means that if $\beta : i \longrightarrow j$ is any morphism then $\alpha_i = \alpha_j \circ \beta$.

2. Morphisms from $(X, \{\alpha_i\})$ to $(Y, \{\beta_i\})$ are morphisms $\varphi : X \longrightarrow Y$ such that $\beta_i = \varphi \circ \alpha_i$ for every $i \in \mathcal{I}$.

Dually, one can define the category of **objects in $\mathcal{C}$ over** f, denoted $\mathcal{C}_{/f}$, which is defined analogously only with the α_i's being morphisms from X to $f(i)$ instead of the other way around.

Now recall that an object X of a category $\mathcal{C}$ is called **initial** if there is a unique morphism from X to any other object in $\mathcal{C}$. Similarly, an object is called **terminal** if there is a unique morphism into it from any other object in $\mathcal{C}$. It is a simple exercise to show that if X, X' are both initial objects in $\mathcal{C}$ then there is a **unique isomorphism** $X \cong X'$ in $\mathcal{C}$, and similarly for final objects.

Definition 2.2. Let $\mathcal{C}$ be a category, $\mathcal{I}$ a small category and $f : \mathcal{I} \longrightarrow \mathcal{C}$ a diagram. A **colimit** of f is an initial object in the category $\mathcal{C}_{f/}$ of objects under f. Similarly, a **limit** of f is a terminal object in the category $\mathcal{C}_{/f}$ of objects over f.

Remark 2.3. As initial and final objects colimits and limits don't have to exist. However, when they exist they are unique up to a unique isomorphism.

Example 1. If $\mathcal{I}$ is the empty category then a colimit of $\mathcal{I}$ is simply an initial object in $\mathcal{C}$ and a limit of $\mathcal{I}$ is a final object in $\mathcal{C}$.

Example 2. If $\mathcal{I}$ is a set, i.e. a small category with no non-identity morphisms, then an $\mathcal{I}$-shaped diagram in $\mathcal{C}$ is simply given by a collection of objects $C_i \in \mathcal{C}$ indexed by $\mathcal{I}$. A colimit for such a diagram is then called a **coproduct** and a limit over such a diagram is called a **product**. For the category of sets a coproduct is given by taking the disjoint union $\coprod_{i\in\mathcal{I}} C_i$ and product by the Cartesian product $\prod_{i\in\mathcal{I}} C_i$. The same is true for the category of topological spaces only know we endow the disjoint union of the disjoin union topology and the Cartesian product with the Cartesian product topology.

Example 3. Let $\mathcal{I}$ be the category with three objects $0, 1, 2$ such that $\mathrm{Hom}(0, 1) = \mathrm{Hom}(0, 2) = *$ and $\mathrm{Hom}(1, 2) = \mathrm{Hom}(2, 1) = \varnothing$. Then an $\mathcal{I}$-shaped diagram in a category $\mathcal{C}$ is a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle f}\big\downarrow & & \\ Y & & \end{array}$$

A colimit for such a diagram is given by extending this diagram to a commutative square

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ Y & \longrightarrow & P \end{array}$$

in a way that is initial among all possible choices. This type of colimit is often called a **pushout**. If $\mathcal{C}$ is the category of sets then P is given by $[X \coprod Y]/\sim$ where the equivalence relation is the equivalence relation generated by $f(a) \sim g(a)$ for every $a \in A$. If $\mathcal{C}$ is the category of topological spaces then P is given by the same formula, where $[X \coprod Y]/\sim$ is now endowed with the quotient topology. Geometrically, we often think of this procedure as **gluing X to Y along A**.

Remark 2.4. Most categories you met have all limits and all colimits. Examples of such categories include the categories of sets, topological spaces, groups, rings and many others. Some categories don't admit arbitrary limits and colimits but only do so when $\mathcal{I}$ is finite, i.e. has finitely many objects and finitely many morphisms. For example the category of finitely generated groups admits finite colimits (but not finite limits) and the category of finite groups admits finite limits but not finite colimits (in general). The category of smooth manifolds admits finite products and finite coproducts but any other shape of diagram might fail to have a limit/colimit.

Definition 2.5. Let $\mathcal{I}$ be a small category and let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. We will say that $\mathcal{F}$ **preserves colimits of $\mathcal{I}$-shaped diagrams** if for every diagram $f : \mathcal{I} \longrightarrow \mathcal{C}$ the induced a functor

$$\mathcal{F}_{f/} : \mathcal{C}_{f/} \longrightarrow \mathcal{D}_{\mathcal{F}\circ f/}$$

sends initial objects to initial objects. Similarly, we will say that $\mathcal{F}$ **preserves limits of $\mathcal{I}$-shaped diagrams** if the induced functor

$$\mathcal{F}_{/f} : \mathcal{C}_{/f} \longrightarrow \mathcal{D}_{/\mathcal{F}\circ f}$$

sends terminal objects to terminal objects.

Example 4.

1. The archetypical example of functors which preserve limits are **representable functors**. Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$ an object. The representable functor $R_X : \mathcal{C} \longrightarrow \texttt{Set}$ is given by the formula

   $$R_X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

   Then it is not hard to show that R_X preserves all limits. This claim is usually written informally as

   $$\mathrm{Hom}_{\mathcal{C}}\left(X, \lim_{i\in\mathcal{I}} f(i)\right) \cong \lim_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X, f(i))$$

   This property is actually strong enough to characterize limits, i.e., we could have taken the above property as a **definition** for limits.

   Applying the above claim to $\mathcal{C}^{\mathrm{op}}$ we can produce a similar statement for the **corepresentable functors** $R^X : \mathcal{C}^{\mathrm{op}} \longrightarrow \texttt{Set}$ given by

   $$\mathrm{R}^X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

   Considering R^X as a contravariant functor from $\mathcal{C}$ to $\texttt{Set}$ one can say that R^X maps colimits to limits, i.e.,

   $$\mathrm{Hom}_{\mathcal{C}}\left(\mathrm{colim}_{i\in\mathcal{I}} f(i), X\right) \cong \lim_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(f(i), X)$$

   Again, this property completely characterizes colimits.

2. The functor $\mathcal{D} : \texttt{Set} \longrightarrow \texttt{Top}$ from sets to topological spaces which associates to each set X the discrete space with point set X preserves all colimits. However, it doesn't preserve limits. For example, an infinite product of discrete spaces with the product topology is not discrete.

3. The full inclusion $\texttt{Ab} \subseteq \mathrm{Gr}$ preserves limits but not colimit. For example, the coproduct of two abelian group in the category Gr is given by the free product which is almost never commutative.

4. The abelianization functor Gr $\longrightarrow$ Ab from groups to abelian groups preserves colimits.

5. Let $\mathcal{I}$ be a small category and $\mathcal{C}$ an arbitrary category. Given an object $i \in \mathcal{I}$ we can construct the evaluation functor $\mathrm{ev}_i : \mathcal{C}^{\mathcal{I}} \longrightarrow \mathcal{C}$ given by

$$\mathrm{ev}_i(f) = f(i)$$

Then ev_i preserves all limits and all colimits. In other words, limits and colimits in functors categories are computed **objectwise** (this is especially useful for us as we will be working in the category of simplical sets which is a functor category).

6. Let $\mathcal{I}$ be a small category and $\mathcal{C}$ a category which admits $\mathcal{I}$-shaped colimits. Then the construction of colimits can be organized into a functor

$$\mathrm{colim} : \mathcal{C}^{\mathcal{I}} \longrightarrow \mathcal{C}$$

and this functor preserves all colimits. This statement is usually phrased as saying that colimits commute with colimits. The analogous statement for limits is true as well. However, note that colimits and limits need not, in general, commute with each other.

2.3 Functor Categories

Let $\mathcal{I}$ be a small category. In this section we will recall some basic properties of the functor category $\mathtt{Set}^{\mathcal{I}}$ of functors from $\mathcal{I}$ to $\mathtt{Set}$. We first observe that the construction of representable functors induces a functor

$$\iota : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathtt{Set}^{\mathcal{I}}$$

given by $\iota(i) = R_i$. Now let $i \in \mathcal{I}$ be an object and $f \in \mathtt{Set}^{\mathcal{I}}$ a functor. Given a natural transformation $T : R_i \longrightarrow f$ we can associate with it the element $T_i(\mathrm{Id}_i) \in f(i)$ where $\mathrm{Id}_i \in R_i(i) = \mathrm{Hom}_{\mathcal{I}}(i,i)$ is the identity element. We then have the famous Yoneda lemma:

Lemma 2.6 (Yoneda)**.** *The association $T \mapsto T_i(\mathrm{Id}_i) \in f(i)$ determines a bijection*

$$Hom_{\mathtt{Set}^{\mathcal{I}}}(R_i, f) \xrightarrow{\cong} f(i)$$

Proof. It is not hard to construct an inverse to the map $T \mapsto T_i(\mathrm{Id}_i)$. Given an element $a \in f(i)$ and an element $\varphi \in R_i(j) = \mathrm{Hom}_{\mathcal{I}}(i,j)$ we can construct the element

$$f(\varphi)(a) \in f(b)$$

The association $\varphi \mapsto f(\varphi)(a)$ determines natural maps

$$T^a_j : R_i(j) \longrightarrow f(j)$$

which fit together to form a natural transformation

$$T^a : R_i \longrightarrow f$$

It is then straightforward to check that the association $a \mapsto T^a$ is inverse to the association $T \mapsto T_i(\mathrm{Id}_i)$. □

Next we show that every functor $f \in \mathtt{Set}^{\mathcal{I}}$ can be obtained canonically as a colimit of representables. Let $f \in \mathtt{Set}^{\mathcal{I}}$ be a functor. Define the category $\mathcal{I}^{\mathrm{op}}_{/f}$ as follows: the objects of $\mathcal{I}^{\mathrm{op}}_{/f}$ are pairs (i, a) where $i \in \mathcal{I}^{\mathrm{op}}$ is an object and $a \in f(i)$ is an element. A morphism $(i, a) \longrightarrow (j, b)$ in $\mathcal{I}^{\mathrm{op}}_{/f}$ is a morphism $\varphi : j \longrightarrow i$ in $\mathcal{I}$ (i.e. a morphism $i \longrightarrow j$ in $\mathcal{I}^{\mathrm{op}}$) such that $f(\varphi)(b) = a$. There is a natural functor

$$p : \mathcal{I}^{\mathrm{op}}_{/f} \longrightarrow \mathtt{Set}^{\mathcal{I}}$$

given by $(i, a) \mapsto R_i$. Furthermore, according to the Yoneda lemma the element $a \in f(i)$ determines a natural map $\alpha_{(i,a)} : R_i \longrightarrow f$. These maps are compatible and we obtain an element

$$(f, \{\alpha_{(i,a)}\}) \in \left(\mathtt{Set}^{\mathcal{I}}\right)_{/p}$$

Proposition 2.7. *The object* $(f, \{\alpha_{(i,a)}\})$ *is initial in* $\left(\mathtt{Set}^{\mathcal{I}}\right)_{/p}$.

Proof. Consider a functor $g \in \mathtt{Set}^{\mathcal{I}}$ and a compatible family of morphisms

$$\beta_{(i,a)} : R_i \longrightarrow g$$

Using the Yoneda lemma we can identify a map $\beta_{(i,a)} : R_i \longrightarrow g$ with an element $b_{(i,a)} \in g(i)$. For each i the association $a \mapsto b_{(i,a)}$ gives a map

$$T_i : f(i) \longrightarrow g(i)$$

and these maps fit together to form a natural transformation

$$T : f \longrightarrow g$$

which is compatible with the structure maps $\{\alpha_{(i,a)}$ and $\{\beta_{(i,a)}\}$, hence giving a map

$$T : (f, \{\alpha_{(i,a)}\}) \longrightarrow (g, \{\beta_{(i,a)}\})$$

in $\mathtt{Set}^{\mathcal{I}}_{/p}$. On the other hand, if

$$S : (f, \{\alpha_{(i,a)}\}) \longrightarrow (g, \{\beta_{(i,a)}\})$$

is any competing natural transformation then the compatibility constraint at the object (i, a) will imply that

$$S_i(a) = b_{(i,a)} = T_i(a)$$

and so we get that $S = T$. □

2.4 Geometric Realization

Specializing to the case of $\mathcal{I} = \Delta^{\mathrm{op}}$ we know get $\mathtt{Set}^{\Delta^{\mathrm{op}}} = \mathtt{Set}_\Delta$, our desired category of **simplicial sets**. For each $[n] \in \Delta$, the representable functor $R_{[n]} : \Delta^{\mathrm{op}} \longrightarrow \mathtt{Set}$ will be denoted by Δ^n and will be referred to as the **standard simplex** (now as a simplicial set, as opposed to a topological space). More explicitly, we have

$$\Delta^n([k]) = \mathrm{Hom}_\Delta([k],[n])$$

is the set of simplicial maps from $|\Delta^k|$ to $|\Delta^n|$. For each k, this set contains a subset which corresponds to injective simplicial maps. We call these the **non-degenerate** simplices of Δ^n. These exist in each dimension k between 0 and n and correspond to the k-dimensional faces of Δ^n (as well as to subsets of $[n]$).

Now the Yoneda lemma tells us that for an arbitrary simplicial set X one has a natural identification

$$X([n]) = \mathrm{Hom}_{\mathtt{Set}_\Delta}(\Delta^n, X)$$

We will refer to this set as the set of **n-simplices of** X. We will also usually denote $X([n])$ simply by X_n. When X is the singular simplicial set of a topological space the the set of n-simplices of X is the set of maps from $|\Delta^n|$ to that topological space.

Now Lemma 2.7 tells us that each simplicial set X can be built as a colimit of simplices by forming the category $\Delta^{\mathrm{op}}_{/X}$ of **simplices over** X (this category is also sometimes called the **simplex category** of X). The elements in this category can be identified with pairs (Δ^n, f) where $f : \Delta^n \longrightarrow X$ is a map of simplicial sets. Lemma 2.7 then states that

$$X \cong \mathrm{colim}_{\Delta^n \longrightarrow X} \Delta^n$$

In other words, X can be built out of standard simplices, and the recipe for how to do this is located in the collection of all maps $\Delta^n \longrightarrow X$, once this information is properly organized into a category.

We are now ready to define the **geometric realization** functor $|\bullet| : \mathtt{Set}_\Delta \longrightarrow \mathtt{Top}$. The way we define this functor is essentially by choosing its value on the standard simplices to be $\Delta^n \mapsto |\Delta^n|$ and then extending by colimits. More explictly, one defines

$$|X| \stackrel{\mathrm{def}}{=} \mathrm{colim}_{\Delta^n \longrightarrow X} |\Delta^n| \in \mathtt{Top}$$

where the colimit is taken over the category $\Delta^{\mathrm{op}}_{/X}$ of simplices over X and is evaluated in the category $\mathtt{Top}$. Note that this definition is consistent with what we started from as

$$\mathrm{colim}_{\Delta^n \longrightarrow \Delta^k} |\Delta^n| \cong |\Delta^k|$$

due to the fact that the simplex category of Δ^k has a terminal object given by the identity map $\Delta^k \longrightarrow \Delta^k$.

2.5 Adjunctions

Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}, \mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$ a pair of functors. We will say that a a natural transformation

$$u_X : X \longrightarrow \mathcal{G}(\mathcal{F}(X))$$

is a **unit of an adjunction** $\mathcal{F} \dashv \mathcal{G}$ if for any $X \in \mathcal{C}, Y \in \mathcal{D}$ the composed map

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(\mathcal{F}(X)), \mathcal{G}(Y)) \xrightarrow{u_X^*} \mathrm{Hom}_{\mathcal{C}}(X, \mathcal{G}(Y))$$

is a bijection of sets. One then says that $\mathcal{F}$ is **left adjoint** to $\mathcal{G}$ and that $\mathcal{G}$ is right adjoint to $\mathcal{F}$.

Note that by setting $X = \mathcal{G}(Y)$ above we obtain a bijection

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(\mathcal{G}(Y)), Y) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}(Y), \mathcal{G}(Y))$$

The morphisms $\nu_Y : \mathcal{F}(\mathcal{G}(Y)) \longrightarrow Y$, which correspond to the identity $\mathcal{G}(Y) \longrightarrow \mathcal{G}(Y)$ under the bijection above, fit together to form a natural transformation $\mathcal{F} \circ \mathcal{G} \longrightarrow \mathrm{Id}_{\mathcal{D}}$, which is called the **counit** of the adjunction.

Example 5.

1. Let $\mathcal{F} : \mathtt{Set} \longrightarrow \mathrm{Gr}$ be the functor that associates the each set X the free group $< X >$ generated by the elements of X. Let $\mathcal{U} : \mathrm{Gr} \longrightarrow \mathtt{Set}$ be the functor that associates to each group its underlying set of elements. Then the natural map
$$X \longrightarrow \mathcal{U}(\mathcal{F}(X))$$
which associates to each element of X the corresponding generator of $\mathcal{F}(X)$ is a unit for an adjunction $\mathcal{F} \dashv \mathcal{U}$.

2. Let $\mathcal{D} : \mathtt{Set} \longrightarrow \mathtt{Top}$ be the functor that associates to each set X the discrete topological space with point set X. Let $\mathcal{P} : \mathtt{Top} \longrightarrow \mathtt{Set}$ be the functor which associates to each topological space its underlying set of points. Then the natural map
$$X \longrightarrow \mathcal{P}(\mathcal{D}(X))$$
is a unit of an adjunction $\mathcal{D} \dashv \mathcal{P}$.

3. The abelianization functor $\mathrm{Gr} \longrightarrow \mathtt{Ab}$ is left adjoint to the full inclusion $\mathtt{Ab} \hookrightarrow \mathrm{Gr}$.

4. Let $\mathcal{I}$ be a small category and $\mathcal{C}$ a category which admits colimits for $\mathcal{I}$-shaped diagram. Then the construction of colimits for all $\mathcal{I}$-shaped diagrams fits into a functor
$$\mathrm{colim} : \mathcal{C}^{\mathcal{I}} \longrightarrow \mathcal{C}$$

from the functor category $\mathcal{C}^I$ to $\mathcal{C}$. This functor is **left adjoint** to the constant diagram functor

$$\text{const} : \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{I}}$$

which associates to each object $X \in \mathcal{C}$ the constant diagram with value X. A similar construction can be done for limits, in which case the construction of limits will be **right adjoint** to the constant diagram functor.

Proposition 2.8. *Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor which admits a right adjoint $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{C}$. Then $\mathcal{F}$ preserves all colimits. Similarly if $\mathcal{F}$ admits a left adjoint then $\mathcal{F}$ preserves all limits.*

Proof. Let $f : \mathcal{I} \longrightarrow \mathcal{C}$ be a diagram and let $g = \mathcal{F} \circ f : \mathcal{I} \longrightarrow \mathcal{D}$ be the composed diagram. The functor $\mathcal{F}$ induces a functor

$$\mathcal{F}_{f/} : \mathcal{C}_{f/} \longrightarrow \mathcal{D}_{g/}$$

We need to show that this functor preserves initial objects. Let $(X, \{\alpha_i\})$ be an initial object. We need to show that $(\mathcal{F}(X), \{\mathcal{F}(\alpha_i)\})$ is initial in $\mathcal{D}_{g/}$. Let $(Y, \{\beta_i\}) \in \mathcal{D}_{g/}$ be an arbitrary element. We need to show that there exists a unique map

$$(\mathcal{F}(X), \{\mathcal{F}(\alpha_i)\}) \longrightarrow (Y, \{\beta_i\})$$

in $\mathcal{D}_{g/}$. In other words, we need to show that there exists a unique map

$$\varphi : \mathcal{F}(X) \longrightarrow Y$$

such that $\varphi \circ \mathcal{F}(\alpha_i) = \beta_i$ for every i. Using adjunction and naturality, we observe that this is equivalent to saying that there exists a unique map

$$\psi : X \longrightarrow \mathcal{G}(Y)$$

such that

$$\psi \circ \alpha_i = \mathcal{G}(\beta_i) \circ u_{f(i)}$$

But this is just a consequence of the fact that $(X, \{\alpha_i\})$ is initial in $C_{f/}$. □

In certain situations Proposition 2.8 admits an inverse, i.e., if $\mathcal{C}$ is a **presentable category** then any colimit preserving functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ admits a right adjoint. Most categories that you know (which have all colimits) are presentable. A prominent example of a category which has all colimits but is not presentable is the category of topological spaces. This is one of the ways in which the category of topological spaces is not so good from a category theoretic point of view.

3 Lecture 3

Recall from last time:

Definition 3.1. Let $X \in \mathtt{Set}_\Delta$. The **category of elements** $\Delta \downarrow X$ has objects the maps of simplicial sets $\Delta^n \longrightarrow X$ and maps the commutative triangles

$$\begin{array}{ccc} \Delta^n & \longrightarrow & \Delta^m \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

The **realization** is the functor $|{-}| : \mathtt{Set}_\Delta \longrightarrow Top$ given by $|X| = \mathrm{colim}_{\Delta^n \longrightarrow X} |\Delta^n|$.

Goal: for every $X \in \mathtt{Set}_\Delta$, $|X|$ is a CW-complex.

Proposition 3.2. *There is an adjunction* $|-| \dashv Sing$ *(left adjoint on the left).*

Proof. Recall that for any category $\mathcal{C}$ with colimits and any object $c \in \mathcal{C}$, the functor $\mathcal{C}(,c) : \mathcal{C}^{op} \longrightarrow Set$ satisfies $\mathcal{C}(\mathrm{colim}_I D, c) \cong \lim_I \mathcal{C}(D,c)$. Now suppose $X \in \mathtt{Set}_\Delta$ and $Y \in Top$. Then

$$Top(|X|, Y) = Top(\mathrm{colim}_{\Delta^n \longrightarrow X} |\Delta^n|, Y) \cong \lim_{\Delta^n \longrightarrow X} Top(|\Delta^n|, Y) \cong$$

$$\lim_{\Delta^n \longrightarrow X} \mathtt{Set}_\Delta(\Delta^n, Sing(Y)) \cong \mathtt{Set}_\Delta(\mathrm{colim}_{\Delta^n \longrightarrow X} \Delta^n, Sing(Y)) \cong \mathtt{Set}_\Delta(X, Sing(Y)).$$

□

Corollary 3.3. *The functor* $|-|$ *preserves colimits.*

our strategy is to build, for any simplicial set X, a "skeleton" filtration in which every stage is obtained from the previous one by a push-out of standard simplicies along their boundaries, and then apply $|{-}|$ to the filtration and obtain a CW-structure on $|X|$. To construct this filtration, we need to understand a bit better the combinatorics of $\mathtt{Set}_\Delta$.

Recall:

Definition 3.4. The category Δ has as objects the ordered sets $[n] = \{0, ..., n\}$ $(n \geq 0)$ and as maps the nno-decresing maps of sets. The **cofaces** $d^i : [n-1] \longrightarrow [n]$ $(0 \leq i \leq n)$ are given by $d^i(0 \to 1 \to ... \to n-1) = (0 \to 1 \to ... \to i-1 \to i+1 \to ... \to n)$ and the **codegeneracies** are given by $s^j : [n+1] \longrightarrow [n]$ $(0 \leq j \leq n)$ are given by $s^j(0 \to 1 \to ... \to n+1) = (0 \to 1 \to ... \to j \to j \to ... \to n)$.

Proposition 3.5. *The coface and codegeneracies satisfy the following relations:*

- $d^j d^i = d^i d^{j-1}$ *if* $i < j$.
- $s^j d^i = d^i s^{j-1}$ *if* $i < j$.

- $s^j d^j = id = s^j d^{j+1}$.
- $s^j d^i = d^{i-1} s^j$ *if* $i > j + 1$.
- $s^j s^i = s^i s^{j+1}$ *if* $i \leq j$.

Proposition 3.6. *Any map in Δ may be uniquely factorized as a composition of codegeneracies s^j followed by a composition of cofaces d^i. Thus, a simplicial set X can equivalently be described as a collection of sets $\{X_n\}_{n\geq 0}$ and face and degeneracy maps $d_i : X_n \longrightarrow X_{n-1}$, $s^j X_n \longrightarrow X_{n+1}$ (resp.) satisfying the opposite relations to those of Proposition 3.5.*

Definition 3.7. The **nerve** of a category is the simplicial set $N\mathcal{C}$ defined by setting $(N\mathcal{C})_n$ to be the set of all n-composable morphisms in $\mathcal{C}$

$$c_0 \longrightarrow c_1 \longrightarrow ... \longrightarrow c_n.$$

The description of the cofaces and codegeneracies in Definition 3.4 gives a way to define the face maps d_i and degeneracy maps s_j of $N\mathcal{C}$ by (respectively) composing two consecutive maps with common vertex c_i or inserting the identity at vertex j

Example 6. To every monoid M, we can associate a category $\mathbb{B}M$ that has one object and the set M as morphisms (the composition is given by multiplication in M). The realization $|N\mathbb{B}M|$ is known as the **classifying space** of M and plays an important role in algebraic topology.

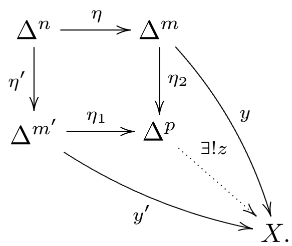
Definition 3.8. Let X be a simplicial set. An n-simplex $x \in X_n$ is called **degenerate** if there is $m < n$, a surjection $\eta[n] \to [m]$ and an m–simplex $y \in X_m$ such that $x = \eta^* y$.

Lemma 3.9 (Eilenberg-Zilber)**.** *Let $X \in \mathtt{Set}_\Delta$ and $x \in X_n$. There is a unique surjection $\eta : [n] \to [m]$ and a unique m-simplex $y \in X_m$ such that $\eta^* y = x$.*

Proof. Existence: if x is non-degenerate, take $\eta = id_{[n]}$. if x is degenerate, choose (η, y) such that $y \in X_m$ $m < n$, η is surjective and $\eta * y = x$. If y is non-degenerate, we are done. Otherwise, choose a similar pair (η', y') for y. This process stop after a finite number of stages. Uniqueness: Assume (η, y) and (η', y') are two such pairs. observe: There is a push-out of simplicial sets

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\eta} & \Delta^m \\ {\scriptstyle \eta'}\downarrow & & \downarrow{\scriptstyle \eta_2} \\ \Delta^{m'} & \xrightarrow{\eta_1} & \Delta^p \end{array}$$

Now, by assumption $\eta^* y = x = \eta'^* y'$ so the universal property of the push-out yields a map $z : \Delta^p \longrightarrow X$ as follows:



Since y and y' are non-degenerate, $\eta_1 = \eta_2 = id$, $y = y'$ and thus $\eta = \eta'$ □

3.1 The skeleton of a simplicial set

Let Δ_n be the full subcategory of Δ spanned by $[0], ..., [n]$. The inclusion $\Delta_n \hookrightarrow \Delta$ induces a **trunction** functor $tr_n : \mathtt{Set}_\Delta \longrightarrow \mathtt{Set}_{\Delta_n} := \mathtt{Set}^{\Delta_n^{op}}$. An object in the target category of this functor is called an **n-truncated** simplicial set. For $p \leq n$ the representable functor in $\mathtt{Set}_{\Delta_n}$ on $[p]$ is denoted by Δ_n^p.

Definition 3.10. The skeleton of an n-truncted simplicial set $X \in \mathtt{Set}_{\Delta_n}$ is given by

$$sk_n X = \operatorname{colim}_{\Delta_n^p \to X} \Delta^p$$

where for $p \leq n$, Δ_n^p is the representable functor $\Delta_n(-, [p])$. This defines a functor $sk_n : \mathtt{Set}_{\Delta_n} \longrightarrow \mathtt{Set}_\Delta$.

Proposition 3.11. *There is an adjunction $sk_n \dashv tr_n$.*

Proof.

$$\mathtt{Set}_\Delta(sk_n X, Y) \cong \mathtt{Set}_\Delta(\operatorname{colim}_{\Delta_n^p \to X} \Delta^p, Y) \cong$$

$$\lim_{\Delta_n^p \to X} Y_p \cong \lim_{\Delta_n^p \to X} \mathtt{Set}_{\Delta_n}(\Delta_n^p, tr_n Y) \cong \mathtt{Set}_{\Delta_n}(X, tr_n Y)$$

□

Observe 3.12. Since $\mathrm{sk}_n X$ is a quotient of a sum of Δ^p's for $p \leq n$, and the m-simplicies of Δ^p are degenerate for $m > n$, it follows that $\mathrm{sk}_n X$ consists only of degenerate simplicies.

Proposition 3.13. *The counit map $\mathrm{sk}_n \mathrm{tr}_n X \longrightarrow X$ is a monomorphism.*

Proof. Clearly, for $m \leq n$, $(\mathrm{sk}_n \mathrm{tr}_n X)_m \longrightarrow X_m$ is a bijection. It will suffice to show that if $f : Y \longrightarrow X$ is a map of simplicial sets which injective for all $Y_m \longrightarrow X_m$ with $m \leq n$ and Y_m consists only of degenerate simplicies for $m > n$ then f is injective. Let $y, y' \in Y_m$ for $m > n$. By E-Z lemma, there are $\eta : [m] \twoheadrightarrow [p]$, $\eta' : [m] \twoheadrightarrow [p']$ and non-degenerate $z \in Y_p$ and $z' \in Y_{p'}$ s.t. $\eta^* z = y$ and $\eta'^* z' = y'$. Since $p, p' \leq n$ and $f_p, f_{p'}$ are injective, it follows that $f_p(z)$ and $f_{p'}(z')$ are non-degenerate. Indeed, if, say, $f_p(z) = \alpha^* x$ w/ $\alpha : [p] \twoheadrightarrow [q]$ and $x \in X_q$, then α has a section $\epsilon : [q] \hookrightarrow [p]$ ($\alpha\epsilon = id$) and then

$\epsilon^* f_p(z) = \epsilon^*\alpha^* x = x$ so $\alpha^*(\epsilon^* f_p(z)) = \alpha^* x = f_p(z)$ so $z = \alpha^*\epsilon^* z$ is degenerate, yielding a contradiction.

Now, since $f_m(y) = \eta^* f_p(z)$ and $f_m(y') = \eta'^* f_{p'}(z')$, if $f_m(y) = f_m(y')$ then by E-Z $\eta = \eta'$ and $f_p(z) = f_{p'}(z')$ so we must have $p = p'$ and since f_p is injective, $z = z'$. □

Notation 3.14. Let $\operatorname{Sk}_n := \operatorname{sk}_n \circ \operatorname{tr}_n : \mathtt{Set}_\Delta \longrightarrow \mathtt{Set}_\Delta$ be the composite. We shall say that $X \in SS$ is of **dimension n** if $\operatorname{Sk}_n X = X$.

Observe 3.15.

- The pair $sk_n \dashv tr_n$ provides an equivalence between the category of n-truncated simplicial sets and the full subcategory of $\mathtt{Set}_\Delta$ spanned by the objects of dimension n.
- Any $X \in \mathtt{Set}_\Delta$ is the union of its skeleta

$$\operatorname{Sk}_0 X \subseteq \operatorname{Sk}_1 X \subseteq \ldots \subseteq \cup Sk_n X = X.$$

Definition 3.16. Let $\partial^i\Delta^n$ be the subsimplicial set

$$\partial^i\Delta^n := \operatorname{Im}(d^i : \Delta^{n-1} \longrightarrow \Delta^n).$$

This is the ı-th **face** of Δ^n. The **boundary** is then $\partial\Delta^n = \cup_{i=0}^n \partial^i\Delta^n$. Alternatively, $(\partial\Delta^n)_m := \{\alpha : [m] \longrightarrow [n] | \, \alpha \text{ is not surjective}\}$.

Observe 3.17. Δ^n has one non-degenerate simplex in dimension n and $n+1$ non-degenerate simplicies in dimension $n-1$, corresponding to the maps $d^i : [n-1] \to [n]$. Thus, $\operatorname{Sk}_{n-1}\Delta^n = \partial\Delta^n$.

Let ∂_n be the diagram of simplicial sets with objects $(\Delta^{n-2})_{ij} \equiv \Delta^{n-2}$, $0 \leq i < j \leq n$ and $(\Delta^{n-1})_i$, $0 \leq i \leq n$ and with morphisms the maps

$$(\Delta^{n-2})_{ij} \xrightarrow{d^{j-1}} (\Delta^{n-1})_i$$

and

$$(\Delta^{n-2})_{ij} \xrightarrow{d^i} (\Delta^{n-1})_j$$

Proposition 3.18. $\partial\Delta^n = \operatorname{colim}\partial_n$.

Proof. For each $i < j$, there is a commutative diagram

$$\begin{array}{ccc} \Delta^{n-2} & \xrightarrow{d^i} & \Delta^{n-1} \\ {\scriptstyle d^{j-1}}\downarrow & & \downarrow{\scriptstyle d^j} \\ \Delta^{n-1} & \xrightarrow{d^i} & \partial\Delta^n \end{array}$$

which is depicted in ∂_n as the diagram

$$\begin{array}{ccc} (\Delta^{n-2})_{ij} & \xrightarrow{d^i} & (\Delta^{n-1})_j \\ {\scriptstyle d^{j-1}}\downarrow & & \downarrow{\scriptstyle d^j} \\ (\Delta^{n-1})_i & \xrightarrow{d^i} & \partial\Delta^n \end{array}$$

so that $\partial\Delta^n$ is a **cone** over the diagram ∂_n and we have a map $\varphi : \operatorname{colim}\partial_n \longrightarrow \partial\Delta^n$. φ is surjective since every simplex of $\partial\Delta^n$ factors through some d^i. To see that φ is injective, let $\theta \in (\Delta^{n-1})_i$ and $\eta \in (\Delta^{n-1})_j$ be such that $\varphi(\theta) = \varphi(\eta) = \alpha$. Then $\alpha : [m] \to [n]$ avoids i and j and thus θ avoids $j-1$ and η avoids i. Hence, $\alpha = d^j \circ d^i \circ \overline{\alpha}$ and then $d^{j-1} : \overline{\alpha} \mapsto \theta$ and $d^i : \alpha \mapsto \eta$ so that $\theta = \eta$ in the colimit. ☐

Observe that by definition $|d^i : \Delta^{n-1} \longrightarrow \Delta^n| = d^i : |\Delta^{n-1}| \longrightarrow |\Delta^n|$. Since $|-|$ preserve colimits we get

Corollary 3.19.

$$|\partial\Delta^n| \cong \partial|\Delta^n|$$

and

$$|\partial\Delta^n \hookrightarrow \Delta^n| \cong \partial|\Delta^n| \hookrightarrow |\Delta^n|.$$

For $X \in \mathtt{Set}_\Delta$ let $e(X)_n$ denote the set of non-degenerate n-simplicies. For each $x \in e(X)_n$ we have

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow{\scriptstyle x} \\ \operatorname{Sk}_{n-1} X & \longrightarrow & \operatorname{Sk}_n X \end{array}$$

and summing all of that we get

$$\begin{array}{ccc} \coprod_{e(X)_n} \partial\Delta^n & \longrightarrow & \coprod_{e(X)_n} \Delta^n \\ \downarrow & & \downarrow \\ \operatorname{Sk}_{n-1} X & \longrightarrow & \operatorname{Sk}_n X \end{array} \tag{1}$$

Proposition 3.20. *The square 1 is a pushout.*

Proof. Since all the simplicial sets are of dimension n, it is enough to show that this diagram is a pushout after applying tr_n, i.e. that

$$\begin{array}{ccc} (\coprod_{e(X)_n} \partial\Delta^n)_m & \longrightarrow & (\coprod_{e(X)_n} \Delta^n)_m \\ \downarrow & & \downarrow \\ (\operatorname{Sk}_{n-1} X)_m & \longrightarrow & (\operatorname{Sk}_n X)_m \end{array}$$

is a pushout of sets for all $m \leq n$. For $m \leq n-1$ this is clear since then the two horizontal maps are isomorphims. For $m = n$ the complement of $\partial\Delta^n$ in Δ^n consists of one element $id_{[n]}$. Thus, the complement of $\coprod_{e(X)_n} (\partial\Delta^n)_n$ in $\coprod_{e(X)_n} (\Delta^n)_n$ is isomorphic to $e(X)_n$. But $(\operatorname{Sk}_n X)_m = (\operatorname{Sk}_{n-1} X)_n \cup e(X)_n$ so that the diagram is indeed a pushout. □

Since $|-|$ commutes with colimits we finally obtain:

Corollary 3.21. *For any simplicial set X, $|X|$ is a CW-complex with filtration* $|\operatorname{Sk}_0 X| \subseteq |\operatorname{Sk}_1 X| \subseteq \ldots \subseteq |\operatorname{Sk}_n X| \subseteq \ldots$.

4 Lecture 4

4.1 Kan complexes

The **k-th horn** $(0 \leq k \leq n)$ is defined to be $\Lambda^n_k = \cup_{i \neq k} \operatorname{Im}(d^i : \Delta^{n-1} \to \Delta^n)$.

Observe 4.1.

$$\mathtt{Set}_\Delta(\Lambda^n_k, X) = \{(x_0, \ldots, \hat{x_k}, \ldots, x_n) \in (X_{n-1})^n | d_i x_j = d_{j-1} x_i \ \forall i < j,\ i \neq k \text{ and} j \neq k\}.$$

Definition 4.2. A map $X \longrightarrow Y$ of simplicial sets is called a **Kan fibration** if for every commutative diagram of solid arrows

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \downarrow & \exists \nearrow & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

the doted arrow exists $(0 \leq k \leq n)$.

Recall:

Definition 4.3. A map of topological space $U \longrightarrow V$ is called a **Serre fibration** if every commutative diagram of solid arrows

$$\begin{array}{ccc} |\Lambda^n_k| & \longrightarrow & U \\ \downarrow & \exists \nearrow & \downarrow \\ |\Delta^n| & \longrightarrow & V \end{array}$$

the dotted arrow exists.

Observe 4.4. $f : U \longrightarrow V$ is a Serre fibration iff $S(f) : S(U) \longrightarrow S(V)$ is a Kan fibration. This follows by the adjunction $|-| \dashv S$.

Warning 4.5. It is not trivial to show that the realization of a Kan fibration is a Serre fibration and we shall do so in the future.

Lemma 4.6. *For every $U \in Top$, $S(U) \longrightarrow *$ is a Kan fibration.*

Proof. The inclusion $i : |\Lambda^n_k| \longrightarrow |\Delta^n|$ has a section $r : |\Delta^n| \longrightarrow \Lambda^n_k$ that makes the domain a strong deformation retract of the codomain. Thus, in a diagram of the form

$$\begin{array}{ccc} |\Lambda^n_k| & \xrightarrow{\varphi} & U \\ \downarrow & \overset{\exists\theta}{\nearrow} & \downarrow \\ |\Delta^n| & \longrightarrow & * \end{array}$$

the dotted arrow can be defined as $\theta = \varphi \circ r$ showing that $U \longrightarrow *$ is a Serre fibration and hence that $S(U) \longrightarrow *$ is a Kan fibration. □

Recall that a **groupoid** is a category in which every morphism is an isomorphism. A second class of kan complexes can be obtained via taking the nerve of a groupoid, as the following proposition asserts:

Proposition 4.7. *For every groupoid $\mathcal{G}$, $N\mathcal{G}$ is a Kan complex.*

Proof. Observe first that a map $X \longrightarrow N\mathcal{G}$ is determined by a map $tr_2 X \longrightarrow N\mathcal{G}$ since it is enough to know to which objects do the vertices X_0 are sent, to which arrows do the edges (1-simplicies) X_1 are being sent and that this assignment is compatible with composition, domain and codomain. Alternatively, X is a colimit of its simplicies Δ^n and so it is enough to see that this claim is true for $X = \Delta^n$. But maps $\Delta^n \longrightarrow N\mathcal{G}$ are precisely n-composable morphisms in $\mathcal{G}$ and thus these maps are completely determined by a map $tr_2\Delta^n \longrightarrow \mathrm{tr}_2\, N\mathcal{G}$ or, equivalently by a map $\mathrm{Sk}_2\, X \longrightarrow N\mathcal{G}$.

We consider the lifting problem

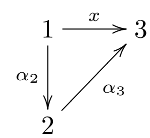
$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\varphi} & N\mathcal{G} \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

For $n > 3$ this is completely formal: we only need to provide a lift $\mathrm{Sk}_2\,\Delta^n \longrightarrow N\mathcal{G}$ but since $\mathrm{Sk}_2\,\Lambda^n_k \longrightarrow \mathrm{Sk}_2\,\Delta^n = id$ for $n > 3$, the lifting problem is trivial.

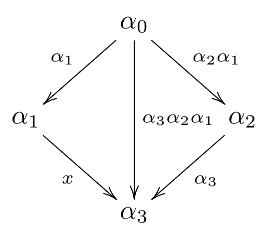
For $n = 1$ we can find a lift by associating to a vertex $x : \Delta^0 \longrightarrow N\mathcal{G}$ the identity arrow $s_0 x : \Delta^1 \longrightarrow N\mathcal{G}$. For $n = 2$ and $k = 1$, a map $\Lambda^n_k \longrightarrow N\mathcal{G}$ is precisely a pair of composable arrows in $\mathcal{G}$ so that we can solve the lifting problem for Λ^2_1 by defining a map $\Delta^2 \longrightarrow N\mathcal{G}$ via the element $(N\mathcal{G})_2$ given by that very pair of composable morphisms. For ($n = 2$) $k = 0$ and $k = 2$ we solve the corresponding lifting problem by inverting one of the arrows (recall that $\mathcal{G}$ contains only iso's) and then associating the pair of composable morphisms as before. We are left with $n = 3$. we consider only the lifting problem of

$$\begin{array}{ccc} \Lambda^3_0 & \xrightarrow{\alpha} & N\mathcal{G} \\ \downarrow & \nearrow & \\ \Delta^3 & & \end{array}$$

since the other cases are similar. Since $\mathrm{Sk}_1 \Lambda^3_0 = \mathrm{Sk}_1 \Delta^3$ we are given arrows $\alpha_1 : 0 \to 1$, $\alpha_2 : 1 \to 2$, $\alpha_3 : 2 \to 3$ and $x : 1 \to 3$. Note that the arrows $0 \to 2$ and $0 \to 3$ must be $\alpha_2\alpha_1$ and $\alpha_3\alpha_2\alpha_1$ since $\Lambda^3_0 \longrightarrow N\mathcal{G}$ must restrict to a commutative triangle on the faces d^3 and d^2 (respectively). However, the face $d^0\Delta^3$ is missing in Λ^3_0 so that we don't know that the triangle

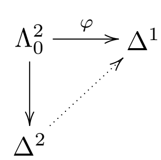


is commutative, and knowing this is precisely knowing that the lift we are looking for exist (note that in this case finding a lift only requires checking that the given data satisfies a certain condition and the is no need of constructing new data out of the given data). So we need to know that $x = \alpha_3\alpha_2$. However, the faces $d^1\Delta^3$ and $d^2\Delta^3$ together, are mapped into a commutative diagram in $N\mathcal{G}$ as follows:



so that $x\alpha_1 = \alpha_3\alpha_2\alpha_1$ and since $\mathcal{G}$ has only isomorphisms, $x = \alpha_3\alpha_2$ as required. □

Observe 4.8. The standard n-simplex Δ^n, $n > 0$, does not satisfy the Kan condition. Consider Δ^1 and the horn Λ^2_0 , which consists of the edges $0 \to 2$ and $0 \to 1$ of Δ^2 and their degeneracies. Now consider the simplicial map $\varphi : \Lambda^2_0 \longrightarrow \Delta^1$ that takes $0 \to 2 \in \Lambda^2_0$ to $0 \to 0 \in \Delta^1$ and $0 \to 1 \in Lam^2_0$ to $0 \to 1 \in \Delta^1$. There is a unique such simplicial map since weve specified what happens on all the nondegenerate simplices of Λ^2_0. Notice that this is perfectly well-defined as a simplicial map since all functions on all simplices are order-preserving. Thus, we have defined a lifting problem



However, this cannot be extended to a map $\Delta^2 \longrightarrow \Delta^1$ since we have already prescribed that on vertexes, $0 \mapsto 0$, $1 \mapsto 1$ and $2 \mapsto 0$, which is clearly not order-preserving on Δ^2.

A third class of Kan complexes is given by the following

Proposition 4.9. *The underlying simplicial set of a simplicial group*

$$G : \Delta^{op} \longrightarrow Set$$

is a Kan complex.

Proof. Let $x_0, ..., \hat{x_k}, ..., x_n \in G_{n-1}$ be $n-1$-simplicies satisfying $d_i x_j = d_{j-1} x_i$ for all $i < j$ and $i, j \neq k$. Then se set

- $w_0 = s_0 x_0$
- $w_i = w_{i-1}(s_i d_i w_{i-1})^{-1} s_i x_i$, $0 < i < k$.
- $w_n = w_{k-1}(s_{n-1} d_n w_{k-1})^{-1} s_{n-1} x_n$,
- $w_i = w_{i+1}(s_{i-1} d_i w_{i+1})^{-1} s_i x_i$, $k < i < n$

Then $x := w_{k+1}$ satisfy $d_i x = x_i$ for all $i \neq k$. □

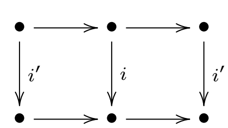
5 Lecture 5

5.1 The small object argument

Developing the homotopy theory of simplicial sets based solely on Kan fibrations involves long combinatorics. The theory of anodyne extensions provides a shortcut.

Definition 5.1. A class of mono's $\mathcal{M} \subseteq \mathtt{Set}_\Delta$ is called **saturated** if the following conditions hold:

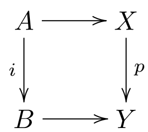
- $\mathrm{Iso}(\mathtt{Set}_\Delta) \subseteq \mathcal{M}$
- $\mathcal{M}$ is closed under pushouts.
- $\mathcal{M}$ is closed under retracts – namely, if $i \in \mathcal{M}$ and i' is a retract of i i.e. there is a commutative digram in $\mathtt{Set}_\Delta$ of the form

  in which the two horizontal composites are the identity,
  then i' is in $\mathcal{M}$ as well.
- $\mathcal{M}$ is closed under coproduct (of morphisms).
- $\mathcal{M}$ is closed under ω-compositions, i.e. if $F : \omega \longrightarrow \mathtt{Set}_\Delta$ is a diagram indexed by the first countable ordinal and such that for any $F(n \to n+1) \in \mathcal{M}$ for any $n \to n+1 \in \omega$, then the map $F(0) \longrightarrow \mathrm{colim}_\omega D$ (the "countable composition") is in $\mathcal{M}$.

We are going to have a lot of lifting problems around, so it's worth to have some terminology for that.

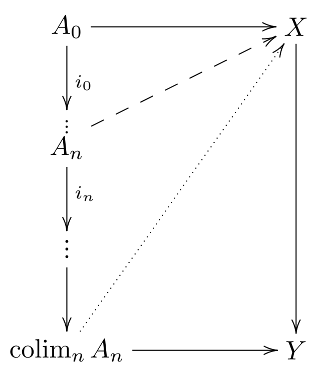
Definition 5.2. Given a pair of maps $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ we will say that p has the **right lifting property** (RLP) wrt i and that i has the **left lifting property** (LLP) wrt to p if every diagram of solid arrows as below admits a dotted lift



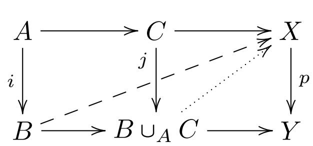
A map $p : X \to Y$ is said to have the right lifting property wrt a **class** of mono's $\mathcal{M} \subseteq \mathtt{Set}_\Delta$ if it has the RLP wrt to every $i \in \mathcal{M}$.

Lemma 5.3. *The class of all mono's that have a LLP wrt a fixed map $p : X \longrightarrow Y$ is saturated.*

Proof. Closness under countable composition: consider the lifting problem



since for each n the dashed arrow exists, we can view X as a cocone over the diagram $\{A_n\}$. By the universal property of colimit, we obtain the dotted map $\operatorname{colim}_n A_n \longrightarrow X$ which renders the diagram commutative. The map $A_0 \longrightarrow \operatorname{colim}_n A_n$ is clearly a mono. closness under pushouts: consider a morphism $i : A \longrightarrow B \in \mathcal{M}$ and a pushout diagram



The map j is clearly a mono and we need to check the lifting property. The dashed arrow exists since $i \in \mathcal{M}$ and the dotted arrow exist by the universal property of a pushout. Thus, $j \in \mathcal{M}$.

The rest of the conditions are verified in a similar manner. □

Definition 5.4. Let $\Gamma \subseteq \mathtt{Set}_\Delta$ be a class of morphisms. The intersection of all saturated classes containing Γ is called the **saturated class generated by** Γ.

For example, if $m : A \longrightarrow X$ is a mono, then we have a pushout

$$\begin{array}{ccc} \coprod_{e(X-A)_n} \partial\Delta^n & \longrightarrow & \coprod_{e(X-A)_n} \Delta^n \\ \downarrow & & \downarrow \\ \mathrm{Sk}_{n-1}(X) \cup A & \longrightarrow & \mathrm{Sk}_n(X) \cup A \end{array} \qquad (2)$$

which is constructed as before. Moreover, $X = \operatorname*{colim}_{n\geq -1} \mathrm{Sk}_n(X) \cup A$ and the map$\mathrm{Sk}_{-1}(X) \cup A \longrightarrow \operatorname*{colim}_{n\geq -1} \mathrm{Sk}_n(X) \cup A$ is precisely $m : A \longrightarrow X$. Thus the saturated class generated by the maps $\partial\Delta^n \hookrightarrow \Delta^n$ is all monomorphisms.

Definition 5.5. The saturated class generated by $\{\Lambda^n_k \hookrightarrow \Delta^n | n \geq 1,\ 0 \leq k \leq n\}$ is called the class of **anodyne extensions** and is denoted by $\mathcal{A}$.

Proposition 5.6. *A map $p : X \longrightarrow Y$ is a Kan fibration iff it has the RLP wrt all anodyne extensions.*

Proof. A class of maps that is defined as the maps with a LLP wrt to any class of maps $\mathcal{F} \subset \mathtt{Set}_\Delta$ is saturated. When $\mathcal{F}$ is taken to be the class of all Kan fibrations, we get from minimality of $\mathcal{A}$ that any anodyne extension has the LLP wrt Kan fibrations. The other direction follows from the definitions. □

Definition 5.7. A map $E \longrightarrow X$ is called a **trivial fibration** if it has the RLP wrt $\{\partial\Delta^n \hookrightarrow \Delta^n | n \geq 0\}$.

Observe 5.8. Any trivial fibration is a fibration: the saturated class generated by $\{\partial\Delta^n \hookrightarrow \Delta^n | n \geq 0\}$ is the class of all mono's and thus any trivial fibration has the RLP wrt to $\{\Lambda^n_k \hookrightarrow \Delta^n | n \geq 1,\ 0 \leq k \leq n\}$.

Theorem 5.9. *Any map $f : X \longrightarrow Y$ in $\mathtt{Set}_\Delta$ can be factored as*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle p} & \\ & E & \end{array}$$

where i is an anodyne extension and p is a fibration.

Proof. Consider the set L of all commutative diagrams of the form

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \Delta^n & \longrightarrow & Y \end{array}$$

summing over L gives

$$\begin{array}{ccc} \coprod_L \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \coprod_L \Delta^n & \longrightarrow & Y \end{array}$$

and i is anodyne since it is a coproduct of such. Consider now the pushout

$$\begin{array}{ccc} \coprod_L \Lambda^n_k & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_0} \\ \coprod_L \Delta^n & \longrightarrow & X_1. \end{array}$$

We see that i_0 is anodyne as a pushout of such and we have a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i_0"'] & & Y \\ & X_1 \arrow[ur, "f_1"'] & \end{tikzcd}$$

We now repeat the process with f_1 instead of f – summing over all the lifting problems and taking the pushouts. Let $E = \operatorname{colim}_{n>0} X_n$ and $p : E \longrightarrow Y$ the induced map. We have a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

in which i is anodyne as a countable composition of such. Let us show that $p : E \longrightarrow Y$ is a fibration: we consider a lifting problem

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{h} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \longrightarrow & Y. \end{array}$$

Since Λ^n_k has only finitely many non-degenerate simplicies, h must factor through some X_n. But if we extend this lifting problem to X_{n+1} it must have a solution because of how X_{n+1} was constructed

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "h"] \arrow[d] & X_n \arrow[r] \arrow[d, "f_n"'] & X_{n+1} \arrow[dl, "f_{n+1}"] \\ \Delta^n \arrow[r] \arrow[urr, dashed] & Y & \end{tikzcd} .$$

We thus obtain a lift to E. □

Corollary 5.10. *Any map that has the LLP wrt the class of all Kan fibrations is anodyne.*

Proof. Let $i : A \longrightarrow B$ be such, and factor it in the form

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & B \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle p} & \\ & E & \end{array}$$

where p is a fibration and j is anodyne. Since i and p are such, we have a lift

$$\begin{array}{ccc} A & \xrightarrow{\;j\;} & E \\ {\scriptstyle i}\downarrow & \overset{k}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\;id_B\;} & B \end{array}$$

but then, i is a retract of j via

$$\begin{array}{ccccc} A & \xrightarrow{\;id_A\;} & A & \xrightarrow{\;id_A\;} & A \\ {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow[\;k\;]{} & E & \xrightarrow[\;p\;]{} & B \end{array}$$

so that i is anodyne. □

Remark 5.11. Let $\mathtt{Set}_\Delta^{[1]}$ be the arrow category of simplicial sets, i.e. the category whose objects are maps of simplicial sets and whose morphisms are the corresponding commutative squares. Furthermore, the source and target of morphisms can be thought of as functors

$$\mathtt{Set}_\Delta^{[1]} \underset{t}{\overset{s}{\rightrightarrows}} \mathtt{Set}_\Delta .$$

We can then consider the pullback $\mathtt{Set}_\Delta^{[1]}\,{}_s\times_t \mathtt{Set}_\Delta^{[1]}$ as the set of pairs of composable morphisms. Thus a factorization of the identity functor $\mathtt{Set}_\Delta^{[1]} \longrightarrow \mathtt{Set}_\Delta^{[1]}$ as

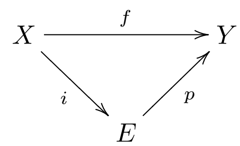
$$\mathtt{Set}_\Delta^{[1]} \longrightarrow \mathtt{Set}_\Delta^{[1]}\,{}_s\times_t \mathtt{Set}_\Delta^{[1]} \xrightarrow{\circ} \mathtt{Set}_\Delta^{[1]}$$

yields a factorization of any morphism into a composable pair of morphism. If we denote the class of Kan fibrations by $\mathcal{F}$, then $\mathcal{A}, \mathcal{F} \subseteq \mathtt{Set}_\Delta^{[1]}$ are the full subcategories spanned by the anodyne extensions and the Kan fibrations. We see that the factorization of Theorem 5.9 can be extended into a factorization of $id_{\mathtt{Set}_\Delta^{[1]}}$ as

$$\mathtt{Set}_\Delta^{[1]} \longrightarrow \mathcal{F}_s \times_t \mathcal{A} \xrightarrow{\circ} \mathtt{Set}_\Delta^{[1]}$$

and we say that our **factorizations are functorial**.

Theorem 5.12. *Any map $f : X \longrightarrow Y$ in $\mathtt{Set}_\Delta$ can be (functorially) factored*



where i is a mono and p is a trivial fibration.

Proof. Repeat the previous proof with $\{\partial\Delta^n \hookrightarrow \Delta^n | n \geq 0\}$ instead of the horn inclusions. □

6 Lecture 6

Our goal in this lecture is define a suitable notion of a **mapping space** in the setting of simplicial sets. Given two simplicial sets $X, Y \in \mathtt{Set}_\Delta$, we will want to construct a new simplicial set $\mathrm{map}(X, Y) \in \mathtt{Set}_\Delta$ which will be well-behaved in the following sense. For simplicial sets Y which are **Kan** we will want

1. The mapping spaces $\mathrm{map}(X, Y)$ to be Kan for every Y.
2. To be able to use the mapping spaces to give a good notion of homotopy for maps $X \longrightarrow Y$.
3. To be able to use the mapping spaces to give a good notion of **homotopy groups** for Y.

Having all the above we will be in a good position to study simplicial sets as if they were topological spaces, and to formulate in what sense they are equivalent to spaces. Let us begin with the definition

Definition 6.1. Let $X, Y \in \mathtt{Set}_\Delta$ be simplicial sets. We define $\mathrm{map}(X, Y) \in \mathtt{Set}_\Delta$ to be the simplicial set whose n-simplices are given by

$$\mathrm{map}(X, Y)_n = \mathrm{Hom}_{\mathtt{Set}_\Delta}(X \times \Delta^n, Y)$$

The map $\rho^* : \mathrm{map}(X, Y)_n \longrightarrow \mathrm{map}(X, Y)_k$ associated to a morphism $\rho : [k] \longrightarrow [n]$ is given by pre-composition with the induced map $\mathrm{Id} \times \rho : X \times \Delta^k \longrightarrow X \times \Delta^k$.

Remark 6.2. The simplicial set $\mathrm{map}(X, Y)$ depends covariantly on Y and contravariantly on X. Put formally, we have a functor

$$\mathrm{map}(-,-) : \mathtt{Set}_\Delta^{\mathrm{op}} \times \mathtt{Set}_\Delta \longrightarrow \mathtt{Set}_\Delta$$

Before we proceed to questions (1) – (3) above, let us point out a very important formal property of the mapping space constructed above. We have an evaluation map

$$\mathrm{ev}_{X,Y} : X \times \mathrm{map}(X, Y) \longrightarrow Y$$

which is defined as follows. Given an n-simplex $\sigma \in X_n$ and a map $f : X \times \Delta^n \longrightarrow Y$ (considered as an n-simplex $f \in \mathrm{map}(X,Y)_n$ we define $\mathrm{ev}_{X,Y}(\sigma, f) \in Y_n$ to be the n-simplex $f(\sigma, \mathrm{Id})$ where

$$(\sigma, \mathrm{Id}) : \Delta^n \longrightarrow X \times \Delta^n$$

is considered as an n-simplex of $X \times \Delta^n$.

Proposition 6.3 (The exponential law)**.** *The function*

$$\mathrm{ev}_* : Hom_{\mathtt{Set}_\Delta}(K, map(X,Y)) \longrightarrow Hom_{\mathtt{Set}_\Delta}(X \times K, Y)$$

which sends a map $K \longrightarrow map(X,Y)$ to the composition

$$X \times K \longrightarrow X \times map(X,Y) \stackrel{\mathrm{ev}_{X,Y}}{\longrightarrow} Y$$

is a bijection which is natural in $K, X \in \mathtt{Set}_\Delta^{\mathrm{op}}$ and $Y \in \mathtt{Set}_\Delta$.

Proof. An explicit inverse to ev_* can be constructed by sending a map $g : X \times K \longrightarrow Y$ to the map $\mathrm{ev}_*^{-1}(g) : K \longrightarrow \mathrm{Hom}(X,Y)$ which maps an n-simplex $\sigma : \Delta^n \longrightarrow K$ of K to the composition

$$X \times \Delta^n \stackrel{\mathrm{Id} \times \sigma}{\longrightarrow} X \times K \stackrel{g}{\longrightarrow} Y$$

□

Remark 6.4. In terms of adjoint functors one can phrase Proposition 6.3 by saying that for each X the functor

$$\mathrm{map}(-,Y) : \mathtt{Set}_\Delta \longrightarrow \mathtt{Set}_\Delta$$

is **right adjoint** to the functor

$$X \times (-) : \mathtt{Set}_\Delta \longrightarrow \mathtt{Set}_\Delta$$

This claim can be formalized for all X's at once by considering a suitable notion of an **adjunction in two variables**. We will not make this idea precise here.

Exercise 1. Let X, Y, Z be simplicial set. The diagonal maps $\Delta^n \longrightarrow \Delta^n \times \Delta^n$ induce a composition product

$$\mathrm{map}(X,Y) \times \mathrm{map}(Y,Z) \longrightarrow \mathrm{map}(X,Z)$$

Show that this composition is unital and associative, so that we can consider $\mathtt{Set}_\Delta$ as a category enriched over $\mathtt{Set}_\Delta$.

Our first goal now is to address issue (1) above, i.e., we want to verify that $\mathrm{map}(X,Y)$ is Kan whenever Y is Kan. For this it will be useful to tackle a more general situation.

Theorem 6.5. *Let $K \hookrightarrow L$ be an inclusion of simplicial sets and let $X \longrightarrow Y$ be a Kan fibration. Then the induced map*

$$map(L,X) \longrightarrow map(K,X) \times_{map(K,Y)} map(L,Y)$$

is a Kan fibration.

Corollary 6.6. *Mapping spaces into a Kan simplicial set are Kan.*

Proof. Theorem 6.5 applied to the case $K = \varnothing$ and $Y = *$ implies that $\mathrm{map}(L,X)$ is Kan whenever X is Kan. □

In order to tackle Theorem 6.5 above it will be useful to reformulate it in terms of anodyne maps. In view of the exponential law 6.3, Theorem 6.5 is equivalent to the following assertion:

Theorem 6.7. *If $f : K \hookrightarrow L$ is a monomorphism and $g : S \longrightarrow T$ is an anodyne map then the induced map*

$$f \Box g : [S \times L] \coprod_{S \times K} [T \times K] \longrightarrow T \times L$$

is an anodyne map.

The map $f \Box g$ above is usually called the **pushout-product** of f and g. Note that the formation of pushout products is **associative**: if we have three maps f, g, h then we have a natural isomorphism:

$$(f \Box g) \Box h \cong f \Box (g \Box h)$$

In order to prove Theorem 6.7 we first observe a few reductions:

1. The class of maps g for which Theorem 6.7 holds for arbitrary f's is saturated. Hence it will be enough to prove for g being a horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$.

2. The class of monomorphisms is closed under pushout-products. Hence we have the following consequence: if Theorem 3 is true for a particular anodyne map g then it is true for any anodyne map of the form $h \Box g$ where h is a monomorphism.

3. For $i = 0, 1$ let $\sigma_i : \Delta^0 \longrightarrow \Delta^1$ denote the inclusion which maps Δ^0 to the i'th vertex of Δ^1. Then it is relatively straightforward to show the map $h^n_k : \Lambda^n_k \hookrightarrow \Delta^n$ is a retract of the map $h^n_k \Box \sigma_i$ (we leave this part as an exercise). Hence it will be enough to prove Theorem 3 for $g = \sigma_i$.

4. The class of maps f for which Theorem 6.7 holds for $g = \sigma_i$ is saturated. Hence it will be enough to prove for f being an inclusion of the form $\partial \Delta^n \hookrightarrow \Delta^n$.

We are hence left with proving the following claim: for $i = 0, 1$ the map

$$\left[\Delta^{\{i\}} \times \Delta^n\right] \coprod_{\Delta^{\{i\}} \times \partial\Delta^n} \left[\Delta^1 \times \partial\Delta^n\right] \longrightarrow \Delta^1 \times \Delta^n \tag{3}$$

is an anodyne map. This we will do by hand. For simplicity let us just do the case $i = 1$ (the case $i = 0$ is completely analogous). We start by listing all the non-degenerate simplices of $\Delta^1 \times \Delta^n$ which are not contained in the LHS of 3. We first observe that every $(n-1)$-simplex of $\Delta^1 \times \Delta^n$ is contained in $\left[\Delta^1 \times \partial\Delta^n\right]$ (because its image in Δ^n is contained in $\partial\Delta^n$). Furthermore, $\Delta^1 \times \Delta^n$ has non-degenerate simplices only up to dimension $\leq n+1$ while the LHS only has up to dimension n. In dimension n, the non-degenerate simplices of $\Delta^1 \times \Delta^n$ which are not contained the the LHS are exactly those whose projection to the Δ^n is surjective (i.e. the identity) and whose projection to the Δ^1 coordinate is not contained in $\Delta^{\{1\}}$. Those are given by

$$(\rho_i, \mathrm{Id}) : \Delta^n \longrightarrow \Delta^1 \times \Delta^n$$

where $\rho_i : [n] \longrightarrow [1]$ for $i = 0, ..., n-1$ is the map given by

$$\rho_i(j) = \begin{cases} 0 & j \leq i \\ 1 & j > i \end{cases}$$

In particular, there are exactly n non-degenerate n-simplices in $\Delta^1 \times \Delta^n$ which are not contained in the LHS of 3. Now the non-degenerate $(n+1)$-simplices of $\Delta^1 \times \Delta^n$ are given by maps of the form

$$\tau_i : \Delta^{n+1} \longrightarrow \Delta^1 \times \Delta^n$$

where τ_i can be described by its action on vertices as follows. The sequence of vertices $\tau_i(0), ..., \tau_i(n+1)$ is the sequence

$$(0,0), (0,1), ..., (0,i), (1,i), (1,i+1), ..., (1,n)$$

in the Cartesian product $[1] \times [n]$. We now observe the following:

1. The only face of τ_0 which is not contained in the LHS of 3 is ρ_0.
2. For $i > 0$, there are exactly two faces of τ_i which are not contained in the LHS of 3, namely ρ_{i-1} and ρ_i.

This means that if we add the $(n+1)$-simplices $\tau_0, ..., \tau_{n+1}$ to the LHS of 3 **in that order**, then each addition will be realized as a pushout along a horn inclusion, and hence an anodyne map. This means that the map 3 is anodyne as desired.

Now that we know that the mapping space into any Kan simplicial set is Kan (and in fact a much stronger claim), we can proceed to our next order of business, which is to define the notion of homotopies in the setting of simplicial set. We start with the basic definition:

Definition 6.8. Let $f, g : X \longrightarrow Y$ be two maps of simplicial sets. A **homotopy** from f to g is a map

$$H : X \times \Delta^1 \longrightarrow Y$$

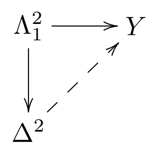
such that $H|_{X\times\Delta^{\{0\}}} = f$ and $H|_{X\times\Delta^{\{1\}}} = g$. We will say that f is **homotopic** to g if there exists a homotopy from f to g, and denote $f \sim g$.

Remark 6.9. In light of the exponential law (Proposition 6.3) we see that a homotopy from f to g is the same as a **1-simplex** in the $\mathrm{map}(X, Y)$ from f to g. This observation is always useful to keep in mind.

Our first task is to verify that the notion of homotopy is well behaved, at least when the target is a Kan simplicial set:

Lemma 6.10. *Let X, Y be simplicial sets such that Y is Kan. Then the relation $f \sim g$ is an equivalence relation.*

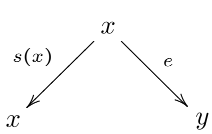
Proof. In light of Corollary 12.13 we may reduce to the case $X = *$. In other words, we need to show that the relation on vertices of Y given by the existence of edges is an equivalence relation. Now the degenerate edges give us reflexivity, and transitivity follows by the existence of lifts to diagram of the form



Let us now show symmetry. Let $e : \Delta^1 \longrightarrow Y$ be an edge from $x = e|_{\Delta^{\{0\}}}$ to $y = e|_{\Delta^{\{1\}}}$. By combining e with the degenerate edge $s(x)$ from x to itself we may construct a map

$$f : \Lambda^2_2 \longrightarrow Y$$

which can be diagrammatically depicted as



Since Y is Kan the map f can be extended to the whole 2-simplex, yielding an edge back from y to x. This finishes the proof of the lemma. □

Definition 6.11. Given two simplicial sets X, Y such that Y is Kan we will denote by $[X, Y]$ the set of homotopy classes of maps from X to Y.

Definition 6.12. The **homotopy category** of simplicial sets is the category whose objects are **Kan simplicial sets** and whose morphisms are homotopy classes of maps as above. We will denote this category by $\mathtt{Ho}(\mathtt{Set}_\Delta)$.

The object of this course is to explain in what way the homotopy theory of simplicial sets is equivalent to that of nice topological spaces. At least one aspect of this we can already formulate (although not prove) now: the homotopy category of simplicial sets is equivalent to the homotopy category of CW complexes.

7 Lecture 7

Recall from last time:

Proposition 7.1. *If $i : K \hookrightarrow L$ is an inclusion of simplicial sets and $p : X \longrightarrow Y$ a (Kan) fibration then the map*

$$map_{\square}(i,p) \longrightarrow map(K,X) \times_{map(K,Y)} map(L,Y)$$

induced from the (commutative) square

$$\begin{array}{ccc} map(L,X) & \longrightarrow & map(K,X) \\ \downarrow & & \downarrow \\ map(L,Y) & \longrightarrow & map(K,Y) \end{array}$$

is a fibration.

Corollary 7.2.

(a) If $p : X \longrightarrow Y$ is a fibration and $K \in \mathtt{Set}_{\Delta}$ an arbitrary simplicial set then $p_* : map(K,X) \longrightarrow map(K,Y)$ is a fibration.

(b) If $X \in \mathtt{Kan}$, and $i : K \hookrightarrow$ is an inclusion of simplicial sets then

$$i^* : map(L,X) \longrightarrow map(K,X)$$

is a fibration.

Proof.

Consider the mono $\varnothing \hookrightarrow K$. The Square

$$\begin{array}{ccc} \mathrm{map}(K,X) & \longrightarrow & \mathrm{map}(K,Y) \\ \downarrow & & \downarrow \\ * = \mathrm{map}(\varnothing,X) & \longrightarrow & \mathrm{map}(\varnothing,Y) = * \end{array}$$

is already a pullback square so that $p_* = \mathrm{map}_{\square}(i,p)$ is a fibration by Proposition 7.1.

(b) If $X \in \texttt{Kan}$, $p : X \longrightarrow *$ is a fibration. The square

$$\begin{array}{ccc} \mathrm{map}(L,X) & \longrightarrow & \mathrm{map}(K,X) \\ \downarrow & & \downarrow \\ * = \mathrm{map}(L,*) & \longrightarrow & \mathrm{map}(K,*) = * \end{array}$$

is already a pullback so that we can use Proposition 7.1 again and deduce that $i^* = \mathrm{map}_\square(i,p)$ is a fibration.

□

For this lecture it would be useful to establish a stronger version of proposition 7.1, namely:

Proposition 7.3. *In the setting of Proposition 7.1, we have in addition that* $i : K \hookrightarrow L$ *is anodyne, then* $map_\square(i,p)$ *is a trivial fibration.*

Proof. We need to check a lifting property wrt boundary inclusions of standard simplicies, i.e. to consider the lifting problem:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathrm{map}(L,X) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{map}(K,X) \times_{\mathrm{map}(K,Y)} \mathrm{map}(L,Y) \end{array}$$

This lifting problem in turn, is equivalent to the following lifting problem:

$$\begin{array}{ccc} \partial\Delta^n \times L \coprod_{\partial\Delta^n \times K} \Delta^n \times K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta^n \times L & \longrightarrow & Y \end{array}$$

(this is a good exercise in applying the exponential law and using the universal properties of pushouts and pullbacks).

The vertical map on the LHS is anodyne since it was obtained as the pushout product of an anodyne map $K \hookrightarrow L$ and a mono $\partial\Delta^n \hookrightarrow \Delta^n$ and thus a lifting exist since p is a fibration.

□

Now that we know that the mapping space into any Kan simplicial set is Kan (and in fact a much stronger claim), we can proceed to our next order of business, which is to define the notion of homotopies in the setting of simplicial set. We start with the basic definition:

Definition 7.4.

(a) Let $f, g : K \longrightarrow X$ be two maps of simplicial sets. A **homotopy** from f to g, $f \xrightarrow{\simeq} g$ is a map
$$h : K \times \Delta^1 \longrightarrow X$$
such that $h|_{K\times\Delta^{\{0\}}} = f$ and $H|_{K\times\Delta^{\{1\}}} = g$. We will say that f is **homotopic** to g if there exists a homotopy from f to g, and denote $f \simeq g$.

(b) Suppose $i : A \hookrightarrow K$ is an inclusion of simplicial sets and $f, g : K \longrightarrow X$ satisfy $f|_A = g|_A$. We will say that $f \simeq g$ (rel A) if there is a homotopy $h : K \times \Delta^1 \longrightarrow X$ such that $h|_{K\times\Delta^{\{0\}}} = f$ and $H|_{K\times\Delta^{\{1\}}} = g$ (as before) and in addition "h is stationary on A" in the sense that the following square commutes:
$$\begin{array}{ccc} A \times \Delta^1 & \xrightarrow{pr_A} & A \\ {\scriptstyle i\times id}\downarrow & & \downarrow{\scriptstyle f|_A = g|_A} \\ K \times \Delta^1 & \xrightarrow[h]{} & X \end{array}$$

Remark 7.5. In light of the exponential law we see that a homotopy from f to g is the same as a 1**-simplex** in the $\mathrm{map}(X, Y)$ from f to g. This observation is always useful to keep in mind.

The notion of a homotopy between maps enables us to talk about a homotopy between vertices since a vertex of X is simply a map of simplicial sets $\Delta^0 \longrightarrow X$.

Lemma 7.6. *For $X \in \mathtt{Kan}$ homotopy between vertices is an equivalence relation.*

Proof. We need to show that the relation on vertices of X given by the existence of edges is an equivalence relation. Now the degenerate edges give us reflexivity, and transitivity follows by the existence of lifts to diagram of the form
$$\begin{array}{ccc} \Lambda^2_1 & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^2 & & \end{array}$$

Let us now show symmetry. Let $e : \Delta^1 \longrightarrow X$ be an edge from $x = e|_{\Delta^{\{0\}}}$ to $y = e|_{\Delta^{\{1\}}}$. By combining e with the degenerate edge $s(x)$ from x to itself we may construct a map
$$f : \Lambda^2_2 \longrightarrow K$$
which can be diagrammatically depicted as
$$\begin{array}{ccccc} & & x & & \\ & {\scriptstyle s(x)}\swarrow & & \searrow{\scriptstyle e} & \\ x & & & & y \end{array}$$

Since X is Kan the map f can be extended to the whole 2-simplex, yielding an edge back from y to x. This finishes the proof of the lemma. □

We can thus define

Definition 7.7. For a Kan simplicial set X, the set of connected components is the quotient $\mathtt{Set}_\Delta(\Delta^0, X)/\simeq$ of vertices of X up to homotopy of vertices.

The above-mentioned lemma is in fact all we need in order to show that homotopy of maps is an equivalence relation:

Corollary 7.8. *Suppose X is fibrant and $A \hookrightarrow K$ is an inclusion. Then:*

(a) Homotopy between maps $K \longrightarrow X$ is an equivalence relation.

(b) Homotopy of maps $K \longrightarrow X$ (rel A) is an equivalence relation.

Proof. By Corollary 7.2 the map $i^*\mathrm{map}(K, X) \longrightarrow \mathrm{map}(A, X)$ is a fibration and hence its fiber over any vertex $v \in \mathrm{map}(A, X)$, $F_v := (i^*)^{-1}(v)$ is a Kan complex since a pullback of a fibration is a fibration. Observe now that given two maps $f, g : K \longrightarrow X$ that satisfy $f|_A = g|_A$, a homotopy $f \simeq g$ (rel A) is precisely a homotopy between the corresponding vertices of f and g in the fiber F_v over $v = f|_A = g|_A$. Since F_v is Kan, the later is an equivalence relation and hence the former. □

Definition 7.9. Given two simplicial sets X, K such that K is Kan we will denote by $[X, K]$ the set of homotopy classes of maps from X to K.

Definition 7.10. The **homotopy category** of simplicial sets is the category whose objects are **Kan simplicial sets** and whose morphisms are homotopy classes of maps as above. We will denote this category by $\mathtt{Ho}(\mathtt{Set}_\Delta)$.

The object of this course is to explain in what way the homotopy theory of simplicial sets is equivalent to that of nice topological spaces. At least one aspect of this we can already formulate (although not prove) now: the homotopy category of simplicial sets is equivalent to the homotopy category of CW complexes.

The next definition is a central one in this course. Recall that in topological spaces we have notions of a weak equivalence and a homotopy equivalence. In order to compare topological spaces and simplicial sets we need to have an analogous notion, intrinsic to $\mathtt{Set}_\Delta$.

Definition 7.11. A map $f : X \longrightarrow Y$ is called a **weak equivalence** if for any Kan complex K, $[f^*] : [Y, K] \longrightarrow [X, K]$ is an isomorphism.

Observe 7.12. If X and Y are Kan complexes, a map $X \longrightarrow Y$ is a weak equivalence iff it is a homotopy equivalence where a map $f : X \longrightarrow Y$ is called a **homotopy equivalence** if there is a map $g : Y \longrightarrow X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

As said, weak equivalences will be a key notion in this course. In order to relate it to the previous ones, we would like to understand which of the classes of maps that we talked about is a weak equivalence.

Proposition 7.13. *any anodyne map $i : S \hookrightarrow T$ is a weak equivalence.*

Proof. Let K be a Kan complex and denote by $p : K \longrightarrow *$ the corresponding fibration. Then by Proposition 7.3, the map

$$\mathrm{map}_{\square}(i, p) : \mathrm{map}(T, K) \longrightarrow \mathrm{map}(S, K)$$

is a trivial fibration. We now claim that any trivial fibration between Kan complexes induces an isomorphism on π_0, as will be show below. Given this, we are done since $\pi_0\mathrm{map}(T, K) = [T, K]$ and $\pi_0\mathrm{map}(S, K) = [S, K]$. ☐

Lemma 7.14. *A trivial fibration of Kan complexes $X \longrightarrow Y$ induces an isomorphism on π_0.*

Proof. It is enough to show thatany vertex in Y has a source and that any homotopy between vertices in Y can be lifted to a homotopy in X between their sources. This is obtained by solving the lifting problems

$$\begin{array}{ccc} \varnothing = \partial\Delta^0 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^0 & \longrightarrow & Y \end{array}$$

$$\begin{array}{ccc} \partial\Delta^1 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^1 & \longrightarrow & Y \end{array}$$

which can be solved since a trivial fibration has a right lifting property wrt any mono. ☐

Next, we want to establish an analogous result for trivial fibrations. We first claim that:

Proposition 7.15. *Any trivial fibration $p : X \longrightarrow Y$ admits a section $s : Y \longrightarrow X$ ($ps = id_Y$) which is also a homotopy inverse.*

Proof. We construct the section s as the solution to the lifting problem

$$\begin{array}{ccc} \varnothing & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ Y & \xrightarrow[id_Y]{} & Y \end{array}$$

so that we immediately have $ps = id_Y$. It remains to show that $id_X \simeq sp$. That homotopy is obtained as the solution to the lifting problem

$$\begin{array}{ccc} X \times \partial\Delta^1 & \xrightarrow{id_X \amalg sp} & X \\ \downarrow & \nearrow & \downarrow p \\ X \times \Delta^1 & \xrightarrow[p \circ pr_X]{} & Y \end{array}$$

Note that the homotopy h is also "compatible" with p in that the following triangle commutes:

$$\begin{array}{ccc} X \times \Delta^1 & \xrightarrow{h} & X \\ & {\scriptstyle p\circ pr} \searrow \quad \swarrow {\scriptstyle p} & \\ & Y & \end{array}$$

□

Corollary 7.16. *Every trivial fibration $p : X \longrightarrow Y$ is a (fibration and a) weak equivalence.*

Proof. As before, a homotopy inverse $s : Y \longrightarrow X$ induces, for every Kan complex K, an inverse to

$$[p^*] : [Y, K] \longrightarrow [X, K]$$

□

7.1 Simplicial homotopy groups

Having the notion of homotopy classes of maps, we can proceed to the homotopy groups. Of course, as we saw above we have to restrict ourselves to Kan complexes.

Definition 7.17. Let X be a Kan complex, $v \in X_0$ a vertex and $n \geq 1$. Consider the set $\texttt{Set}_\Delta\left((\Delta^n, \partial\Delta^n), (X, v)\right)$ of maps $\alpha : \Delta^n \longrightarrow X$ that send $\partial\Delta^n$ to v. The **n-th homotopy set** of X wrt to v is the quotient set

$$\pi_n(X, v) \coloneqq \texttt{Set}_\Delta\big((\Delta^n, \partial\Delta^n), (X, v)\big)/\sim$$

where $\alpha \sim \beta$ iff $\alpha \simeq \beta$ (rel $\partial\Delta^n$).

The n-th homotopy set of a Kan complex is meant to constitute a simplicial version of the homotopy groups of topological spaces. Hence, the next thing in order is to establish the group structure.

Definition 7.18. Given $\alpha, \beta \in \pi_n(X, v)$, consider the n-simplicies

$$v_i = \begin{cases} v & if \;\; 0 \le i \le n-2 \\ \alpha & i = n-1 \\ \beta & i = n. \end{cases} \tag{4}$$

where v stands for $s_0 ... s_0 v$. These simplicies satisfy $d_i v_j = d_{j-1} v_i$, $i < j$, $i, j \neq n$ and there is thus a diagram of solid arrows

$$\begin{array}{ccc} \Lambda^{n+1}_n & \xrightarrow{(v_0, ..., v_{n-1}, \, , v_{n+1})} & X. \\ \big\downarrow & \nearrow_{\exists \omega} & \\ \Delta^{n+1} & & \end{array}$$

we define the **product** to be $[\alpha] \cdot [\beta] := [d_n \omega]$.

Observe 7.19. For an n-simplex $\alpha \in X$, let us denote by $\partial(\alpha) = (d_0 \alpha, ..., d_n \alpha)$ which we shall call the boundary of α. Then

$$\partial(d_n \omega) = (d_0 d_n \omega, ..., d_{n-1} d_n \omega, d_n d_n \omega) = (d_{n-1} d_0 \omega, ..., d_{n-1} d_{n-1} \omega, d_n d_{n+1} \omega) = (v, ..., v)$$

so that $d_n \omega$ indeed defines an element $[d_n \omega] \in \pi_n(X, v)$.

Of course, Definition 7.18 is only valid once we prove:

Lemma 7.20. *The homotopy class of $d_n \omega$ is independent of the choice of representatives α, β, and the lift ω.*

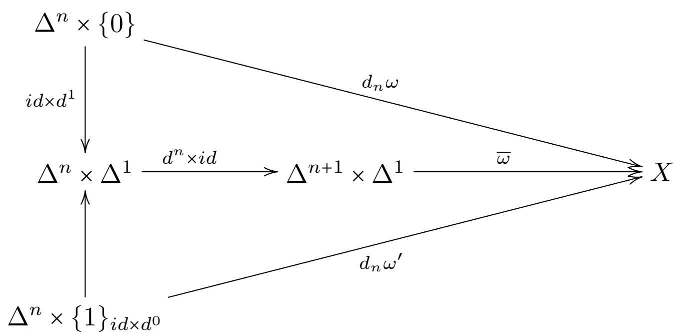
Proof. Suppose h_{n-1} is a homotopy $\alpha \xrightarrow{\simeq} \alpha'$ (rel $\partial \Delta^n$) and $h_{n+1} : \beta \xrightarrow{\simeq} \beta'$ (rel $\partial \Delta^n$). Choose lifts ω and ω' such that $\partial(d_n \omega) = (v, ..., v, \alpha, d_n \omega, \beta)$ and $\partial(d_n \omega') = (v, ..., v, \alpha', d_n \omega', \beta')$ (i.e. $d_n \omega = \alpha \cdot \beta$ and $d_n \omega' = \alpha' \cdot \beta'$). We then get a map

$$\begin{array}{ccc} \Delta^{n+1} \times \partial \Delta^1 \cup \Lambda^{n+1}_n \times \Delta^1 & \xrightarrow{(\omega', \omega, (v, ..., v, h_{n-1}, \, , h_{n+1}))} & X \\ \big\downarrow & \nearrow_{\overline{\omega}} & \\ \Delta^{n+1} \times \Delta^1 & & \end{array}$$

and the composite

$$\Delta^n \times \Delta^1 \xrightarrow{d^n \times id} \Delta^{n+1} \times \Delta^1 \xrightarrow{\overline{\omega}} X$$

makes the following diagram commutative



(note that the order of ω', ω was chosen so that this diagram will commute) and hence constitutes a homotopy $d_n\omega \xrightarrow{\simeq} d_n\omega'$ (rel $\partial\Delta^n$). ☐

Having that in mind, we now claim

Theorem 7.21. *The product of Definition 7.18 defines a group structure on* $\pi_n(X, v)$*.*

Proof. Let us prove that the product is associative:

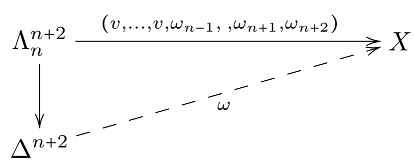
Let $\alpha, \beta, \gamma : \Delta^n \longrightarrow X$ be representatives of elements of $\pi_n(X, v)$. Choose $n+1$-simplicies $\omega_{n-1}, \omega_{n+1}, \omega_{n+2}$ as liftings with boundaries:

$$\partial\omega_{n-1} = (v, ..., v, \alpha, d_n\omega_{n-1}, \beta)$$

$$\partial\omega_{n+1} = (v, ..., v, d_n\omega_{n-1}, d_n\omega_{n+11}, \gamma)$$

$$\partial\omega_{n+2} = (v, ..., v, \beta, d_n\omega_{n+2}, \gamma)$$

i.e. $[d_n\omega_{n-1}] = [\alpha] \cdot [\beta]$, $[d_n\omega_{n+1}] = ([\alpha][\beta])\gamma$ and $[d_n\omega_{n+2}] = [\beta][\gamma]$. Then there is a map:



and the extension ω satisfies:

$$\partial(d_n\omega) = (v, ..., v, d_{n-2}d_n\omega, d_{n-1}d_n\omega, d_nd_n\omega, d_{n+1}d_n\omega)$$

$$= (v, ..., v, d_{n-1}d_{n-2}\omega, d_{n-1}d_{n-1}\omega, d_nd_{n+1}\omega, d_nd_{n+2}\omega) = (v, ..., v, \alpha, d_n\omega_{n+1}, d_n\omega_{n+2})$$

and this in turn means that

$$([\alpha][\beta])[\gamma] = [d_n\omega_{n-1}][\gamma] = [d_n\omega_{n+1}] = [d_nd_n\omega] = [\alpha][d_n\omega_{n+2}] = [\alpha]([\beta][\gamma])$$

As for the rest of it, the unit is of course (the homotopy class of) the constant simplex $e : \Delta^n \longrightarrow \Delta^0 \xrightarrow{v} X$. Let us show right divisibility. Given two elements $\alpha, \beta : \Delta^n \longrightarrow X$, we solve the lifting problem

$$\begin{array}{ccc} \Lambda^{n+1}_{n-1} & \xrightarrow{(v,\ldots,v,\ ,\beta,\alpha)} & X \\ \big\downarrow & \overset{\omega}{\nearrow} & \\ \Delta^{n+1} & & \end{array}$$

this shows that $[\beta] = [d_{n-1}\omega][\alpha]$. Left divisibility can be verified in a similar manner.

□

8 Lecture 8

Let X, Y be Kan simplicial sets and let $p : X \longrightarrow Y$ be a Kan fibration. Let $x_0 \in X_0$ be a base vertex and $y_0 = p(x_0) \in Y_0$ its image in Y. Let $F = p^{-1}(y_0) \subseteq X$ be the fiber of p over y_0, so that F is a Kan simplicial set as well (which contains x_0). Like in classical algebraic topology, the homotopy groups of X, Y and F can be related via a long exact sequence. In order to set up this long exact sequence we need to construct a boundary map

$$\partial : \pi_n(Y, y_0) \longrightarrow \pi_{n-1}(F, x_0)$$

An element $[\alpha] \in \pi_n(Y, y_0)$ can be represented by a map

$$\alpha : \Delta^n \longrightarrow Y$$

such that $\alpha|_{\partial\Delta^n}$ is constant with value y_0. Consider the diagram

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{f} & X \\ \big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow p \\ \Delta^n & \xrightarrow{\alpha} & Y \end{array} \qquad (5)$$

Where f is constant with image x_0. The lift θ exists because the left vertical map is an anodyne map and the right vertical map is a Kan fibration. One then defines

$$\partial([\alpha]) = [\theta|_{\Delta^{\{1,\ldots,n\}}}] \in \pi_{n-1}(F, x_0)$$

where $\theta|_{\Delta^{\{1,\ldots,n\}}}$ is considered as a map from Δ^n to $F \subseteq X$. The element $[\theta|_{\Delta^{\{1,\ldots,n\}}}]$ can be shown to be independent of the choice of the representative α (because any homotopy $\alpha \sim \alpha'$ can be lifted to θ) and of the lift θ (because the space of lifts satisfies the extension property with respect to $\partial\Delta^1 \hookrightarrow \Delta^1$ and is hence connected).

Let us spell out the situation for $n = 1$. In this case an element of $\pi_1(Y, y_0)$ can be represented by a path $\alpha : \Delta^1 \longrightarrow Y$ which starts and ends at y_0. We then lift α to a path $\theta : \Delta^1 \longrightarrow X$ which starts at x_0 and ends at some other point, which must lie over y_0. The image $\partial[\alpha] \in \pi_0(F, x_0)$ is then the connected component containing the end point of θ. Note that we may lift θ in many different ways, but the connected component of the end point of γ will not change. One way to explain this is as follows: by replacing each fiber of p with its set of connected components one would obtain a **covering map** over Y. It is then well known that covering maps admit unique lifts to paths given a choice of starting point.

Remark 8.1. In the case of $n = 1$ we see that there is actually an **action** of $\pi_1(Y, y_0)$ on the set of connected components $\pi_0(F)$ of the fiber over y_0. The map ∂ is then given by sending a class $[\alpha]$ to $[\alpha][x_0]$ where $[x_0] \in \pi_0(F)$ denotes the component of x_0.

Our next goal is to show that ∂ is a homomorphism of groups when $n \geq 2$. Let $\alpha_1, \alpha_2, \alpha_3 : (\Delta^n, \partial\Delta^n) \longrightarrow (Y, y_0)$ be maps. Recall that $[\alpha_3] = [\alpha_1] \cdot [\alpha_2]$ in $\pi_n(Y, y_0)$ if and only if there exists a map

$$\omega : \Delta^{n+1} \longrightarrow Y$$

such that the n-faces of ω are given by $(y_0, y_0, ..., y_0, \alpha_1, \alpha_3, \alpha_2)$. For $i = 1, 2, 3$ let $\theta_i : \Delta^n \longrightarrow X$ be a lift as in 5 so that $[\alpha_i] = [\theta_i|_{\Delta^{\{1,...,n\}}}]$. Observe that θ_i sends the $(n-1)$-skeleton of Δ^n to x_0. We can hence construct a commutative diagram of the form

$$\begin{array}{ccc} \Lambda^{n+1}_0 & \xrightarrow{g} & X \\ \downarrow & \overset{\gamma}{\nearrow} & \downarrow p \\ \Delta^{n+1} & \xrightarrow{\omega} & Y \end{array}$$

where g sends the faces of Λ^{n+1}_0 to $(x_0, x_0, ..., x_0, \theta_1, thet_3, \theta_2)$ (note that since $n \geq 2$ the horn Λ^{n+1}_0 has at least 3 faces). We now observe that $\gamma|_{\Delta^{\{1,...,n\}}}$ is an n-simplex in F whose $(n-1)$-faces are exactly

$$(x_0, ..., x_0, (\theta_1)|_{\Delta^{\{1,...,n\}}}, (\theta_3)|_{\Delta^{\{1,...,n\}}}, (\theta_2)|_{\Delta^{\{1,...,n\}}})$$

which means that

$$\partial([\alpha_3]) = \partial([\alpha_1])\partial([\alpha_2])$$

Our next goal is to show that ∂ enables one to relate the homotopy groups of X, Y and F in a long exact sequence

Proposition 8.2. *The sequence*

$$... \longrightarrow \pi_n(F, x_0) \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0) \xrightarrow{\partial}$$

$$\pi_{n-1}(F, x_0) \longrightarrow ... \longrightarrow \pi_1(Y, y_0) \xrightarrow{\partial} \pi_0(F, x_0) \longrightarrow \pi_0(X, x_0) \longrightarrow \pi_0(Y, y_0)$$

is an exact sequence of pointed sets.

Proof. We will now prove the exactness of

$$\pi_n(X, x_0) \xrightarrow{p_*} \pi_n(Y, y_0) \xrightarrow{\partial} \pi_n(F, x_0)$$

The exactness at the other places is left as an exercise. We need to show that an element $[\alpha] \in \pi_n(Y, y_0)$ comes from $\pi_n(X, x_0$ if and only if $\partial([\alpha]) = 0$. If α comes from $\pi_n(X, x_0)$ then there exists a $\beta : (\Delta^n, \partial\Delta^n) \longrightarrow (X, x_0)$ such that $p\beta : \Delta^n \longrightarrow Y$ represents the class $[\alpha]$. In this case we are allowed to choose β as a lift of α for the purpose of constructing $\partial[\alpha]$. But $\beta|_{\Delta^{1,\dots,n}}$ is constant on x_0 and so $\partial([\alpha]) = 0$.

On the other direction, assume that $\partial([\alpha]) = 0$. Let $\theta : \Delta^n \longrightarrow X$ be a lift such that $\partial([\alpha]) = \theta|_{\Delta^{\{1,\dots,n\}}}$. The triviality of $\partial([\alpha])$ Then there exists a homotopy

$$h_0 : \Delta^{n-1} \times \Delta^1 \longrightarrow F$$

from $\theta|_{\Delta^{\{1,\dots,n\}}}$ to the constant map on x_0 such that h_0 is constant along $\partial\Delta^{n-1} \times \Delta^1$. Hence we can trivially extend h_0 to a homotopy h_0' from $\theta|_{\partial\Delta^n}$ to the constant map on x_0. We then obtain a diagram of the form

$$\begin{array}{ccc} (\Delta^n \times \Delta^{\{0\}}) \cup (\partial\Delta^n \times \Delta^1) & \xrightarrow{(\theta, h_0')} & X \\ \downarrow & \overset{h}{\nearrow} & \\ \Delta^n \times \Delta^1 & & \end{array}$$

in which the lift exists because the vertical map is anodyne and X is Kan. We hence obtain a homotopy from θ to a map $\theta' = h|_{\Delta^n \times \Delta^{\{1\}}}$ which is constant on $\partial\Delta^n$. Then θ' defines an element of $\pi_n(X, x)$ which is mapped to the same class as $[\alpha]$ in $\pi_n(Y, y_0)$, as the homotopy $p \circ h$ shows. □

9 Lecture 9

We have talked quite a bit about homotopy groups and it is perhaps good to have a simple example in mind.

Example 7. Let G be a group, thought of as a category. We saw that the simplicial set $\mathbb{B}G := NG$ is a Kan complex (this was true for the nerve of any groupoid). We have $\mathbb{B}G_0 = e := e_G$, $\mathbb{B}G_1 = G$ and $\mathbb{B}G_2 = G \times G$. Since $\mathbb{B}G$ has only one vertex, $\pi_0\mathbb{B}G = 0$ and in addition, any pair of 1-simplicies $\alpha, \beta : \Delta^1 \longrightarrow \mathbb{B}G$ must send $\partial\Delta^1$ to e and thus define elements in $\pi_1(\mathbb{B}G, e)$. The maps α, β are homotopic iff there is $h : \Delta^1 \times \Delta^1 \longrightarrow \mathbb{B}G$ s.t. $h|_{\Delta^1 \times \Delta^{\{0\}}} = \alpha$, $h|_{\Delta^1 \times \Delta^{\{1\}}} = \beta$ and $h|_{\partial\Delta^1 \times \Delta^1 = e}$. Thus, such h gives rise to a commutative square

$$\begin{array}{ccc} e & \xrightarrow{\alpha} & e \\ {\scriptstyle s_0 e}\downarrow & & \downarrow{\scriptstyle s_0 e} \\ e & \xrightarrow[\beta]{} & e \end{array}$$

in G which means that $e \cdot \alpha = \beta \cdot e$ or $\alpha = \beta$. Thus, $\pi_1(\mathbb{B}G) = G$ and we can also say that $\pi_n(\mathbb{B}G) = 0$ for $n > 1$ – this is because there is only one n-simplex $\Delta^n \longrightarrow \mathbb{B}G$ that sends $\partial\Delta^n$ to e. The simplicial set $\mathbb{B}G$ is called the **classifying space** of G is sometimes denoted $K(G,1)$ – an **Eilenberg-Maclane space of type (G,1)**.

Recall from last time:

Proposition 9.1. *Let X be a Kan complex. Then $X \longrightarrow *$ is a trivial fibration iff for every vertex $x_0 \in X$, $\pi_n(X, x_0) = 0$ for all $n \geq 0$.*

Definition 9.2. A map of Kan complexes $f : X \longrightarrow Y$ is called a **homotopy isomorphism** is $\pi_0 f : \pi_0 X \longrightarrow \pi_0 Y$ is an isomorphism and for any vertex $x_0 \in X$, $\pi_n f : \pi_n(X, x_0) \longrightarrow \pi_n(Y, f(x_0)$ is an isomorphism.

Using this and the LES for a fibration, we can prove the following:

Proposition 9.3. *If a map of Kan complexes $p : X \longrightarrow Y$ is a fibration and a homotopy isomorphism then it is trivial fibration.*

Proof. Assume p is a fibration and a homotopy isomorphism. we consider a lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^n & \xrightarrow[f]{} & Y. \end{array}$$

We denote by $v_0 \in \Delta^n$ the vertex associated to the zero map $[0] \longrightarrow [n]$ and by $f(v_0)$ the constant map $\Delta^n \longrightarrow Y$.

Claim: f is homotopic to $f(v_0)$: we first define a homotopy we need to define a map $\Delta^n \times \Delta^1 \longrightarrow \Delta^n$ and we thus define a map of posets $[n] \times [1] \longrightarrow [n]$ via the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 1 & \longrightarrow \cdots \longrightarrow & n. \end{array}$$

This is clearly a homotopy from $v_0 : \Delta^n \longrightarrow \Delta^n$ to id_{Δ^n}. Post-composing this homotopy with f yields a homotopy from $f(v_0)$ to f.

Let us denote by $h : f \xrightarrow{\simeq} f(v_0)$ "the" inverse homotopy to the one we just constructed and by abuse of notation we write $h : \partial\Delta^n \times \Delta^1 \longrightarrow Y$ its restriction. The lifting problem

$$\begin{array}{ccc} \partial\Delta^n \times \Delta^{\{0\}} & \longrightarrow & X \\ \downarrow & \overset{\theta}{\nearrow} & \downarrow \\ \partial\Delta^n \times \Delta^1 & \longrightarrow & Y \end{array}$$

admits a solution since the left vertical map is anodyne. Moreover, the composite

$$p \circ \theta|_{\partial\Delta^n \times \Delta^{\{1\}}}$$

is

$$h|_{\partial\Delta^n \times \Delta^{\{1\}}} = f(v_0)$$

so that we can write

$$\theta|_{\partial\Delta^n \times \Delta^{\{1\}}} : \partial\Delta^n \times \Delta^{\{1\}} \longrightarrow F := p^{-1}(f(v_0))$$

. However, since p is a homotopy isomorphism, the LES tells us that $\pi_n(F, f_0) = 0$ for any vertex $f_0 \in F$ and 9.1 implies that $F \longrightarrow *$ is a trivial fibration. We thus obtain a lift of the form

$$\begin{array}{ccc}
\partial\Delta^n \times \Delta^{\{1\}} & \longrightarrow & X \\
\downarrow & \overset{\varphi}{\nearrow} & \\
\Delta^n \times \Delta^{\{1\}}. & &
\end{array}$$

We now combine all that data into a lifting problem of solid arrows:

$$\begin{array}{ccccc}
\partial\Delta^n \times \Delta^{\{0\}} & \dashrightarrow & \left(\partial\Delta^n \times \Delta^1\right) \coprod_{\partial\Delta^n \times \Delta^{\{1\}}} \left(\Delta^n \times \Delta^{\{1\}}\right) & \longrightarrow & X \\
\vdots & & \downarrow & \overset{l}{\nearrow} & \downarrow \\
\Delta^n \times \Delta^{\{0\}} & \dashrightarrow & \Delta^n \times \Delta^1 & \longrightarrow & Y
\end{array}$$

which admits a lift since the left vertical map is anodyne and $p : X \longrightarrow Y$ is a fibration. Precomposing l with the dotted arrows we obtain a lift to the original problem. Thus, $p : X \longrightarrow Y$ is a trivial fibration.

□

Let X be a Kan complex and $x_0 \in X$ a vertex.

Definition 9.4. The **path space** of (X, x_0) is the pullback

$$\begin{array}{ccc}
PX & \longrightarrow & \mathrm{map}(\Delta^1, X) \\
\downarrow & & \downarrow{\scriptstyle ev_0 = (d^1)^*} \\
\Delta^0 & \longrightarrow & \mathrm{map}(\Delta^0, X).
\end{array}$$

This comes with a map $\pi : PX \longrightarrow X$ which is the composite

$$PX \hookrightarrow \mathrm{map}(\Delta^1, X) \xrightarrow{(d^0)^*} \mathrm{map}(\Delta^0, X).$$

Observe 9.5. The map $PX \longrightarrow \Delta^0$ is a trivial fibration, so that PX is a Kan complex with $\pi_n(PX, x) = 0$ for any vertex $x \in X$. Moreover, The map π :

$PX \longrightarrow X$ is a fibration since we can obtain PX as a two-stage pullback

$$\begin{array}{ccc} PX & \longrightarrow & \operatorname{map}(\Delta^1, X) \\ \downarrow{\scriptstyle \pi=(d^0)^*} & & \downarrow{\scriptstyle ((d^0)^*,(d^1)^*)} \\ X & \longrightarrow & \operatorname{map}(\partial\Delta^1, X) = X \times X \\ \downarrow & & \downarrow{\scriptstyle pr_2} \\ \Delta^0 & \longrightarrow & X \end{array}$$

in which the top-right vertical map is a fibration. This is indeed a two-stage pullback since the bottom square is a clear pullback and the entire rectangle is the defining pullback square of PX (use the pasting lemma for pullbacks).

Definition 9.6. The loop space of (X, x_0) is the fiber $\Omega X := \pi^{-1}(*)$. In other words, $\Omega X = \operatorname{map}\big((\Delta^1, \partial\Delta^1), (X, x_0)\big)$

We obtain, for any pointed Kan complex (X, x_0) a fiber sequence $\Omega X \longrightarrow PX \longrightarrow X$. Moreover, ΩX is a Kan complex since the map $\Omega X \longrightarrow *$ is a pullback of the fibration $\pi : PX \longrightarrow X$.

Corollary 9.7. *For every $n \geq 1$, $\pi_n(X) \cong \pi_{n-1}(\Omega X)$.*

Proof. Use the LES for a fibration and the fact that $\pi_n(PX) = 0$ for any n. □

We now claim that the loop space on X admits a binary operation "up to homotopy". To see this, consider the map

$$\operatorname{map}\big((\Delta^2, (\Delta^2)_0), (X, x_0)\big) \xrightarrow{(i_1)^*} \operatorname{map}\big((\Lambda^2_1, (\Lambda^2_1)_0), (X, x_0)\big)$$

that is induced by the inclusion $i_1 : \Lambda^2_1 \longrightarrow \Delta^2$, or in other words

$$\operatorname{map}\big((\Delta^2, (\Delta^2)_0), (X, x_0)\big) \xrightarrow{(i_1)^*} \operatorname{map}\big((\Delta^1, \partial\Delta^1), (X, x_0)\big) \times \operatorname{map}\big((\Delta^1, \partial\Delta^1), (X, x_0)\big) = \Omega X \times \Omega X.$$

We also have a map

$$\operatorname{map}\big((\Delta^2, (\Delta^2)_0), (X, x_0)\big) \xrightarrow{(d^1)^*} \operatorname{map}\big((\Delta^1, (\Delta^1)_0), (X, x_0)\big) = \Omega X$$

The map $(d^1)^*$ is a trivial fibration since it is a pullback of

$$\operatorname{map}(\Delta^2, X) \longrightarrow \operatorname{map}(\Lambda^2_1, X)$$

(which is a trivial fibration since i_1 is anodyne) along the map

$$\operatorname{map}\big((\Lambda^2_1, (\Lambda^2_1)_0), (X, x_0)\big) \longrightarrow \operatorname{map}(\Lambda^2_1, X).$$

Thus, $(d^1)^*$ admits a section $s : \Omega X \times \Omega X \longrightarrow \operatorname{map}\big((\Delta^2, (\Delta^2)_0), (X, x_0)\big)$ and we obtain a map

$$\Omega X \times \Omega X \xrightarrow{\star := (d^1)^* \circ s} \Omega X .$$

The map $\star$ induces a binary operation ΩX upon passage to $\texttt{Ho}(\texttt{Set}_\Delta)$ and this does not depend on our choice of section since we saw that any two sections of a trivial fibration are homotopic. It can easily be verified that $\star$ is unital wrt the constant path $x_0 : \Delta^1 \longrightarrow X$.

Consider now $\pi_1(\Omega X, x_0) = [\Delta^1/\partial\Delta^1, \Omega X]_*$. This set admits another operation, induced from that of $\star$,

$$\star : [\Delta^1/\partial\Delta^1, \Omega X]_* \times [\Delta^1/\partial\Delta^1, \Omega X]_* \longrightarrow [\Delta^1/\partial\Delta^1, \Omega X]_*$$

(in other words for $\alpha, \beta : \Delta^1 \longrightarrow \Omega X$ representatives of elements of $\pi_1(X, x_0)$ define $\alpha \star \beta(x) = \alpha(x) \star \beta(x)$). This last map is a group homomorphism since π_1 is a functor to groups. If we denote the group structure of $\pi_1(\Omega X, x_0)$ by $\cdot$, this means that for any $\alpha, \beta, \gamma, \delta \in \pi_1(X, x_0)$,

$$(\alpha \cdot \beta) \star (\gamma \cdot \delta) = (\alpha \star \gamma) \cdot (\beta \star \gamma).$$

Lemma 9.8 (Eckmann-Hilton)**.** *If A is a pointed set with two unital binary operations $\cdot$ and $\star$ such that for any $\alpha, \beta, \gamma, \delta \in A$,*

$$(\alpha \cdot \beta) \star (\gamma \cdot \delta) = (\alpha \star \gamma) \cdot (\beta \star \gamma)$$

then $\cdot = \star$ and the operation is commutative and associative.

Proof.

$$\alpha \star \beta = (\alpha \cdot e) \star (e \cdot \beta) = (\alpha \star e) \cdot (e \star \beta) = \alpha \cdot \beta$$

and also

$$\beta \star \alpha = (e \cdot \beta) \star (\alpha \cdot e) = (e \star \alpha) \cdot (\beta \star e) = \alpha \cdot \beta.$$

Associativity follows similarly. □

Corollary 9.9. *For any pointed Kan complex (X, x_0), the higher homotopy groups $\pi_n(X, x_0)$ $(n \geq 2)$ are abelian.*

Proof. We saw that $\pi_2(X, x_0) \cong \pi_1(\Omega X)$ is abelian by the last lemma. But for $n \geq 2$, $\pi_n(X) = \pi_1(\Omega(...(\Omega X))$ so it is abelian as well. □

10 Lecture 10

Let $p : X \longrightarrow Z$ be a Kan fibration. Geometrically, we would like to think of p as describing a family of spaces which is parameterized "continuously" by the space Z. To make this intuition precise, we would like to prove that if $v, u \in Z_0$ are two vertices which are connected by an edge, then the fibers $p^{-1}(v)$ and $p^{-1}(u)$ are weakly equivalent. This will be achieved by the following lemma:

Lemma 10.1. *Let $p : X \longrightarrow Z$ be a Kan fibration and let $e : \Delta^1 \longrightarrow Z$ be an edge. Then the natural map*

$$\iota : p^{-1}(e(0)) = X \times_Z \Delta^{\{0\}} \longrightarrow X \times_Z \Delta^1$$

induced by the inclusion $\iota : \Delta^{\{0\}} \subseteq \Delta^1$ admits a ***homotopy inverse****. In particular, ι is a weak equivalence.*

Proof. We will prove for $i = 0$. The proof for $i = 1$ is completely analogous. We would like to construct a homotopy inverse for ι. Let $\pi : X \times_Z \Delta^1 \longrightarrow \Delta^1$ be the natural projection. Let $h : \Delta^1 \longrightarrow \Delta^1 \longrightarrow \Delta^1$ be a homotopy from the constant map $h|_{\Delta^1 \times \Delta^{\{1\}}} = s_0$ on $\Delta^{\{0\}}$ to the identity $h|_{\Delta^1 \times \Delta^{\{0\}}} = \mathrm{Id}$ which is constant on $\Delta^{\{0\}}$. By composing h with π we then obtain a homotopy

$$h_1 : \pi' \sim \pi : X \times_Z \Delta^1 \longrightarrow \Delta^1$$

where π' is the constant map taking the value $\Delta^{\{0\}}$. Now the restriction of h_1 to $X \times_Z \Delta^{\{0\}}$ is a constant homotopy and so we can lift it to the constant homotopy from

$$C_\iota : (X \times_Z \Delta^{\{0\}}) \times \Delta^1 \longrightarrow X \times_Z \Delta^1$$

from ι to itself. We can then extend this homotopy to all of $X \times_Z \Delta^1$. Formally speaking, we choose a lift in the following diagram:

$$\begin{array}{ccc} \left[(X \times_Z \Delta^{\{1\}}) \times \Delta^1\right] \coprod_{(X \times_Z \Delta^{\{0\}}) \times \Delta^{\{0\}}} \left[(X \times_Z \Delta^1) \times \Delta^{\{1\}}\right] & \longrightarrow & X \times_Z \Delta^1 \\ \downarrow & \overset{H}{\nearrow} & \downarrow \\ (X \times_Z \Delta^1) \times \Delta^1 & \xrightarrow{\quad h_1 \quad} & \Delta^1 \end{array}$$

The map H can be interpreted as a homotopy from some map $g : X \times_Z \Delta^1 \longrightarrow X \times_Z \Delta^1$ to the identity map, such that the image of g is contained in $X \times_Z \Delta^{\{0\}}$ and such that $g|_{X \times_Z \Delta^{\{0\}}} = \iota$. We then see that we can interpret g is a map from $X \times_Z \Delta^1$ to $X \times_Z \Delta^{\{0\}}$ which is a homotopy inverse of ι. □

Corollary 10.2. *Let $p : X \longrightarrow Z$ be a Kan fibration. Then any edge $e : \Delta^1 \longrightarrow X$ induces a homotopy equivalence*

$$p^{-1}(e(0)) \longrightarrow p^{-1}(e(1))$$

Proof. Compose the map $\iota : p^{-1}(e(0)) \hookrightarrow X \times_Z \Delta^1$ with the homotopy equivalence

$$g : X \times_Z p^{-1}(e(1))$$

constructed in 10.1. □

The above statements can be generalized to pullbacks along arbitrary maps $A \longrightarrow Z$:

Exercise 2. Let $p : X \longrightarrow Z$ be a Kan fibration. Let $f, g : A \longrightarrow X$ be two maps and let $f^*Z \longrightarrow A$ and $g^*X \longrightarrow A$ denote the two pullbacks. Then any homotopy $H : A \times \Delta^1 \longrightarrow X$ from f to g induces a fiberwise weak equivalence

$$\begin{array}{ccc} f^*X & \xrightarrow{\quad \simeq \quad} & g^*X \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

In algebra topology the notion of a fibration has rigid analogue which is known as a **fiber bundle** maps. Fiber bundle maps satisfy the property that the fibers over close enough points in the base have **isomorphic** fibers. Our purpose in this lecture is to generalize this rigid analogue to the world of simplicial sets.

Definition 10.3. Let $p : X \longrightarrow Z$ be a Kan fibration. Let $\sigma, \sigma' : \Delta^n \longrightarrow X$ be two n-simplices. We will say that σ is p-equivalent to τ (written $\sigma \sim_p \sigma'$) if there exists a homotopy $h : \Delta^n \times \Delta^1 \longrightarrow X$ from σ to τ such that $h|_{\partial\Delta^n}$ and $p \circ h$ are constant homotopies.

Let $p : X \longrightarrow Z$ be a Kan fibration. Then we know that the map

$$p_* : \mathrm{map}(\Delta^n, X) \longrightarrow \mathrm{map}(\Delta^n, Z) \times_{\mathrm{map}(\partial\Delta^n, Z)} \mathrm{map}(\partial\Delta^n X)$$

is a Kan fibration. Given a map $\tau : \partial\Delta^n \longrightarrow X$ (considered as a vertex in $\mathrm{map}(\partial\Delta^n, X)$) and an n-simplex $\rho : \Delta^n \longrightarrow Z$ (considered as a vertex in $\mathrm{map}(\Delta^n, Z)$) such that $\rho|_{\partial\Delta^n} = p \circ tau$ we can consider the fiber $p_*^{-1}(\tau, \rho)$ of the map p_* over the vertex (τ, ρ) (which is always a Kan simplicial set). One can then rephrase definition 10.3 as follows: two n-simplices $\sigma, \sigma' \in X$ are p-equivalent if $\sigma|_{\partial\Delta^n} = \sigma_{\partial\Delta^n}$, $p \circ \sigma = p \circ p'$ and σ, σ' are related by a path inside $p_*^{-1}(\tau, \rho)$. This way we can conclude from previous lectures that the relation $\sim_p$ is an equivalence relation.

Definition 10.4. Let $p : X \longrightarrow Z$ be a Kan fibration. We will say that p is a **minimal Kan fibration** if every p-equivalence class contains exactly one element. We will say that a simplicial set X is **minimal Kan complex** if the trivial map $X \longrightarrow *$ is a minimal Kan fibration.

Remark 10.5. The property of being a minimal Kan fibration is stable under pullbacks. In particular, the fibers of a minimal Kan fibration are minimal Kan complexes.

Our next goal is to show that minimal fibrations indeed behave like fiber bundles. For this it will be useful to use the following lemma, whose proof is left as an exercise:

Exercise 3. Let $p : Y \longrightarrow Z$ be a minimal Kan fibration. Let $H_1, H_2 : \Delta^n \times \Delta^1 \longrightarrow Y$ be two homotopies such that

$$H_1|_{\partial\Delta^n \times \Delta^1} = H_2|_{\partial\Delta^n \times \Delta^1}$$

and

$$p \circ H_1 = p \circ H_2$$

Then $H_1|_{\Delta^n \times \Delta^{\{0\}}} = H_2|_{\Delta^n \times \Delta^{\{0\}}}$ if and only if $H_1|_{\Delta^n \times \Delta^{\{1\}}} = H_2|_{\Delta^n \times \Delta^{\{1\}}}$.

We are now ready to prove the key property of minimal Kan fibrations:

Proposition 10.6. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

be a map of minimal Kan fibrations over Z which is a fiberwise weak equivalence. Then f is a fiberwise isomorphism. In particular, f is an isomorphism.

We can assume without loss of generality that $Z = *$. Since weak equivalences between Kan complexes have homotopy inverses the desired result will now follow from the following lemma:

Lemma 10.7. *Let $f, g : X \longrightarrow Y$ be two homotopic maps between minimal Kan complexes. Assume that g is an isomorphism. Then f is an isomorphism as well.*

Proof. We first prove that $f_n : X_n \longrightarrow Y_n$ is injective by induction on the n. Assume that f_i was injective for $i \leq n-1$ and let $\sigma_1, \sigma_2 : \Delta^n \longrightarrow X$ be two n-simplices such that $f(\sigma_1) = f(\sigma_2)$. Then f identifies the boundaries $\sigma_1|_{\partial\Delta^n}$ and $\sigma_1|_{\partial\Delta^n}$ so our induction hypothesis implies that $\sigma_1|_{\partial\Delta^n} = \sigma_2|_{\partial\Delta^n}$. Let $H : X \times \Delta^1 \longrightarrow Y$ be a homotopy from f to g. By restricting H to σ_1 and σ_2 we obtain two homotopies

$$H_1, H_2 : \Delta^n \times \Delta^1 \longrightarrow Y$$

such that H_i is a homotopy from $f(\sigma_i)$ to $g(\sigma_i)$. Since $\sigma_1|_{\partial\Delta^n} = \sigma_2|_{\partial\Delta^n}$ we see that H_1 and H_2 agree on $\partial\Delta^n \times \Delta^1$. Since $f(\sigma_1) = f(\sigma_2)$ we may use Exercise 3 to conclude that $g(\sigma_1) = g(\sigma_2)$. Since g was injective it follows that $\sigma_1 = \sigma_2$.

Let us now prove that f is surjective. Assume by induction that f_i is surjective for $i \leq n-1$ and let $\sigma : \Delta^n \longrightarrow Y$ be an n-simplex. Since we already know that f_i is injective for $i \leq n-1$ we can conclude that f maps $\mathrm{sk}_{n-1}(X)$ isomorphically to $\mathrm{sk}_{n-1}(Y)$. Hence there exists a map $\tau : \partial\Delta^n \longrightarrow X$ such that $f\tau = \sigma|_{\partial\Delta^n}$.

By restricting the homotopy H along τ we obtain a homotopy

$$H_\tau : \partial\Delta^n \times \Delta^1 \longrightarrow Y$$

from $f\tau = \sigma|_{\partial\Delta^n}$ to $g\tau$. We can then choose an extension of H_τ to a homotopy

$$\overline{H}_\tau : \Delta^n \times \Delta^1 \longrightarrow Y$$

such that $\overline{H}_\tau|_{\Delta^n \times \Delta^{\{0\}}} = \sigma$. Let $\sigma' = \overline{H}_\tau|_{\Delta^n \times \Delta^{\{1\}}}$. Then $\sigma'|_{\partial\Delta^n} = g\tau$. Since g is an isomorphism there exists a $\rho : \Delta^n \longrightarrow X$ such that $g\rho = \sigma'$ and such that $\rho|_{\partial\Delta^n} = \tau$. Finally, if we restrict the homotopy H to ρ we obtain a homotopy

$$H_\rho : \Delta^n \times \Delta^1 \longrightarrow Y$$

from $f\rho$ to $g\rho = \sigma'$. But now $\overline{H}_\tau$ and H_ρ are two homotopies which end at σ' and agree on $\partial\Delta^n$. From exercise 3 we conclude that $\overline{H}_\tau$ and H_ρ also start at the same simplex. This means that $f\rho = \sigma$ and so we found a pre-image for σ as desired. □

Corollary 10.8. *Let $p : X \longrightarrow Z$ be a minimal Kan fibration. Let $f, g : A \longrightarrow X$ be two maps and let $f^*Z \longrightarrow A$ and $g^*X \longrightarrow A$ denote the two pullbacks. Then any homotopy $H : A \times \Delta^1 \longrightarrow X$ from f to g induces an isomorphism*

$$\begin{array}{ccc} f^*X & \xrightarrow{\cong} & g^*X \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

Corollary 10.9 (The fiber bundle property)**.** *Let $p : Y \longrightarrow Z$ be a minimal Kan fibration and let $\sigma : \Delta^n \longrightarrow Z$ be a simplex. Let $v_0 \in (\Delta^n)_0$ denote the 0'th vertex and let $z_0 = \sigma(v_0)$ be its image under σ. Then we have an isomorphism*

$$\begin{array}{ccc} \sigma^*Y & \xrightarrow{\cong} & \Delta^n \times p^{-1}(z_0) \\ & \searrow \quad \swarrow & \\ & \Delta^n & \end{array}$$

Proof. This follows directly from Corollary 10.8 because σ is homotopic to the constant map $c : \Delta^n \longrightarrow Y$ which sends everything to z_0, and the pullback $Y \times_Z \Delta^n$ along the constant map $c : \Delta^n \longrightarrow X$ is isomorphic to the product $\Delta^n \times p^{-1}(z_0)$. □

Corollary 10.10. *Let $Y \longrightarrow Z$ be a minimal Kan fibration and assume that Z has finitely many non-degenerate simplices. Then the map $|Y| \longrightarrow |Z|$ is a fiber bundle.*

Proof. We can assume without loss of generality that Z is connected. Let $z_0 \in Z$ be a vertex and define $F = p^{-1}(z_0)$. We will show that $|Y| \longrightarrow |Z|$ is a fiber bundle with fiber $|F|$. By using induction on the simplices of Z it will be enough to prove the following. Suppose we are given a pushout square of the form

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^n \\ \downarrow{\scriptstyle f} & & \downarrow \\ Z_0 & \longrightarrow & Z_1 \end{array}$$

and suppose that $E \longrightarrow |Z_0|$ is a fiber bundle. Suppose that we are given an isomorphism $\rho : |\partial\Delta^n| \times |F| \cong |f|^{-1}E$ of bundles over $|\partial\Delta^n|$. We can then form the space E' as the pushout in the diagram.

$$\begin{array}{ccc} |\partial\Delta^n| \times |F| & \longrightarrow & |\Delta^n| \times |F| \\ \downarrow & & \downarrow \\ E & \longrightarrow & E' \end{array}$$

where the left vertical map is given by composing ρ with the natural map $f^{-1}E \longrightarrow E$ over $|Z_0|$. We then have a natural map $E' \longrightarrow |Z_1|$. In order to finish the proof we need to show that this map is a fiber bundle. We leave the topological details to the reader. ☐

Our next goal is to show that minimal fibrations exist in abundance. More precisely, every Kan fibration is equivalent in rather strong way to a minimal fibration. We begin with an auxiliary definition

Definition 10.11. Let $p : X \longrightarrow Z$ be a Kan fibration and let $\iota : Y \hookrightarrow X$ be an inclusion. We will say that Y is p**-minimal** if whenever we have two n-simplices $\sigma_1, \sigma_2 : \Delta^n \longrightarrow Y$ such that $\iota(\sigma_1), \iota(\sigma_2)$ are p-equivalent then $\sigma_1 = \sigma_2$. We will say that Y is **maximal** p-minimal if it is p-minimal and is not properly contained in any other p-minimal sub-complex of X.

Remark 10.12. Note that in the definition of p-minimality we do not require that $p \circ \iota$ will be a Kan fibration.

Now let $p : X \longrightarrow Z$ be a Kan fibration. Note that the empty set $\varnothing \subseteq X$ is always p-minimal in X. Furthermore, if we have an ascending chain $\{Y_\alpha\}$ of p-minimal sub-complexes then their union is p-minimal as well. By Zorn lemma we get that a maximal p-minimal sub-complex always exist.

Lemma 10.13. *Let $p : X \longrightarrow Z$ be a Kan fibration and let $\iota : Y \hookrightarrow X$ be a maximal p-minimal sub-complex. Let $\sigma : \Delta^n \longrightarrow X$ be an n-simplex whose boundary $\sigma|_{\partial\Delta^n}$ is contained in Y. Then σ is p-equivalent to a unique n-simplex which is contained in Y.*

Proof. If there exists a $\sigma : \Delta^n \longrightarrow X$ such that $\sigma|_{\partial\Delta^n}$ is contained in Y and such that σ is not p-equivalent to any n-simplex in Y then we can add σ to Y and form a larger p-minimal subcomplex. Since Y is maximal this is impossible. The uniqueness part is a direct consequence of the p-minimality of Y. ☐

Our final goal is to show that if $Y \hookrightarrow X$ is a maximal p-minimal sub-complex then Y is a fiberwise strong deformation retract of X. In particular, $p \circ \iota : Y \longrightarrow Z$ is a minimal Kan fibration which is equivalent to p.

Proposition 10.14. *Let $p : X \longrightarrow Z$ be a Kan fibration and let $\iota : Y \hookrightarrow X$ be a maximal p-minimal sub-complex. Then Y is a fiberwise deformation retract of X.*

Proof. We need to construct a map $r : X \longrightarrow Y$ over Z such that $r \circ \iota = \mathrm{Id}$ and such that $\iota \circ r$ is fiberwise homotopic to the identity on X. We shall construct r by induction on the skeletons of X. First consider the set of vertices X_0. According to Lemma 10.13, for each $x \in X_0$ there exists a unique $y \in Y_0$ such that x is p-equivalent to y. By setting $r_0(x) = y$ we obtain a map $r_0 : X_0 \longrightarrow Y_0$ over Z_0 such that $r_0 \circ \iota_0 = \mathrm{Id}$. On the other hand, since x is p-equivalent to $r_0(x)$ we can choose a path from x to $r_0(x)$ which is contained in the fiber over $p(x)$. This gives a fiberwise homotopy from the identity to $\iota_0 \circ r_0$.

Now let $n \geq 1$ and assume that a map $r_{n-1} : \mathrm{sk}_{n-1}(X) \longrightarrow \mathrm{sk}_{n-1}(Y)$ over $\mathrm{sk}_{n-1}(Z)$ has been constructed such that $r_{n-1} \circ \iota_{n-1} = \mathrm{Id}$. Assume in addition that we are provided with a fiberwise homotopy H_{n-1} from the identity on $\mathrm{sk}_{n-1}(X)$ to $\iota_{n-1} \circ r_{n-1}$. We wish to extend r_{n-1} to $\mathrm{sk}_n(X)$. Now for each non-degenerate n-simplex $\sigma : \Delta^n \longrightarrow X$, we can consider the restriction of H_{n-1} to the boundary of σ. This gives a homotopy over Z from $\sigma|_{\partial\Delta^n}$ to $\tau = \iota_{n-1}(r_{n-1}(\sigma|_{\partial\Delta^n}))$. We can then choose an extension of this homotopy to a homotopy (over Z)

$$H'_\sigma : \Delta^n \times \Delta^1 \longrightarrow X$$

which goes from σ to some σ' satisfying $\sigma'|_{\partial\Delta^n} = \tau$. Now since τ is contained in Y we get from Lemma 10.13 that there exists a unique n-simplex $\sigma'' : \Delta^n \longrightarrow Y$ which is p-equivalent ot σ'. We can then choose a homotopy H''_σ over Z from σ' to σ'' which is constant on $\partial\Delta^n$. By setting $r_n(\sigma) = \sigma''$ we obtain a map $r_n : X_n \longrightarrow Y_n$ extending r_{n-1}. Furthermore, by gluing together all the homotopies H'_σ and H''_σ we obtain, respectively, a pair of homotopies

$$H'_n, H''_n : \mathrm{sk}_n(X) \times \Delta^1 \longrightarrow \mathrm{sk}_n(X)$$

such that H'_n extends H_{n-1} and H''_n is constant on sk_{n-1}. We can then construct a fiberwise composition H_n of H'_n and H''_n which extends H_{n-1}. Then H''_n gives the desired fiberwise homotopy from the identity to $\iota_n \circ r_n$. □

Corollary 10.15. *Let $p : X \longrightarrow Z$ be a Kan fibration and let $\iota : Y \hookrightarrow X$ be a maximal p-minimal sub-complex. Then $q = p \circ \iota$ is a minimal Kan fibration.*

Proof. From the fact that q is a retract of p we can deduce two things:

1. q is a Kan fibration.
2. Two n-simplices σ, σ' of Y are q-equivalent if and only if $\iota(\sigma), \iota(\sigma')$ are p-equivalent.

since Y is p-minimal we deduce that it is q-minimal as well, i.e., $q : Y \longrightarrow Z$ is a minimal Kan fibration. □

11 Lecture 11

Goal 11.1. At the point we're at, there are five classes of maps in $\mathtt{Set}_\Delta$: monomorphisms, fibrations, weak equivalences, anodyne and trivial fibrations. The goal which we have been revolving around in the past few weeks is to show that these five classes are actually three, namely that a map is a trivial fibration iff it is a fibration and a weak equivalence, and that it is an anodyne iff it is a monomorphism and a weak equivalence.

Recall from last time:

Proposition 11.2. *Let $p : X \longrightarrow Y$ be a Kan fibration. Then p is a fiberwise deformation retract of a minimal fibration. In other words, there exist a* *factorization:*

$$\begin{array}{ccccc} Z & \xrightarrow{i} & X & \xrightarrow{r} & Z \\ & {\scriptstyle q}\searrow & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle q} & \\ & & Y & & \end{array}$$

such that

1. $ri = id_Z$
2. $q : Z \longrightarrow Y$ *is a minimal fibration, and*
3. *There is a homotopy* $h : i \circ r \Rightarrow id_X$ *which renders the following trianghle commutative:*

$$\begin{array}{ccc} X \times \Delta^1 & \xrightarrow{h} & X \\ & {\scriptstyle pr}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y & \end{array}$$

Our key claim today is the following

Proposition 11.3. *The map* $r : Z \longrightarrow Y$ *of Proposition 11.2 is a trivial fibration.*

Proof. The following proof is somewhat technical but does not depend on the explicit construction of r. We consider a lifting problem of the form:

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{u} & X \\ \downarrow & & \downarrow{\scriptstyle r} \\ \Delta^n & \xrightarrow[v]{} & Z. \end{array}$$

Then we obtain commutative diagrams

$$\begin{array}{ccccc} \partial\Delta^n \times \Delta^1 & \xrightarrow{u\times 1} & X \times \Delta^1 & \xrightarrow{h} & X \\ \downarrow & & & & \downarrow{\scriptstyle p} \\ \Delta^n \times \Delta^1 & \xrightarrow[pr]{} & \Delta^n & \xrightarrow[qv]{} & Y \end{array}$$

and

$$\begin{array}{ccccccc} \partial\Delta^n & \xrightarrow{u} & X & \longrightarrow & Z & \xrightarrow{i} & X \\ \downarrow & & & & & & \downarrow{\scriptstyle p} \\ \Delta^n & & \xrightarrow[v]{} & & Z & \xrightarrow[q]{} & Y \end{array}$$

and thus a lifting problem with a lift h':

$$
\begin{array}{ccccc}
(\partial\Delta^n \times \Delta^1) \cup (\Delta^n \times \Delta^{\{0\}}) & & \xrightarrow{(h(u\times 1), iv)} & & X \\
\downarrow & & \overset{\exists h'}{\nearrow} & & \downarrow{\scriptstyle p} \\
\Delta^n \times \Delta^1 & \xrightarrow[pr]{} & \Delta^n & \xrightarrow[qv]{} & Y.
\end{array}
\tag{6}
$$

Note that iv agrees with $h(u \times 1)$ on the intersection of the two factors in the top-left corner since on that $iv = iru = h_0 u$.

Let v_1 be the n-simplex defined by the composite $v_1 : \Delta^n \times \Delta^{\{1\}} \longrightarrow \Delta^n \times \Delta^1 \xrightarrow{h'} X$. We wish to show that v_1 is our desired lift i.e. that the upper and lower triangles in

$$
\begin{array}{ccc}
\partial\Delta^n & \xrightarrow{u} & X \\
\downarrow & \overset{v_1}{\nearrow} & \downarrow{\scriptstyle r} \\
\Delta^n & \xrightarrow[v]{} & Z
\end{array}
$$

commutes. The upper triangle commutes since $h'|_{\partial\Delta^n \times \Delta^{\{1\}}} = h|_{\partial\Delta^n \times \Delta^{\{1\}}} \circ u = u$.

Let us show that the lower triangle commutes. The composite

$$\Delta^n \times \Delta^1 \xrightarrow{v_1 \times 1} X \times \Delta^1 \xrightarrow{h} X \xrightarrow{r} Z$$

is a homotopy

$$rirv_1 = rv_1 \Rightarrow rv_1$$

which restricts to $rh(u \times 1)$ on $\partial\Delta^n \times \Delta^1$ since the upper triangle commutes. We can thus formulate the lifting problem:

$$
\begin{array}{ccccc}
(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_2) & & \xrightarrow{(s_0(rh(u\times 1)), (rh', rh(u\times 1), -))} & & Z \\
\downarrow & & \overset{H}{\nearrow} & & \downarrow{\scriptstyle q} \\
\Delta^n \times \Delta^2 & \xrightarrow[pr]{} & \Delta^n & \xrightarrow[qv]{} & Y
\end{array}
$$

in which the upper-horizontal map is well-defined since

$$rh'|_{\partial\Delta^n \times \Delta^1} = rh(v_1 \times 1)$$

by 6. Taking the extended edge that arises from H we get a homotopy $H(1 \times d^2)$ from $d^1(H(1 \times d^2)) = rh_0 v_1 = rirv_1 = rv_1$ to $d^0(H(1 \times d^2)) = rh'_0 = riv = v$. We now claim that $H(1 \times d^2)$ is in fact a fiberwise homotopy relative to the boundary in that

$$
\begin{array}{ccc}
\partial\Delta^n \times \Delta^1 & \xrightarrow{pr} & \partial\Delta^n \\
\downarrow & & \downarrow{\scriptstyle ru} \\
\Delta^n \times \Delta^1 & \xrightarrow{H(1\times d^2)} & Z \\
\downarrow & & \downarrow{\scriptstyle q} \\
\Delta^n & \xrightarrow[qv]{} & Y
\end{array}
$$

commutes. The lower square clearly commutes and the upper one does since $H(1 \times d^2)|_{\partial\Delta^2\times\Delta^1} = rh(u \times 1) = riru = ru$ where the next to last equality comes from the fact that h is a fiberwise homotopy.

We conclude that $v = rv_1$ and so r is a trivial fibration.

□

There are two types of ways to continue from here to accomplish goal 11.1. One of them is intrinsic to $\mathtt{Set}_\Delta$ and the other uses topological spaces; we shall choose the latter since it is shorter. In order to do so, we shall need to restrict ourselves to the category of "nice" topological spaces as follows:

Definition 11.4. A topological space X is called **compactly generated weak Hausdorff** (CGWH) space if

1. a subset $U \subseteq X$ is open iff $t^{-1}(U)$ is open for all continuous maps $t : C \longrightarrow X$ from a compact Hausdorff space into X (CG).
2. For every compact Hausdorff space C and a continuous map $t : C \longrightarrow X$, the image $t(C)$ is closed in X (WH).

We denote by Top_c the full subcategory spanned by compactly generated weak Hausdorff spaces.

We refer the reader to Strickland to a self contained account on CGWH-spaces. Let us mention the highlights.

- The class of CGWH-spaces includes: CW-complexes, metric spaces and locally compact Hausdorff spaces.
- The product of $X, Y \in Top_c$ is calculated by applying $k : Top \longrightarrow Top_c$ (the left adjoint to the inclusion) on $X \times Y \in Top$; we shall denote it by $X \times_k Y$. Similarly, the fiber product of $X \longrightarrow Z \longleftarrow Y \in Top_c$ is $X \times_{kZ} Y$.
- Although it is not true that $|X \times Y| \cong |X| \times |Y|$, it is true that $|X \times Y| \cong |X| \times_k |Y|$ and this is a key advantage that Top_c has over Top. Thus, we think of the realization as a functor $|-| : \mathtt{Set}_\Delta \longrightarrow Top_c$ (since the realization is a CW-complex and any CW-complex is a CGWH-space) and it is then true that $|-|$ commutes with fiber products.

From now on, we shall restrict our attention to Top_c and think of the realization functor as landing there instead of in Top. Under these assumptions, thing become very easy. For example:

Lemma 11.5. *If $p : X \longrightarrow Y$ is a trivial fibration, $|p| : |X| \longrightarrow |Y|$ is a Serre fibration.*

Proof. The solution to the lifting problem in $\mathtt{Set}_\Delta$:

$$\begin{array}{ccc} X & \longrightarrow & X \\ {\scriptstyle (1,p)}\downarrow & & \downarrow{\scriptstyle p} \\ X \times Y & \xrightarrow[pr_Y]{} & Y \end{array}$$

presents p as a retract of the projection pr_Y. It follows that $|p|$ is a retract of the projection $|X| \times_k |Y| \longrightarrow |Y|$ and hence a Serre fibration. □

Corollary 11.6 (Quillen)**.** *The realization of a Kan fibration is a Serre fibration.*

Proof. Decompose the Kan fibration into a retract followed be a minimal fibration. We have seen that this retract is a trivial fibration and thus realizes into a Serre fibration. We have also seen that the realization of a minimal fibration is a Serre fibration and thus the composite is so. □

Our next task is to compare the simplicial and topological homotopy groups:

Proposition 11.7. *Suppose X is a Kan complex. Then the unit map $X \longrightarrow S|X|$ is a homotopy isomorphism.*

Proof. Observe that the definition of realization can be formulated as $|X| = \coprod_{n\geq 0} X_n \times |\Delta^n|/\sim$ where $(x, s^j t) \sim (s_j x, t)$ and $(x, d^i t) \sim (d_i x, t)$. Thus, every map $v : |\Delta^0| \longrightarrow |X|$ factors through a realization of a simplex in X, $|\sigma| : |\Delta^n| \longrightarrow |X|$ so that every vertex of $S|X|$ has a source in X_0 and in particular $\pi_0(\eta_X) : \pi_0 X \longrightarrow \pi_0 S|X|$ is onto. On the other hand, X is a disjoint union of its path components and $S|-|$ preserves disjoint unions and thus $\pi_0(\eta_X)$ is a monomorphism. Assume by induction that for all Kan complexes X, all base-points $x \in X$ and all $i \leq n$, $\pi_i(\eta_X) : \pi_n(X, x) \longrightarrow \pi_i(S|X|, \eta_X(x))$ is an isomorphism. Using the fibration $\Omega X \longrightarrow PX \longrightarrow X$ we get

$$\begin{array}{ccc} \pi_{n+1}(X,x) & \longrightarrow & \pi_{n+1}(S|X|, \eta_X(x)) \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ \pi_n(\Omega X, x) & \xrightarrow{\cong} & \pi_n(S|\Omega X|, \eta_X(x)) \end{array}$$

so that it is enough to show that $S|\Omega X| \longrightarrow S|PX| \longrightarrow S|X|$ is a fibration and that $S|PX|$ has trivial homotopy groups. The first claim follows from adjunction and the fact that $|\Omega X| \longrightarrow |PX| \longrightarrow |X|$ is a Serre fiber sequence (note that $|-|$now preserves fiber sequnces since it commutes with fiber products). The second claim follows from the fact that PX is homotopy equivalent to $*$ using the homotopy h in

$$\begin{array}{ccc} (\Delta^0 \times \Delta^1) \cup (PX \times \partial\Delta^1) & \xrightarrow{(x,(1_{PX},x))} & PX \\ \downarrow & \overset{h}{\nearrow} & \downarrow \\ PX \times \Delta^1 & \longrightarrow & \Delta^0 \end{array}$$

□

Observe that

$$\pi_n(S|X|, \eta_X(x)) = \left\{ \begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^0 \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & S|X| \end{array} \right\} / \simeq$$

and

$$\pi_n(|X|, |x|) = \left\{ \begin{array}{ccc} |\partial\Delta^n| & \longrightarrow & |\Delta^0| \\ \downarrow & & \downarrow \\ |\Delta^n| & \longrightarrow & |X| \end{array} \right\} / \simeq .$$

Moreover, a homotopy $h : \Delta^n \times \Delta^1 \longrightarrow S|X|$ between $\alpha, \beta : \Delta^n \longrightarrow S|X|$ gives rise to and arises from a homotopy $h : |\Delta^n| \times_k |\Delta^1| \longrightarrow |X|$ between the adjoints of α and β. Using this observation and the previous proposition gives:

Corollary 11.8. *For a pointed Kan complex (X, x),*

$$\pi_n(X, x) \cong \pi_n(|X|, |x|)$$

and this isomorphism is natural in X. Thus, a map of Kan complexes $X \longrightarrow Y$ is a homotopy isomorphism iff $|X| \longrightarrow |Y|$ is a weak equivalence.

We can now acomplish our declared goal:

Theorem 11.9. *A fibration $p : X \longrightarrow Y$ is trivial iff it is a weak equivalence.*

Proof. We have seen that if p is a trivial fibration then it has a homotopy inverse and hence is a weak equivalence. Conversely, by Proposition 11.3 it is enough to show that a minimal fibration which is a weak equivalence is a trivial fibration so we assume p is minimal. A lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow{\beta} & Y \end{array}$$

is the same as a lifting problem

$$\begin{array}{ccccc} \partial\Delta^n & \longrightarrow & \Delta^n \times_Y X & \xrightarrow{\cong} & \Delta^n \times F_y \\ \downarrow & & \downarrow & \swarrow & \\ \Delta^n & \xrightarrow{1} & \Delta^n & & \end{array}$$

where F_y is the fiber over $y = \beta(v_0)$ and $v_0 \in (\Delta^n)_0$ the zero vertex. The isomorphism in the last diagram is obtained from minimality of p. We see that it is enough (in fact equivalent) to solve the lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & F_y \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

<u>Claim:</u> $|p|$ has a homotopy inverse.
<u>Proof:</u> As we saw, $[|Y|, |X|] \cong [Y, S|X|]$ and since p is a weak equivalence, this is isomorphic to $[X, S|X|] \cong [|X|, |X|]$. One can check that the class in $[|Y|, |X|]$ that corresponds to $id_{|X|}$ is a homotopy inverse to $|p|$.

Now, $|p|$ is a homotopy isomorphism so that $\pi_n |F_y| = 0$ for all $n \geq 0$. But F_y is a Kan complex so $\pi_n F_y = 0$ fro all $n \geq 0$. It follows from a previous proposition that $F_y \longrightarrow *$ is a trivial fibration and hence the desired lift exist. □

Corollary 11.10. *A monomorphism $i : A \longrightarrow B$ is anodyne iff it is a weak equivalence.*

Proof. If i is anodyne, then for any Kan complex K, $\mathrm{map}(B, K) \longrightarrow \mathrm{map}(A, K)$ is a trivial fibration of Kan complexes which is thus a homotopy isomorphism. In particular, it is an isomorphism on π_0 which precisely means that $[B, K] \longrightarrow [A, K]$ is an isomorphism so that i is a weak equivalence. Suppose i is a weak equivalence and factor it as an anodyne followed by a fibration

$$\begin{array}{ccccc} A & & \xrightarrow{\quad i \quad} & & B \\ & {\scriptstyle j}\searrow & & \nearrow{\scriptstyle p} & \\ & & E & & \end{array}$$

It follows that p is a weak equivalence since j and i are such, and so p is a trivial fibration by Theorem **??**. But then solving the lifting problem

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle i}\downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1]{} & B \end{array}$$

presents p as a retract of j and hence an anodyne. □

12 Lecture 12

12.1 Model categories

In this lecture we will finally give a formal meaning to our desired claim that the homotopy theory of simplicial sets is equivalent to the homotopy theory of topological spaces. This will be done through Quillen's notion of **model categories**.

Definition 12.1. A **model category** is a category $\mathcal{M}$ equipped with three distinguished subcategories $\mathcal{W}, \mathcal{C}of, \mathcal{F}ib \subseteq \mathcal{M}$ such that:

1. $\mathcal{M}$ has all small limits and colimits.

2. $\mathcal{W}, \mathcal{C}of, \mathcal{F}ib$ contain all isomorphisms (hence in particular all objects) and are closed under retracts.

3. $\mathcal{M}$ satisfies the 2-out-of-3 property.

4. Every morphism $f : X \longrightarrow Y$ in $\mathcal{M}$ can be factored (not uniquely) as

   (a) $X \xrightarrow{i} X' \xrightarrow{p} Y$ with $i \in \mathcal{C}of \cap \mathcal{W}$ and $p \in \mathcal{F}ib$.

   (b) $X \xrightarrow{i} X' \xrightarrow{p} Y$ with $i \in \mathcal{C}of$ and $p \in \mathcal{F}ib \cap \mathcal{W}$.

5. Any square

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ X & \longrightarrow & B \end{array}$$

   in which $i \in \mathcal{C}of, p \in \mathcal{F}ib$ and at least one of i, p is in $\mathcal{W}$ admits a lift $f : X \longrightarrow Y$.

The morphisms in $\mathcal{W}$ will be called **weak equivalences**, the morphisms in $\mathcal{C}of$ will be called **cofibrations** and the morphisms in $\mathcal{F}ib$ will be called **fibrations**. A cofibration which is also a weak equivalence will be called a **trivial cofibration**, and similarly for fibrations. An object X will be called **cofibrant** if the map from the initial object $\varnothing \longrightarrow X$ is a cofibration. Similarly, an object will be called **fibrant** if the map to the terminal object $X \longrightarrow *$ is a fibration.

Remark 12.2. The axioms of model categories are very restrictive, and hence small pieces of data often determine the whole structure. For example, any two of the trio $\mathcal{W}, \mathcal{C}of, \mathcal{F}ib$ determines the third (if exists). In particular, the classes $\mathcal{C}of, \mathcal{F}ib \cap \mathcal{W}$ determine each other via the relation

$$\mathcal{F}ib \cap \mathcal{W} = (\mathcal{C}of)^{\perp}$$

$$\mathcal{C}of = ^{\perp}(\mathcal{F}ib \cap \mathcal{W})$$

and similarly for $\mathcal{C}of \cap \mathcal{W}$ and $\mathcal{F}ib$.

Examples:

1. The category $\mathtt{Set}_{\Delta}$ of simplicial set together with $\mathcal{W}$ = weak equivalences of simplicial sets, $\mathcal{C}of$ = injective maps and $\mathcal{F}ib$ = Kan fibrations forms a model category, called the **Kan model category**. We will prove this fact later in this lecture.

2. The category $\mathtt{CG}$ of compactly generated spaces with $\mathcal{W}$ = weak equivalences of spaces, $\mathcal{C}of$ = saturated class generated from $\{S^{n-1} \hookrightarrow D^n\}$ and $\mathcal{F}ib$ = Serre fibrations forms a model category, called the **Quillen model category**. We will prove this fact in the next lecture.

3. The category $\mathtt{CG}$ of compactly generated spaces with $\mathcal{W}$ = homotopy equivalences, $\mathcal{C}of$ = closed Hurewitz cofibrations and $\mathcal{F}ib$ = Hurewitz fibrations forms a model category, called the **strom model category**. We will not prove this fact in this course.

4. Let R be a ring and let $\mathtt{Ch}^{\geq 0}(R)$ be the category of non-negatively graded chain complexes. Then the category $\mathtt{Ch}^{+} \geq 0(R)$ together with $\mathcal{W}$ = quasi-isomorphisms, $\mathcal{C}of$ = injective maps with projective cokernel and $\mathcal{F}ib$ = maps which are surjective in degrees ≥ 1 forms a model category. This model category is standardly used in homological algebra for the study of derived functors of $\otimes$ and Hom.

To get some basic intuition for the notion let us start by explaining how to do basic "homotopy theory" in a general model category. Let $f, g : X \longrightarrow Y$ be two morphisms in a model category $\mathcal{M}$. We claim that the model structure on $\mathcal{M}$ enables us to define the notion of a **homotopy** from f to g.

Definition 12.3. Let $\mathcal{M}$ be a model category and $X \in \mathcal{M}$ an object. We will say that a diagram of the form

$$X \coprod X \xrightarrow{i} C \xrightarrow{p} X$$

exhibits C as a **cylinder object** for X if the map i is a cofibration, the map p is a weak equivalence and the composition $p \circ i : X \coprod X \longrightarrow X$ is the identity on each X component.

Now observe that by factoring the natural map

$$f : X \coprod X \longrightarrow X$$

as a cofibration followed by a trivial fibration we can produce a cylinder object for every X. Also note that every two cylinder obejcts for X are weakly equivalence. Now suppose that we are given two maps $f, g : X \longrightarrow Y$ and a cylinder object C for X. A **homotopy** from f to g with respect to C is a map

$$H : C \longrightarrow Y$$

such that the composed map

$$X \coprod X \longrightarrow C \longrightarrow Y$$

is equal to (f, g). We will say that f is **homotopic** to g if there exists a homotopy between them with respect to some cylinder object of X.

Lemma 12.4. *If X is cofibrant and Y is fibrant then the relation of homotopy between maps $X \longrightarrow Y$ is an equivalence relation. We will denote by $[X, Y]$ the set of homotopy classes of maps from X to Y.*

Definition 12.5. Let $\mathcal{M}$ be a model category. The **homotopy category** $\mathtt{Ho}(\mathcal{M})$ is the category whose objects are the objects of $\mathcal{M}$ and whose morphism sets are $[X^{\mathrm{cof}}, Y^{\mathrm{fib}}]$ where X^{cof} is a cofibrant replacement for X and Y^{fib} is a firant replacement for Y.

Remark 12.6. Given two maps $f, g : X \longrightarrow Y$ and two homotopies H_1, H_2 between them we may talk of **relative homotopies** from H_1 to H_2. One can then talk about homotopies between homotopies between homotopies and so on. In fact, one can construct a whole homotopy type describing the "space of maps" from X to Y. When X is cofibrant and Y is fibrant this space of maps is well-behaved, and in particular its set of connected components can be identified with $[X, Y]$. Note that up to weak equivalence one can always replace X with a cofibrant substitute $X' \xrightarrow{\simeq} X$ and Y with a fibrant substitute $Y \xrightarrow{\simeq} Y'$. The resulting mapping space is usually referred to as the **derived mapping space** from X to Y.

Remark 12.7. The technology of model categories enables one to do much more than just constructing derived mapping space. It enables one to construct suitable homotopy theoretic analogous to the notions of limits and colimits as well, yielding the notions of **homotopy limits and colimits**. We will not develop this idea further in this course but we recommend the curious student to read more about this.

Now that we know what model categories are, let us explain how to map one model category to another.

Definition 12.8. Let $\mathcal{M}$,$\mathcal{N}$ be model categories. A map from $\mathcal{M}$ to $\mathcal{N}$ is an adjunction

$$\mathcal{M} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{N} \quad (L \dashv R)$$

such that L preserves cofibrations and trivial cofibrations (or equivalently (see Remark 12.2), R preserves fibrations and trivial fibraitons). Such an adjunction is called a **Quillen adjunction**.

Now let

$$\mathcal{M} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{N} \quad (L \dashv R)$$

be a Quillen adjunction. Then for any The association $X \mapsto L(X^{\mathrm{cof}})$ determines a well defined functor

$$\mathbb{L}L : \mathtt{Ho}(M) \longrightarrow \mathtt{Ho}(N)$$

which is called the **left derived functor** of L. Similarly, the association $Y \mapsto R(Y^{\mathrm{fib}})$ determines a well defined functor

$$\mathbb{R}R : \mathtt{Ho}(N) \longrightarrow \mathtt{Ho}(M)$$

which is called the **right derived functor** of R. The pair $\mathbb{L}L, \mathbb{R}R$ always forms an adjunction between $\mathtt{Ho}(M)$ and $\mathtt{Ho}(N)$.

Definition 12.9. We will say that a Quillen adjunction

$$\mathcal{M} \underset{R}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{N}$$

is a **Quillen equivalence** if the derived adjunction

$$\mathtt{Ho}(\mathcal{M}) \underset{\mathbb{R}R}{\overset{\mathbb{L}L}{\underset{\perp}{\rightleftarrows}}} \mathtt{Ho}(\mathcal{N})$$

is an equivalence of categories.

In this course we will show that the adjunction

$$\mathtt{Set}_\Delta \underset{\mathrm{Sing}}{\overset{|\bullet|}{\underset{\perp}{\rightleftarrows}}} \mathtt{CG}$$

is a Quillen equivalence.

12.2 The Kan model structure

In this section we will set up a model structure on the category $\mathtt{Set}_\Delta$ is simplicial sets, known as the Kan-Quillen model structure. In order to do this we will need the following notion:

Definition 12.10. Let $\mathcal{C}$ be a category which admits small colimits, $X \in \mathcal{C}$ and object and I a class of maps in $\mathcal{C}$. Let $\overline{I}$ be smallest saturated class containing I. We will say that X is **small** with respect to I if whenever we have an infinite sequence

$$Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} Y_2 \longrightarrow ...$$

such that $f_i \in \overline{I}$ for every i then the natural map

$$\mathrm{colim}_i \mathrm{Hom}_{\mathcal{C}}(X, Y_i) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{colim}_i Y)$$

is a bijection of sets.

Remark 12.11. If I, J are two collections of maps in $\mathcal{C}$ such that $\overline{J} \subseteq \overline{I}$ then any object which is small with respect to I is small with respect to J.

Example 8. Let $\mathcal{C} = \mathtt{Set}_\Delta$ be the category of simplicial set and let $X \in \mathcal{C}$ be an object with finitely many non-degenerate simplices. Then X is small with respect to any class of maps I (i.e., it is small with respect to $I = \mathcal{C}$).

Example 9. Let $\mathcal{C} = \mathtt{CG}$ be the category of compactly generated Hausdorf spaces and let $X \in \mathcal{C}$ be a compact space. Let $I = \{S^{n-1} \hookrightarrow \mathcal{D}^n\}_{n \geq 0}$. Then X is small with respect to I.

The relevance of small objects to the construction of model structures is explained in the following lemma:

Lemma 12.12. *Let $\mathcal{C}$ be a category which admits small colimits and let I be a* ***set*** *of maps. Assume that the domain of every morphism in $\mathcal{I}$ is small with respect to I. Let $\overline{I}$ denote the saturated class generated from I and by let $I^{\perp}$ be the set of all maps which have the right lifting property with respect to I. Then any map $f : X \longrightarrow Y$ in $\mathcal{C}$ can be factored as $X \xrightarrow{i} Z \xrightarrow{p} Y$ such that $i \in \overline{I}$ and $p \in I^{\perp}$.*

Proof. Let $f : X \longrightarrow Y$ be a map in $\mathcal{C}$. Given a morphism $i : A \longrightarrow B$ let us denote by $S_{i,f}$ the set of commutative squares of the form

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

We then define

$$\mathcal{F}_i = \coprod_{S_{i,f}} A$$

$$\mathcal{G}_i = \coprod_{S_{i,f}} B$$

and we defined H_1 to be the pushout in the natural diagram

$$\begin{array}{ccc} \coprod_{i \in I} \mathcal{F}_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod_{i \in I} \mathcal{G}_i & \longrightarrow & H_1 \end{array}$$

We observe that we have a natural factorization

$$\begin{array}{ccccc} & & H_1 & & \\ & \nearrow & & \searrow & \\ X & & \xrightarrow{\quad f \quad} & & Y \end{array}$$

and that the map $X \longrightarrow H_1$ is in $\overline{I}$. We can now apply the above procedure to the map $H_1 \longrightarrow Y$. Iterating this repeatedly we obtain a sequence

$$\begin{array}{ccccccc} X & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ Y & = & Y & = & Y & = & \cdots \end{array}$$

Let $H = \operatorname{colim}_n H_n$ be the colimit of the above sequence. Then we have a natural factorization

$$\begin{array}{ccccc} & & H & & \\ & \nearrow & & \searrow & \\ X & & \xrightarrow{\quad f \quad} & & Y \end{array}$$

and the map $X \longrightarrow H$ is in $\overline{I}$. We claim that the map $H \longrightarrow Y$ is in $I^{\perp}$. To see this, consider a square of the form

$$\begin{array}{ccc} A & \xrightarrow{g} & H \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ B & \longrightarrow & Y \end{array} \tag{7}$$

with $i \in I$. Since A is small with respect to I we get that there exists an n such that g factors as the composition

$$A \xrightarrow{g'} H_n \longrightarrow H$$

and so we obtain a commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{g'} & H_n \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ B & \longrightarrow & Y \end{array}$$

By the definition of H_{n+1} we obtain a diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & H_{n+1} \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow \\ B & \longrightarrow & Y \end{array}$$

yielding a lift for the diagram 7. □

Corollary 12.13. *Let $\mathcal{M}$ be a category with all small limits and colimits. Let $\mathcal{W} \subseteq \mathcal{M}$ be a subcategory containing all isomorphisms, closed under retracts and closed under 2-out-of-3. Let $I, J \in \mathcal{M}$ be two sets of maps whose domains are small with respect to I, such that*

(i) $\overline{J} = \overline{I} \cap \mathcal{W}$

(ii) $I^{\perp} = J^{\perp} \cap \mathcal{W}$

Then $(\mathcal{M}, \mathcal{W}, \mathcal{C}of, \mathcal{F}ib)$ is a model category, where $\mathcal{C}of = \overline{I}$ and $\mathcal{F}ib = J^{\perp}$.

Proof. First note that $\mathcal{C}of$ and $\mathcal{F}ib$ contain all isomorphisms and are closed under retracts. In light of (i) and (ii) above we see that axiom (4) follows directly from Lemma 12.12 and axiom (5) holds by definition. □

Corollary 12.14. *Let $\mathcal{M} = \mathtt{Set}_\Delta$ with $\mathcal{W}$ the collection of weak equivalences, $I = \{\partial\Delta^n \longrightarrow \Delta^n\}_{n\geq 0}$ and $J = \{\Lambda^n_i \longrightarrow \Delta^n\}_{n\geq 1, i\leq n}$. Then I and J satisfy the assumptions of Theorem 12.13. This proves the existence of the* ***Kan model structure****.*

13 Lecture 13

Summary 13.1. In this lecture we shall establish the equivalence of the "homotopy theories" of simplicial sets and topological spaces.

Our first part will be to replace the category of all topological spaces with a (full) subcategory of "nice" spaces. The key property we require from this subcategory is that the realization functor $|-| : \mathtt{Set}_\Delta \longrightarrow \mathtt{Top}$ will factor through it and will preserve finite limits.

Our second part will be to show "transfer" the model structure from simplicial sets to that of (nice) topological spaces. Once we do that, we will be able to show easily that the two model structures are actually equivalent.

13.1 Compactly generated spaces

In this part we shall focus on the point-set topology level. For convenience, we view a topology τ on a set X as a collection of **closed** subsets of X which is closed under finite unions and arbitrary intersections. Thus, when we say a topological space (or just a space) in this section we really mean an arbitrary topological space. Furthermore, **our arrows will be assumed by default to be continuous** and we will explicitly say "function" when there is no continuity assumption.

Let $\mathtt{CH}$ the category of compact Hausdorff spaces.

Definition 13.2. Let (X, τ) be a space. A subset $Y \subseteq X$ is k-closed if for every $K \in \mathtt{CH}$ and every continuous map $u : K \longrightarrow X$, $u^{-1}(Y)$ is closed in K. We write $k(\tau)$ for the collection of all k-closed sets. Clearly, $\tau \subseteq k(\tau)$ and $k(\tau)$ is a topology on X.We write $kX = (X, k(\tau))$ and say that X is **compactly generated** if $X = kX$. We denote by $\mathtt{Top}_c$ the full subcategory of $\mathtt{Top}$ spanned by the compactly generated spaces.

Observe 13.3. Let $(X, \tau) \in \mathtt{Top}$ and $K \in \mathtt{CH}$. Then a function $u : K \longrightarrow X$ is continuous iff $u : K \longrightarrow kX$ is continuous. We see that $k^2X = kX$.

Proposition 13.4. *Let $X \in \mathtt{Top}_c$ and $Y \in \mathtt{Top}$. Then a function $f : X \longrightarrow Y$ is continuous $\Leftrightarrow$ $f : X \longrightarrow kY$ is continuous.*

Proof. $\underline{\Leftarrow}$: If $Z \subseteq Y$ is closed in Y then Z is closed in kY so $f^{-1}(Z)$ is closed in X. $\underline{\Rightarrow}$: Let $Z \subseteq Y$ be k-closed, $K \in \mathtt{CH}$ and $u : K \longrightarrow X$ a map. Then fu

is continuous so $(fu)^{-1}(Z) = u^{-1}(f^{-1}(Z))$ is closed in K. Since $X \in \mathtt{Top}_c$ it follows that $f^{-1}Z$ is k-closed and thus closed in X. □

The previous proposition can actually be phrased in a nicer way – For any $X \in \mathtt{Top}_c$, $Y \in \mathtt{Top}$ and a continuous map $f : X \longrightarrow Y$, there is a unique lift as follows:

$$\begin{array}{ccc} & & kY \\ & \overset{\exists! f}{\nearrow} & \downarrow{\scriptstyle id} \\ X & \xrightarrow{f} & Y \end{array}$$

In other words:

Corollary 13.5. *There is an adjunction*

$$\mathtt{Top}_c \underset{k}{\overset{i}{\rightleftarrows}} Top \quad (i \dashv k)$$

It is perhaps good to show first that $\mathtt{Top}_c$ contains enough "nice" spaces. There are many classes of spaces that can be considered, but what we actually need in this course just the CW-complexes. Since the latter are always Hausdorff, it is good to first see:

Proposition 13.6. *Let X be a Hausdorff space. Then a subset $Y \subseteq X$ is k-closed $\Leftrightarrow$ $Y \cap K$ is closed for any compact Hausdorff $K \subseteq X$.*

Proof. $\underline{\Rightarrow:}$ Trivial. $\underline{\Leftarrow:}$ If $K \in \mathtt{CH}$ and $u : K \longrightarrow X$ a map then $u(K)$ is compact Hausdorff so $u(K) \cap Y$ is closed and thus $u^{-1}(Y) = u^{-1}(u(K) \cap Y)$ is closed. □

We assume that the students are familiar with the following statement. Proof can be found for example in Hatcher.

Proposition 13.7. *Let X be a CW-complex. Then $Y \subseteq X$ is closed iff $Y \cap \overline{e_n^\alpha}$ is closed for any cell e_n^α.*

Corollary 13.8. *Any CW-complex X is compactly generated.*

Proof. If $Y \subseteq X$ is k-closed, $Y \cap e_n^\alpha$ is closed since e_n^α is compact and thus Y is closed in X. □

Next we want to have quotients in $\mathtt{Top}_c$.

Proposition 13.9. *If $X \in \mathtt{Top}_c$ and E is an equivalence relation on X then $X/E \in \mathtt{Top}_c$.*

Proof. $q : X \longrightarrow k(X/E)$ is continuous so if $Z \subseteq X/E$ is k-closed then $q^{-1}(Z)$ is k-closed in X so it is closed in X since $X \in \mathtt{Top}_c$. By the definition of the quotient topology, this means that Z is closed in X/E. □

Having done that, we take care of the coproducts:

Proposition 13.10. *If* $\{X_i\}_{i\in I} \subseteq \mathtt{Top}_c$ *then* $X := \coprod_i X_i \in \mathtt{Top}_c$.

Proof. If $Z \subseteq X$ is k-closed write $Z = \coprod_i Z_i$ where $Z_i := Z \cap X_i$. It is then enough to show that $Z_i \subseteq X_i$ is closed but since $X_i \in \mathtt{Top}_c$ it is enough to show that Z_i is k-closed. Let $K \in \mathtt{CH}$ and $u : K \longrightarrow X_i$ a map. then, for $j_i : X_i \longrightarrow X$, $(u \circ j_i)^{-1}(Z) = u^{-1}(Z_i)$ so $u^{-1}(Z_i)$ is closed since $Z \subseteq X \in \mathtt{Top}_c$. □

Having done that, we now examine products:

Definition 13.11. Given $X, Y \in \mathtt{Top}$, write $X \times_o Y$ for their product in $\mathtt{Top}$. We denote by $X \times Y := k(X \times_0 Y)$ and similarly for a family of spaces $\{X_i\}$ we write $\prod_i X_i := k(\prod_{0,i} X_i)$.

Proposition 13.12. *Let* $\{X_i\}_{i\in I} \subseteq \mathtt{Top}_c$. *Then the projection maps* $\pi_i \prod_i X_i \longrightarrow X_i$ *are continuous and, moreover, for any* $Y \in \mathtt{Top}_c$, $f : Y \longrightarrow \prod_i X_i$ *is continuous iff* $\pi_i \circ f$ *is so for each* i.

Proof. The projections π_i are continuous since the topology of $\prod_i X_i$ is bigger than that of $\prod_{0,i} X_i$. The second part follows from the fact that the map $f : Y \longrightarrow \prod_{0,i} X_i$ is continuous iff $\pi_i \circ f$ is so for each i and from Proposition 13.4. □

In other words, the previous proposition shows that $\prod_i X_i$ is the product in the category $\mathtt{Top}_c$.

Theorem 13.13. *The category* $\mathtt{Top}_c$ *is bicomplete.*

1. *Limits are computed by applying the functor* k *to the limit in* $\mathtt{Top}$.
2. *Colimits are computed as in* $\mathtt{Top}$.

Proof. Let I be a small category and $\underset{\sim}{X} : I \longrightarrow \mathtt{Top}_c$ a diagram.

1. For any $W \in \mathtt{Top}_c$,

   $$Top_c(W, k(\lim_i \iota X_i)) \cong \mathtt{Top}(\iota W, \lim_i \iota X_i) \cong \lim_i Top(\iota W, \iota X_i) \cong \lim_i \mathtt{Top}_c(W, X_i)$$

   so $k(\lim_i \iota X_i)$ is the limit in $\mathtt{Top}_c$.

2. $\mathrm{colim}_i\, \iota X_i$ is a quotient of a coproduct of compactly generated spaces and so compactly generated itself. Moreover, for any $Y \in \mathtt{Top}_c$,

   $\mathtt{Top}_c(\mathrm{colim}_i\, \iota X_i, Y) \cong \mathtt{Top}(\iota X, \iota Y) \cong \mathtt{Top}(\mathrm{colim}\, \iota X_i, \iota Y) \cong \lim_i \mathtt{Top}(\iota X_i, \iota Y)$
   $\cong \lim_i \mathtt{Top}_c(X_i, Y)$

   so $\mathrm{colim}_i\, \iota X_i$ is the colimit in $\mathtt{Top}_c$.

□

Observe 13.14.

1. Since $S^n \in \mathtt{Top}_c$, for any $X \in \mathtt{Top}$, $\pi_n(X) = \mathtt{Top}_*(S^n, X)/\simeq \cong \mathtt{Top}_{c,*}(S^n, kX)/\simeq$ $= \pi_n(kX)$.

2. A map $p : X \longrightarrow Y$ is a Serre fibration iff $kX \longrightarrow kY$ is a Serre fibration (this is again because $|\Lambda^n_k|, |\Delta^n| \in \mathtt{Top}_c$ and k is a right adjoint).

Thus, every Serre fibration in $\mathtt{Top}_c$ induces a LES on homotopy groups where the fiber is now taken in Top_c – the fiber in $\mathtt{Top}_c$ is always weakly equivalent to that in $\mathtt{Top}$.

13.2 A model structure on $\mathtt{Top}_c$

Recall from last time:

Definition 13.15. We define the following classes of morphisms in $\mathtt{Top}_c$

1. $\mathcal{W}$ the class of weak homotopy equivalence of topological spaces.
2. $I = \{|\partial\Delta^n| \longrightarrow |\Delta^n|\}_n$
3. $J = \{|\Lambda^n_k| \longrightarrow |\Delta^n|\}_{n,k}$.
4. $\mathcal{C}of = \overline{I}$ the saturated class generated from I
5. Anodyne=$\overline{J}$ the saturated class generated from J.
6. $\mathcal{F}ib = J^\perp$ the maps with the right lifting property wrt J
7. trivial fibrations=$I^\perp$ the maps with the right lifting property wrt I.

As was shown in the last talk, the small object argument applies to this case since the domains of I and J are small relative to I. This means that we have a factorization of any map into an anodyne followed by a fibration and into a cofibration followed by a trivial fibration.

In order to show that the abovementioned classes of maps define a model structure on $\mathtt{Top}_c$ it is thus left to show:

- $\overline{J} = \overline{I} \cap \mathcal{W}$ and
- $I^\perp = J^\perp \cap \mathcal{W}$.

We proceed as follows

Proposition 13.16. *$\overline{J} \subseteq \overline{I} \cap \mathcal{W}$.*

Proof. First, $J \subseteq \overline{I}$ since we saw that any horn inclusion can be obtained as a transfinite composition of pushouts along $\partial\Delta^n \longrightarrow \Delta^n$ and $|-|$ preserves these as a left adjoint. By definition, $\overline{J} \subseteq \overline{I}$. Second, $i : |\Lambda^n_k| \longrightarrow |\Delta^n|$ is a deformation retract and thus any pushout $\bar{i}$ of it is such, so $\bar{i} \in \mathcal{W}$. Now, a transfinite composition of maps in $PO(J) \cap \mathcal{W} \subseteq \overline{I} \cap \mathcal{W}$ is in $\mathcal{W}$ since $\pi_n(\operatorname{colim}_i X_i) \cong \operatorname{colim} \pi_n(X_i)$ for any sequence $X_1 \longrightarrow X_2 \longrightarrow ...$ of maps in $\overline{I}$ since S^n is small relative to I. Finally, $\mathcal{W}$ is closed under retracts and so $\overline{J} \subseteq \overline{I} \cap \mathcal{W}$. $\square$

Proposition 13.17. *$I^{\perp} = J^{\perp} \cap \mathcal{W}$ i.e. a Serre fibration is trivial iff it is a weak equivalence.*

Proof. If $p : X \longrightarrow Y$ is a trivial fibration then for any $y \in Y$, $F_y \longrightarrow *$ is a trivial fibration and hence $\pi_n F_y = 0$ directly from the lifting property. Thus p is a weak equivalence by the LES for homotopy groups. Conversely, if p is a weak equivalence and

$$\begin{array}{ccc} |\partial\Delta^n| & \longrightarrow & X \\ \downarrow & & \downarrow \\ |\Delta^n| & \xrightarrow{f} & Y \end{array}$$

then f is homotopic to the constant map $f(v_0)$ where $v_0 \in |\Delta^n|$ and we can proceed exactly as in the case of simplicial sets. □

The next proposition shows that the inclusion $\overline{I} \cap \mathcal{W} \subseteq \overline{J}$ is a formal consequence of what we did so far and has nothing to do with topological spaces.

Proposition 13.18. $\overline{I} \cap \mathcal{W} \subseteq \overline{J}$

Proof. Let $f : X \longrightarrow Y$ be a cofibration ($= \in \overline{I}$) and a weak equivalence. Factor f as $f = (X \xrightarrow{j} Z \xrightarrow{p} Y)$ where j is anodyne and p is a fibration. As we saw, j is a weak equivalence so by the 2-out-of-three property, p is a weak equivalence which, as we saw, implies that it is a trivial fibration. This means that p has a right lifting property with respect to maps in I and hence wrt maps in $\overline{I}$. But $f \in \overline{I}$ so we can solve the lifting problem

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle \exists g} & \downarrow{\scriptstyle p} \\ Y & = & Y \end{array}$$

and we can thus present f as a retract of j via

$$\begin{array}{ccccc} X & = & X & = & X \\ {\scriptstyle f}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle f}\downarrow \\ Y & \xrightarrow[g]{} & Z & \xrightarrow[p]{} & Y \end{array}$$

and we conclude that f is anodyne.

□

Corollary 13.19. *The tuple $(\mathtt{Top}_c, \mathcal{W}, \mathcal{F}ib, \mathcal{C}of)$ is a model category.*

13.3 Equivalence of homotopy theories

Observe 13.20. The adjunction

$$\mathtt{Top}_c \underset{S}{\overset{|-|}{\rightleftarrows}} \mathtt{Set}_\Delta$$

now becomes a Quillen adjunction since clearly the realization of a cofibration or a trivial cofibration is so. In order to prove that this is in fact a Quillen equivalence, we need to show that the derived unit and counit are weak equivalences. But since every object in $\mathtt{Set}_\Delta$ is cofibrant and every object in $\mathtt{Top}_c$ is fibrant this amounts to proving that the actual unit and counit are weak equivalences.

Proposition 13.21. *For every* $X \in \mathtt{Set}_\Delta$*,* $X \longrightarrow S|X|$ *is a weak equivalence.*

Proof. Factorize $X \longrightarrow *$ to an anodyne followed by a fibration $X \longrightarrow \overline{X} \longrightarrow *$ so that $\overline{X}$ is Kan. Then

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \overline{X} \\ \downarrow & & \downarrow \sim \\ S|X| & \longrightarrow & S|\overline{X}| \end{array}$$

so that it is enough to show that $S|X| \longrightarrow S|\overline{X}|$ is a weak equivalence. But $|X| \longrightarrow |\overline{X}|$ is an anodyne between CW-complexes hence a homotopy equivalence so $S|X| \longrightarrow S|\overline{X}|$ is a homotopy equivalence hence a weak equivalence. □

Proposition 13.22. *For any* $X \in \mathtt{Top}_c$*, the counit map* $\epsilon : |SX| \longrightarrow X$ *is a weak equivalence.*

Proof. By comparing the fibration sequences $\Omega X \longrightarrow PX \longrightarrow X$ and $|S\Omega X| \longrightarrow |SPX| \longrightarrow |SX|$ we see that it is enough to show that for any $X \in \mathtt{Top}_c$, ϵ is an iso on π_0. But it is clearly onto vertices (since $(SX)_0 = X$) and onto homotopies between vertices and so an isomorphism indeed. □

14 Lecture 15

Today we are going to see some applications of the theory of homotopy (co)limits. The main point to promote, is that when we take homotopy limits or colimits, we get cleaner and more general statements. We start with

14.1 Postnikov towers

Let X be a topological space.

Definition 14.1. A Postnikov tower for X is a diagram of spaces

$$\begin{array}{ccc} & & \vdots \\ & & P_2X \\ & \overset{f_2}{\nearrow} & \downarrow q_2 \\ & \overset{f_1}{\nearrow} & P_1X \\ & & \downarrow q_1 \\ X & \xrightarrow{f_0} & P_0X \end{array}$$

s.t.

1. For all i, and for all $k \leq i$, $\pi_k(f_i) : \pi_k(X) \longrightarrow \pi_k(P_iX)$ is an isomorphism.
2. $\pi_k(P_iX) = 0$ for all $k > i$.

Observe 14.2.

1. In such a situation, it follows that $\pi_k(P_{i+1}X) \longrightarrow \pi_k(P_iX)$ is an isomorphism for all $k \neq i+1$ and trivial for $k = i+1$.
2. Thus, the LES for a homotopy fibration sequence implies that the homotopy fiber $F_h(q_{i+1})$ is an Eilenberg-Maclane space of type $K(\pi_{i+1}X, i+1)$.

Construction 14.3. We construct P_iX as a cw-complex relative to X. We set $(P_iX)_{i+1} = X$ and we attach an $(i+2)$-cell to $(P_iX)_{i+1}$ for every map $S^{i+1} \longrightarrow (P_iX)_{i+1}$. At the next stage, we attach an $(i+3)$ cell to $(P_iX)_{i+2}$ for every map $S^{i+2} \longrightarrow (P_iX)_{i+2}$. The maps $f_i : X \longrightarrow P_iX$ are the skeleton inclusion, and the maps $q_{i+1} : P_{i+1}X \longrightarrow P_iX$ are constructed as X-cellular maps as follows. $(P_{i+1}X)_{i+1} = X = (P_iX)_{i+1}$ so we define q_{i+1} to be the identity on the $(i+1)$-skeleton. On the $(i+2)$-skeleton, the map $(P_{i+1}X)_{i+2} = X \longrightarrow (P_iX)_{i+2}$ is the inclusion of skeleta.

On the $(i+3)$-skeleton, let $\Phi : D^{i+3} \longrightarrow (P_{i+1}X)_{i+3}$ be a cell of $P_{i+1}X$ with an attaching map $\varphi : S^{i+2} \longrightarrow (P_{i+1}X)_{i+2} = X$. The composite $q_{i+1} \circ \varphi$ is an attaching map in $(P_iX)_{i+2}$ and so underlines a map $\mathring{D}^{i+3} \cong \Phi(\mathring{D}^{i+3}) \longrightarrow (P_iX)_{i+3}$ and so we can extend q_{i+1} to the $i+3$ skeleton and repeat the process. It follows from our construction that $q_{i+1} \circ f_{i+1} = f_i$ since this is true on each skeleton and the two properties of Definition 14.1 are immediate to check.

Proposition 14.4. *In* $\left(\mathtt{Top}^{\mathbb{N}^{op}}\right)_{inj}$*, a tower* $\{X_n\}_n$ *is fibrant if* $X_n \longrightarrow X_{n-1}$ *is a fibration for any* n*.*

Proof. Let $\{A_n\} \longrightarrow \{B_n\}$ be a trivial cofibration if $\left(\mathtt{Top}^{\mathbb{N}^{op}}\right)_{inj}$ and consider a lifting problem

$$\begin{array}{ccc} \{A_n\} & \longrightarrow & \{X_n\} \\ \downarrow & \nearrow & \\ \{B_n\} & & \end{array}$$

. The assumption that $\{A_n\} \longrightarrow \{B_n\}$ is a trivial cofibration in the injective model structure on towers of spaces means simply that for each n, $A_n \longrightarrow B_n$ is a trivial cofibration in $\mathtt{Top}$. Now, since X_0 is fibrant, we get a lift of the form

$$\begin{array}{ccc} A_0 & \longrightarrow & X_0 \\ \downarrow & \nearrow & \\ B_0 & & \end{array}$$

. Then, we can solve the lifting problem

$$\begin{array}{ccccc} A_1 & \longrightarrow & & \longrightarrow & X_1 \\ \downarrow & & \nearrow & & \downarrow \\ B_1 & \longrightarrow & B_0 & \longrightarrow & X_0 \end{array}$$

(since $A_1 \longrightarrow B_1$ is a trivial cofibration and $X_1 \longrightarrow X_0$ is a fibration) and a trivial induction gives a lift for any n and hence a lift of towers. □

Remark 14.5. The converse implication of the previous Proposition is also true, i.e., any tower of fibrations is a fibrant object in the injective model structure on $\left(\mathtt{Top}^{\mathbb{N}^{op}}\right)_{inj}$.

Construction 14.6. Let X be a space and $\{P_nX\}$ its Postnikov tower of Construction 14.3. Set $\overline{P_0X} \coloneqq P_0X$ factor the map $q_1 : P_1X \longrightarrow \overline{P_0X}$ to

$$P_1X \xrightarrow{\overline{c_1}} \overline{P_1X} \xrightarrow{\overline{q_1}} \overline{P_0X}$$

where the first map is trivial cofibration and the second is a fibration. Then, factor the composite $P_2X \longrightarrow P_1X \longrightarrow \overline{P_1X}$ into

$$P_2X \xrightarrow{\overline{c_2}} \overline{P_2X} \xrightarrow{\overline{q_1}} \overline{P_1X}$$

where again the first map is a trivial cofibration and the second is a fibration. The various $\overline{P_n}$'s assemble into a tower together with a map of towers $\overline{c} : \{P_nX\} \longrightarrow \{\overline{P_nX}\}$ which is a weak equivalence in $\left(\mathtt{Top}^{\mathbb{N}^{op}}\right)_{inj}$.

Corollary 14.7. $\operatorname{holim} P_nX \sim \lim \overline{P_n}$.

Proof. Since each $\overline{q_{n+1}} : \overline{P_{n+1}X} \longrightarrow \overline{P_nX}$ is a fibration, and $\overline{c} : \{P_nX\} \longrightarrow \{\overline{P_nX}\}$ is a weak equivalence, $\{\overline{P_nX}\}$ is a fibrant replacement of $\{P_nX\}$ in $\left(\mathtt{Top}^{\mathbb{N}^{op}}\right)_{inj}$. By definition, $\operatorname{holim} P_nX \sim \lim \overline{P_n}$. □

We now turn into the comparison between a space and the homotopy limit of its Postnikov tower.

Proposition 14.8. *For any sequence of fibrations of pointed spaces* $\cdots \longrightarrow X_2 \xrightarrow{q_2} X_1 \xrightarrow{q_1} X_0$*, the map* $\lambda_i : \pi_i(\lim_n X_n) \longrightarrow \lim_n \pi_i(X_n)$ *is surjective. Moreover,* λ_i *is injective if* $\pi_{i+1}(q_n) : \pi_{i+1}(X_n) \longrightarrow \pi_{i+1}X_{n-1}$ *is surjective for sufficiently large* n*.*

The proof of Proposition 14.8 is not hard and will be omitted. Using it, we can immediately conclude:

Corollary 14.9. *For any $X \in \mathtt{Top}$, $X \sim \operatorname{holim} P_n X$.*

Proof. The map

$$\lambda_i : \pi_i(\lim_n P_n X) \longrightarrow \lim_n \pi_i(P_n X)$$

satisfy the strong hypothesis of Proposition 14.8 an isomorphism. Thus, the map

$$\pi_i(\lim f_n) : \pi_i X \longrightarrow \pi_i(\lim P_n X) \cong \lim \pi_i(P_n X)$$

is an isomorphism since the right hand stabilzes on $\pi_i X$ after a finite number of stages. □

We see that the notion of homotopy limit enables us to formulate clean statements. Note that Corollary 14.9 applies to any "abstract" Postnikov tower of X as in Definition 14.1 so that we do not have to commit to a model.

14.2 Homotopy colimits

We now turn into investigating the notion of a homotopy colimit. Recall that a homotopy colimit of a diagram $D : I \longrightarrow \mathtt{Set}_\Delta$ (or $D : I \longrightarrow \mathtt{Top}$) was defined by the colimit of a cofibrant replacement of D in $\left(\mathtt{Top}^I\right)_{proj}$.

Theorem 14.10 (Bousfield-Kan)**.** *Let $D : I \longrightarrow \mathtt{Set}_\Delta$ be a diagram of simplicial sets. Consider the diagram $S : I^{op} \longrightarrow \mathtt{Set}_\Delta$ given by $S(i) = N(I^{op}/i)$. Then* $\operatorname{hocolim}_I D \simeq \coprod S(i) \times D(i)/\sim$ *where for any $\alpha : i \longrightarrow j$, $(c_j, \alpha_* d_i) \sim (\alpha^* c_j, d_i)$. If $D : I \longrightarrow Top$ then replacing S with $|S|$ in the above formula give the homotopy colimit in* $\mathtt{Top}$.

Example 10. Let $I = (0 \longrightarrow 2 \longleftarrow 1)$. Then $I^{op} = (0 \longleftarrow 2 \longrightarrow 1)$ and we see that $N(I^{op}/0) = N(I^{op}/1) = \Delta^0$ and that $N(I^{op}/2) = \Lambda^2_2$. Now, if $D = (A \xrightarrow{f} X \xleftarrow{g} Y)$ is an I-diagram in $\mathtt{Top}$, then the Bousfield-Kan formula implies that $\operatorname{hocolim}_I D \simeq X \coprod A \times \Lambda^2_2 \coprod Y/\sim$ where $(a, 0) \sim f(a)$ and $(a, 1) \sim g(a)$. Thus, we obtain a model for the homotopy pushout $X \coprod^h_A Y$ in Topological spaces which is usually referred to as the **double mapping cylinder**.

In general one could obtain a model for the homotopy pushout by replacing the maps f and g by **any** cofibration and then take the corresponding pushout.

Example 11. Let $D = (* \longrightarrow X \longleftarrow *)$. Then the Bousfield-Kan formula gives $\operatorname{hocolim} D \simeq \Sigma X$.

It will be convenient to have a notion of a square which is commutative and models a homotopy pushout.

Definition 14.11. A (commutative) square

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & P \end{array}$$

is called **homotopy coCartesian**, if the canonical map $X \coprod_A^h Y \longrightarrow P$ is a weak equivalence.

It now turns out, that Van-Kampen Theorem works in much higher generality.

Theorem 14.12. *For any homotopy coCartesian square of pointed connected topological spaces*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & P \end{array}$$

the square

$$\begin{array}{ccc} \pi_1 A & \longrightarrow & \pi_1 X \\ \downarrow & & \downarrow \\ \pi_1 Y & \longrightarrow & \pi_1 P \end{array}$$

is a coCartesian (i.e. a pushout) square in $\mathtt{Gp}$.

Another application of homotopy pushouts is obtained via the following

Proposition 14.13. *The category* $\mathtt{Ch}_{\geq 0}(\mathbb{Z})$ *admits an "injective" model structure for which the weak equivalences are quasi-isomorphisms and the cofibrations are the monomorphisms. The functor* $Ch : \mathtt{Set}_\Delta \longrightarrow \mathtt{Ch}_{\geq 0}(\mathbb{Z})$ *defined as the composite of the free simplicial abelian group functor followed by the functor to chain complexes which take the alternating sum of the face maps is a* ***left Quillen functor****.*

Corollary 14.14. *The functor* Ch *preserves homotopy coCartesian squares.*

Proof. Any left Quillen functor preserves homotopy colimits. □

Thus, if

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & P \end{array}$$

is a homotopy coCartesian square of simplicial sets, the square

$$\begin{array}{ccc} Ch(A) & \longrightarrow & Ch(X) \\ \downarrow & & \downarrow \\ Ch(Y) & \longrightarrow & Ch(P) \end{array}$$

is homotopy coCartesian in $\mathtt{Ch}_{\geq 0}(\mathbb{Z})$ and this means that

$$\begin{array}{ccc} Ch(A) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ Ch(X) \oplus Ch(Y) & \longrightarrow & Ch(P) \end{array}$$

is a homotopy coCartesian square, i.e.,

$$Ch(A) \longrightarrow Ch(X) \oplus Ch(Y) \longrightarrow Ch(P)$$

is a homotopy cofibration sequence in $\mathtt{Ch}_{\geq 0}(\mathbb{Z})$ and so induces a LES on homology. We thus get a generalized version of the Mayer-Vietoris sequence for homotopy pushouts:

Corollary 14.15 (Mayer-Vietoris)**.** *For any homotopy coCartesian square of simplicial sets*

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & P \end{array}$$

there is a LES

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \oplus H_n(Y) \longrightarrow H_n(P) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

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