Integral points on log K3 surfaces

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Let $\mathcal{X}$ be $\mathbb{Z}$-scheme, i.e., a smooth separated scheme of finite type over $\mathbb{Z}$. In this talk most schemes of interest will be **affine**, and so given by a collection of polynomial equations with integral coefficients. A prominent goal of Diophantine geometry is to understand the set $\mathcal{X}(\mathbb{Z})$ of **integral points** of $\mathcal{X}$. If $\mathcal{X}$ is affine this is simply the set of solutions in integral numbers to the equations defining $\mathcal{X}$. More specifically, one may consider the following two (very general) questions:

1. Given a $\mathbb{Z}$-scheme $\mathcal{X}$, does it have an integral point?
2. If integral points exist, are they in any sense "abundant"?

The second question above is somewhat vague, and is a bit of as a catch-all terminology for various specific questions. For example, one might ask if there exist integral points which further satisfy various approximation conditions arbitrarily well. In another direction, one might ask if $\mathcal{X}(\mathbb{Z})$ is **Zariski dense**. Moving from the qualitative to the quantitative, one may use a suitable **height function** $H : \mathcal{X}(\mathbb{Z}) \longrightarrow \mathbb{R}_{\geq 0}$ (if we assume that $\mathcal{X} \subseteq \mathbb{A}^n$ is affine, we may simply use the function $(x_1, ..., x_n) \mapsto \max_i |x_i|$), and ask for the asymptotic growth of the counting function

$$N(\mathcal{X}, B) = \{P \in \mathcal{X}(\mathbb{Z}) | H(P) \leq B\}.$$

In this context, one often speaks of **polynomial growth** when $N(\mathcal{X}, B)$ is comparable to a polynomial in B, and of **polylogarithmic growth** when $N(\mathcal{X}, B)$ is comparable to a polynomial in $\log(B)$. It is often also desirable to consider the counting function $N(\mathcal{U}, B)$ where $\mathcal{U} \subseteq \mathcal{X}$ is a small enough open subset (thus removing, for example, the contribution of certain subvarieties, which may have a different behaviour than $\mathcal{X}$ as a whole).

When $\mathcal{X}$ is projective the set $\mathcal{X}(\mathbb{Z})$ of integral points coincides with the set of **rational points** of the $\mathbb{Q}$-variety $X = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Q}$. In this case a fundamental paradigm in Diophantine geometry asserts that the behaviour of rational points should be strongly controlled by the geometry of X. Focusing on simply connected varieties, three geometric classes have been singled out and intensively studied:

1. Rationally connected varieties. For this class it is expected that rational points are very well-behaved. For example, a conjecture of Colliot-Thélène ([**?**, p. 174]) predicts that the closure of the set of rational points inside the space of adelic points coincides with the **Brauer set** of X. When rational points exist, they are conjectured to be Zariski dense, and exhibit a polynomial growth (Manin, Batyrev).

2. Simply connected general type varieties. For this class rational points are expected to be rather scarce. For example, a conjecture of Lang predicts that rational points will never be Zariski dense. Furthermore, one does not expect to obtain good sufficient conditions for the existence of rational points.

3. Calabi-Yau varieties. This is an intermediary class, whose arithmetic is considered very subtle and largely unknown. In dimension 2 they are also known as **K3 surfaces**, in which case a bit more is known. Growth of rational points is believed to be subpolynomial (Manin, Batyrev) and even polylogarithmic in suitably generic cases (Swinnerton-Dyer, Van-Luijk).

When X is not projective, the situation is more subtle. To this end it is often convenient to consider a smooth compactification $X \subseteq \overline{X}$ such that the complement is a simple normal crossing divisor. Having the pair $(\overline{X}, D)$ one can access many properties which are relevant to the behaviour of integral points on X:

1. Having the pair $(\overline{X}, D)$ we can study it using the framework of **log geometry**. This enables one to find suitable integral counterparts to the three geometric classes of varieties described above, in the form of log rationally connected pairs, log Calabi-Yau pairs and log general type pairs.

2. The behavior of real points on D has a strong influence on $\mathcal{X}(\mathbb{Z})$. For example, if $D(\mathbb{R}) = \emptyset$ then $X(\mathbb{R})$ is compact. If, in addition, X is affine then $\mathcal{X}(\mathbb{Z})$ is automatically finite. In this case we could see this directly from $X(\mathbb{R})$. However, there can be more subtle behaviours. For example, it could be that some components of D have real points and some not, and the same can be said for components of the intersections of components and so on. It turns out that even small differences in the configuration of real points on D can have an impact on the behaviour of integral points, and hence it is useful to have direct access to D.

3. Having the pair $(\overline{X}, D)$ enables us to study questions of **deformations** and **moduli spaces** for our objects. While the open variety X itself is not so amenable to deformations, the pair $(\overline{X}, D)$ can be handled with the usual set of tools. This enables us to get a basic idea of "what is out there".

In this talk we wish to focus out attention on **log K3 surfaces** and their integral points. To give the basic definitions let us consider a general base field

k with a fixed algebraic closure $\overline{k}$. Given a variety X over k we will denote by $X \otimes_k \overline{k}$ the base change of X from k to $\overline{k}$.

Definition 1. Let X be a smooth geometrically integral surface over k. A **simple compactification** of X is a smooth compactification $\iota : X \hookrightarrow \overline{X}$ (defined over k) such that $D = \overline{X} \setminus X$ is a simple normal crossing divisor.

Definition 2. Let X be a smooth geometrically integral surface over k. A **log K3 structure** on X is a simple compactification (X, D, ι) such that $[D] = -K_{\overline{X}}$ (where $K_{\overline{X}} \in \mathrm{Pic}(\overline{X})$ is the canonical class of $\overline{X}$). A **log K3 surface** is a smooth, geometrically integral, **simply connected** surface X equipped with a log K3 structure $(\overline{X}, D, \iota)$.

Let X be a log K3 surface. Since $X_{\overline{k}}$ is simply connected it follows that $k^*[X] = k^*$ and that $\mathrm{Pic}(X_{\overline{k}})$ is torsion free, hence isomorphic to $\mathbb{Z}^r$ for some r. We shall call the integer $r = \mathrm{rank}(\mathrm{Pic}(X_{\overline{k}}))$ the **geometric Picard rank** of X. Furthermore, if $(\overline{X}, D, \iota)$ is a log K3 structure on X then there exists a short exact sequence

$$0 \longrightarrow \mathbb{Z}|D| \longrightarrow \mathrm{Pic}(\overline{X}_{\overline{k}}) \longrightarrow \mathrm{Pic}(X_{\overline{k}}) \longrightarrow 0$$

where $\mathbb{Z}|D|$ is the free abelian group generated by the geometric components of D. In particular, the geometric Picard rank of X is given by $\mathrm{rank}(\mathrm{Pic}(\overline{X}_{\overline{k}})) - \sharp D$.

Claim 3. *Let X be a log K3 surface and $(\overline{X}, D, \iota)$ a log K3 structure on X. Then*

1. *Either $D = \emptyset$ and $X = \overline{X}$ is a (proper) K3 surface or $D \neq \emptyset$ and $\overline{X}_{\overline{k}}$ is a rational surface.*
2. *If D is non-empty then D is either a smooth projective genus 1 curve or a cycle of genus 0 curves.*

When $X = \overline{X}$ and $D = \emptyset$ we obtain the usual notion of a K3 surface. We may hence suppose that the behaviour of integral points on log K3 surfaces is close in spirit to the behaviour of rational points on smooth projective K3 surfaces. The following is one of the variants of a conjecture appearing in [VL]:

Conjecture 4 ([VL])**.** *Let X be a K3 surface over a number field k, and let H be a height function associated to an ample divisor. Suppose that X has geometric Picard number 1. Then there exists a Zariski open subset $U \subseteq X$ such that*

$$\mathrm{N}(U, B) = \#\{P \in U(k) | H(P) \leq B\} = O(\log(B)).$$

We note that Conjecture 4 can be backed by various counting heuristics, such as heuristics based on the circle method. If we apply these heuristics to more general K3 surfaces they will predict that $\mathrm{N}(U, B)$ grows as $\log^r(B)$, where r is the rank of the $\mathrm{Pic}(X_{\overline{k}})^{\Gamma_k}$. However, in this generality this heuristic is known to break down in special cases. For example, if X admits infinitely many maps

$\mathbb{P}^1 \longrightarrow X$ defined over k then the growth will be polynomial on every open set. (In this case it is conjectured that the exponent can then be made arbitrarily small by decreasing U.) Another special case is when X admits an elliptic fibration with a section (defined over k). In this case there may be infinitely many fibers whose rank is larger then $\mathrm{Pic}(X_{\overline{k}})^{\Gamma_k}$, yielding a polylogarithmic growth with too big an exponent. Conjecturally, these issues do not occur when $\mathrm{rank}\,\mathrm{Pic}(X_{\overline{k}}) = 1$ (for example, in this case X cannot admit an elliptic fibration, and has a finite automorphism group).

Returning to the case of log K3 surfaces we may cautiously expect that the growth of integral points on a small enough open subset will be, at least in suitable cases, logarithmic. We note that the minimal geometric Picard number a non-proper smooth log K3 surface may attain is 0, corresponding to the case where the components of D form a basis of $\mathrm{Pic}(\overline{X}_{\overline{k}})$ (although this is by no means the "generic" case). We then consider the following (possibly overly optimistic) conjecture:

Conjecture 5. *Let $\mathcal{X}$ be a smooth, separated scheme over $\mathbb{Z}$ such that $X = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a log K3 surface with $\mathrm{Pic}(X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) = 0$. Let H be a height function associated to an ample divisor. Then there exists a Zariski open subset $\mathcal{U} \subseteq \mathcal{X}$ and a constant b such that*

$$\#\{P \in \mathcal{U}(\mathbb{Z}) | H(P) \leq B\} \simeq O\left(\log(B)^b\right).$$

We may try to use circle method heuristics in order to estimate the constant b. Given a log K3 surface $(X, \overline{X}, D, \iota)$ over $\mathbb{Q}$, let us define s to be 0 if $D(\mathbb{R}) = \emptyset$, to be 2 if D contains a component defined over $\mathbb{R}$ which contains an intersection point defined over $\mathbb{R}$, and 1 otherwise. In the terminology of [CLY10], s is equal to 1 plus the dimension of the **analytic Clemens complex** of D over $\mathbb{R}$ (where we agree then the empty complex has dimension -1). The circle method heuristic will then predict that $\mathrm{N}(\mathcal{U}, B)$ should grow as $\log^b(B)$, where $b = \mathrm{rank}(\mathrm{Pic}(X_{\overline{k}})) + s$. As we will see below, unlike in the case of projective K3 surfaces, the value of b predicted by this heuristic seems to be **higher** then the true asymptotics in a few particular examples. The reason for this discrepancy is currently completely mysterious.

Let us now review a few examples of log K3 surfaces.

Example 6. Let $\overline{X} \subseteq \mathbb{P}^3$ be a cubic surface and $D \subseteq \overline{X}$ a hyperplane section which is a simple normal crossing divisor. Then $\overline{X}$ is a del Pezzo surface of degree 3 and $[D] = K_{\overline{X}}$. Since we assumed D to have simple normal crossings there are three possibilities: either D is a smooth curve of genus 1 or a cycle of genus 0 curves whose length is either 2 or 3. In all cases one can show that X is simply connected and hence a log K3 surface. Since the geometric Picard number of $\overline{X}$ is 7 we get that the geometric Picard number of X is then either $6, 5$ or 4, accordingly. Such log K3 surfaces always admit an affine cubic equation in three variable. The much studied surfaces

$$x^3 + y^3 + z^3 = a$$

are examples of such log K3 surfaces with D a smooth genus 1 curve. An example with D a cycle of three genus 0 curves is given by the (modified) Markoff-Rosenberger equation

$$ax^2 + by^2 + cz^2 = Dxyz + 1$$

which is smooth as soon as $\frac{D^2}{abc} \neq 4$.

Example 7. Let $\overline{X} \longrightarrow \mathbb{P}^1$ be a conic bundle of the form

$$f(t,s)x^2 + g(t,s)y^2 = z^2$$

where $f(t,s), g(t,s)$ are separable homogeneous polynomials of degree 2 without common factors, and $(t:s)$ are homogeneous coordinates on $\mathbb{P}^1$. Then $\overline{X}$ is a del Pezzo surface of degree 4 and the bisection $D \subseteq \mathbb{P}^1$ given by $z = 0$ is a smooth curve of genus 1 whose class is the canonical class. One can then show that $X = \overline{X} \setminus D$ is simply connected, and hence a log K3 surface of geometric Picard rank 5. We may write X as a bundle of affine conics of the form

$$f(t,s)x^2 + g(t,s)y^2 = 1$$

where this equation should be interpreted as defining X inside the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$. As affine conics can be considered as analogues of elliptic curves, we may say that X is a **conic log K3 surface**, in analogy with the terminology of elliptic K3 surfaces in the projective case.

Remark 8. In Example 7 the compactifying variety $\overline{X}$ is a del Pezzo surface of degree 4 equipped with a conic bundle structure. In general del Pezzo surfaces of degree 4 need not admit such a structure. However, they always admit a presentation as complete intersections of two quadrics in $\mathbb{P}^4$. Cutting out a smooth hyperplane one obtains a log K3 surface which admits an affine presentation of the form

$$f(x,y,z,w) = 0$$
$$g(x,y,z,w) = 0$$

where f, g are (non-homogeneous) polynomials of degree 2. In their recent paper [JS16] Janel and Damaris compute the Brauer groups and Brauer Manin obstructions for many examples of this form.

Example 9. Let $\overline{X}$ be the blow-up of $\mathbb{P}^2$ at a point $P \in \mathbb{P}^2(k)$. Let $C \subseteq \mathbb{P}^2$ be a quadric passing through P and let $L \subseteq \mathbb{P}^2$ be a line which does not contain P and meets C at two distinct points $Q_0, Q_1 \in \mathbb{P}^2(k)$, both defined over k. Let $\widetilde{C}$ be the strict transform of C in $\overline{X}$. Then $D = \widetilde{C} \cup L$ is a simple normal crossing divisor, and it is straightforward to check that $[D] = -K_{\overline{X}}$. One can show that the smooth variety $X = \overline{X} \setminus D$ is simply connected, and so X is a log K3 surface. Since $[L]$ and $[\widetilde{C}]$ form a basis for $\mathrm{Pic}(\overline{X}_{\overline{k}})$ we get that X has Picard rank 0.

To construct explicit equations for X, let x, y, z be projective coordinates on $\mathbb{P}^2$ such that L is given by $z = 0$. Let $f(x,y,z)$ be a quadratic form vanishing on C and let $g(x,y,z)$ be a linear form such that the line $g=0$ passes through P and Q_1. Then X is isomorphic to the affine variety given by the equation

$$f(x,y,1)t = g(x,y,1).$$

By a suitable linear change of coordinates (over k) we may always rewrite our equation as

$$(xy-1)t = x-1.$$

We note that this particular surface, unique as it is over $\mathbb{Q}$, admits infinitely many pairwise non-isomorphic forms when working over the ring of integers $\mathbb{Z}$.

Example 10. Let $L = k(\sqrt{a})$ be a quadratic extension of k. Let $P_0, P_1 \in \mathbb{P}^2(L)$ be a $\mathrm{Gal}(L/k)$-conjugate pair of points and let $\overline{X}$ be the blow-up of $\mathbb{P}^2$ at P_0 and P_1. Let $L \subseteq \mathbb{P}^2$ be a line defined over k which does not meet $\{P_0, P_1\}$ and let $L_1, L_2 \subseteq \mathbb{P}^2$ be a $\mathrm{Gal}(L/k)$-conjugate pair of lines such that L_1 contains P_1 but not P_2 and L_2 contains P_2 but not P_1. Assume that the intersection point of L_1 and L_2 is not contained in L. Let $\widetilde{L}_1, \widetilde{L}_2$ be the strict transforms of L_1 and L_2 in $\overline{X}$. Then $D = L \cup \widetilde{L}_1 \cup \widetilde{L}_2$ is a simple normal crossing divisor, and it is straightforward to check that $[D] = -K_{\overline{X}}$. One can show that the smooth variety $X = \overline{X} \setminus D$ is simply connected and so X is a log K3 surface. Since $[L], [\widetilde{L}_1]$ and $[\widetilde{L}_2]$ form a basis for $\mathrm{Pic}(\overline{X}_{\overline{k}})$, we get that that X has Picard rank 0.

To construct explicit equations for X, let x, y, z be projective coordinates on $\mathbb{P}^2$ such that L is given by $z = 0$. Let $f_1(x,y,z)$ and $f_2(x,y,z)$ be linear forms defined over L which vanish on L_1 and L_2 respectively. Let $g(x,y,z)$ be a linear form defined over k such that the line $g = 0$ passes through P_1 and P_2. Then X is isomorphic to the affine variety given by the equation

$$f_1(x,y,1)f_2(x,y,1)t = g(x,y,1).$$

By a suitable linear change of variables (over k) we may always write our equation as

$$(x^2 - ay^2)t = y - 1.$$

As above, for each $a \in \mathbb{Z}$ this surface admits infinitely pairwise non-isomorphic forms over $\mathbb{Z}$.

Remark 11. It may seem surprising that in Examples 8 and 9 we could always bring the equation to a canonical form, while this does not seem to be the case for Examples 6 and 7. This is because the dimension of the space of first order deformations of the pair $(\overline{X}, D)$ which fix D is equal to $\mathrm{rank}(\mathrm{Pic}(X_{\overline{k}})) = \mathrm{rank}(\mathrm{Pic}(\overline{X}_{\overline{k}})) - \sharp D$ (see [Fr15]). Hence while examples 9 and 8 are rigid, Examples 6 and 7 have a positive dimensional moduli space.

Our current understanding of the behaviour of integral points in these cases is rather preliminary. Let us review some of what is known about the examples above. Concerning Example 6, it is not known if there exists an a such that the set of integral points on $x^3 + y^3 + z^3 = a$ is Zariski dense, and it is not known if there exists an a such that this set is not Zariski dense. The circle method heuristic (see [HB92]) predicts that integral points should grow as $\log(B)$. A result of Colliot-Thélène and Wittenberg ([CTW12]) gives sufficient conditions for the vanishing of the integral Brauer-Manin obstruction. For example when $a = 33$ the integral Brauer-Manin obstruction vanishes, but it is not known if integral points exist or not.

For the modified Markoff-Rosenberger equation, if one assumes that $a, b, c|D$ then the variety
$$ax^2 + by^2 + cz^2 = Dxyz + 1$$
is acted upon (over $\mathbb{Z}$) by $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$. A Theorem of [YS01] states that there are only six choices of (a, b, c, D) with $a, b, c|D$ and $\frac{D^2}{abc} \neq 4$ for which the set of integral points is not empty. These cases were later analysed in [BU04], where it was shown that in each case there are finitely many orbits, and the asymptotic growth of integral points inside each orbit grows as $\log^2(B)$.

Let us now consider Example 7. The following theorem is one of the main results of [Ha15b]. Let S be a finite set of places of $\mathbb{Q}$. Recall that for a set S of places of $\mathbb{Q}$ we denote by $\mathbb{Z}_S$ the ring of **S-integers**, i.e., rational numbers whose denominator is divisible only by primes in S. Let $f(t, s), g(t, s) \in \mathbb{Z}_S[t, s]$ be separable homogeneous polynomials of degrees 2 which split completely over $\mathbb{Z}_S$, so that we can write $f(t, s) = a(c_1t + d_1s)(c_2t + d_2s)$ and $g(t, s) = b(c_1t + d_1s)(c_2t + d_2s)$ where each c_i, d_i is a coprime pair of S-integers. We will denote by $\Delta_{i,j} = c_jd_i - c_id_j$ the respective resultants, which we assume to be non-zero.

Theorem 12. *Let S be a finite set of places of $\mathbb{Q}$ containing $2, \infty$. Let $f(t, s), g(t, s) \in \mathbb{Z}_S[t, s]$ be homogeneous polynomials of degree 2 which split completely over $\mathbb{Z}_S$ and let $\mathcal{Y} \longrightarrow \mathbb{P}^1_S$ be the pencil of affine conics determined inside $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ by the equation*
$$f(t, s)x^2 + g(t, s)y^2 = 1.$$
Let a, b and $\Delta_{i,j} \neq 0$ be as above. Assume the following:

1. *The S-integers a, b are square-free and not divisible by 3 or 5.*
2. *The classes of the elements $\{-1, a, b\} \cup \{\Delta_{i,j}\}_{i>j}$ are linearly independent in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.*
3. *$\mathcal{Y}$ has an S-integral adelic point.*

Then $\mathcal{Y}$ has a Zariski dense set of S-integral points.

The proof of Theorem 11 uses an integral point adaptation of a method which was pioneered by Swinnerton-Dyer to study rational points on pencils of elliptic curves.

Let us now consider Example 8.

Theorem 13 ([Ha15a])**.** *Let $\mathcal{X}/\mathbb{Z}$ be as in Example 8. Then the set of integral points $\mathcal{X}(\mathbb{Z})$ is not Zariski dense.*

Proof. Applying a suitable coordinate change we have that X is isomorphic over $\mathbb{Q}$ (though not over $\mathbb{Z}$) to the affine surface in $\mathbb{A}^3$ given by the equation

$$(xy-1)t = x-1 \tag{1}$$

Hence the coordinates x, y, t determine three rational functions f_x, f_y, f_t on the scheme $\mathcal{X}$ which are regular when restricted to X. It follows that the poles of f_x, f_y and f_t are all "vertical", i.e., they are divisors of the form $M = 0$ for some $M \in \mathbb{Z}$. In particular, there exists divisible enough M such that for every $P \in \mathcal{X}(\mathbb{Z})$ the values $Mf_x(P), Mf_y(P)$ and $Mf_t(P)$ are all integers. Given an $M \in \mathbb{Z}$, we will say that a number $x \in \mathbb{Q}$ is M-integral if $Mx \in \mathbb{Z}$. We will say that a solution (x, y, t) of 1 is M-integral if each of x, y and t is M-integral. Now by the above there exists an M such that $(f_x(P), f_y(P), f_z(P))$ is an M-integral solution of 1 for every $P \in \mathcal{X}(\mathbb{Z})$. It will hence suffice to show that for every $M \in \mathbb{Z}$, the set of M-integral solutions of 1 is not Zariski dense (in the affine variety 1).

Since the function $f(y) = \frac{y-1}{y}$ on $\mathbb{R}$ converges to 1 as y goes to either $\pm\infty$ it follows that there exists a positive constant $C > 0$ such that $\left|\frac{y}{y+1}\right| < C$ for every M-integral number $-1 \neq y \in \mathbb{Q}$. We now claim that if (x, y, t) is a M-integral solution then either $|y| \leq 2M$ or $|x-1| \leq 2C$ or $t = 0$. Indeed, suppose that (x, y, t) is an M-integral triple such that $|y| > 2M$, $|x-1| > 2C$, and $t \neq 0$. Then $|x-1| > 2\left|\frac{y-1}{y}\right|$ and hence

$$|(xy-1)t| = |(x-1)y + (y-1)||t| > \frac{1}{2M}|(x-1)y| > |x-1|,$$

which means that (x, y, t) is not a solution to 1. It follows that all the Q-integral solutions of 1 lie on either the curve $t = 0$, or on the curve $x - 1 = i$ for $|i| \leq 2C$ an M-integral number, or the curve $y = j$ for $|j| \leq 2M$ an M-integral number. Since this collection of curves is finite it follows that M-integral solutions to 1 are not Zariski dense. □

Theorem 12 raises the following question:

Question 14. *Is it true that $\mathcal{X}(\mathbb{Z})$ is not Zariski dense for any integral model of any log K3 surface of Picard rank 0?*

We shall now show that the answer to question 13 is negative. Theorem I of [Na88] implies, in particular, that there exists a real quadratic number field $L = \mathbb{Q}(\sqrt{a})$, ramified at 2 and with trivial class group, such that the reduction map $\mathcal{O}_L^* \longrightarrow (\mathcal{O}_L/\mathfrak{p})^*$ is surjective for infinitely many prime ideals $\mathfrak{p} \subseteq \mathcal{O}_L$ of degree 1 over $\mathbb{Q}$. Given such an L, we may find a square-free positive integer $a \in \mathbb{Q}$ such that $L = \mathbb{Q}(\sqrt{a})$. Now let $\mathcal{X}$ be the ample log K3 surface over $\mathbb{Z}$ given by the equation

$$(x^2 - ay^2)t = y - 1. \tag{2}$$

We now claim the following:

Proposition 15. *The set $\mathcal{X}(\mathbb{Z})$ of integral points is Zariski dense.*

Proof. Let $\mathfrak{p} = (\pi) \subseteq \mathcal{O}_L$ be an odd unramified prime ideal of degree 1 such that $\mathcal{O}_L^* \longrightarrow (\mathcal{O}_L/\mathfrak{p})^*$ is surjective and let $p = N_{L/\mathbb{Q}}(\pi)$. Let $r \in \mathbb{F}_p^*$ be the image of the residue class of $\sqrt{a}$ under the (unique) ring isomorphism $\mathcal{O}_L/\mathfrak{p} \cong \mathbb{F}_p$. Let $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ be a generator. Then the image of the residue class of $\sqrt{a}$ under the isomorphism $\mathcal{O}_L/\sigma(\mathfrak{p}) \cong \mathbb{F}_p$ is necessarily $-r$.

By our assumption on L there exists a $u \in \mathcal{O}_L^*$ such that the residue class of $u\sigma(\pi)$ mod $\mathfrak{p}$ is equal to $2r$. Since L is ramified at 2 there exists $x_0, y_0 \in \mathbb{Z}$ such that $u\sigma(\pi) = x_0 + \sqrt{a}y_0$. Let $\overline{x}_0, \overline{y}_0 \in \mathbb{F}_p$ be the reductions of x_0 and y_0 mod p respectively. By construction we have $\overline{x}_0 - r\overline{y}_0 = 0$ and $\overline{x}_0 + r\overline{y}_0 = 2r$ and hence $\overline{x}_0 = r$ and $\overline{y}_0 = 1$. It follows that $y_0 - 1$ is divisible by p and since $x_0^2 - ay_0^2 = N_L(u\sigma(\pi)) = \pm p$ there exists a $t_0 \in \mathbb{Z}$ such that

$$(x_0^2 - ay_0^2)t_0 = y_0 - 1.$$

In particular, the triple (x_0, y_0, t_0) is a solution for 2. Let $\mathcal{C}_p \subseteq \mathcal{X}$ be the curve given by the additional equation $x^2 - ay^2 = p$. We have thus found an integral point on either $\mathcal{C}_p$ or $\mathcal{C}_{-p}$. By multiplying u with units whose image in $\mathcal{O}_L/\mathfrak{p}$ is trivial we may produce in this way infinitely many integral points on $\mathcal{C}_p$. Now any irreducible curve in X is either equal to $C_p = \mathcal{C}_p \otimes_{\mathbb{Z}} \mathbb{Q}$ for some p or intersects each C_p at finitely many points. Our construction above produces infinitely many p's for which $\mathcal{C}_p$ has infinitely many integral points, and hence $\mathcal{X}(\mathbb{Z})$ is Zariski dense, as desired. $\square$

Question 16. *Does conjecture 5 hold for the surface 2? If so, what is the appropriate value of b?*

Remark 17. The circle method heuristic discussed above suggests that both Example 8 and Example 9 should exhibit a growth of $\mathrm{N}(X, B) \sim \log^2(B)$. This is clearly wrong for 8, and for 9 it also does not seem to agree with some preliminary simulations (which tend to favour the estimate $\mathrm{N}(X, B) \sim \log(B)$). Other heuristic simulations suggest that the when all the components of D, all the intersection points of D are defined over $\mathbb{Q}$, and the Galois action on $\mathrm{Pic}(X)_{\overline{k}}$ is trivial, then the correct estimate should be $\mathrm{N}(X, B) \sim \log^r(B)$ where r is the geometric Picard rank.

A natural question that may arise at this point is to what extent do examples 8 and 9 represent the class of log K3 surfaces of Picard rank 0? To answer this question it will be useful to introduce the following terminology:

Definition 18. Let X be a log K3 surface. We shall say that X is **ample** if it admits a log K3 structure $(\overline{X}, D, \iota)$ such that $K_{\overline{X}}$ is ample (i.e., such that $\overline{X}$ is a del Pezzo surface).

Theorem 19 ([Ha15a]). *Any ample log K3 surface of Picard rank* 0 *over a field k of characteristic* 0 *admits a log K3 structure of the form* $(\overline{X}, D, \iota)$ *where* $\overline{X}$ *is a del Pezzo surface of degree* 5 *and D is a cycle of five* (-1)*-curve.*

Given a log K3 structure $(\overline{X}, D, \iota)$ of the form considered in Theorem 18 we may consider the Galois action on the dual graph of D, yielding an invariant $\alpha \in H^1(k, \mathbb{D}_5)$, where $\mathbb{D}_5$ is the dehidral group of order 10, considered here as the automorphism group of a cyclic graph of length 5. We then obtain the following classification theorem:

Theorem 20 ([Ha15a])**.** *Let k be a field of characteristic 0. The association $X \mapsto \alpha_X$ determines a bijection between the set of k-isomorphism classes of ample log K3 surfaces of Picard rank 0 and the Galois cohomology set $H^1(k, \mathbb{D}_5)$.*

Theorem 21 ([Ha15a])**.** *Let k be a field of characteristic 0. An ample log K3 surface X of Picard rank 0 over a field k is isomorphic to one of the log K3 surface of Example 8 if and only if its characteristic class $\alpha_X \in H^1(k, \mathbb{D}_5)$ is trivial. It is isomorphic to one of the log K3 surface of Example 9 if and only if α_X is in the image of the map $H^1(k, \mathbb{Z}/2) \longrightarrow H^1(k, \mathbb{D}_5)$, where $\mathbb{Z}/2 \subseteq \mathbb{D}_5$ is generated by a reflection.*

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