Pro-categories in homotopy theory

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Many categories which arise in nature, such as the categories of sets, groups, rings and others, are **large**: they contain a proper class of objects, even when the objects are considered up to isomorphism. However, in each of the examples above, the large category is in some sense determined by a much smaller subcategory. To see this, let us consider the example of the category Set of sets. Let $\mathrm{Set}^{\mathrm{fin}} \subseteq \mathrm{Set}$ denote the category of finite sets. Given an arbitrary set $A \in \mathrm{Set}$, let $P_{\mathrm{fin}}(A)$ denote the partially ordered set of finite subsets of A, where we say that $A' \leq A''$ iff $A' \subseteq A''$. We may naturally treat $P_{\mathrm{fin}}(A)$ as a **category** (whose morphism set $\mathrm{Hom}(A', A'')$ is a singleton if $A' \leq A''$ and empty otherwise). As we can functorially associate to each $P_{\mathrm{fin}}(A)$ a finite set equipped with an embedding in A we obtain a canonical functor $P_{\mathrm{fin}}(A) \longrightarrow \mathrm{Set}^{\mathrm{fin}}$ and a natural map

$$\operatorname*{colim}_{A' \in P_{\mathrm{fin}}(A)} A' \longrightarrow A.$$

It is not hard to verify that this map is in fact an **isomorphism**. This is a first sense in which we can say that the object $A \in \mathrm{Set}$ is built out of objects in $\mathrm{Set}^{\mathrm{fin}}$, namely, it is the colimit of its finite subsets. However, even more is true. Suppose that A and B are two sets. Then we can write A as $\mathrm{colim}_{A' \in P_{\mathrm{fin}}(A)} A'$ and B as $\mathrm{colim}_{B' \in P_{\mathrm{fin}}(B)} B'$. But can we also describe all the maps $f : A \longrightarrow B$ in terms of the diagrams $\{A'\}_{A' \in P_{\mathrm{fin}}(A)}$ and $\{B'\}_{B' \in P_{\mathrm{fin}}(B')}$? Well, since A is a colimit of the diagram $\{A'\}_{A' \in P_{\mathrm{fin}}(A)}$ we may immediately write

$$\mathrm{Hom}_{\mathrm{Set}}(A, B) = \lim_{A' \in P_{\mathrm{fin}}(A)} \mathrm{Hom}_{\mathrm{Set}}(A', B).$$

A-priori, this is as good as it gets. Indeed, the fact that B may be written as a suitable colimit cannot, in general, be exploited to describe maps into B. However, the poset $P_{\mathrm{fin}}(B')$ satisfies a special property: for any two elements $B', B'' \in P_{\mathrm{fin}}(B)$, there exists a third element $C \in P_{\mathrm{fin}}(B)$ such that $B', B'' \leq C$ (In other words, every two finite subsets of B are contained in a common finite subset of B). When this property holds we say that a poset is **filtered**. Now even though it's not in general easy to say something smart about maps into colimits, when mapping **finite sets** into **filtered colimits** the behaviour is completely determined by maps into the individual components. More precisely, if C is a finite set, P is a filtered poset and $\{X_\alpha\}_{\alpha \in P}$ is a diagram of sets indexed by P then the natural map

$$\operatorname*{colim}_{\alpha\in P}\mathrm{Hom}(C,X_\alpha)\xrightarrow{\cong}\mathrm{Hom}\left(C,\operatorname*{colim}_{\alpha\in P}X_\alpha\right)$$

is an **isomorphism**. It follows, in particular, that if A,B are two sets then the natural map

$$\lim_{A'\in P_{\mathrm{fin}}(A)}\operatorname*{colim}_{B'\in P_{\mathrm{fin}}(B)}\mathrm{Hom}_{\mathrm{Set}^{\mathrm{fin}}}(A',B')\xrightarrow{\cong}\mathrm{Hom}_{\mathrm{Set}}(A,B)$$

is an isomorphism. We hence see that not only can we describe every set as a colimit of a diagram of finite sets, we can also use these diagrams to determine all the maps between two sets. It is hence reasonable to say that the entire large category of sets is determined by the small subcategory of finite sets. We may then ask if we can do this procedure **formally**. Suppose, for the matter of argument, that we didn't know that Set is the category of sets and $\mathrm{Set}^{\mathrm{fin}}$ is the category of finite sets, but we only knew them as abstract categories, and that we wanted to describe a general construction which, when applied to the category $\mathrm{Set}^{\mathrm{fin}}$, yields the category Set. Mimicking the above we may say the following. Define $\mathrm{Ind}(\mathrm{Set}^{\mathrm{fin}})$ to be the category whose objects are diagrams $\{X_\alpha\}_{\alpha\in P}$ in $\mathrm{Set}^{\mathrm{fin}}$, indexed by a filtered poset P. We call such formal diagrams **Ind objects** in $\mathrm{Set}^{\mathrm{fin}}$. Define the morphism set between two Ind-objects by the formula

$$\mathrm{Hom}(\{X_\alpha\}_{\alpha\in P},\{Y_\beta\}_{\beta\in Q})\stackrel{\mathrm{def}}{=}\lim_{\alpha\in P}\operatorname*{colim}_{\beta\in Q}\mathrm{Hom}_{\mathrm{Set}^{\mathrm{fin}}}(X_\alpha,Y_\beta).$$

One may then see that there is an straightforward composition that can be defined. We refer to $\mathrm{Ind}(\mathrm{Set}^{\mathrm{fin}})$ as the **Ind-category** of $\mathrm{Set}^{\mathrm{fin}}$. There is a natural functor $\mathrm{Ind}(\mathrm{Set}^{\mathrm{fin}})\longrightarrow\mathrm{Set}$ which sends the abstract diagram $\{X_\alpha\}_{\alpha\in P}$ to its colimits $\mathrm{colim}_{\alpha\in P}X_\alpha$. The above considerations show that this functor is fully-faithful and essentially surjective, hence an **equivalence of categories**. We may hence formally say that the large category Set is determined by the small subcategory $\mathrm{Set}^{\mathrm{fin}}$ in the sense that Set is naturally equivalent to the category $\mathrm{Ind}(\mathrm{Set}^{\mathrm{fin}})$ of Ind-objects in $\mathrm{Set}^{\mathrm{fin}}$. It turns out that this kind of determination by small data holds in many natural examples. For example, the category of abelian groups is the Ind-category of the category of finitely generated abelian groups. The category of all groups is equivalent to the Ind-category of finitely presented groups. The category of rings is equivalent to the Ind-category of finitely presented rings. Such categories are called **ω-accessible categories**.

To achieve better flexibility it is convenient to allow for Ind-objects to be indexed by small categories which are not necessarily posets. This can be done by generalizing the notion of being filtered from posets to categories, and requires adding an extra condition concerning morphisms, which is automatically satisfied in the case the category comes from a poset. Fortunately, it can be shown that using general filtered categories as indexing diagrams to define Ind-objects yields an equivalent definition of $\mathrm{Ind}(\mathcal{C}_0)$.

One advantage of this added flexibility is that now one can quite easily show that for any category $\mathcal{C}_0$, the category $\mathrm{Ind}(\mathcal{C}_0)$ has filtered colimits (i.e., colimits indexed by filtered indexing categories), and that these can be computed in some sense formally. We note that there is a canonical fully-faithful embedding $\iota : \mathcal{C}_0 \hookrightarrow \mathrm{Ind}(\mathcal{C}_0)$ where $\iota(X)$ is the constant diagram with value X indexed by the trivial category. Moreover, the category $\mathrm{Ind}(\mathcal{C}_0)$ is the universal category with filtered colimits receiving a functor $\iota : \mathcal{C}_0 \longrightarrow \mathrm{Ind}(\mathcal{C}_0)$: if $\mathcal{D}$ is any other category with filtered colimits then restriction along ι identifies the category of filtered colimit preserving functors $\mathrm{Ind}(\mathcal{C}_0) \longrightarrow \mathcal{D}$ with the category of all functors $\mathcal{C}_0 \longrightarrow \mathcal{D}$. In that sense one may consider $\mathrm{Ind}(\mathcal{C}_0)$ as the category obtained by freely adding filtered colimits to $\mathcal{C}_0$.

For a category $\mathcal{C}$ to be the category of Ind-objects on some small subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$, one needs, in particular, that every object in $\mathcal{C}$ could be built from objects in $\mathcal{C}_0$ via the process of taking colimits. Some naturally arising categories do not exhibit such a favourable behaviour with respect to colimits, but instead it is the limits which play the crucial role. For example, in the category of pro-finite groups, every object is an inverse limit of a diagram of finite groups. Furthermore, the maps between two pro-finite groups are completely determined by the corresponding diagrams. To be able to work with such categories as well let us observe that all our constructions above can be easily dualized. Let us say that a small category $\mathcal{I}$ is **cofiltered** if the inverse category $\mathcal{I}^{\mathrm{op}}$ is filtered. A **pro-object** in a category $\mathcal{C}_0$ is a diagram $\{X_\alpha\}_{\alpha\in\mathcal{I}^{\mathrm{op}}}$ in $\mathcal{C}_0$ indexed by a small cofiltered category. We may formally define the morphism set between two pro-objects by the formula

$$\mathrm{Hom}(\{X_\alpha\}_{\alpha\in P}, \{Y_\beta\}_{\beta\in Q}) \stackrel{\mathrm{def}}{=} \lim_{\beta\in Q} \operatorname*{colim}_{\alpha\in P} \mathrm{Hom}_{\mathrm{Set}^{\mathrm{fin}}}(X_\alpha, Y_\beta).$$

The resulting category $\mathrm{Pro}(\mathcal{C}_0)$ is known as the **Pro-category** of $\mathcal{C}_0$. The category $\mathrm{Pro}(\mathcal{C}_0)$ enjoys the dual universal property of $\mathrm{Ind}(\mathcal{C}_0)$. It is the universal category with cofiltered limits receiving a map from $\mathcal{C}_0$, and can be considered as the category obtained from $\mathcal{C}_0$ by freely adding cofiltered limits. Examples of Pro-categories which arise naturally include:

1. The category of pro-finite groups is naturally equivalent to the pro-category of finite groups.

2. The category of totally disconnected compact Hausdorff spaces is naturally equivalent to the pro-category of finite sets.

3. The category of compact Hausdorff topological groups is naturally equivalent to the pro-category of lie groups (this is non-trivial, and can be considered as a compact variant of Hilbert's fifth problem).

Working in homotopy theory, one often requires constructions such as Pro-categories to work in a higher categorical setting. Here are two examples where such a need arises:

1. The notion of pro-finite completion of groups can be extended to pro-finite completion of **spaces**. The homotopy theory of such spaces is most naturally associated with a (higher categorical) pro-category of a suitable category of **π-finite spaces** (these are the truncated spaces whose homotopy groups are all finite). Similarly, given a prime p, the homotopy theory of **p-complete spaces** is associated to the pro-category of p-finite spaces.

2. Given a locally-connected Grothendieck site, such as the site of open subsets in a locally-connected topological spaces or the étale site of an algebraic variety, one may define its **shape** as a cofiltered system of homotopy types. On the étale side this construction was first considered by Artin and Mazur and was called the **étale homotopy type**. To study this construction with modern tools one needs to lift it to a suitable cofiltered system of spaces, which should live in a suitable higher categorical pro-category of spaces.

The higher categorical avatar of pro-categories was developed in the literature in two parallel paths. These two paths correspond to the two general approaches driving modern homotopy theory.

The older and more established approach can be traced back to Quillen seminal work [Qu67], where he defined the notion of a **model category**. A model category is an ordinary category $\mathcal{M}$, equipped with a suitable additional structure (more precisely, three distinguished classes of morphisms satisfying certain conditions), which allows one to perform homotopy theoretical constructions in $\mathcal{M}$. For example, the category of topological spaces can be endowed with a model structure, and the homotopy theoretical constructions which were classically performed on spaces (such as taking homotopy limits and colimits) could be neatly formulated in this setting. Further examples of model categories include simplicial sets, symmetric spectra and chain complexes (the latter yielding a unified framework for all classical constructions of homological algebra). Similar notions which are based on categories with extra structure include fibration/cofibration categories (Anderson, Brown, Cisinski and others, see [An78],[Ci10a],[Ci10b],[RB06] and [Sz14])) and relative categories (Dwyer, Kan, Barwick and others, see [DHKS14], [BK12]). These variants carry less structure and are easier to set up, but the associated homotopy theoretical constructions are often less accessible.

The second approach, which was developed in recent years in the groundbreaking works of Lurie ([Lu09],[Lu11]), built on previous work of Joyal ([Jo08]), Rezk and others, aims to establish a new notion, the notion of an **∞-category**, which should be the correct homotopy theoretical analogue of the notion of a category. Unlike a model category, an ∞-category is not an ordinary category with extra structure, and its definition is more subtle (in fact, several suitably equivalent definitions exist). To any model category (or fibration/cofibration category, relative category), one can associate a corresponding ∞-category which it **models**. The ∞-category is always where the relevant homotopy theoretical information lies, but the more rigid models (when they exist) can often be used for concrete manipulations and computations. The two approaches should hence be considered as completing each other, as opposed to competing with each other. A lot of the work in the field consists of relating the two approaches, for example, by showing that various constructions on model categories indeed model their ∞-categorical analogues.

Returning to the notion of pro-categories, in the realm of **∞-categories**, one can define pro-categories by adapting their universal property to the ∞-categorical setting. This was done in [Lu09] for $\mathcal{C}_0$ a small ∞-category and in [Lu11] for $\mathcal{C}_0$ an accessible ∞-category with finite limits. On the other hand, when $\mathcal{C}_0$ is a model category, one may attempt to construct a model structure on $\mathrm{Pro}(\mathcal{C}_0)$ which is naturally inherited from that of $\mathcal{C}_0$. This was indeed established in [EH76] when $\mathcal{C}_0$ satisfies certain conditions ("Condition N") and later in [Is04] when $\mathcal{C}_0$ is a proper model category. In [BS15a] it was observed that a much simpler structure on $\mathcal{C}_0$ is enough to construct, under suitable hypothesis, a model structure on $\mathrm{Pro}(\mathcal{C}_0)$. Recall that

Definition 1. A **weak fibration category** is a category $\mathcal{C}$ equipped with two subcategories $\mathcal{F}ib, \mathcal{W} \subseteq \mathcal{C}$ containing all the isomorphisms, such that the following conditions are satisfied:

1. $\mathcal{C}$ has all finite limits.
2. $\mathcal{W}$ has the 2-out-of-3 property.
3. For every pullback square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \longrightarrow & W \end{array}$$

   with $f \in \mathcal{F}ib$ (resp. $f \in \mathcal{F}ib \cap \mathcal{W}$) we have $g \in \mathcal{F}ib$ (resp. $g \in \mathcal{F}ib \cap \mathcal{W}$).

4. Every morphism $f : X \longrightarrow Y$ can be factored as $X \xrightarrow{f'} Z \xrightarrow{f''} Y$ where $f' \in \mathcal{W}$ and $f'' \in \mathcal{F}ib$.

Example 2.

1. Every model category is a weak fibration category.
2. Let $\mathcal{C}$ be a Grothendieck site. Then the category of simplcial sheaves on $\mathcal{C}$ with local weak equivalences and local fibrations is a weak fibration category.
3. The category of simplicial sets with finitely many non-degenerate simplices satisfies the dual axioms of Definition 1, and is hence a **weak cofibration category**.

The main result of [BS15a] is the construction of a model structure on the pro-category of a weak fibration category $\mathcal{C}$, under suitable hypothesis. In this project we give a general definition of what it means for a model structure on $\mathrm{Pro}(\mathcal{C})$ to be **induced** from a weak fibration structure on $\mathcal{C}$. In order to formulate this notion we need to establish some terminology.

Let $\mathcal{C}$ be a category. Given a pro-object $X = \{X_i\}_{i \in \mathcal{I}}$ in $\mathcal{C}$ and a functor $p : \mathcal{J} \longrightarrow \mathcal{I}$ we will denote by $p^*X \stackrel{\mathrm{def}}{=} X \circ p$ the restriction (or reindexing) of X along p. We note that there is a natural map $X \longrightarrow p^*X$. Objects which are indexed by the trivial category (i.e., objects in the image of $\iota : \mathcal{C} \longrightarrow \mathrm{Pro}(\mathcal{C})$) will be called **simple objects**. If $X, Y : \mathcal{I} \longrightarrow \mathcal{C}$ are two pro-objects indexed by $\mathcal{I}$ then any natural transformation: $T_i : X_i \longrightarrow Y_i$ gives rise to a morphism $T_* : X \longrightarrow Y$ in $\mathrm{Pro}(\mathcal{C})$. We call such morphisms **levelwise maps**. If $\mathcal{M}$ is a class of maps in $\mathcal{C}$ then the maps $T_* : X \longrightarrow Y$ which are induced by a natural transformations $T_i : X_i \longrightarrow Y_i$ taking values in $\mathcal{M}$ will be called **levelwise $\mathcal{M}$-maps**.

Recall that a functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ is **coinitial** if the comma-category $\mathcal{C}_{/d}$ is weakly contractible for every $d \in \mathcal{D}$. The following special case of the above construction is well-known.

Lemma 3. *Let $p : \mathcal{J} \longrightarrow \mathcal{I}$ be a coinitial functor between small cofiltered categories, and let $X = \{X_i\}_{i \in \mathcal{I}}$ be a pro-object indexed by $\mathcal{I}$. Then the morphism of pro-objects $X \longrightarrow p^*X$ determined by p is an* ***isomorphism****.*

We may now define what we mean for a model structure on $\mathrm{Pro}(\mathcal{C})$ to be induced from a weak fibration structure on $\mathcal{C}$.

Definition 4. Let $(\mathcal{C}, \mathcal{W}, \mathcal{F}ib)$ be a weak fibration category. We say that a model structure $(\mathbf{W}, \mathbf{Cof}, \mathbf{Fib})$ on $\mathrm{Pro}(\mathcal{C})$ is **induced** from $\mathcal{C}$ if the following conditions are satisfied:

1. The cofibrations $\mathbf{Cof}$ are the maps satisfying the left lifting property with respect to $\mathcal{F}ib \cap \mathcal{W}$.

2. The trivial cofibrations $\mathbf{Cof} \cap \mathbf{W}$ are the maps satisfying the left lifting property with respect to $\mathcal{F}ib$.

3. If $f : Z \longrightarrow X$ is a morphism in $\mathcal{C}^{\mathcal{T}}$, with $\mathcal{T}$ a cofiltered category, then there exists a cofiltered category $\mathcal{J}$, a coinitial functor $\mu : \mathcal{J} \longrightarrow \mathcal{T}$ and a factorization
$$\mu^* Z \xrightarrow{g} Y \xrightarrow{h} \mu^* X$$
in $\mathcal{C}^{\mathcal{J}}$ of the map $\mu^* f : \mu^* Z \longrightarrow \mu^* X$ such that g is a cofibration in $\mathrm{Pro}(\mathcal{C})$ and h is both a trivial fibration in $\mathrm{Pro}(\mathcal{C})$ and a levelwise trivial fibration.

Proposition 5. *Let $(\mathcal{C}, \mathcal{W}, \mathcal{F}ib)$ be a weak fibration category. If the induced model structure on* $\mathrm{Pro}(\mathcal{C})$ *exists then every levelwise weak equivalence is a weak equivalence in* $\mathrm{Pro}(\mathcal{C})$*. In the other direction, if $f : X \longrightarrow Y$ is either a trivial cofibration or a trivial fibration in the induced model structure then f is isomorphic to a levelwise weak equivalence.*

Example 6. The induced model structure on $\mathrm{Pro}(\mathcal{C})$ exists in any of the following cases:

1. $\mathcal{C}$ is the underlying weak fibration category of a proper model category.
2. $\mathcal{C}$ is "homotopically small" in the sense of [BS15a] (e.g. $\mathcal{C}$ is small) and the class of maps in $\mathrm{Pro}(\mathcal{C})$ which are isomorphic to levelwise weak equivalences is closed under 2-out-of-3.

In particular, our approach unifies the constructions of [EH76], [Is04] and [BS15a], and also answers a question posed by Edwards and Hastings in [EH76]. Having constructed a model structure on $\mathrm{Pro}(\mathcal{C})$, a most natural and urgent question is the following: is $\mathrm{Pro}(\mathcal{C})$ a model for the ∞-category $\mathrm{Pro}(\mathcal{C}_\infty)$? Our main theorem is the following:

Theorem 7. *Assume that the induced model structure on* $\mathrm{Pro}(\mathcal{C})$ *exists. Then the natural map*

$$\mathcal{F} : \mathrm{Pro}(\mathcal{C})_\infty \longrightarrow \mathrm{Pro}(\mathcal{C}_\infty)$$

is an equivalence of ∞*-categories.*

In order to prove Theorem 7 one needs, in particular, to be able to compare the mapping spaces on both sides. Recall that when $\mathcal{C}$ be an ordinary category, the set of morphisms from $X = \{X_i\}_{i\in\mathcal{I}}$ to $Y = \{Y_j\}_{j\in\mathcal{J}}$ in $\mathrm{Pro}(\mathcal{C})$ is given by the formula

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X,Y) = \lim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j)$$

The validity of this formula can be phrased as a combination of the following two statements:

1. The compatible family of maps $Y \longrightarrow Y_j$ induces an isomorphism

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X,Y) \xrightarrow{\cong} \lim_{j\in\mathcal{J}} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X,Y_j)$$

2. For each simple object $Y \in \mathcal{C} \subseteq \mathrm{Pro}(\mathcal{C})$ the compatible family of maps $X \longrightarrow X_i$ (combined with the inclusion functor $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$) induces an isomorphism

$$\operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X_i, Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(X,Y)$$

In the ∞-categorical construction of Pro-categories, the mapping spaces can be described by a similar formula as in the ordinary case, by replacing limits and colimits of Hom sets by the corresponding homotopy limits and colimits of mapping spaces. The first step towards proving Theorem 7 is to obtain a similar formula for the mapping space on the left hand side. For this, we need to a good way to describe mapping spaces in weak fibration categories.

Definition 8. Let $\mathcal{C}$ be a weak fibration category. Let $X, Y \in \mathcal{C}$ two objects. We denote by $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ the category of diagrams of the form

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \longrightarrow & * \end{array}$$

where $* \in \mathcal{C}$ is the terminal object and $p : Z \longrightarrow X$ belongs to $\mathcal{W} \cap \mathcal{F}ib$.

There is a natural map from the nerve $\mathrm{N}\,\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ to the simplicial set $\mathrm{Map}_{L^H(\mathcal{C},\mathcal{W})}(X, Y)$ where $L^H(\mathcal{C}, \mathcal{W})$ denotes the hammock localization of $\mathcal{C}$ with respect to $\mathcal{W}$. We hence obtain a natural map

$$\mathrm{N}\,\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Map}^h_{\mathcal{C}}(X, Y). \tag{1}$$

Proposition 9 (Cisinski)**.** *Let $\mathcal{C}$ be a weak fibration category. Then for every $X, Y \in \mathcal{C}$ with Y fibrant the map 1 is a weak equivalence.*

The first step towards the proof of Theorem 7 is to prove that when $\mathcal{C}$ is a weak fibration category, statements (1) and (2) above hold for **derived mapping spaces** in $\mathrm{Pro}(\mathcal{C})$, as soon as one replaces limits and colimits with their respective homotopy limits and colimits. In particular, we wish to obtain an explicit formula of the form

$$\mathrm{Map}^h_{\mathrm{Pro}(\mathcal{C})}(X, Y) = \mathrm{holim}_{j \in \mathcal{J}}\, \mathrm{hocolim}_{i \in \mathcal{I}}\, \mathrm{Map}^h_{\mathcal{C}}(X_i, Y_j).$$

We first observe that assertion (1) above is equivalent to the statement that the maps $Y \longrightarrow Y_j$ exhibit Y as the limit, in $\mathrm{Pro}(\mathcal{C})$, of the diagram $j \mapsto Y_j$. The analogous statement for homotopy limits in the setting of the induced model structure is essentially a consequence of Proposition 5:

Proposition 10. *Let $\mathcal{C}$ be a weak fibration category and let $Y = \{Y_j\}_{j \in \mathcal{J}} \in \mathrm{Pro}(\mathcal{C})$ be a pro-object. Let $\overline{\mathcal{F}} : \mathcal{J}^{\triangleleft} \longrightarrow \mathrm{Pro}(\mathcal{C})$ be the limit diagram extending $\mathcal{F}(j) = Y_j$ so that $\overline{\mathcal{F}}(*) = Y$ (where $* \in \mathcal{J}^{\triangleleft}$ is the cone point). Then the image of $\overline{\mathcal{F}}$ in $\mathrm{Pro}(\mathcal{C})_\infty$ is a limit diagram. In particular, for every $X = \{X_i\}_{i \in \mathcal{I}}$ the natural map*

$$\mathrm{Map}^h_{\mathrm{Pro}(\mathcal{C})}(X, Y) \longrightarrow \mathrm{holim}_{j \in \mathcal{J}}\, \mathrm{Map}^h_{\mathrm{Pro}(\mathcal{C})}(X, Y_j)$$

is a weak equivalence.

Sktech of proof. Using general arguments we may assume that Y is indexed by a Reedy poset $\mathcal{T}$. Consider the Reedy weak fibration structure on $\mathcal{C}^{\mathcal{T}}$. We may then replace $t \mapsto Y_t$ with a Reedy fibrant **levelwise equivalent** diagram $t \mapsto Y'_t$. By Proposition 5 the pro-object $Y' = \{Y'_t\}_{t \in \mathcal{T}}$ is weakly equivalent to Y in $\mathrm{Pro}(\mathcal{C})$, and so it is enough to prove the claim for Y'. This, in turn, can be seen by interpreting $\{Y'_t\}$ as a Reedy fibrant diagram in $\mathrm{Pro}(\mathcal{C})^{\mathcal{T}}$ and using the compatibility of the model categorical and ∞-categorical colimits. $\square$

The main step towards Theorem 7 then becomes the following:

Proposition 11. *Let $X = \{X_i\}_{i \in \mathcal{I}}$ be a pro-object and $Y \in \mathcal{C} \subseteq \mathrm{Pro}(\mathcal{C})$ a simple object. Then the compatible family of maps $X \longrightarrow X_i$ induces a weak equivalence*

$$\mathrm{hocolim}_{i \in \mathcal{I}} \mathrm{Map}^h_{\mathcal{C}}(X_i, Y) \longrightarrow \mathrm{Map}^h_{\mathrm{Pro}(\mathcal{C})}(X, Y) \tag{2}$$

Sketch of Proof. The idea of the proof is to use the mapping space description of Proposition 9 to relate the mapping space $\mathrm{Map}^h_{\mathrm{Pro}(\mathcal{C})}(X, Y)$, which depends on trivial fibrations in $\mathrm{Pro}(\mathcal{C})$ of the form $X' \overset{\simeq}{\twoheadrightarrow} X$, to the various mapping spaces $\mathrm{Map}^h_{\mathrm{Pro}(\mathcal{C})}(X_i, Y)$, which, in turn, depend on trivial fibrations in $\mathcal{C}$ of the form $Y \overset{\simeq}{\twoheadrightarrow} X_i$. These two types of data could be related if one could restrict to using trivial fibrations which are simultaneously **levelwise trivial fibrations**. One hence needs to know that levelwise trivial fibrations are in some sense sufficiently common. This, in turn, is essentially guaranteed by Condition (3) of Definition 4. □

We give two applications of our general comparison theorem. Our first application involves the theory of **shapes of topoi**. In [AM69], Artin and Mazur defined the **étale homotopy type** of an algebraic variety. This is a pro-object in the homotopy category of spaces, which depends only on the étale site of X. Their construction is based on the construction of the **shape** of a topological space X, which is a similar type of pro-object constructed from the site of open subsets of X. More generally, Artin and Mazur's construction applies to any **locally connected** site.

In [BS15a] the first author and Schlank used their model structure to define what they call the **topological realization** of a Grothendieck topos. Their construction works for any Grothendieck topos and **refines** the previous constructions form a pro-object in the homotopy category of spaces to a pro-object in the category of simplicial sets. On the ∞-categorical side, Lurie constructed in [Lu09] an ∞-categorical analogue of shape theory and defined the shape assigned to any ∞-topos as a pro-object in the ∞-category $\mathcal{S}_\infty$ of spaces. A similar type of construction also appears in [TV03]. One then faces the same type of pressing question: Is the topological realization constructed in [BS15a] using model categories equivalent to the one defined in [Lu09] using the language of ∞-categories? We give a positive answer to this question:

Theorem 12. *For any Grothendieck site $\mathcal{C}$ there is a weak equivalence*

$$|\mathcal{C}| \simeq \mathrm{Sh}(\widehat{\mathcal{S}\mathrm{hv}}_\infty(\mathcal{C}))$$

of pro-spaces, where $|\mathcal{C}|$ is the topological realization constructed in [BS15a] and $\mathrm{Sh}(\widehat{\mathcal{S}\mathrm{hv}}_\infty(\mathcal{C})) \in \mathrm{Pro}(\mathcal{S}_\infty)$ is the shape of the hyper-completed ∞-topos $\widehat{\mathcal{S}\mathrm{hv}}_\infty(\mathcal{C})$ constructed in [Lu09].

Combining the above theorem with [BS15a, Theorem 1.11] we obtain:

Corollary 13. *Let X be a locally Noetherian scheme, and let $X_{ét}$ be its étale site. Then the image of* $\mathrm{Sh}(\widehat{\mathcal{S}\mathrm{hv}}_\infty(X_{ét}))$ *in* $\mathrm{Pro}(\mathrm{Ho}(\mathcal{S}_\infty))$ *coincides with the étale homotopy type of X.*

Our second application is to the study of **profinite homotopy theory**. Let $\mathcal{S}$ be the category of simplicial sets, equipped with the Kan-Quillen model structure. The existence of the induced model structure on $\mathrm{Pro}(\mathcal{S})$ (in the sense above) follows from the work of [EH76] (as well as [Is04] and [BS15a] in fact). In [Is05], Isaksen showed that for any set K of fibrant object of $\mathcal{S}$, one can form the maximal left Bousfield localization $L_K \mathrm{Pro}(\mathcal{S})$ of $\mathrm{Pro}(\mathcal{S})$ for which all the objects in K are local. The weak equivalences in $L_K \mathrm{Pro}(\mathcal{S})$ are the maps $X \longrightarrow Y$ in $\mathrm{Pro}(\mathcal{S})$ such that the map

$$\mathrm{Map}^h_{\mathrm{Pro}(\mathcal{S})}(Y, A) \longrightarrow \mathrm{Map}^h_{\mathrm{Pro}(\mathcal{S})}(X, A)$$

is a weak equivalence for every A in K. When choosing a suitable candidate $K = K^\pi$, the model category $L_{K^\pi} \mathrm{Pro}(\mathcal{S})$ can be used as a theoretical setup for **profinite homotopy theory**.

On the other hand, one may define what profinite homotopy theory should be from an ∞-categorical point of view. Recall that a space X is called **π-finite** if it has finitely many connected components, and finitely many non-trivial homotopy groups which are all finite. The collection of π-finite spaces can be organized into an ∞-category $\mathcal{S}^\pi_\infty$, and the associated pro-category $\mathrm{Pro}(\mathcal{S}^\pi_\infty)$ can equally be considered as the natural realm of profinite homotopy theory. One is then yet again faced with the salient question: is $L_{K^\pi} \mathrm{Pro}(\mathcal{S})$ a model for the ∞-category $\mathrm{Pro}(\mathcal{S}^\pi_\infty)$? We give a positive answer to this question:

Theorem 14. *The underlying ∞-category $L_{K^\pi} \mathrm{Pro}(\mathcal{S})$ is naturally equivalent to the ∞-category $\mathrm{Pro}(\mathcal{S}^\pi_\infty)$ of profinite spaces.*

A similar approach was undertaken for the study of **p-profinite homotopy theory**, when p is a prime number. Choosing a suitable candidate $K = K^p$, Isaksen's approach yields a model structure $L_{K^p} \mathrm{Pro}(\mathcal{S})$ which can be used as a setup for p-profinite homotopy theory. On the other hand, one may define p-profinite homotopy theory from an ∞-categorical point of view. Recall that a space X is called **p-finite** if it has finitely many connected components and finitely many non-trivial homotopy groups which are all finite p-groups. The collection of p-finite spaces can be organized into an ∞-category $\mathcal{S}^p_\infty$, and the associated pro-category $\mathrm{Pro}(\mathcal{S}^p_\infty)$ can be considered as a natural realm of p-profinite homotopy theory (see [Lu11] for a comprehensive treatment). Our results allow again to obtain the desired comparison:

Theorem 15. *The underlying ∞-category $L_{K^p} \mathrm{Pro}(\mathcal{S})$ is naturally equivalent to the ∞-category $\mathrm{Pro}(\mathcal{S}^p_\infty)$ of p-profinite spaces.*

Isaksen's approach is not the only model categorical approach to profinite and p-profinite homotopy theory. In [Qu11] Quick constructs a model structure on the category $\widehat{\mathcal{S}}$ of **simplicial profinite sets** and uses it as a setting to perform profinite homotopy theory. His construction is based on a previous construction of Morel ([Mo96]), which endowed the category of simplicial profinite sets with a model structure aimed at studying p-profinite homotopy theory. We show that Quick and Morel's constructions are Quillen equivalent to the corresponding Bousfield localizations studied by Isaksen.

Theorem 16. *There are* ***Quillen equivalences***

$$\Psi_{K^\pi} : L_{K^\pi} \operatorname{Pro}(\mathcal{S}) \rightleftarrows \widehat{\mathcal{S}}_{\text{Quick}} : \Phi_{K^\pi}$$

and

$$\Psi_{K^p} : L_{K^p} \operatorname{Pro}(\mathcal{S}) \rightleftarrows \widehat{\mathcal{S}}_{\text{Morel}} : \Phi_{K^p}$$

These Quillen equivalences appear to be new. A weaker form of the second equivalence was proved by Isaksen in [Is05, Theorem 8.7], by constructing a length two zig-zag of adjunctions between $L_{K^p} \operatorname{Pro}(\mathcal{S})$ and $\widehat{\mathcal{S}}_{\text{Morel}}$ where the middle term of this zig-zag is not a model category but only a relative category.

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