Pro-categories in homotopy theory

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Many categories which arise in nature, such as the categories of sets, groups, rings and others, are **large**: they contain a proper class of objects, even when the objects are considered up to isomorphism. However, in each of the examples above, the large category is in some sense determined by a much smaller subcategory. There are two ways in which a small subcategory can determine an entire large category. The first is via the process of taking **colimits**, and the second via the process of taking **limits**. Let us exemplify these two possibilities in two typical examples. Consider on one hand the category Set of sets, and on the other hand the category $\mathrm{Gr}^{\mathrm{profin}}$ of pro-finite groups. The small subcategories are going to be the full subcategory $\mathrm{Set}^{\mathrm{fin}} \subseteq \mathrm{Set}$ of finite sets and the full subcategory $\mathrm{Gr}^{\mathrm{fin}} \subseteq \mathrm{Gr}^{\mathrm{profin}}$ of finite groups.

Given an arbitrary set $A \in \mathrm{Set}$, let $P_{\mathrm{fin}}(A)$ denote the partially ordered set of finite subsets of A, where we say that $A' \leq A''$ iff $A' \subseteq A''$. Similarly, given an arbitrary pro-finite group G, let $Q_{\mathrm{fin}}(G)$ denote the partially ordered set of finite quotients $G \twoheadrightarrow G'$, where we say that $G' \leq G''$ if the quotient map $G \twoheadrightarrow G''$ factors through G'. We may naturally treat any partially ordered set as a **category** (whose morphism set $\mathrm{Hom}(x,y)$ is a singleton if $x \leq y$ and empty otherwise). Then for every set A and every pro-finite group G we have natural maps

$$\operatorname*{colim}_{A' \in P_{\mathrm{fin}}(A)} A' \longrightarrow A$$

$$G \longrightarrow \lim_{G' \in Q_{\mathrm{fin}}(G)} G'$$

and one may verify that these are in fact (quite familiar) **isomorphisms**. The former identifies any set with the colimit (or "union") of its finite subsets, while the second one identifies a pro-finite group with the inverse limit of its finite quotients.

In both cases, the story doesn't end here. To say that a small subcategory generates all objects under limits or colimits is not enough to satisfactorily say that the entire large category is **determined** by the small data. For this one would need to know that morphisms between any two "large" objects can be recovered from their presentation as limits or colimits of small objects. For this one may observe that if $A \longrightarrow B$ is a map of sets, then for every finite subset $A' \subseteq A$, the restricted map $A' \longrightarrow B$ factors through a finite subset of B. Similarly, if $G \longrightarrow H$ is a map of pro-finite groups and $H \twoheadrightarrow H'$ is a finite quotient of H then the composite $G \longrightarrow H \longrightarrow H'$ factors through a finite quotient of G. This quickly leads to the formulas

$$\mathrm{Hom}_{\mathrm{Set}}(A,B) \cong \lim_{A' \in P_{\mathrm{fin}}(A)} \operatorname*{colim}_{B' \in P_{\mathrm{fin}}(B)} \mathrm{Hom}_{\mathrm{Set}^{\mathrm{fin}}}(A',B')$$

and

$$\mathrm{Hom}_{\mathrm{Gr}^{\mathrm{profin}}}(G,H) \cong \lim_{H' \in Q_{\mathrm{fin}}(H)} \operatorname*{colim}_{G' \in Q_{\mathrm{fin}}(G)} \mathrm{Hom}_{\mathrm{Gr}^{\mathrm{fin}}}(G',H').$$

One may hence justifiably say that the category Set is determined by $\mathrm{Set}^{\mathrm{fin}}$ and that $\mathrm{Gr}^{\mathrm{profin}}$ is determined by $\mathrm{Gr}^{\mathrm{fin}}$. To make this idea formal let us give some definitions. Since our main motivation comes from cases which look like $\mathrm{Gr}^{\mathrm{profin}}$ let us formulate all the definitions in this direction.

Definition 1. Let $\mathcal{I}$ be a small category. We will say that $\mathcal{I}$ is **cofiltered** if the following two conditions hold:

1. For every $x, y \in \mathcal{I}$ there exists an object $z \in \mathcal{I}$ equipped with maps $z \longrightarrow x$ and $z \longrightarrow y$.
2. For every two maps $f, g : x \longrightarrow y$ in $\mathcal{I}$ there exists an object $z \in \mathcal{I}$ and a map $h : z \longrightarrow x$ such that $f \circ h = g \circ h$.

Dually, we say that $\mathcal{I}$ is filtered if $\mathcal{I}^{\mathrm{op}}$ is cofiltered.

Definition 2. Let $\mathcal{C}$ be a category. A **pro-object** in $\mathcal{C}$ is a diagram $\{X_\alpha\}_{\alpha \in \mathcal{I}}$ in $\mathcal{C}$ indexed by a small cofiltered category $\mathcal{I}$. We define the morphism set between two pro-objects by the formula

$$\mathrm{Hom}(\{X_\alpha\}_{\alpha\in\mathcal{I}}, \{Y_\beta\}_{\beta\in\mathcal{J}}) \overset{\mathrm{def}}{=} \lim_{\beta\in\mathcal{J}} \operatorname*{colim}_{\alpha\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X_\alpha, Y_\beta).$$

We let $\mathrm{Pro}(\mathcal{C})$ denote the category whose objects are the pro-objects in $\mathcal{C}$ and whose morphism sets are given as above (composition is defined in a strightforward way). The category $\mathrm{Pro}(\mathcal{C})$ is called the **pro-category** of $\mathcal{C}_0$.

Remark 3. Dualizing Definition 2 using filtered diagrams instead of cofiltered fiagrams we obtain the notions of an **ind-object** and of the **ind-category** $\mathrm{Ind}(\mathcal{C})$ of $\mathcal{C}$. In particular, an ind-object in $\mathcal{C}$ is the same thing as a pro-object in $\mathcal{C}^{\mathrm{op}}$ and there is a natural equivalence of categories $\mathrm{Ind}(\mathcal{C}) \simeq \mathrm{Pro}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Examples:

1. The category of pro-finite groups is naturally equivalent to the pro-category of finite groups.
2. The category of totally disconnected compact Hausdorff spaces is naturally equivalent to the pro-category of finite sets.
3. The category of compact Hausdorff topological groups is naturally equivalent to the pro-category of lie groups (this is non-trivial, and can be considered as a compact variant of Hilbert's fifth problem).

4. The category of sets is naturally equivalent to the ind-category of finite sets.

5. The category of groups is naturally equivalent to the ind-category of finitely presented groups. A similar statement holds for rings, modules and other types of algebraic categories.

For any category $\mathcal{C}$, the category $\mathrm{Pro}(\mathcal{C})$ admits cofiltered limits (i.e., limits indexed by cofiltered indexing categories), and these can be computed in some sense formally. We note that there is a canonical fully-faithful embedding $\iota : \mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ where $\iota(X)$ is the constant diagram with value X indexed by the trivial category. Moreover, the category $\mathrm{Pro}(\mathcal{C})$ is the universal category with cofiltered limits receiving a functor $\iota : \mathcal{C} \longrightarrow \mathrm{Pro}(\mathcal{C})$: if $\mathcal{D}$ is any other category with cofiltered limits then restriction along ι identifies the category of cofiltered limit preserving functors $\mathrm{Pro}(\mathcal{C}) \longrightarrow \mathcal{D}$ with the category of all functors $\mathcal{C} \longrightarrow \mathcal{D}$. In that sense one may consider $\mathrm{Pro}(\mathcal{C})$ as the category obtained by **freely adding** cofiltered limits to $\mathcal{C}$. Similarly, the category $\mathrm{Ind}(\mathcal{C})$ is the category obtained by freely adding filtered colimits to $\mathcal{C}$.

The higher categorical avatar of pro-categories was developed in the literature in two parallel paths. These two paths correspond to the two general approaches driving modern homotopy theory. The classical approach can be traced back to Quillen seminal work [Qu67], where he defined the notion of a **model category**. A model category is an ordinary category $\mathcal{M}$, equipped with a suitable additional structure, which allows one to perform homotopy theoretical constructions in $\mathcal{M}$. Similar notions which are based on categories with extra structure include fibration/cofibration categories and relative categories. The second approach, which was developed in recent years in the ground-breaking works of Lurie building on previous work of Joyal, Rezk and others, establishes the notion of an **∞-category**, which should be the correct homotopy theoretical analogue of the notion of a category. Unlike a model category, an ∞-category is not an ordinary category with extra structure, and its definition is more subtle. To any model category (or fibration/cofibration category, relative category), one can associate a corresponding ∞-category which it **models**.

Returning to the notion of pro-categories, in the realm of **∞-categories**, one can define pro-categories by adapting their universal property to the ∞-categorical setting. This was done in [Lu09] for $\mathcal{C}$ a small ∞-category and in [Lu11] for $\mathcal{C}$ an accessible ∞-category with finite limits. On the other hand, when $\mathcal{C}$ is a model category, one may attempt to construct a model structure on $\mathrm{Pro}(\mathcal{C})$ which is naturally inherited from that of $\mathcal{C}$. This was indeed established in [EH76] when $\mathcal{C}$ satisfies certain conditions ("Condition N") and later in [Is04] when $\mathcal{C}$ is a proper model category. This was generalized to the case when $\mathcal{C}$ is only a weak fibration category (under suitable hypothesis) in [BS15a]. Recall that

Definition 4. A **weak fibration category** is a category $\mathcal{C}$ equipped with two subcategories $\mathcal{F}ib, \mathcal{W} \subseteq \mathcal{C}$ containing all the isomorphisms, such that the following conditions are satisfied:

1. $\mathcal{C}$ has all finite limits.

2. $\mathcal{W}$ has the 2-out-of-3 property.

3. The subcategories $\mathcal{F}ib$ and $\mathcal{F}ib \cap \mathcal{W}$ are stable under base change.

4. Every morphism $f : X \longrightarrow Y$ can be factored as $X \xrightarrow{f'} Z \xrightarrow{f''} Y$ where $f' \in \mathcal{W}$ and $f'' \in \mathcal{F}ib$.

To unify the above mentioned approaches it is useful to consider a general definition specifying what it means for a model structure on $\mathrm{Pro}(\mathcal{C})$ to be **induced** from a weak fibration structure on $\mathcal{C}$. When $\mathcal{C}$ is **small** the definition simple: we say that a model structure on $\mathrm{Pro}(\mathcal{C})$ is induced from a weak fibration structure $(\mathcal{F}ib, \mathcal{W})$ on $\mathcal{C}$ if it is **fibrantly generated** by the images of $\mathcal{F}ib$ and $\mathcal{W} \cap \mathcal{F}ib$ in $\mathrm{Pro}(\mathcal{C})$. This means that the cofibrations are exactly those map which have the left lifting property with respect to $\mathcal{F}ib \cap \mathcal{W}$ and the trivial cofibrations are exactly those map which have the left lifting property with respect to $\mathcal{F}ib$. If $\mathcal{C}$ is not a small then a slightly more elaborate definition is required, which we will not spell out explicitly.

Our main theorem is the following:

Theorem 5. *Assume that the induced model structure on* $\mathrm{Pro}(\mathcal{C})$ *exists. Then the natural map*

$$\mathcal{F} : \mathrm{Pro}(\mathcal{C})_\infty \longrightarrow \mathrm{Pro}(\mathcal{C}_\infty)$$

is an equivalence of ∞*-categories.*

Examples:

1. If $\mathcal{M}$ is a proper model category then the induced model structure on $\mathrm{Pro}(\mathcal{M})$ exists by the work of [Is04]. For example, if $\mathcal{M}$ is the category of simplicial sets with the Kan-Quillen model structure then $\mathcal{M}$ is a model for the ∞-category of spaces and by Theorem 5 we get that $\mathrm{Pro}(\mathcal{M})$ with the induced model structure is a model for the ∞-category of pro-spaces. We will return to this example when we discuss pro-finite homotopy theory.

2. Let $\mathcal{T}$ be a Grothendieck site. Then the category $\mathcal{C}$ of simplicial sheaves on $\mathcal{T}$ with local weak equivalences and local fibrations is a weak fibration category, which is a model for the ∞-category of sheaves of spaces on $\mathcal{T}$. By the results of [BS15a] the induced model structure on $\mathrm{Pro}(\mathcal{C})$ exists in this case, and so by Theorem 5 we get that $\mathrm{Pro}(\mathcal{C})$ is a model for the ∞-category of pro-sheaves of spaces on $\mathcal{T}$. We will return to this example when we discuss the étale homotopy type.

In order to prove Theorem 5 one needs, in particular, to be able to compare the mapping spaces on both sides. If $\mathcal{C}$ is an ∞-category then the mapping spaces in $\mathrm{Pro}(\mathcal{C})$ can be described by a similar formula as in the ordinary case, by replacing limits and colimits by the corresponding homotopy limits and colimits of mapping spaces:

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{C})}(X, Y) = \mathrm{holim}_{j \in \mathcal{J}} \, \mathrm{hocolim}_{i \in \mathcal{I}} \, \mathrm{Map}_{\mathcal{C}}(X_i, Y_j). \qquad (1)$$

The first step towards proving Theorem 5 is to obtain a similar formula for the derived mapping space associated to an induced model structure on $\mathrm{Pro}(\mathcal{C})$ when $\mathcal{C}$ is a weak fibration category. For this, we need to a good way to describe mapping spaces in weak fibration categories.

Definition 6. Let $\mathcal{C}$ be a weak fibration category. Let $X, Y \in \mathcal{C}$ two objects. We denote by $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \subseteq \mathcal{C}_{/X\times Y}$ consisting of those objects $X \xleftarrow{p} Z \xrightarrow{f} Y$ such that p is a trivial fibration.

There is a natural map from the nerve $\mathrm{N}\,\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$ to the simplicial set $\mathrm{Map}_{L^H(\mathcal{C},\mathcal{W})}(X,Y)$ where $L^H(\mathcal{C},\mathcal{W})$ denotes the hammock localization of $\mathcal{C}$ with respect to $\mathcal{W}$. We hence obtain a natural map

$$\mathrm{N}\,\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Map}^h_{\mathcal{C}}(X,Y). \tag{2}$$

Proposition 7 (Cisinski)**.** *Let $\mathcal{C}$ be a weak fibration category. Then for every $X, Y \in \mathcal{C}$ with Y fibrant the map 2 is a weak equivalence.*

The first step towards the proof of Theorem 5 is to prove that when $\mathcal{C}$ is a weak fibration category, formula 2 holds for the derived mapping spaces in the induced model structure on $\mathrm{Pro}(\mathcal{C})$. We first observe that the limit part of 2 is equivalent to the statement that the maps $Y \longrightarrow Y_j$ exhibit Y as the limit, in $\mathrm{Pro}(\mathcal{C})$, of the diagram $j \mapsto Y_j$. The analogous statement for homotopy limits in the setting of the induced model structure is essentially a consequence of the following proposition

Proposition 8. *Let $(\mathcal{C},\mathcal{W},\mathcal{F}ib)$ be a weak fibration category. If the induced model structure on $\mathrm{Pro}(\mathcal{C})$ exists then every levelwise weak equivalence is a weak equivalence in $\mathrm{Pro}(\mathcal{C})$.*

Corollary 9. *Let $\mathcal{C}$ be a weak fibration category and let $Y = \{Y_j\}_{j\in\mathcal{J}} \in \mathrm{Pro}(\mathcal{C})$ be a pro-object. Let $\overline{\mathcal{F}} : \mathcal{J}^{\triangleleft} \longrightarrow \mathrm{Pro}(\mathcal{C})$ be the limit diagram extending $\mathcal{F}(j) = Y_j$ so that $\overline{\mathcal{F}}(*) = Y$ (where $* \in \mathcal{J}^{\triangleleft}$ is the cone point). Then $\overline{\mathcal{F}}$ is also a homotopy limit diagram.*

The main step towards Theorem 5 then becomes the following:

Proposition 10. *Let $X = \{X_i\}_{i\in\mathcal{I}}$ be a pro-object and $Y \in \mathcal{C} \subseteq \mathrm{Pro}(\mathcal{C})$ a simple object. Then the compatible family of maps $X \longrightarrow X_i$ induces a weak equivalence*

$$\mathrm{hocolim}_{i\in\mathcal{I}}\,\mathrm{Map}^h_{\mathcal{C}}(X_i,Y) \longrightarrow \mathrm{Map}^h_{\mathrm{Pro}(\mathcal{C})}(X,Y) \tag{3}$$

Sketch of Proof. The idea of the proof is to use the mapping space description of Proposition 7 to relate the mapping space $\mathrm{Map}^h_{\mathrm{Pro}(\mathcal{C})}(X,Y)$, which depends on trivial fibrations in $\mathrm{Pro}(\mathcal{C})$ of the form $X' \overset{\simeq}{\twoheadrightarrow} X$, to the various mapping spaces $\mathrm{Map}^h_{\mathrm{Pro}(\mathcal{C})}(X_i,Y)$, which, in turn, depend on trivial fibrations in $\mathcal{C}$ of the form $Y \overset{\simeq}{\twoheadrightarrow} X_i$. These two types of data could be related if one could restrict to using trivial fibrations which are simultaneously **levelwise trivial fibrations**.

One hence needs to know that levelwise trivial fibrations are in some sense sufficiently common. When $\mathcal{C}$ is small this can be achieved by the dual version of the small object argument. In general one needs to require certain conditions which we have builtin to our definition of when a model structure on $\mathrm{Pro}(\mathcal{C})$ is induced from $\mathcal{C}$. These conditions can be shown to hold in all the examples we consider. $\square$

We give two applications of our general comparison theorem. Our first application involves the theory of **shapes of topoi**. In [AM69], Artin and Mazur defined the **étale homotopy type** of an algebraic variety. This is a pro-object in the homotopy category of spaces, which depends only on the étale site of X. Their construction is based on the construction of the **shape** of a topological space X, which is a similar type of pro-object constructed from the site of open subsets of X. More generally, Artin and Mazur's construction applies to any **locally connected** site.

In [BS15a] the first author and Schlank used their model structure to define what they call the **topological realization** of a Grothendieck topos. Their construction works for any Grothendieck topos and **refines** the previous constructions form a pro-object in the homotopy category of spaces to a pro-object in the category of simplicial sets. On the ∞-categorical side, Lurie constructed in [Lu09] an ∞-categorical analogue of shape theory and defined the shape assigned to any ∞-topos as a pro-object in the ∞-category $\mathcal{S}_\infty$ of spaces. A similar type of construction also appears in works of Toën and Vezzosi. One then faces the same type of pressing question: Is the topological realization constructed in [BS15a] using model categories equivalent to the one defined in [Lu09] using the language of ∞-categories? We give a positive answer to this question:

Theorem 11. *For any Grothendieck site $\mathcal{C}$ there is a weak equivalence*

$$|\mathcal{C}| \simeq \mathrm{Sh}(\widehat{\mathcal{S}\mathrm{hv}}_\infty(\mathcal{C}))$$

of pro-spaces, where $|\mathcal{C}|$ is the topological realization constructed in [BS15a] and $\mathrm{Sh}(\widehat{\mathcal{S}\mathrm{hv}}_\infty(\mathcal{C})) \in \mathrm{Pro}(\mathcal{S}_\infty)$ is the shape of the hyper-completed ∞-topos $\widehat{\mathcal{S}\mathrm{hv}}_\infty(\mathcal{C})$ constructed in [Lu09].

Combining the above theorem with [BS15a, Theorem 1.11] we obtain:

Corollary 12. *Let X be a locally Noetherian scheme, and let $X_{\acute{e}t}$ be its étale site. Then the image of $\mathrm{Sh}(\widehat{\mathcal{S}\mathrm{hv}}_\infty(X_{\acute{e}t}))$ in $\mathrm{Pro}(\mathrm{Ho}(\mathcal{S}_\infty))$ coincides with the étale homotopy type of X.*

Our second application is to the study of **profinite homotopy theory**. Let $\mathcal{S}$ be the category of simplicial sets, equipped with the Kan-Quillen model structure. The existence of the induced model structure on $\mathrm{Pro}(\mathcal{S})$ (in the sense above) follows from the work of [EH76] (as well as [Is04] and [BS15a] in fact). In [Is05], Isaksen showed that for any set K of fibrant object of $\mathcal{S}$, one can form the "maximal" left Bousfield localization $L_K \mathrm{Pro}(\mathcal{S})$ of $\mathrm{Pro}(\mathcal{S})$ for which all the objects in K are local. When choosing a suitable candidate $K = K^\pi$, the model category $L_{K^\pi}\operatorname{Pro}(\mathcal{S})$ can be used as a theoretical setup for **profinite homotopy theory**.

On the other hand, one may define what profinite homotopy theory should be from an ∞-categorical point of view. Recall that a space X is called **π-finite** if it has finitely many connected components, and finitely many non-trivial homotopy groups which are all finite. The collection of π-finite spaces can be organized into an ∞-category $\mathcal{S}^\pi_\infty$, and the associated pro-category $\operatorname{Pro}(\mathcal{S}^\pi_\infty)$ can equally be considered as the natural realm of profinite homotopy theory. One is then yet again faced with the salient question: is $L_{K^\pi}\operatorname{Pro}(\mathcal{S})$ a model for the ∞-category $\operatorname{Pro}(\mathcal{S}^\pi_\infty)$? We give a positive answer to this question:

Theorem 13. *The underlying ∞-category $L_{K^\pi}\operatorname{Pro}(\mathcal{S})$ is naturally equivalent to the ∞-category $\operatorname{Pro}(\mathcal{S}^\pi_\infty)$ of profinite spaces.*

Isaksen's approach is not the only model categorical approach to profinite homotopy theory. In [Qu11] Quick constructs a model structure on the category $\widehat{\mathcal{S}}$ of **simplicial profinite sets** and uses it as a setting to perform profinite homotopy theory. His construction is based on a previous construction of Morel, which endowed the category of simplicial profinite sets with a model structure aimed at studying p-profinite homotopy theory. We then have the following result:

Theorem 14. *There is a Quillen equivalences*

$$\Psi_{K^\pi} : L_{K^\pi}\operatorname{Pro}(\mathcal{S}) \rightleftarrows \widehat{\mathcal{S}}_{\mathrm{Quick}} : \Phi_{K^\pi}$$

In particular, Quick's model category is indeed a model for the ∞-category of profinite spaces.

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