Integral points on log K3 surfaces

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Let k be a number field, S a finite set of places of k and $\mathcal{X}$, $\mathcal{O}_S$ the ring of S-integers in k and $\mathcal{X}$ a nice scheme (smooth, separated, finite type) over $\mathcal{O}_S$. A prominent goal of Diophantine geometry is to understand the set $\mathcal{X}(\mathcal{O}_S)$ of **integral points** of $\mathcal{X}$. When $k = \mathbb{Q}$, $\mathcal{O}_S = \mathbb{Z}$ and X is affine then we are really talking about solutions in integer numbers to a set of polynomial equation (most of the examples appearing in this talk will be of this type).

Assuming S-integral points exist, a natural qualitative question is whether or not $\mathcal{X}(\mathcal{O}_S)$ is **Zariski dense**. On the quantitative side, one may use a suitable **height function** $H : \mathcal{X}(\mathbb{Z}) \longrightarrow \mathbb{R}_{\geq 0}$, choose a suitable open subset $\mathcal{U} \subseteq \mathcal{X}$, and ask for the asymptotic growth of the counting function

$$N(\mathcal{U}, B) = \{P \in \mathcal{X}(\mathbb{Z}) | H(P) \leq B\}.$$

In this context, one often speaks of **polynomial growth** when $N(\mathcal{U}, B)$ is comparable to a polynomial in B, and of **polylogarithmic growth** when $N(\mathcal{U}, B)$ is comparable to a polynomial in $\log(B)$. It can also be that for every $\varepsilon > 0$ there exists an open set $\mathcal{U} \subseteq \mathcal{X}$ such that $N(\mathcal{U}, B) = O(B^{\varepsilon})$, in which case one may talk about **subpolynomial growth**.

If $\mathcal{X}$ is **projective** the set $\mathcal{X}(\mathcal{O}_S)$ of S-integral points coincides with the set of **rational points** of the k-variety $X = \mathcal{X} \otimes_{\mathcal{O}_S} k$. In this case a fundamental paradigm in Diophantine geometry (attributed to Weil) asserts that the behavior of rational points should be strongly controlled by the geometry of X. More precisely, if we let $X_{\overline{k}} = X \otimes_k \overline{k}$ denote the base change of X to $\overline{k}$, then the canonical class $K_X \in \mathrm{Pic}(X_{\overline{k}})$ is one of the geometric features which effects the arithmetic very strongly. Roughly speaking, the more positive $-K_X$ is (i.e., effective, nef, big, ample, etc.) the more we expect rational points to be abundant and their existence to be effectively controlled. On the other hand, the more positive K_X is the more we expect rational points to be scarce, and the less we expect their existence to be effectively controlled. A conjectural framework of Manin and Batyrev translates this picture into a quantitative estimate for the growth of rational points. For example, when K_X is big one expects a polynomial growth. On the other hand, when X is of general type (i.e., when $-K_X$ is sufficiently positive), Lang's conjecture asserts that rational points are not Zariski dense, which means that the asymptotic growth (on a small enough open subset) is trivial.

An interesting and subtle case is when the canonical class is **trivial**. When X is furthermore simply connected not much is known. In dimension 2 such varieties are called **K3 surfaces**. The conjecutral picture of Batyrev and Manin asserts that in this case the number of rational points (if they exist) should grow subpolynomially. When X is a K3 surface of Picard number one might even expect to have a **logarithmic growth**, see [VL].

When X is not projective, the situation is more subtle. To this end it is often convenient to consider a smooth compactification $X \subseteq \overline{X}$ such that the complement is (geometrically) a simple normal crossing divisor. Having the pair $(\overline{X}, D)$ one can access many properties which are relevant to the behavior of integral points on X. For example, one can study it using the framework of **log geometry**. In particular, one has log analogues of the cotangent bundle (sheaf of 1-forms with logarithmic singularities) and the log canonical class is $K_{\overline{X}} + [D]$. This enables one to find suitable integral counterparts of various geometric classes of varieties familiar from the projective case. Another source of relevant information lies in the behavior of real points on D. For example, if $D(\mathbb{R}) = \emptyset$ then $X(\mathbb{R})$ is compact. If, in addition, X is affine then $\mathfrak{X}(\mathbb{Z})$ is automatically finite. In this case we could see this directly from $X(\mathbb{R})$. However, there can be more subtle behaviors. For example, it could be that some components of D have real points and some not, and the same can be said for components of the intersections of components and so on. It turns out that even small differences in the configuration of real points on D can have an impact on the behavior of integral points, and hence it is useful to have direct access to D.

The conjectural picture of Batyrev and Manin can be translated to the realm of integral points, although the situation becomes more complicated, and subtle features of the pair $(\overline{X}, D)$ need be taken into account. Results giving explicit formulas for the growth of integral points were established for toric varieties (Tschinkel, Chambert-Loir) and partial compatifications of split semi-simple algebraic groups (Tschinkel, Takloo-Bighash) under suitable conditions. These constitute instances where the log anti-canonical class is sufficiently positive. In this talk we wish to focus instead on the intermediate case of **log K3 surfaces** and their integral points. As we will see, not much is known in terms of growth of integral points.

Definition 1. Let X be a smooth geometrically integral surface over a field k. A **log K3 structure** on X is a smooth compactification (X, D, ι) such that D is (geometrically) a smooth normal crossing divisor and $[D] = -K_{\overline{X}}$. A **log K3 surface** is a smooth, geometrically integral, **simply connected** surface X equipped with a log K3 structure $(\overline{X}, D, \iota)$.

Let X be a log K3 surface. Since $X_{\overline{k}}$ is simply connected it follows that $k^*[X] = k^*$ and that $\mathrm{Pic}(X_{\overline{k}})$ is torsion free, hence isomorphic to $\mathbb{Z}^r$ for some r. We shall call the integer $r = \mathrm{rank}(\mathrm{Pic}(X_{\overline{k}}))$ the **geometric Picard rank** of X. Furthermore, if $(\overline{X}, D, \iota)$ is a log K3 structure on X then there exists a short exact sequence

$$0 \longrightarrow \mathbb{Z}|D| \longrightarrow \mathrm{Pic}(\overline{X}_{\overline{k}}) \longrightarrow \mathrm{Pic}(X_{\overline{k}}) \longrightarrow 0$$

where $\mathbb{Z}|D|$ is the free abelian group generated by the geometric components of D. In particular, the geometric Picard rank of X is given by $\mathrm{rank}(\mathrm{Pic}(\overline{X}_{\overline{k}}))-\sharp D$. It can be showed that if X be a log K3 surface and $(\overline{X}, D, \iota)$ a log K3 structure on X then either $D = \emptyset$ and $X = \overline{X}$ is a (proper) K3 surface or $D \neq \emptyset$ and $\overline{X}_{\overline{k}}$ is a rational surface. Furthermore, if D is non-empty then D is either a smooth projective genus 1 curve or a cycle of genus 0 curves.

Example 2. Let $\overline{X} \subseteq \mathbb{P}^3$ be a cubic surface and $D \subseteq \overline{X}$ a hyperplane section which is a simple normal crossing divisor. Then $\overline{X}$ is a del Pezzo surface of degree 3 and $[D] = K_{\overline{X}}$. Since we assumed D to have simple normal crossings there are three possibilities: either D is a smooth curve of genus 1 or a cycle of genus 0 curves whose length is either 2 or 3. In all cases one can show that X is simply connected and hence a log K3 surface. Since the geometric Picard number of $\overline{X}$ is 7 we get that the geometric Picard number of X is then either $6, 5$ or 4, accordingly. Such log K3 surfaces always admit an affine cubic equation in three variable. The much studied surfaces

$$x^3 + y^3 + z^3 = a \tag{1}$$

are examples of such log K3 surfaces with D a smooth genus 1 curve. An example with D a cycle of three genus 0 curves is given by the (modified) Markoff-Rosenberger equation

$$ax^2 + by^2 + cz^2 = Dxyz + e \tag{2}$$

which is smooth as soon as $a, b, c, d, e \neq 0$ and $\frac{D^2}{abc} \neq \frac{4}{e}$.

Our current understanding of the behaviour of integral points in these cases is rather preliminary. For example, it is not known if there exists an $a \in \mathbb{Z}$ such that the set of integral points on 2 is Zariski dense, and it is not known if there exists an $a \in \mathbb{Z}$ such that this set is not Zariski dense. The circle method heuristic (see [HB92]) predicts that integral points should grow as $\log(B)$. A result of Colliot-Thélène and Wittenberg ([CTW12]) gives sufficient conditions for the vanishing of the integral Brauer-Manin obstruction. For example when $a = 33$ the integral Brauer-Manin obstruction vanishes, but it is not known if integral points exist or not.

For the modified Markoff-Rosenberger equation, if one assumes that $a, b, c|D$ then the variety 2 is acted upon (over $\mathcal{O}_S$) by $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$. Yuan and Schmidt (see [YS01]) considered the case where $k = \mathbb{Q}$, $\mathcal{O}_S = \mathbb{Z}$ and a, b, c, d are all positive and $e = 1$. They found that there are only six choices of such $a, b, c, D \in \mathbb{Z}$ with $a, b, c|D$ and $\frac{D^2}{abc} \neq 4$ for which the set of integral points is not empty. These cases were later analysed by Baragar and Umeda ([BU04]), where it was shown that in each case there are finitely many orbits, and the asymptotic growth of integral points inside each orbit grows as $\log^2(B)$.

Example 3. Let $\overline{X} \longrightarrow \mathbb{P}^1$ be a conic bundle of the form

$$f(t,s)x^2 + g(t,s)y^2 = z^2$$

where $f(t,s), g(t,s)$ are separable homogeneous polynomials of degree 2 without common factors, and $(t:s)$ are homogeneous coordinates on $\mathbb{P}^1$. Then $\overline{X}$ is a del Pezzo surface of degree 4 and the bisection $D \subseteq \mathbb{P}^1$ given by $z = 0$ is a smooth curve of genus 1 whose class is the canonical class. One can then show that $X = \overline{X} \setminus D$ is simply connected, and hence a log K3 surface of geometric Picard rank 5. We may write X as a bundle of affine conics of the form

$$f(t,s)x^2 + g(t,s)y^2 = 1 \tag{3}$$

where this equation should be interpreted as defining X inside the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$. As affine conics can be considered as analogues of elliptic curves, we may say that X is a **conic log K3 surface**, in analogy with the terminology of elliptic K3 surfaces in the projective case.

Let S be a finite set of places of $\mathbb{Q}$. Conic log K3 surfaces such as 3 can be attacked using an integral point adapation of a metohd invented by Swinnerton-Dyer. Suppose that $f(t,s) = a(c_1t + d_1s)(c_2t + d_2s)$ and $g(t,s) = b(c_1t + d_1s)(c_2t + d_2s)$, where each c_i, d_i is a coprime pair of S-integers. Denote by $\Delta_{i,j} = c_jd_i - c_id_j$ the respective resultants, which one assumes to be non-zero for 3 to be smooth. The following theorem is one of the main results of [Ha15b].

Theorem 4. *Let S be a finite set of places of $\mathbb{Q}$ containing $2, \infty$. Let $f, g \in \mathbb{Z}_S[t,s]$ be homogeneous polynomials of the above form with a, b and $\Delta_{i,j}$ defined as above. Assume that the S-integers a, b are square-free and not divisible by 3 or 5, and that the classes of the elements $\{-1, a, b\} \cup \{\Delta_{i,j}\}_{i>j}$ are linearly independent in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.*

Let $\mathcal{Y} \longrightarrow \mathbb{P}^1_S$ be the pencil of affine conics 3. If $\mathcal{Y}$ has an S-integral adelic point then $\mathcal{Y}$ has a Zariski dense set of S-integral points.

Remark 5. More examples of log K3 surfaces which are embedded in del Pezzo surfaces of degree 4 (without assuming a conic bundle structure) are described in a recent paper [JS16] Janel and Damaris, which also contains computations of Brauer groups and Brauer Manin obstructions for various such surfaces.

Definition 6. Let X be a log K3 surface. We shall say that X is **ample** if it admits a log K3 structure $(\overline{X}, D, \iota)$ such that $K_{\overline{X}}$ is ample (i.e., such that $\overline{X}$ is a del Pezzo surface).

Arguably, the simplest type of log K3 surface is an ample log K3 surface whose Picard number is 0. It turns out that these can be completely classified.

Theorem 7 ([Ha15a])**.** *Any ample log K3 surface of Picard rank 0 over a field k of characteristic 0 admits a log K3 structure of the form $(\overline{X}, D, \iota)$ where $\overline{X}$ is a del Pezzo surface of degree 5 and D is a cycle of five (-1)-curve.*

Let us say a few words about the proof of Theorem 7. One begins by observing that log K3 surfaces of Picard rank 0 can only admit ample log K3 structures $(\overline{X}, D, \iota)$ in which D is a cycle of rational curves. The choice of such a compactification is not unique. However, two different compactifications of this kind can always be related by a sequence of **corner blow-ups** and **corner blow-downs**. These consist of blowing up and intersection point of D, or blowup-down a component of D of self-intersection -1. Let $a_1, ..., a_n$ denote the self intersections of the components of D. The adjunction formula then implies that

$$\sum_i a_i = 3d + 2r - 20$$

where $d = [D] \cdot [D]$ is the degree of $[D] = K_{\overline{X}}$ and r is the geometric Picard rank of X. In particular, if $r = 0$ then $\sum_i a_i = 3d - 20$. Since we are looking only at compactifications with $\overline{X}$ ample we also have $a_i \geq -1$ for every i. This gives enough combinatorial rigidity to show that when $r = 0$ one can always arrive at $d = 5$ and $a_i = -1$ for every i by performing a sequence of corner blow-ups and corner blow-downs.

Given a log K3 structure $(\overline{X}, D, \iota)$ of degree 5 such that D is a cycle of five (-1)-curves we may consider the Galois action on the dual graph of D, yielding an invariant $\alpha \in H^1(k, \mathbb{D}_5)$, where $\mathbb{D}_5$ is the dehidral group of order 10, considered here as the automorphism group of a cyclic graph of length 5. We then obtain the following classification theorem:

Theorem 8 ([Ha15a])**.** *Let k be a field of characteristic 0. The association $X \mapsto \alpha_X$ determines a bijection between the set of k-isomorphism classes of ample log K3 surfaces of Picard rank 0 and the Galois cohomology set $H^1(k, \mathbb{D}_5)$.*

Theorem 9 ([Ha15a])**.** *Let k be a field of characteristic 0. An ample log K3 surface X over k of Picard rank 0 whose invariant α_X is trivial is k-isomorphic the affine surface*

$$(xy - 1)t = x - 1 \tag{4}$$

Theorem 10 ([Ha15a])**.** *Let k be a field of characteristic 0. An ample log K3 surface X over k of Picard rank 0 whose invariant α_X is the image of $[a] \in H^1(k, \mathbb{Z}/2)$ under the map induced y an inclusion $\mathbb{Z}/2 \subseteq \mathbb{D}_5$ is k-isomorphic to the affine surface*

$$(x^2 - ay^2)t = y - 1. \tag{5}$$

Proposition 11 ([Ha15a])**.** *Let $\mathcal{X}/\mathbb{Z}$ be such that $X = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ is $\mathbb{Q}$-isomorphic to the log K3 surface appearing in Theorem 9. Then the set of integral points $\mathcal{X}(\mathbb{Z})$ is not Zariski dense.*

Sketch. Applying a suitable coordinate change we have that X is isomorphic over $\mathbb{Q}$ (though not over $\mathbb{Z}$) to the affine surface 4. It is then not hard to find a constant C such that any real solutions (x, y, t) of 4 will satisfy $\min(|x|, |y|, |t|) \leq C$. This can then be used to find a finite collection of curves containing all integral points. $\square$

The proof of Theorem 11 has lead Jahnel and Schindler to define the following notion:

Definition 12 ([JS16])**.** Let X be a smooth algebraic variety over $\mathbb{Q}$ and let $C \subseteq X(\mathbb{R})$ be a connected component. We will say that C is **strongly unobstructed at** ∞ if for every finite collection of non-constant regular functions $f_1, ..., f_1 \in \mathbb{Q}[X]$ the real function $\min(|f_1|, ..., |f_n|) : C \longrightarrow \mathbb{R}$ is unbounded.

Arguing as in the proof of 11 Jahnel and Schindler show that if $C \subseteq X(\mathbb{R})$ is a component that is **not** strongly unobstructed then integral points which lie on C are not Zariski dense. An example in which every component is strongly unobstructed at ∞ is given by the log K3 surface X appearing in Theorem 10 when $a \in \mathbb{Z}$ is **positive**. We shall now show that this class of surfaces contains cases where integral points are Zariski dense. Theorem I of [Na88] implies, in particular, that there exists a real quadratic number field $L = \mathbb{Q}(\sqrt{a})$, ramified at 2 and with trivial class group, such that the reduction map $\mathcal{O}_L^* \longrightarrow (\mathcal{O}_L/\mathfrak{p})^*$ is surjective for infinitely many prime ideals $\mathfrak{p} \subseteq \mathcal{O}_L$ of degree 1 over $\mathbb{Q}$. Given such an L, we may find a square-free positive integer $a \in \mathbb{Q}$ such that $L = \mathbb{Q}(\sqrt{a})$. Now let $\mathcal{X}$ be the ample log K3 surface over $\mathbb{Z}$ given by the equation 5. We now claim the following:

Proposition 13 ([Ha15a])**.** *The set $\mathcal{X}(\mathbb{Z})$ of integral points is Zariski dense.*

Sketch. Let $\mathfrak{p} = (\pi) \subseteq \mathcal{O}_L$ be an odd unramified prime ideal of degree 1 such that $\rho_{\mathfrak{p}} : \mathcal{O}_L^* \longrightarrow (\mathcal{O}_L/\mathfrak{p})^*$ is surjective and let $p = N_{L/\mathbb{Q}}(\pi)$. Using the surjectivity of $\rho_{\mathfrak{p}}$ one may then construct a point on the curve $\mathcal{C}_p \subseteq \mathcal{X}$ given by the additional equation $x^2 - ay^2 = \pm p$. By multiplying with units whose image in $\mathcal{O}_L/\mathfrak{p}$ is 1 we may produce in this way infinitely many integral points on $\mathcal{C}_p$, and this can be done for infinitely many values of p's. This implies that $\mathcal{X}(\mathbb{Z})$ is Zariski dense. $\square$

Question 14. *Let $\mathcal{X}$ be a smooth, separated scheme over $\mathbb{Z}$ such that $X = \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a log K3 surface with* $\mathrm{Pic}(X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) = 0$*. Should we expect integral points on a small enough open subset to grow as* $\log(B)^b$ *for some $b > 0$? If so, what is the correct value of b?*

We may try to use circle method heuristics in order to "guess" the constant b. This was explained to the author by Tim Browning in personal communication. Given a log K3 surface $(X, \overline{X}, D, \iota)$ over $\mathbb{Q}$, let us define s to be equal to 1 plus the dimension of the **analytic Clemens complex** of D over $\mathbb{R}$ (where we agree then the empty complex has dimension -1). More explicitly s is 0 if $D(\mathbb{R}) = \emptyset$, s is 2 if D contains a component defined over $\mathbb{R}$ which contains an intersection point defined over $\mathbb{R}$, and s is 1 otherwise. In cases which admit suitably simple equations (such as the cases in Theorems 10 and 9), the circle method heuristic will predict that $\mathrm{N}(\mathcal{U}, B)$ should grow as $\log^b(B)$, where $b = \mathrm{rank}(\mathrm{Pic}(X_{\overline{k}})) + s$. This means that both Example 9 and Example 10 should exhibit a growth of $\mathrm{N}(X, B) \sim \log^2(B)$. However, Theorem 9 asserts that we actually have a trivial growth in this case, i.e., $b = 0$. Preliminary simulations for the case considered in Theorem 10 show that $\mathrm{N}(X, B)$ seems to grow slower than $\log(B)^2$ in this case as well, but not enough data has been gathered to reach a definitive conclusion.

Question 15. *Is there a natural modification of the formula* $b = \mathrm{rank}(\mathrm{Pic}(X_{\overline{k}})) + s$ *which would account for the phenomenon of "obstruction at* ∞*" (in its various incarnations)?*

Let us consider the notion of a strongly unobstructed component $C \subseteq X(\mathbb{R})$ suggested by Jahnel and Schindler. When C is such a component then, in particular, for every finite collection $f_1, ..., f_n \in k[X]$ of non-constant regular functions there exists a sequence of points $\{P_i\} \subseteq C$ such that $\lim_i |f_j(P_i)| = \infty$ for every $j = 1, ..., n$. Since $\overline{X}(\mathbb{R})$ is compact, we may assume that the sequence $\{P_i\}$ has a limit $Q \in \overline{X}(\mathbb{R})$, which consequently must lie in $D(\mathbb{R})$. Then all the $|f_j|$ are unbounded functions on $W \cap C$ for every neighborhood W of Q in $X(\mathbb{R})$. This leads to the following definition:

Definition 16. Let $C \subseteq X(\mathbb{R})$ be a connected component and let $Q \in D(\mathbb{R})$ be a point. We will say that Q is a **universal pole** for C if for every non-constant regular function $f \in \mathbb{Q}[X]$ and every neighborhood $Q \in W \subseteq X(\mathbb{R})$, the real function $|f|$ is unbounded on $W \cap C$.We will say that a geometric component D_0 of D defined over $\mathbb{R}$ is a universal pole for C if every point in $D_0(\mathbb{R})$ is a universal pole for C.

We may now attempt to use Definition 16 in order to modify the circle method formula $b = \mathrm{rank}(\mathrm{Pic}(X_{\overline{k}})) + s$, or more precisely the quantity s. Let us fix a connected component $C \subseteq X(\mathbb{R})$ on which we would like to count points. Instead of considering all the real Clemens complex of D let us only take into account intersection points and components which are universal poles for C. In particular, let us set s' to be 0 if D has not universal poles for C, to be 2 if D has a geometric component which is a universal pole and contains an intersection point which is a universal pole, and set $s' = 1$ otherwise.

Question 17. *Let* X *be a log K3 surface of Picard rank* 0*. Should we expect that integral points on a small enough open subset to grow as* $\log(B)^{s'}$ *where* s' *is defined as above?*

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