Basic Notions in Algebraic Topology 1

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Remark 1. In these notes when we say "map" we always mean continuous map.

1 The Spaces of Algebraic Topology

One of the main difference in passing from point set topology to algebraic topology is the vast focusing on very "nice" spaces. In order to understand these guys it is useful to think of the following 3 principles:

1. All discs D^n are "nice".

2. The gluing of two nice spaces along a sub "nice" space is "nice".

3. If $X = \cup_n X_n$ for an ascending sequence of "nice" closed subspaces $X_n \subseteq X$ and X has the strong topology with the respect to the inclusions $X_n \hookrightarrow X$ then X is also "nice". This principle will allow us to do the gluing operation above infinitely many times. It will usually not be a crucial point in this course, but I mention it here nonetheless.

Note that each D^n has as a subspace S^{n-1} which is its boundary when considered as a subset $D^n \subseteq \mathbb{R}^n$. We will many times write $S^{n-1} = \partial D^n$ even though we think of D^n as an abstract space and not as a subset of $\mathbb{R}^n$.

Note that S^n can be obtained by gluing two copies of D^n_1, D^n_2 of the n-dimensional ball along their common boundaries

$$\partial D^n_1 \cong \partial D^n_2 \cong S^{n-1}$$

Hence if S^{n-1} is nice then so should be S^n. Since S^0 is finite discrete we get that all spheres are nice.

In order to understand better what kind of spaces we can build by gluing we should make a more precise definition of the kind of basic gluing step we are interested in.

1.1 Pushouts

Let A, X, Y be topological spaces and $f : A \longrightarrow X, g : A \longrightarrow Y$ two maps. We describe this information by a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \\ Y & & \end{array}$$

We define the **pushout** of this diagram to be the space

$$P = X \coprod Y / \sim$$

where $\sim$ is the equivalence relation which is generated by the pairs $(f(a), g(a))$, i.e. it is the smallest equivalence relation in which $f(a) \sim g(a)$. Note that if both f, g are injective then this equivalence relation just identifies $f(a)$ with $g(a)$ for each a.

The quotient map $X \coprod Y \longrightarrow P$ gives us two natural maps $\psi : X \longrightarrow P$ and $\varphi : Y \longrightarrow P$. Further more the map $\psi \circ f$ from A to P is **equal** to the map $\varphi \circ g$, because $f(a), g(a) \in X \coprod Y$ are mapped by definition to the same point in P. All this information can be written in a square diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \psi} \\ Y & \xrightarrow{\varphi} & P \end{array}$$

We say that this diagram **commutes** (or is **commutative**) because $\psi \circ f = \varphi \circ g$. In general the notion of commuting square diagram (or more general diagrams) will be very useful in this course, mostly in the second half. We will sometimes call the whole diagram above a **pushout diagram**.

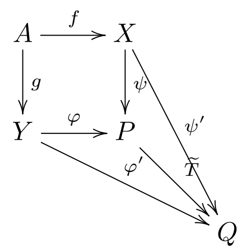
It is worth while to note that P satisfies a very important property, which actually determines it completely (in this context we usually call such properties **universal** properties). This property is the following: suppose we are given another **commutative** diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \psi'} \\ Y & \xrightarrow{\varphi'} & Q \end{array}$$

Then the maps ψ' and φ' determine a map $T : X \coprod Y \longrightarrow Q$. From the commutativity of the diagram we see that for each $a \in A$, $T(f(a)) = T(g(a))$. This means that T induces a well defined map

$$\widetilde{T} : P \longrightarrow Q$$

This whole situation can be described by an even bigger diagram



This diagram is commutative not only in the main square, but also in the two adjacent triangles, i.e. we have $\widetilde{T} \circ \varphi = \varphi'$ and $\widetilde{T} \circ \psi = \psi'$. In fact it is not hard to show that $\widetilde{T} : P \longrightarrow Q$ is the **unique** such map, i.e. the unique map making the diagram above commute. This property of P determines it up to a unique homeomorphism, and can be used to define the notion of a pushout in a general category.

For example the gluing of discs into spheres discussed above can be described by the pushout diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & S^n \end{array}$$

1.2 CW complexes

We can describe more precisely what a very general family of nice spaces:

Definition 2. A **CW complex** is a space X together with a sequence of subspaces of X

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq ... \subseteq X_n \subseteq ...$$

such that $X = \cup_n X_n$, and for each $k = 0, ..., n$ a set I_k and a pushout diagram

$$\begin{array}{ccc} \coprod_{\alpha \in I_k} S^{k-1} & \longrightarrow & \coprod_{\alpha \in I_k} D^k \\ \downarrow & & \downarrow \\ X_{k-1} & \longrightarrow & X_k \end{array}$$

In particular we have for each $\alpha \in I_k$ a map $\varphi^k_\alpha : S^{k-1} \longrightarrow X_k$ and a map $\widetilde{\varphi}^k_\alpha : D^k \longrightarrow X_k$ such that the diagram

$$\begin{array}{ccc} S^{k-1} & \longrightarrow & D^k \\ \downarrow{\scriptstyle \varphi_\alpha} & & \downarrow{\scriptstyle \widetilde{\varphi}_\alpha} \\ X_{k-1} & \longrightarrow & X_k \end{array}$$

commutes. The maps φ_α^k are called the **gluing maps**, the maps $\widetilde{\varphi}_\alpha^k$ are called the **cell maps** and their images $\widetilde{\varphi}_\alpha^k(D^k)$ are called the **cells** of X. The subspace X_k is called the k-skeleton of X.

We say that a CW complex X is **finite** if it has finitely many cells. We say that X is n-dimensional if it doesn't have cells above dimension n. Note that most CW's you will encounter in this course will be finite.

Note that CW complexes are always Hausdorf, and the cell maps $\widetilde{\varphi}_\alpha^k : D^k \longrightarrow X_k$ satisfy the following properties:

1. Each $\widetilde{\varphi}_\alpha^k$ is injective on the interior $\text{Int}(D^k) \subseteq D^k$.
2. The subsets $\widetilde{\varphi}_\alpha^k(\text{Int}(D^k)) \subseteq X$ are disjoint and their union is all of X.
3. The set $\widetilde{\varphi}_\alpha^k(\text{Int}(D^k))$ meets only finitely many cells of dimension less then k.
4. A subset $U \subseteq X$ is open if and only if $(\widetilde{\varphi}_\alpha^k)^{-1}(U)$ is open in D^k for each $k, \alpha \in I_k$.

Remark 3. Note that the maps $\widetilde{\varphi}_\alpha^k$ completely determine the CW structure on X. In fact given a Hausdorf space X and collections of maps $\widetilde{\varphi}_\alpha^k : D^k \longrightarrow X$ they will correspond to a CW structure on a X if and only if they satisfy the 4 properties above.

Examples:

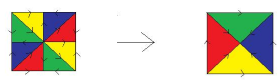
1. 0-dimensional CW complexes are just discrete spaces.
2. 1-dimensional CW complexes are obtained from discrete spaces by attaching segments along their end points. Hence a 1-dimensional CW complex is the underlying space of a **graph** (in the generalized sense: we allow multiple edges between each two vertices and edges between a vertex and itself). The homotopy type of finite 1-dimensional CW complexes depends only on the difference $E-V$ between the number of edges and the number of vertices. This will be proven in this course.
3. Let us now sketch two examples of 2-dimensional CW complexes:

   (a) Let X be the CW complex whose 1-skeleton $X_1 \subseteq X$ is a circle S^1 and X is obtained from X_1 by attaching a single 2-cell D^2 along the gluing map
   $$\varphi : \partial D^2 = S^1 \longrightarrow S^1 = X_1$$
   which is defined as follows: consider the embedding of S^1 in the complex plane $\mathbb{C}$ as
   $$S^1 = \{z \in \mathbb{C} | |z| = 1\}$$
   and let $\varphi : S^1 \longrightarrow S^1$ be given by $\varphi(z) = z^2$. Then φ is a surjective 2-to-1 map such that for each z we have $\varphi(z) = \varphi(-z)$. This means that

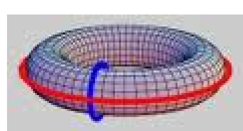
X can be identified with the space obtained from D^2 by identifying antipodal points on the boundary.

This space is know as the 2-dimensional **real projective space** and is denoted by $\mathbb{R}P^2$. It can also be described as the space obtained from the 2-sphere S^2 by gluing antipodal points or as the space of lines through the origin in $\mathbb{R}^3$. There is an analogous construction of $\mathbb{R}P^n$ for every n. We will use this spaces as examples in this course many times.

(b) Consider the 2-dimensional torus $\mathbb{T}^2$. This space can be described as the product $S^1 \times S^1$ (with the product topology). Another description is by taking a square and gluing opposite edges as shown in the figure below:



We would like to describe $\mathbb{T}^2$ as a CW complex. Note that after the gluing of opposite edges all for vertices of the square are identified to a single point $v_0 \in \mathbb{T}^2$. The two red edges are identified into a single edge (from v_0 to itself) and the two blue edges are identified into a single edge B (from v_0 to itself). The results in a a subspace $X_1 \subseteq \mathbb{T}^2$ which is homeomorphic to a bouquet of two circles (i.e. two circles joined at a point):



We will denote by $A \subseteq X_1$ the red circle and by $B \subseteq X_1$ the blue circle.

We claim that $\mathbb{T}^2$ can be obtained from X_1 by attaching a single 2-cell. In order to have a more natural description of the gluing map let us replace the 2-disc with the square $Q = I \times I$ which appears in the first figure (which is homeomorphic to D^2). We denote by ∂Q the union of the 4-edges of the square. Then the identification $Q \cong D^2$ identifies ∂S with $\partial D^2 = S^1$. Hence we can describe a gluing map by describing a map

$$\varphi : \partial Q \longrightarrow X_1$$

Now this map will simply be the map that sends the A-edges to A and B-edges to B (along the directions indicated by the arrows). I suggest going over this construction carefully at home until you understand why this gluing map gives the torus.

We finish this section by adding one last kind of niceness which is new and important in algebraic topology over topology:

4 If a space if homotopy equivalent to a "nice" space, then it is "homotopy-nice". For the homotopy theorists, these are just as good as "nice" spaces.

Remark 4. An important family of interesting spaces that appears in many areas of topology and geometry are the topological manifolds:

Definition 5. A topological space X will be called an n-dimensional **topological manifold** if

1. Every $x \in X$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$.
2. X is Hausdorf and satisfies the second axiom of countability, i.e. the topology on X has a countable basis.

It was first shown by Milnor that every topological manifold is homotopy equivalent to a CW complex, and so our class of homotopy-nice space includes these topological manifolds. One can also ask whether a topological manifold are **homeomorphic** to a CW complex. In this case the full answer is not known, but it is known that **compact** topological manifolds of dimension $n \neq 4$ are homeomorphic to CW complexes. In dimension 4 this is an open problem.

Basic Notions in Algebraic Topology 2

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Remark 1. In these notes when we say "map" we always mean continuous map. If we want to say a map that is not necessarily continuous we will say "function". Sometimes we will say explicitly continuous map to emphasis the continuity.

1 Homotopies Between Maps

Let us recall the basic definitions:

Definition 2. Let X, Y be topological spaces and $f, g : X \longrightarrow Y$ two maps. We say that f is **homotopic** to g (and write $f \sim g$) if there exists a map $H : I \times X \longrightarrow Y$ (where $I = [0, 1]$ is the unit interval) such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$. In this case we call H a **homotopy** from f to g.

Remark 3. You will show in the next exercise that the homotopy relation $f \sim g$ is actually an equivalence relation (it's very straight-forward). We denote by $[X, Y]$ the equivalence classes of maps from X to Y under this relation. We also call these equivalence classes **homotopy classes**.

Examples:

1. Let $f, g : X \longrightarrow \mathbb{R}^n$ be any two maps. Then we can construct a homotopy between them by setting

$$H(t, x) = (1 - t)f(x) + tg(x)$$

   In particular this means that $[X, \mathbb{R}^n] = *$ for every X.

2. Let $X = *$ be the point so that maps from X to Y correspond simply to points in Y. Then a homotopy between two maps $f, g : X \longrightarrow Y$ is simply a path between the corresponding points. In particular $[*, Y]$ is just the set of path-connected components, which we also denote by $\pi_0(Y)$.

The way to think of homotopy is a continuous deformation of the function f into the function g. In fact one can try to make this intuition more precise: for each $t \in [0, 1]$ the composition of the maps

$$X \longrightarrow \{t\} \times X \xrightarrow{H} Y$$

gives a map from X to Y. This gives us a "path" from f to g in the "space of continuous maps" from X to Y. But we need to be careful here: in order for our intuition to make sense we need this path to be continuous in some sense, so we need some reasonable topology on the space of maps from X to Y. This topology is called the **compact-open** topology:

Definition 4. Let X, Y be two topological spaces and denote by $C(X,Y)$ the space of continuous maps from X to Y. The **compact-open** topology on $C(X,Y)$ is the topology generated by the sub-basis

$$W(K,U) = \{f \in C(X,Y) | f(K) \subseteq U\}$$

where $K \subseteq X$ is compact and $U \subseteq Y$ is open.

The following theorem is an exercise in point-set topology:

Theorem 5. *Let $H : Z \times X \longrightarrow Y$ a continuous map. Then the function $\varphi : Z \longrightarrow C(X,Y)$ given by*

$$\varphi(z)(x) = H(z,x)$$

is continuous with respect to the compact-open topology on $C(X,Y)$.

Corollary 6. *Given a homotopy $I \times X \longrightarrow Y$ from f to g we get a continuous path $\varphi : I \longrightarrow C(X,Y)$ from f to g in the space of continuous maps.*

Unfortunately the reverse claim is untrue for general topological spaces, i.e. a homotopy is something a bit stronger then a continuous path in $C(X,Y)$. However if we restrict to "nice" topological spaces then reverse claim will be true.

Definition 7. Let X be a topological space. We say that X is **compactly generated** if X is Hausdorf and in addition if a set $U \subseteq X$ satisfies that $U \cap K$ is open in K for any compact $K \subseteq X$ then U is open in X.

Examples:

1. Any compact Hausdorf space is compactly generated.
2. Any CW-complex is compactly generated.
3. Any locally compact Hausdorf space is compactly generated.

The following is an exercise in point-set topology which is slightly harder than the previous one:

Theorem 8. *Let X be a compactly generated space and Z, Y any topological spaces. Let $\varphi : Z \longrightarrow C(X,Y)$ be a continuous map with respect to the compact-open topology. Then the map*

$$H(z,x) = \varphi(z)(x) \in Y$$

from $Z \times X$ to Y is continuous.

Corollary 9. *If X is a compactly generate space and $f, g : X \longrightarrow Y$ are two maps then homotopies from f to g correspond exactly to continuous paths in the space $C(X,Y)$ from f to g.*

1.1 Pointed Spaces, Maps and Homotopies

It is sometimes more convenient to work with pointed space instead of a space. As the name implies, a pointed space is just a pair (X, x_0) together with a chosen point $x_0 \in X$. A pointed map $f : (X, x_0) \longrightarrow (Y, y_0)$ is a map satisfying $f(x_0) = y_0$.

Let $f, g : (X, x_0) \longrightarrow (Y, y_0)$ be two pointed maps. A **pointed homotopy** from f to g is a map $H : I \times X \longrightarrow Y$ such that $H(0, x) = f(x)$, $H(1, x) = g(x)$ and $H(t, x_0) = y_0$ for all t. As the discussion above we can think of a pointed homotopy as a continuous path between two pointed maps **inside the space of pointed maps**.

It is clear that pointed homotopy is also an equivalence relation, and we denote by $[X, Y]_*$ the set of pointed homotopy classes of pointed maps. In class we have used thses homotopy classes to define interesting invariants of spaces and pointed spaces. In particular we had the unpointed invariants

$$H^0(X) = [X, \mathbb{Z}]$$

$$H^1(X) = [X, S^1]$$

called the 0'th and 1st cohomology groups of X and the pointed invariants

$$\pi^0(X, x_0) = [S^0, X]_*$$

$$\pi^1(X, x_0) = [S^1, X]_*$$

which are called the 0th and 1st homotopy groups of X. $\pi_1(X, x_0)$ is also called the **fundamental group** of X.

Note that since $\mathbb{Z}$ admits a structure of an abelian group we get an abelian group structure on H^0 and H^1 given by point-wise multiplication of maps. On the other hand $\pi^0(X, x_0)$ is just a pointed set and $\pi^1(X, x_0)$ carries a (not necessarily abelian) group structure given by concatenation of paths.

2 The Circle

In this section we are going to compute the set of both pointed and unpointed homotopy classes of maps from S^1 to S^1. We identify S^1 as the unit circle

$$S^1 = \{z \in \mathbb{C} | |z| = 1\} \subseteq \mathbb{C}$$

When we want to consider S^1 as a pointed space we will always use 1 as our base point. Hence when we say **pointed map** $S^1 \longrightarrow S^1$ we mean a map f satisfying $f(1) = 1$. Note that every map $f : S^1 \longrightarrow S^1$ is homotopic to a pointed map by simply rotating the circle slowly until the appropriate angle is achieved.

Remark 10. It will be more convenient for us to thing of a map $f : S^1 \longrightarrow S^1$ as a path $\varphi : I \longrightarrow S^1$ such that $\varphi(0) = \varphi(1)$. To be precise, the exact connection between φ and f is given by $\varphi(x) = f(e^{2\pi i x})$.

Theorem 11. *Let $\varphi : I \longrightarrow S^1$ be a path such that $\varphi(0) = \varphi(1) = 1$. Then there exists a unique path $\widetilde{\varphi} : I \longrightarrow \mathbb{R}$ such that*

$$\varphi(s) = e^{2\pi i \widetilde{\varphi}(s)}$$

for every $s \in I$. We call $\widetilde{\varphi}$ the kilometrage function of φ.

Proof. Consider the open sets $U = S^1 \setminus \{1\}$ and $V = S^1 \setminus \{-1\}$ in S^1. For an ordered pair of points $x, y \in U$ we define the **translation** $d_U(x, y)$ from x to y inside U to be the unique $s \in (-1, 1)$ such that the path $\psi : [0, 1] \longrightarrow S^1$

$$\psi(t) \mapsto xe^{2\pi its}$$

is a path from x to y **which is contained entirely inside** U. Note in particular that

$$e^{2\pi i d_U(x,y)} = y * x^{-1} \in S^1$$

where the product is product of complex numbers. Similarly we define the translation $d_V(x, y)$ inside V. Note that $U \cap V$ has two connected components, one in the upper half plane and one in the lower. It is easy to see that if x, y are in the same connected component of $U \cap V$ then

$$d_V(x, y) = d_U(x, y)$$

and if x and y are in different connected components then

$$|d_V(x, y) - d_U(x, y)| = 1$$

Now let $s \in [0, 1]$ be a point. We say that a partition $0 = s_0 < s_1 < ... < s_n = s$ of the segment $[0, s]$ is **good** if there exist $W_i \in \{U, V\}$ such that $\varphi([s_i, s_{i+1}])$ is contained entirely W_i. From Lebesgue's number theorem and using the fact that $[0, s]$ is compact we see that there exists a good partition. Given such a good partition we define:

$$\widetilde{\varphi}(s) = \sum_{i=1}^{n} d_{W_i}(\varphi(s_{i-1}), \varphi(s_i))$$

It is not hard to show that this definition does not depend on the good partition (because any two good partitions have a common refinement). It is also not hard to show that $\widetilde{\varphi}$ is continuous and by definition we get

$$e^{2\pi i \widetilde{\varphi}(s)} = e^{2\pi i \sum_{j=1}^{n} d_{W_j}(\varphi(s_{j-1}),\varphi(s_j))} =$$

$$\prod_{j=1}^{n} e^{2\pi i d_{W_j}(\varphi(s_{j-1}),\varphi(s_j))} = \prod_{j=1}^{n} \varphi(s_j) * \varphi(s_{j-1})^{-1} = \varphi(s) * \varphi(0)^{-1} = \varphi(s)$$

□

Definition 12. Let $f : S^1 \longrightarrow S^1$ be a pointed map associated with a path $\varphi : I \longrightarrow \mathbb{R}$. We call $\widetilde{\varphi}(1) \in \mathbb{Z}$ the **degree** of f and denote it by $\deg(f)$. We will also denote it sometimes by $\deg(\varphi)$.

Theorem 13. *Let $f, g : S^1 \longrightarrow S^1$ be two pointed maps. Then the following are equivalent:*

1. *f, g are pointed-homotopic.*
2. *f, g are homotopic.*
3. *f, g have the same degree.*

$1 \Rightarrow 2$ *is trivial. Let us prove* $2 \Rightarrow 3$*:*

Let $\varphi, \psi : I \longrightarrow S^1$ be the paths associated with f, g as above (see Remark 10). Let $H : I \times I \longrightarrow S^1$ be a homotopy from φ to ψ satisfying $H(t, 0) = H(t, 1)$ for all t. From lebesgue's number theorem there exist $W_{i,j} \in \{U, V\}$ and partitions

$$0 = t_0 < t_1 < ... < t_n = 1$$

$$0 = s_0 < s_1 < ... < s_n = 1$$

such that $H([t_i, t_{i+1}], [s_i, s_{i+1}])$ is contained entirely in $W_{i,j}$. Let $\gamma_i(s) = H(t_i, s)$. Then we have

$$\deg(\gamma_i) = \sum_{j=0}^{n-1} d_{W_{i,j}}(\gamma_i(s_j), \gamma_i(s_{j+1})) =$$

$$\sum_{j=0}^{n-1} \left[d_{W_{i,j}}(\gamma_i(s_j), \gamma_{i+1}(s_j)) + d_{W_{i,j}}(\gamma_{i+1}(s_j), \gamma_{i+1}(s_{j+1})) + d_{W_{i,j}}(\gamma_{i+1}(s_{j+1}), \gamma_i(s_{j+1}))\right] =$$

$$\sum_{j=0}^{n-1} d_{W_{i,j}}(\gamma_i(s_j), \gamma_i(s_{j+1})) = \deg(\gamma_{i+1})$$

Note that the cancelation occurs because

$$d_{W_{i,j}}(\gamma_i(s_j), \gamma_{i+1}(s_j)) = -d_{W_{i,j}}(\gamma_{i+1}(s_j), \gamma_i(s_j))$$

$$d_{W_{i,j}}(\gamma_i(s_n), \gamma_{i+1}(s_n)) = -d_{W_{i,j}}(\gamma_{i+1}(s_0), \gamma_i(s_0))$$

Hence by induction we get that $\deg(\varphi) = \deg(\gamma_0) = \deg(\gamma_n) = \deg(\psi)$ and we are done.

We now come to proving $3 \Rightarrow 1$*: Suppose that f, g have the same degree n and let $\widetilde{\varphi}, \widetilde{\psi} : I \longrightarrow \mathbb{R}$ be the respective kilometrage functions. Consider the homotopy $\widetilde{H} : I \times I \longrightarrow \mathbb{R}$ given by*

$$\widetilde{H}(t, s) = t\widetilde{\varphi}(s) + (1 - t)\widetilde{\psi}(s)$$

Note that $H(t, 0) = 0$ for all t and $H(t, 1) = n$ for all t. Hence the map $H(t, s) = e^{2\pi i \widetilde{H}(t,s)}$ gives a homotopy from φ to ψ which keeps the end points in fixed. This gives us a pointed homotopy from f to g, and we are done.

Basic Notions in Algebraic Topology 3

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1 Categories and Functors

Let us start by recalling the basic definition:

Definition 1. A **category** C consists of the following data:

1. A collection of objects $Ob(C)$ (we usually just write C instead of $Ob(C)$).
2. For each two objects $X, Y \in C$ a set $\text{Hom}_C(X,Y)$. Elements in $\text{Hom}_C(X,Y)$ are called **morphisms** from X to Y and are written like functions $f : X \longrightarrow Y$.
3. for every 3 objects $X, Y, Z \in C$ a map

$$\text{Hom}_C(X,Y) \times \text{Hom}_C(Y,Z) \longrightarrow \text{Hom}_C(X,Z)$$

called **composition** and denoted by $(f,g) \mapsto g \circ f$.

4. For each $X \in C$ a morphism $Id_X \in \text{Hom}_C(X,X)$ called the **identity** morphism.

This data is required to satisfy the following axioms:

1. The composition is associative, i.e.

$$f \circ (g \circ h) = (f \circ g) \circ h$$

2. The identity elements are neutral, i.e.

$$Id_X \circ f = f$$

$$g \circ Id_X = g$$

Examples:

1. The most prototypical example of a category is the category $\mathcal{S}ets$ whose objects are sets and morphisms from X to Y are just functions from X to Y.
2. Let k be a field. Then we have the category $\mathcal{V}ec/k$ whose objects are vector spaces over k and morphisms from V to U are linear transformations from V to U.

3. The category whose objects are vector spaces over $\mathbb{R}$ with inner product and morphisms are orthonormal linear transformation.
4. The category $\mathcal{G}r$ whose objects are groups and morphisms are homomorphisms.
5. The category $\mathcal{A}b$ whose objects are abelian groups and morphisms are homomorphisms.
6. The category $\mathcal{T}op$ whose objects are topological spaces and morphisms are continuous maps.
7. The category $\mathcal{T}op_*$ whose objects are pointed topological spaces and morphisms are pointed continuous maps.
8. The category $\mathcal{H}o\mathcal{T}op$ whose objects are topological spaces and for each two spaces X, Y we set

   $$\mathrm{Hom}_{\mathcal{H}o\mathcal{T}op}(X, Y) = [X, Y]$$

   Note that the homotopy relation on maps respects composition so we have a well defined composition operation

   $$[X, Y] \times [Y, Z] \longrightarrow [X, Z]$$

9. The category $\mathcal{H}o\mathcal{T}op_*$ whose objects are pointed topological spaces and for each two pointed spaces X, Y we set

   $$\mathrm{Hom}_{\mathcal{H}o\mathcal{T}op_*}(X, Y) = [X, Y]_*$$

Definition 2. Let C, D be categories. A functor $F : C \longrightarrow D$ is a map $Ob(C) \longrightarrow Ob(D)$ together with a collection of maps

$$\mathrm{Hom}_C(X, Y) \longrightarrow \mathrm{Hom}_D(F(X), F(Y))$$

for each $X, Y \in Ob(C)$ which respect composition and send the identities to the identities. Given a morphism $f : X \longrightarrow Y$ we sometimes denote by $F(f)$ the corresponding morphism from $F(X)$ to $F(Y)$. Another common notation for $F(f)$ is f_*. The fact that F respects composition and identity can then be written as

$$F(f \circ g) = F(f) \circ F(g)$$
$$F(Id_X) = Id_{F(X)}$$

1. If the objects in our category are sets with extra structure (like groups, topological space, etc.) and the morphisms are just functions which respect this structure (like in the categories $\mathcal{G}r$,$\mathcal{T}op$, etc.) then we have a natural functor to $\mathcal{S}ets$ obtained by "forgetting" the extra structure. These functors are called **forgetful functors**. For example the forgetful functor from $\mathcal{G}r$ to $\mathcal{S}ets$ which sends each group G to its set of elements and any homomorphism to itself considered as a function between sets.

2. We can also have forgetful functors not into sets by forgetting just part of the structure, like the functor from pointed spaces to spaces which forgets the point.

3. We have the functor $\mathcal{G}r \longrightarrow \mathcal{A}b$ which sends every group G to it abelianization $G/[G,G]$. Note that if $T : G \longrightarrow H$ is a homomorphism of groups then it induces a natural map

$$T_* : G/[G,G] \longrightarrow H/[H,H]$$

because a homomorphism of groups sends commutators to commutators.

4. **Counterexample**: the construction which associates to each group G its center is **not** a functor - given a homomorphism of groups $T : G \longrightarrow H$ there isn't any sensible way to obtain a homomorphism

$$Z(G) \longrightarrow Z(H)$$

5. We have a functor $h : \mathcal{T}op \longrightarrow \mathcal{H}o\mathcal{T}op$ which sends each space X to itself and every map $f : X \longrightarrow Y$ to its homotopy class

$$f \mapsto [f] \in [X,Y] = \mathrm{Hom}_{\mathcal{H}opTop}(X,Y)$$

6. Same as a above but from $\mathcal{T}op_*$ to $\mathcal{H}o\mathcal{T}op_*$.

7. Let C be a category and $X \in C$ an object. We can construct a functor $C \longrightarrow \mathcal{S}ets$ as follows: to the object $Y \in C$ we will associate the set $\mathrm{Hom}_C(X,Y)$ and to each morphism $f : Y \longrightarrow Z$ we will associate the function

$$f_* : \mathrm{Hom}_C(X,Y) \longrightarrow \mathrm{Hom}_C(X,Z)$$

given by

$$g \mapsto f \circ g$$

This functor is called the functor **represented** by X.

8. Let $\mathcal{S}ets_*$ be the category of pointed sets and pointed maps between them. Then we have the functor $\pi_0 : \mathcal{T}op_* \longrightarrow \mathcal{S}ets_*$ given by

$$\pi_0(X, x_0) = [S^0, X]_*$$

This functor is obtained by composing the represented functor of S^0 in $\mathcal{H}o\mathcal{T}op_*$ with the functor $\mathcal{T}op_* \longrightarrow \mathcal{H}o\mathcal{T}op$ and then enriching the set $[S^0, X]_*$ with the structure of a pointed set.

9. The functor $\pi_1 : \mathcal{T}op_* \longrightarrow \mathcal{S}ets_*$ given by

$$\pi_1(X, x_0) = [S^1, X]_*$$

This functor is obtained by composing the represented functor of S^1 in $\mathcal{H}o\mathcal{T}op_*$ with the functor $\mathcal{T}op_* \longrightarrow \mathcal{H}o\mathcal{T}op$ and then enriching the set

$[S^1, X]_*$ with the structure of a group (if you are curious to how this mysterious phenomenon occurs, it's because the object S^1 is a **cogroup** object in the category $\mathcal{H}o\mathcal{T}op_*$. You are welcome to read more on the subject).

Remark 3. Let C, D be categories. Sometimes we would like to consider "opposite functors", i.e. functions $F : Ob(C) \longrightarrow Ob(D)$ together with functions $\mathrm{Hom}_C(X, Y) \longrightarrow \mathrm{Hom}_D(F(X), F(Y))$ such that

$$F(f \circ g) = F(g) \circ F(f)$$

(instead of $F(f) \circ F(g)$ like usual functors). There are 3 ways to describe the difference between these mathematical objects and regular functors:

1. Call them **cofunctors**.
2. Call them **contravariant** functors and call regular functors **covariant** functors.
3. Replace C be a category C^{op} given by

$$Ob(C^{op}) = Ob(C)$$
$$\mathrm{Hom}_{C^{op}}(X, Y) = \mathrm{Hom}_C(Y, X)$$

and then consider them as functors from C^{op} to D.

Examples:

1. Let C be a category and $X \in C$ and object. Then we have a cofunctor from C to $\mathcal{S}ets$ given by

$$F(Y) = \mathrm{Hom}_C(Y, X)$$

This is called the cofunctor **represented** by X.

2. Let k be a field and $\mathcal{V}ec/k$ the category of vectors spaces over k. Then we have a cofunctor $F : \mathcal{V}ec/k \longrightarrow \mathcal{V}ec/k$ given by

$$F(V) = V^*$$

where V^* is the dual space of V, i.e. the space of linear maps from V to k. This cofunctor is obtained by taking the cofunctor represented by k and enriching the set $\mathrm{Hom}_{\mathcal{V}ec/k}(V, k)$ with a structure of a vector space.

3. The cofunctors

$$H^0(X) = [X, \mathbb{Z}]$$
$$H^1(X) = [X, S^1]$$

from the category of topological spaces to the category of abelian groups. These cofunctors are obtained as a composition of the cofunctors represented by $\mathbb{Z}$ and S^1 in $\mathcal{H}o\mathcal{T}op$ with then functor $\mathcal{T}op \longrightarrow \mathcal{H}o\mathcal{T}op$. As in the previous examples we have an additional structure of an abelian group (which comes from the fact that $\mathbb{Z}$ and S^1 are topological abelian groups, or in the spirit of a previous remark, group objects in $\mathcal{T}op$) so we get functors to $\mathcal{A}b$ instead of $\mathcal{S}ets$.

2 Pushouts

Let C be a category and

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \psi} \\ Y & \xrightarrow{\varphi} & P \end{array}$$

a commutative diagram in C (i.e. A, X, Y are objects in C, f, g, φ, ψ are morphisms and $\varphi \circ g = \psi \circ f$). We say that this diagram is a **pushout diagram** (or a **pushout square**) if for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \psi'} \\ Y & \xrightarrow{\varphi'} & P' \end{array}$$

There exists a **unique** morphism $T : P \longrightarrow P'$ such that $\varphi' = T \circ \varphi$ and $\psi' = T \circ \psi$.

Remark 4. In a general category C we call a morphism $f : X \longrightarrow Y$ an **isomorphism** if there exists a $g : Y \longrightarrow X$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$ (examples: for $C = \mathcal{G}r$ you get isomorphism of groups, for $C = \mathcal{T}op$ you get homeomorphism and for $C = \mathcal{H}o\mathcal{T}op$ you get homotopy equivalence). Now if

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \psi} \\ Y & \xrightarrow{\varphi} & P \end{array}$$

and

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \psi'} \\ Y & \xrightarrow{\varphi'} & P' \end{array}$$

are both pushout diagrams then we get maps $T : P \longrightarrow P'$ and $T' : P \longrightarrow P'$. From uniqueness their composition has to be the identity, so we get an isomorphism $P \cong P'$. Again from uniqueness we see that this is the unique isomorphism satisfying $\varphi' = T \circ \varphi$ and $\psi' = T \circ \psi$. This observation is usually phrased by saying that given A, X, Y, f and g the object P is determined **uniquely** up to a **unique** isomorphism.

Examples:

1. Let $C = \mathcal{T}op$. Then the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \psi} \\ Y & \xrightarrow{\varphi} & P \end{array}$$

   is a pushout diagram iff $P = X \coprod Y / f(a) \sim g(a)$ and the maps φ, ψ are the maps which come from the natural inclusions of X and Y in $X \coprod Y$.

2. Let $C = \mathcal{A}b$. Then the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \psi} \\ Y & \xrightarrow{\varphi} & P \end{array}$$

   is a pushout diagram iff $P = X \oplus Y / < f(a) - g(a) >$ and the maps φ, ψ are the maps which come from the natural inclusions of X and Y in $X \oplus Y$.

3. Let $C = \mathcal{G}r$. Then the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \psi} \\ Y & \xrightarrow{\varphi} & P \end{array}$$

   is a pushout diagram iff $P = X * Y / < f(a)g(a)^{-1} >_{normal}$ and the maps φ, ψ are the maps which come from the natural inclusions of X and Y in $X * Y$.

4. **Counterexample**: in the categories $\mathcal{H}o\mathcal{T}op$ and $\mathcal{H}o\mathcal{T}op_*$ there exist diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \\ Y & & \end{array}$$

   which cannot be completed to pushout diagrams. This is one of the reasons why it is necessary to continue working in $\mathcal{T}op$ and $\mathcal{T}op_*$ even though we are really interested in $\mathcal{H}oTop$ and $\mathcal{H}oTop_*$. For example, the diagram

$$\begin{array}{ccc} S^1 & \longrightarrow & D^2 \\ {\scriptstyle 2}\downarrow & & \\ S^1 & & \end{array}$$

   doesn't have a pushout in $\mathcal{H}o\mathcal{T}op_*$. It does, of course, have a pushout it $\mathcal{T}op_*$, which is just $\mathbb{R}P^2$.

3 Van Kampen's Theorem

Theorem 5. *Let X, x_0 be a pointed topological space and $U_1, U_1 \subseteq X$ open subsets containing x_0 such that $X = U_1 \cup U_2$ and $A = U_1 \cap U_2$ is path connected. Then the diagram*

$$\begin{array}{ccc} \pi_1(A, x_0) & \xrightarrow{i_{1*}} & \pi_1(U_1, x_0) \\ \downarrow{\scriptstyle i_{2*}} & & \downarrow{\scriptstyle j_{1*}} \\ \pi_1(U_2, x_0) & \xrightarrow{j_{2*}} & \pi_1(X, x_0) \end{array}$$

is a pushout square (where $i_1 : A \longrightarrow U_1, i_2 : A \longrightarrow U_2, j_1 : U_1 \longrightarrow X$ and $j_2 : U_2 \longrightarrow X$ are all the natural inclusions).

Remark 6. Note that in the situation of the theorem the following diagram

$$\begin{array}{ccc} A & \xrightarrow{i_1} & U_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ U_2 & \xrightarrow{j_2} & X \end{array}$$

is a pushout square in the category $\mathcal{T}op_*$. Hence what Van-Kampen's theorem says is that under certain conditions the functor π_1 maps pushout squares to pushout square.

Remark 7. Note that the assumption that A is path connected is essential. For example if $X = S^1$, $U_1 = S^1 \setminus \{1\}$ and $U_2 = S^1 \setminus \{-1\}$ then U_1, U_2 and A are all simply connected but X has a non-trivial π_1. Hence it is not true that π_1 sends **every** pushout square to a pushout square, i.e. it doesn't preserve pushouts in general.

Let us now show various applications of Van Kampen's theorem.

3.1 The Fundamental Group of CW-complexes

The first application of Van Kampen to the computation of CW-complexes. For simplicity we will prove this for finite CW complexes, but it is true in general.

Theorem 8. *Let X be a CW-complex with skeletons X_n. Let $x_0 \in X_n$ be a point. Then the natural map*

$$\pi_1(X_2, x_0) \longrightarrow \pi_1(X, x_0)$$

induced by the inclusion $X_2 \hookrightarrow X$ is an isomorphism.

Proof. We will show using Van Kampen that adding a cell e^n of dimension $n \geq 3$ will not change the fundamental group. Then by induction we will get the desired result for finite CW complexes. The passage to general CW complexes is through weak topology and compactness arguments (exercise).

Let X be a space and $f : S^{n-1} \longrightarrow X$ with $n \geq 3$. Let $Y = X \cup_f e^n$. Let $p \in e^n$ be a point which lies in the interior of the cell. Let $p \in U$ be a small disc around p. Then

$$Y = U \cup (Y \setminus \{p\})$$

and the intersection

$$U \cap (Y \setminus \{p\}) = U \setminus \{p\}$$

is a punctured disc. Since $n \geq 3$ we get that $U \setminus \{p\} \simeq S^{n-1}$ is simply connected. Let $y_0 \in U \setminus p$ be any point. Since U is contractible we get from Van Kampen's theorem that

$$\pi_1(Y \setminus \{p\}, y_0) \cong \pi_1(Y, y_0)$$

via the map induced by the inclusion $Y \setminus \{p\} \hookrightarrow Y$. This statement will remain valid if we change y_0 to any x_0 which is in the same connected component of y_0. Since $Y \setminus \{p\}$ deformation retracts to X (because a punctured closed disc deformation retracts to its boundary) we can take instead some $x_0 \in X$ and we get the the composition

$$\pi_1(X, x_0) \xrightarrow{\simeq} \pi_1(Y \setminus \{p\}, x_0) \xrightarrow{\simeq} \pi_1(Y, x_0)$$

is an isomorphism as we wanted. □

Hence in order to understand the fundamental group of a CW-complexes it is enough to understand the fundamental groups of 2-dimensional CW-complexes. We begin with 1-dimensional CW complexes:

Theorem 9. *Let X be a 1-dimensional connected CW complex. Then X is homotopy equivalent to a wedge of circles and so its fundamental group is free (with one generators for each circle).*

Proof. Again we prove for finite CW's but the proof can be extended to the general case. A finite connected CW complex is just a finite connected graph. Every connected graph has a spanning tree (i.e. a sub graph containing all the vertices which is connected and has no non-trivial circles). For finite graphs there is a simple argument: if the graph is not a tree then it has a non-trivial circle. Choose one of the edges in that circle and remove it. Clearly the connectivity has not been compromised. Since the graph is finite this process has to stop, giving us a spanning tree.

Hence we can think of X as a tree T with a bunch of extra 1-cells. Note that a tree is always pointedly contractible. Adding one cell is taking the cone of a map from $S^0 \longrightarrow T$. Since T is connected each such map is homotopic to a constant map. Gluing via a constant map is like wedging with a circle. Hence we get that X is homotopy equivalent to T wedge a bunch of circles, and since T is pointedly contractible this is homotopy equivalent to a wedge of circles. □

Now let's see what is the effect of adding a 2-cell. Let $X_1 = S^1 \vee ... \vee S^1$ with $X_0 = \{x_0\}$ being the joining point of all the circles and let $f : S^1 \longrightarrow X_1$

be a gluing map for a new 2-cell. By maybe changing f by a homotopy we can assume that it is pointed, i.e. that $f(1) = x_0$ (use the homotopy extension property - see exercise 2 question 2). Hence we can construct an element

$$[f] \in \pi_1(S^1 \vee ... \vee S^1, x_0) =< a_1, ..., a_n >$$

Let $Y = X_1 \cup_f e^2$ and let $N \triangleleft < a_1, ..., a_n >$ be the minimal normal subgroup containing $[f]$. Then we claim that

$$\pi_1(Y, x_0) =< a_1, ..., a_n > /N$$

This follows again easily from Van Kampen's theorem. As before let U be the interior of the 2-cell of Y and let p be a point in U. Then we can represent Y as

$$Y = (Y \setminus \{p\}) \cup U$$

and the intersection is

$$U \setminus \{p\} \simeq S^1$$

Let $y_0 \in Y$ be a base point contained un $U \setminus \{p\}$. Since U is contractible we get from Van Kampen's theorem that $\pi_1(Y, y_0)$ is isomorphic to the quotient of $\pi_1(Y \setminus \{p\}, y_0)$ by the normal subgroup generated by the image of

$$\pi_1(U \setminus \{p\}, y_0) \longrightarrow \pi_1(Y \setminus \{p\}, y_0)$$

Now $\pi_1(U \setminus \{p\}, y_0) \cong \mathbb{Z}$ and the image of one of the generators is exactly $[f]$.

Hence we see that when understanding 2-dimensional CW complexes we need to treat the 1-skeleton as giving generators and each 2-cell is adding a relation which is given by the gluing map.

Remark 10. Note that we had some freedom in choosing $[f]$ - we could have chosen any other element $[f']$ such that $f' \sim f$ in an **unpointed** homotopy. This would result in $[f], [f']$ being possibly different yet **conjugated** elements in $\pi_1(X_1, x_0)$, so that the minimal normal subgroup they generate is the same.

Corollary 11. *All the sphere S^n of dimension $n \geq 2$ are simply connected.*

3.2 The Fundamental Groups of Surfaces

As an application of the discussion above let us compute the fundamental group of the surface M_g of genus g (surface with g handles). We will do so by funding a CW structure for M_g. This CW structure will have a single vertex x_0 and $2g$ edges, so it's 1-skeleton is a wedge of $2g$ circles and in particular

$$\pi_1((M_g)_1, x_0) =< a_1, b_1, ..., a_g, b_g >$$

In order to explain how to locate this 1-skeleton inside M_g we refer the reader to the example of $g = 2$ showed in the picture below (the 1-skeleton is drawn on the surface and is a wedge of 4 circles, make sure you see how). Note that when we remove the 1-skeleton the remaining open set "unfolds" to an (open) polygon with $4g$ edges. When you glue this polygon back to a surface each edge is mapped to one of the 1-cells. If you follow the order of the 1-cells and the direction of the gluing you get a gluing map

$$[f] = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}$$

Hence by the discussion above we get that $\pi_1(M_g, x_0)$ is the quotient of the free group $< a_1, b_1, ..., a_g, b_g >$ modulu the normal subgroup generated by the element $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}$. In other words, it is the group generated by $a_1, b_1, .., a_g, a_g$ modulue the relation that the product of commutators $[a_i, b_i]$ is trivial. For example in the case $g = 1$ (the 2-torus) we get the abelian group on two generators, i.e. $\mathbb{Z} \oplus \mathbb{Z}$.

Basic Notions in Algebraic Topology 4

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1 The Action of the Fundamental Group on the Fiber

Let (B, b_0) be pointed path connected space. Let $p : E \longrightarrow B$ be a covering map and let $F = f^{-1}(b_0)$. We call F the **fiber** of p (since B is pointed we don't need to say "over b_0"). In class you saw the most basic two properties of covering spaces:

1. Given a path $\gamma : I \longrightarrow B$ starting at b_0 and a point $e \in F$ there exists a unique path $\widetilde{\gamma}_e : I \longrightarrow E$ starting at e such that $p \circ \widetilde{\gamma}_e = \gamma$. Note that if γ happens to be a closed path it doesn't mean that $\widetilde{\gamma}_e$ will be a closed path.
2. Let γ, δ be two paths from b_0 to b_1. Given an end-point-preserving homotopy H from γ to δ and any lift $\widetilde{\gamma} : I \longrightarrow E$ of γ there exists a unique end-point-preserving homotopy $\widetilde{H}$ from $\widetilde{\gamma}$ to $\widetilde{\delta}$ such that $p \circ \widetilde{H} = H$.

Immediate conclusions:

1. given an element $\alpha \in \pi_1(B, b_0)$ represented by a closed path γ the value $\widetilde{\gamma}_e(1) \in E$ is **well-defined** and does not depend on choice if the representative γ. We denote it by
$$\widetilde{\gamma}_e(1) = \alpha(e)$$
2. For any $e \in F$ the induced map $p_* : \pi_1(E, e) \longrightarrow \pi_1(B, p(e))$ is injective. In the rest of the notes we will denote this homomorphism by p_*^e to indicate that it came from taking e as a base point for E.

   Proof. Let $\widetilde{\gamma} : I \longrightarrow E$ be a closed path from e to e such that $\gamma = p \circ \widetilde{\gamma}$ is end-point homotopic to the constant path. By property 2 this homotopy can be lifted to E, resulting in a homotopy from $\widetilde{\gamma}$ to a path δ satisfying the property that $p \circ \delta$ is constant. This means that the image of δ is contained in F. Since F is discrete δ is constant. This means that $\widetilde{\gamma}$ is end-points homotopic to a constant path and we are done. □

Example: When we calculated the fundamental group of S^1 we used the covering map $p : \mathbb{R} \longrightarrow S^1$ given by $p(x) = e^{2\pi i x}$. The "kilometrage" functions that we've defined were nothing but the lifts of paths in S^1 to paths starting at 0. The degree of a closed path φ based at $1 \in S^1$ is just the value $\widetilde{\varphi}(1) \in \mathbb{R}$, and we saw that was indeed invariant to homotopy.

Lemma 1. *1. If $\alpha, \beta \in \pi_1(B, b_0)$ are two elements then $(\alpha * \beta)(e) = \beta(\alpha(e))$ for every $e \in F$.*

2. If $\alpha = 1 \in \pi_1(B, b_0)$ is the neutral element then $\alpha(e) = e$ for every $e \in F$.

Proof. Note that in order to prove the two claims with one stone it is enough to show that

Let $\alpha, \beta \in \pi_1(B, b_0)$ be two elements represented by closed paths γ, δ respectively. Then

$$(\alpha * \beta)(e) = \widetilde{\gamma * \delta}_e(1) = (\widetilde{\gamma}_e * \widetilde{\delta}_{\alpha(e)})(1) = \widetilde{\delta}_{\alpha(e)}(1) = \beta(\alpha(e))$$

Further more if $\alpha = 1 \in \pi_1(B, b_0)$ is the neutral element then it can be represented by a constant path γ. This γ lifts to a constant path $\widetilde{\gamma}$ in E, which means that $\alpha(e) = e$. □

Corollary 2. *Given a element $\alpha \in \pi_1(B, b_0)$ the map $F \longrightarrow F$ given by $e \mapsto \alpha(e)$ is a permutation of F (it has an inverse induced by α^{-1}). Further more these permutations form a **right** action of $\pi_1(B, b_0)$ on F.*

This homomorphism is a basic tool in understanding the fundamental group of a space. In particular one can show that a loop in B is non-trivial by showing that it induces a non-trivial permutation on F. A natural question now is to understand how much of the fundamental group is captured by this action.

Definition 3. Given $e \in E$ and $\alpha \in \pi_1(B, b_0)$ we say that α **fixes** e if $\alpha(e) = e$. The set of all elements $\alpha \in \pi_1(B, p(e))$ that fix e is called the **stabilizer** of e. This is a **subgroup** of $\pi_1(B, b)$.

Now the issue of how much information the action remembers is settled in the following theorem:

Theorem 4. *Let $\alpha \in \pi_1(B, b_0)$ be an element and $e \in F$ a point. Then $\alpha(e) = e$ if and only if there exists a $\beta \in \pi_1(E, e)$ such that $p_*^e(\beta) = \alpha$. In other words the **stabilzer** of e in $\pi_1(B, b_0)$ is exactly the image of the monomorphism p_*^e.*

Proof. The proof is quite immediate - by definition we see that $\alpha(e) = e$ if and only if α can be represented by a closed path γ such that $\widetilde{\gamma}_e(1) = e$, i.e. such that $\widetilde{\gamma}_e$ is a **closed** path. Since lifts are unique this is the same as saying that there **exists** a closed path $\widetilde{\gamma}$ from e to e such that $p \circ \widetilde{\gamma} = \gamma$. But this is equivalent to α being in the image of

$$p_*^e : \pi_1(E, e) \longrightarrow \pi_1(B, b_0)$$

□

Corollary 5. *The* ***kernel*** *of the action (i.e. the subgroup of elements that act as the identity on F) is the intersection*

$$\cup_{e \in F} Im\,(p^e_*)$$

Corollary 6. *If E is simply connected then the action is faithful, i.e. $\alpha \in \pi_1(B, b_0)$ is completely determined by it's action on F. Note that the action might be faithful even when E is not simply connected.*

1.1 Example - the Borsuk-Ulam Theorem in dimension 2

Theorem 7. *Let $f : S^2 \longrightarrow \mathbb{R}^2$ be a continuous map. Then there exist a point $x \in S^2$ such that $f(x) = f(-x)$.*

Proof. Suppose that there was an $f : S^2 \longrightarrow \mathbb{R}^2$ such that $f(x) \neq f(-x)$ for every $x \in S^2$. Consider the map

$$g : S^2 \longrightarrow S^1$$

given by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

This is a well defined continuous function because we've assumed that $f(x) - f(-x) \neq 0$ for every x. Now the function g satisfies $g(-x) = -g(x)$ and so it induces a well-defined map $\overline{g} : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^1$ which fits in a commutative diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{g} & S^1 \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_1} \\ \mathbb{R}P^2 & \xrightarrow{\overline{g}} & \mathbb{R}P^1 \end{array}$$

Let $x_0 \longrightarrow \mathbb{R}P^2$ be a base point and $y_0 = \overline{g}(x_0)$. We claim that $\overline{g}$ must induce an **injective** map $\overline{g}_* : \pi_1(\mathbb{R}P^2, x_0) \longrightarrow \pi_1(\mathbb{R}P^2, y_0)$. This will result in our desired contradiction because $\pi_1(\mathbb{R}P^2, x_0) \cong \mathbb{Z}/2$ and $\pi_1(\mathbb{R}P^1, y_0) \cong \mathbb{Z}$ and there are not injective maps from $\mathbb{Z}/2$ to $\mathbb{Z}$. It is left to show the injectivity of $\overline{g}_*$.

Let $\alpha \in \pi_1(\mathbb{R}P^2, x_0)$ be a non-trivial element. Since S^2 is simply connected the action of α on the fiber of $S^2 \longrightarrow \mathbb{R}P^2$ is non-trivial. Hence $\alpha = [\gamma]$ lifts to an open path $\widetilde{\gamma}$ in S^2 connecting a pair $x, -x$ of antipodal points. This means that $g \circ \widetilde{\gamma}$ is an open path connecting the antipodal points $g(x)$ and $g(-x) = -g(x)$, which implies that $p_1 \circ g \circ \widetilde{\gamma}$ is a closed path in $\mathbb{R}P^1$ representing a non-trivial element in $\pi_1(\mathbb{R}P^1, y_0)$. But

$$[p_1 \circ g \circ \widetilde{\gamma}] = [\overline{g} \circ p_2 \circ \widetilde{\gamma}] = [\overline{g} \circ \gamma] = \overline{g}_* \alpha$$

so $\overline{g}_* \alpha \in \pi_1(\mathbb{R}P^1, y_0)$ is non-trivial. This means indeed that the homomorphism

$$\overline{g}_* : \pi_1(\mathbb{R}P^2, x_0) \longrightarrow \pi_1(\mathbb{R}P^1, y_0)$$

is injective. □

2 Coverings Obtained by Group Actions

Let G be a group acting from the left on a space E. We say that the action of G is a **covering space action** if for every $e \in E$ there exists an open neighborhood $e \in U$ such that for every $g \neq 1 \in G$ we have

$$g(U) \cup U = \emptyset$$

In particular a covering space action is always **free**. It is not hard to show that if we have a covering space action then the resulting map $p : E \longrightarrow E/G \stackrel{def}{=} B$ is a covering map. Let $b_0 \in B$ be a point. We will want to understand what will be the connection between $\pi_1(B, b_0)$ to G and what is the connection their actions on the fiber $F = p^{-1}(b_0)$.

We start with the following observation:

Lemma 8. *The actions of $\pi_1(B, b_0)$ and G on F **commute**, i.e. for each $e \in F$, $\alpha \in \pi_1(B, b_0)$ and $g \in G$ one has*

$$g(\alpha(e)) = \alpha(g(e))$$

Proof. Let γ be a closed path in B representing α. Then by definition one has

$$\alpha(e) = \widetilde{\gamma}_e(1)$$

Now consider the path $g \circ \widetilde{\gamma}_e$ (in this natation we consider g as a transformation from E to E). Then we get that

$$p \circ g \circ \widetilde{\gamma}_e = p \circ \widetilde{\gamma}_e = \gamma$$

and $(g \circ \widetilde{\gamma}_e)(0) = g(e)$. Hence by the uniqueness of path lifts we get that

$$g \circ \widetilde{\gamma}_e = \widetilde{\gamma}_{g(e)}$$

which means that

$$\alpha(g(e)) = \widetilde{\gamma}_{g(e)}(1) = g(\widetilde{\gamma}_e(1)) = g(\alpha(x))$$

□

Now fix a point $e \in F$. For each $\alpha \in \pi_1(B, b_0)$ there is a unique $g \in G$ such that $g(e) = \alpha(e)$. We denote this g by T^e_α. From 8 we get that

$$T^e_{\alpha * \beta}(e) = (\alpha * \beta)(e) = \beta(\alpha(e)) = \beta(T^e_\alpha(e)) =$$

$$T^e_\alpha(\beta(e)) = T^e_\alpha(T^e_\beta(e))$$

which means that $T^e_{\alpha*\beta} = T^e_\alpha \cdot T^e_\beta$ and so the map $\alpha \mapsto T^e_\alpha$ is a homomorphism. We will denote this homomorphism simply by T^e. By Theorem 4 we see that the kernel of T_e is exactly the image of p^e_*. It also not hard to show that the image of T_e is exactly the subgroup of G which preserves the connected component of e (see question 3 in exercise 4). In particular if E is connected we get a "decomposition":

$$\pi_1(E, e) \overset{p_*^e}{\hookrightarrow} \pi_1(B, b_0) \overset{T^e}{\twoheadrightarrow} G$$

We now come to the connection between the action of G versus the action of $\pi_1(B, b_0)$ on F. By choosing an $e \in F$ we can identify F with G via the identification $ge \leftrightarrow g$. Under this identification we see that G acts on F by multiplication on the **left**. Since

$$\alpha(g(e)) = g(T_\alpha^e(e))$$

We see that under this identification $\pi_1(B, b_0)$ acts on $F \leftrightarrow G$ by multiplication on the **right**, through the homomorphism T^e. This gives a full description of the relationship between $\pi_1(B, b_0), G$ and their actions on F.

Basic Notions in Algebraic Topology - 5

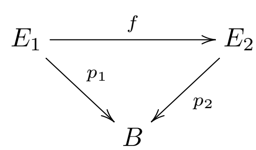
Yonatan Harpaz

1 Classification of Covering Spaces

In this TA class we'll talk about the classification theory of covering spaces. What do we mean by a classification? Consider a topological space B. We want to understand all the covering spaces $E \longrightarrow B$. Note that this is not only a collection of objects. We have interesting **maps** between them. If $p_1 : E_1 \longrightarrow B$ and $p_2 : E_2 \longrightarrow B$ are two covering maps then we would like to consider maps $f : E_1 \longrightarrow E_2$ which respect p_1 and p_2, i.e. which satisfy the property

$$p_2 \circ f = p_1$$

This can be written diagramically as



such maps will be called **maps of covering spaces**. Note that the identity is always a map of covering spaces and that the composition of two maps of covering spaces is again a map of covering spaces. Hence they form what is know as a **category**.

Given a space B we want to understand this category. In this TA class we will prove that if B is a nice enough space then the category of covering spaces over B is equivalent, in a sense we shall explain exactly, to a rather "simple" category constructed from the fundamental group of B.

1.1 Equivalence of Categories

Let $\mathcal{C}, \mathcal{D}$ be two categories. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

We say that F is **fully faithful** if for every $X, Y \in \mathcal{C}$ F gives a bijection

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \stackrel{\simeq}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

where $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ denotes the set of maps (or morphisms) between X and Y in $\mathcal{C}$.

We say that F is **essentially surjective** if for every object $X \in \mathcal{C}$ there is an object $Y \in \mathcal{D}$ such that $F(Y)$ is **isomorphic** to X. Note that we don't require $F(Y)$ to be actually equal to X. Since categories are usually very big (in general one does not require the collection of objects to be set in the set theoretic sense...) and it is somehow not sensible to ask for two objects to be equal, only isomorphic.

We say that F is an **equivalence of categories** if it is fully faithful and essentially surjective. We say that two categories are equivalent if there is a functor between them that is an equivalence. Note that we don't require a functor in the other direction.

1.2 The Category of G Sets

Let G be a group. We define Set_G to be the category whose objects are sets X with an action of G (a.k.a G-sets) and maps are maps of sets which preserve the action of G. Let X be a G-set. We define an equivalence relation on X by saying that $x \sim y$ if there exists a $g \in G$ such that $g(x) = y$.

The equivalence class of a point $x \in X$ is called the **orbit** of x. A G-set in which every two points are equivalent is called transitive. Every G-set can partitioned to a disjoint union of equivalence classes, i.e. to a disjoint union of transitive G-sets. Hence we can construct all G-sets from the transitive ones using the operation of disjoint union.

But what are all the transitive G sets? In order to understand the transitive G-sets we will first understand how do maps out of transitive G-sets look like. Let X, Y be G-sets with X transitive. Fix a point $x \in X$. Since X is transitive any G-map $f : X \longrightarrow Y$ is determined by the value of x. Further more if $f : X \longrightarrow Y$ is a G-map and $s \in \mathrm{St}_x$ then $s(f(x)) = f(s(x)) = f(x)$ so $s \in \mathrm{St}_{f(x)}$. This means that $\mathrm{St}_x \subseteq \mathrm{St}_y$. Now suppose that $y \in Y$ is such that $\mathrm{St}_x \subseteq \mathrm{St}_y$. Is there a G-map f such that $f(x) = y$. The answer is yes.

Given such $y \in Y$ construct f as follows: for every $x' \in X$ there exists some $g \in G$ such that $g(x) = x'$. We then are forced to define $f(x') = g(f(x))$. We need to show that this is well defined. If $g'(x) = g(x) = x'$ then $(g'^{-1}g)(x) = x$ and hence $g'^{-1}g \in \mathrm{St}_x$. By assumption $g'^{-1}g \in \mathrm{St}_y$ and so $g'(y) = g(y)$ as well. Hence our map is well defined.

To conclude we see the following: the set of G-maps from X to Y is in one-to-one correspondence with the set of elements $y \in Y$ such that $\mathrm{St}_x \subseteq \mathrm{St}_y$.

Now consider the following example of a transitive G-set: let $H \subseteq G$ be a subgroup and let X be the set of right cosets of H, i.e. of subsets of the form hH for $h \in G$. We denote this subset by G/H.

G acts G/H by multiplication on the left, i.e. the element $g \in G$ sends the coset hH to the coset ghH. This action is clearly transitive: if we have two cosets h_1H, h_2H then the element $h_2h_1^{-1} \in G$ sends the first to the second. Note that the stabilizer of the coset H is H: if $gH = H$ then $g \in H$.

We claim that every transitive G-set is isomorphic to a G-set of this form. If X is a transitive G-set then choose some $x \in X$. Then we claim that X is isomorphic to the G-set G/St_x. From the criterion above there exists a map $f : X \longrightarrow G/\mathrm{St}_x$ which sends x to the coset $\mathrm{St}_x \in G/\mathrm{St}_x$. There is also a map $g : G/\mathrm{St}_x \longrightarrow X$ which sends the coset St_x to x. Then the composition

$g \circ f$ is a map from X to X which sends x to itself. Since G-maps out of X are determined by the image of x this map has to be the identity. Similarly $f \circ g$ is the identity. Hence X is isomorphic to G/St_x.

To conclude, we see that the category of G sets is fairly simple: every object is a disjoint union of transitive objects and all the transitive objects can be describe by a simple construction from within the group. For example, if G is finite then it has only a finite set of isomorphism classes of transitive objects.

1.3 The Classification of Coverings

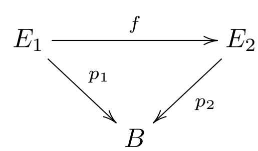
We now come to the main theorem:

Theorem 1. *Let B be a Hausdorff connected, locally connected space which admits a simply connected covering space. Fix a base point $b \in B$ and let $\pi = \pi_1(B, b)$. Then the category of covering of X is equivalent to the category of π-sets.*

Proof. For every covering $p : E \longrightarrow B$ we know that there is an action of π on the fiber $p^{-1}(b)$, i.e. $p^{-1}(b)$ has a structure of a π-set. We define a functor F from the category of coverings of B to the category of π-sets by setting

$$F(E) = p^{-1}(b)$$

We also need to associate to every map



of covering spaces a map $F(f)$ of π-sets. Clearly every such f maps $p_1^{-1}(b)$ to $p_2^{-1}(b)$. We will show that this maps preserves the action of π. If $\alpha \in \pi$ and $x, y \in p_1^{-1}(b)$ are such that α sends x to y then α lifts to a path $\widetilde{\alpha}$ from x to y. Then $f \circ \widetilde{\alpha}$ is a path from $f(x)$ to $f(y)$ which is a lift of α to E_2. Hence α sends $f(x)$ to $f(y)$ and we see that f is a map of π-sets. This means that actually construct F as a functor.

We need to show that F is fully faithful and essentially surjective.

Fully Faithful:

First note that if F maps connected covering spaces to transitive π-sets. Further more if E is a covering space which is a disjoint union of the connected covering space $E_1, ..., E_m$ then F will map E to the disjoint union of the union of the transitive π-sets $F(E_1), ..., F(E_n)$. Note that both in the case of covering spaces and in the case of π-sets, a map out from a disjoint union $X \cup Y$ to some Z is just a pair of maps from X to Z and from Y to Z. This means that it is enough to show that fully faithfulness for pairs of coverings E_1, E_2 where E_1 is connected.

Now that $F(E_1)$ is transitive we can fix an $x \in F(E_1)$ and by the above considerations the maps from $F(E_1)$ to $F(E_2)$ are in one-to-one correspondence with elements $y \in F(E_2)$ such that $\mathrm{St}_x \subseteq \mathrm{St}_y$. Hence what we need to show is the following:

Theorem 2. *Let $p_1 : E_1 \longrightarrow B, p_2 : E_2 \longrightarrow B$ be coverings of a locally connected base B and assume E_1 is connected and Hausdorf. Let $e_1 \in p_1^{-1}(b), e_2 \in p_2^{-1}(b)$. Then there exists a map of covering spaces $f : E_1 \longrightarrow E_2$ sending e_1 to e_2 if and only if $\mathrm{St}_{e_1} \subseteq \mathrm{St}_{e_2}$. Further more in that case there is a unique such f.*

Here St_e means the stabilizer of e in π with respect to the action of π on $p^{-1}(b)$. Since we are very ambitious we will prove an even more general result:

Theorem 3. *Let $p : E \longrightarrow B$ be a covering space and $f : X \longrightarrow B$ a map from a connected locally connected Hausdorf space X. Let $b \in B, x \in f^{-1}(b)$ and $e \in p^{-1}(b)$. Then there exists a lift*

$$\begin{array}{ccc} & & E \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow p \\ X & \xrightarrow{\ f\ } & B \end{array}$$

sending x to e if and only if the image of

$$f_* : \pi_1(X, x) \longrightarrow \pi_1(B, b)$$

is in the stabilizer of e. Further more in that case such a lift is unique.

Proof. Clearly if a lift $\widetilde{f}$ exists then we get a commutative diagram

$$\begin{array}{ccc} & & \pi_1(E, e) \\ & \overset{\widetilde{f}_*}{\nearrow} & \big\downarrow p_* \\ \pi_1(X, x) & \xrightarrow{\ f_*\ } & \pi_1(B, b) \end{array}$$

which means that the image of f_* is contained in the image of p_* and so fixes e. In the other direction assume that the the image of f_* fixes e. We shall construct $\widetilde{f}$. In order to define $\widetilde{f}$ on a point $y \in X$ we choose a path φ from x to y (recall that X is connected). Then $\overline{\varphi} = f \circ \varphi$ is a path from b to $f(y)$. Let $\widetilde{\overline{\varphi}}$ be the lift of $\overline{\varphi}$ that starts at e and define $\widetilde{f}(y) = \widetilde{\overline{\varphi}}(1)$.

We need to show that this is well defined. First of all from the homotopy lifting property we see that $\widetilde{\overline{\varphi}}(1)$ doesn't change if change φ by an end points preserving homotopy. Now suppose that we took some non-homotopic ψ from x to y instead of φ and let $\overline{\psi} = f \circ \psi$. Then ψ is end points homotopic to

$$\psi * \varphi^{-1} * \varphi$$

So we can use it instead. Note that

$$f \circ (\psi * \varphi^{-1} * \varphi) = (f \circ (\psi * \varphi^{-1})) * \overline{\varphi}$$

Now $\psi * \varphi^{-1}$ is a closed path from x to itself so it defines an element $[\psi * \varphi^{-1}] \in \pi_1(X, x)$. The path $f \circ (\psi * \varphi^{-1})$ is the closed path defining the elements $f_*[\psi * \varphi^{-1}]$.

From our assumption $f_*[\psi * \varphi^{-1}]$ needs to stabilize e. Hence $f \circ (\psi * \varphi^{-1})$ lifts to a path starting and ending at e. This means that if we lift $(f \circ (\psi * \varphi^{-1})) * \overline{\varphi}$ to start at e it will end in $\widetilde{\overline{\varphi}}(1)$. Hence $\widetilde{f}(y)$ is well defined.

It is clear that $p \circ \widetilde{f} = f$. We need to show though that $\widetilde{f}$ is at all continuous. Here we will use the fact that X is locally connected. Let $y \in X$ be a point and U be a neighborhood of $f(y)$ such that

$$p^{-1}(U) = \bigcup_i V_i$$

with all V_i's disjoint and $p|_{V_i}$ a homeomorphism. Let W be a connected neighborhood of y such that $f(W) \subseteq U$. Suppose that $\widetilde{f}(y) \in V_i$. Let $q : U \longrightarrow V_i$ be the inverse of $p|_{V_i}$. Then we claim that

$$\widetilde{f}|_W = q \circ f|_W$$

This would imply that $\widetilde{f}$ is continuous because it will be continuous when restricted to some open covering.

We will now prove that claim. Let $y' \in W'$ be any point. Since W is connected there exists a path ρ from y to y' inside W. Hence if φ was the path from x to y used to define $\widetilde{f}(y)$ then we can use $\varphi * \rho$ in order to define $\widetilde{f}(y')$.

Let $\overline{\rho} = f \circ \rho$. Then $\overline{\rho}$ is a path from $f(y)$ to $f(y')$ inside U. Note that $q \circ \overline{\rho}$ constitutes a lift of $\overline{\rho}$ to a path in E starting at $\widetilde{f}(y)$ and so is the unique such lift. Hence

$$\widetilde{\overline{\varphi}} * (q \circ \overline{\rho})$$

is a lift of $\overline{\varphi} * \overline{\rho}$ to a path starting at e. Hence

$$\widetilde{f}(y') = (\widetilde{\overline{\varphi}} * (q \circ \overline{\rho}))(1) = (q \circ \overline{\rho})(1) = q(\overline{\rho}(1)) = q(f(y'))$$

It is left to show uniqueness. Here we will again use the fact that X is hausdorf and connected. Suppose that $\widetilde{f}, \widetilde{f}'$ are two lifts of f sending x to e. Let $S \subseteq X$ be the set of elements $y \in X$ where $\widetilde{f}(y) = \widetilde{f}'(y)$. Since X is Hausdorf S is closed. But from the argument above is $\widetilde{f}, \widetilde{f}'$ agree on y then they agree on some neighborhood W of y. Hence S is also open. Since X is connected and S contains x we see that $S = X$ and $\widetilde{f} = \widetilde{f}'$. □

Essentially surjective: For this we use the fact that B admits a simply connected covering $p : E \longrightarrow B$. Note that since F takes disjoint unions to disjoint unions we only need to show that we can realize in the image of F all the isomorphism types of transitive π-sets.

This is done as follows. For every subgroup $H \subseteq \pi$ we will construct a covering $E_H \longrightarrow B$ such that $F(E_H)$ is isomorphic to π/H. Since E is simply connected it is a normal covering and the group of deck transformations is isomorphic to π. More explicitly by fixing a point $e \in p^{-1}(b)$ we get a specific identification of π with the group of deck transformation. In this identification if T is the deck transformations corresponding to $\alpha \in \pi$ then $T(e) = \alpha(e)$.

Now let E_H be the quotient of E by the action of H obtained by restricting the action of π (under that identification). To be more specific quotient E by the equivalence relation under which $x \sim y$ if there exists an element $h \in H$ such that $h(x) = y$. Since H acts by deck transformations we see that if $x \sim y$ then $p(x) = p(y)$. Hence p induces a well defined map

$$\overline{p} : E_H \longrightarrow B$$

Since p was a covering it is relatively easy to show that $\overline{p}$ is a covering. Also E_H is clearly still connected. Now let $\overline{e}$ be the image of $e \in E$ in E_H. Then by definition we see that the stabilizer of $\overline{e}$ in π is exactly H. Hence we are done.

$\square$

Basic Notions in Algebraic Topology 6

Yonatan Harpaz

1 Pullbacks

We start by reviewing an important concept that appeared in a previous exercise: consider a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

The **pullback** of this diagram is the subspace $P \subseteq X \times Y$ given by

$$P = \{(x, y) \in (X, Y) | f(x) = g(y)\}$$

Note that we have natural maps $p_x : P \longrightarrow X, p_Y : P \longrightarrow Y$ given by projections, and these maps fit together into a diagram

$$\begin{array}{ccc} P & \xrightarrow{p_X} & Y \\ \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & B \end{array}$$

In fact it is not hard to show that P together with p_X, p_Y satisfies a universal property analogous to that of the pushout - for every commutative diagram

$$\begin{array}{ccc} Q & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & B \end{array}$$

there is a unique map $f : Q \longrightarrow P$ making everything commute. In fact this is exactly the same universal property defining pushout, only that the directions of all the maps are reversed. In category theory we say that pullback is the dual notion of pushout.

An important property of pullbacks is that many nice properties of g are inherited to p_X (and the same for f and p_Y). For example if $Y = B \times F$ for some space F and g is the projection on the first coordinate then the pullback would be just $X \times F$ with p_Y being the projection on on the first coordinate.

Applying this argument locally one sees that if $f : Y \longrightarrow B$ is a covering map then so is p_X. In this case we also denote the pullback P by f^*Y.

Now suppose that $(B, b_0), (C, c_0)$ are two nice pointed spaces (see exercise 4) and $f : C \longrightarrow B$ a pointed map. Then the construction $E \mapsto f^*E$ from covering spaces of B to covering spaces of C can be extended to a **functor**: given two covering spaces E_1, E_2 over B and a map $T : E_1 \longrightarrow E_2$ of covering spaces, we have a natural choice for a map

$$f^*T : f^*E_1 \longrightarrow f^*E_2$$

given by

$$(f^*T)(b, e) = (b, T(e))$$

Now assume that both B, C are locally simply-connected. And that case we know that the category of covering spaces over C is equivalent to the category of sets with an action of $\pi_1(B, b_0)$. It is then natural to ask what the functor f^* is in the language of sets with actions. The answer is simple - the map f induces a homomorphism

$$f_* : \pi_1(C, c_0) \longrightarrow \pi_1(B, b_0)$$

Such a homomorphism induced a natural functor from $\pi_1(B, b_0)$-sets to $\pi_1(C, c_0)$-sets - simply pullback the action through f_*. In more explicit terms - if we have a set F with an action of $\pi_1(B, b_0)$ we can construct an action of $\pi_1(C, c_0)$ on the same set through the homomorphism f_*:

$$\alpha(x) \stackrel{def}{=} f_*(\alpha)(x)$$

for every $x \in F, \alpha \in \pi_1(C, c_0)$. Indeed in exercise 4 you show that the action of $\pi_1(C, c_0)$ on the fiber F of f^*E is given through the action of $\pi_1(B, b_0)$ on F in exactly this way.

1.1 Exact Sequences of Pointed Sets

Definition 1. We say that a sequence of maps of pointed sets

$$(A_1, a_1) \xrightarrow{f_1} (A_2, a_2) \xrightarrow{f_2} ... \xrightarrow{f_{n-1}} A_n$$

is **exact** if for each $i = 2, ..., n-1$ the image of f_{i-1} is exactly the pre-image $f_i^{-1}(a_{i+1})$.

Example: Let (B, b) be a path-connected pointed space and $p : (E, e) \longrightarrow (B, b)$ a pointed covering map with fiber $F = p^{-1}(b)$. Then we have an exact sequence of pointed sets:

$$1 \longrightarrow \pi_1(E, e) \xrightarrow{p_*} \pi_1(B, b) \xrightarrow{\varphi} F \xrightarrow{\psi} \pi_0(E, e) \longrightarrow *$$

where the map φ sends an element $\alpha \in \pi_1(B, b)$ to $\alpha(e) \in F$ and the map ψ sends an $f \in F$ to its path-connected component in E. Note that we consider groups as pointed sets by taking the neutral element as the base point.

In order to show exactness we need to check the following, all of which we've already proved:

1. The homomorphism $\pi_1(E, e) \xrightarrow{p_*} \pi_1(B, b)$ is injective.
2. An element $\alpha \in \pi_1(B, b)$ satisfies $\alpha(e) = e$ if and only if it is in the image of p_*.
3. An element $f \in F$ is in the same connected component of e if and only if there exists an $\alpha \in \pi_1(B, b)$ such that $\alpha(e) = f$.
4. Every path-connected components of E contains an element of F.

Now observe that since F is discrete we can identify (F, e) as a pointed set with $\pi_0(F, e)$. Further more we have $\pi_1(F, e) = 1$. Hence we see that we can write the above exact sequence more nicely as:

$$\pi_1(F, e) \longrightarrow \pi_1(E, e) \xrightarrow{p_*} \pi_1(B, b) \xrightarrow{f} \pi_0(F, e) \xrightarrow{g} \pi_0(E, e) \xrightarrow{p_*} \pi_0(B, b)$$

You will see later that this exact sequence can be generalized to include more general maps then covering maps.

Let us now return to the case of pullbacks of covering spaces:

$$\begin{array}{ccc} f^*E & \xrightarrow{p_X} & E \\ \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & B \end{array}$$

Choosing a base point $e \in E$ induces a choice of base point $(x, e) \in f^*E$ making the diagram above a diagram of pointed spaces. We claim that we have an exact sequence of pointed sets connecting the π_1's and π_0's of all these spaces. This exact sequence looks as follows:

$$\{1\} \longrightarrow \pi_1(f^*E, (x, e)) \longrightarrow \pi_1(X, x) \times \pi_1(E, e) \xrightarrow{\varphi} \pi_1(B, b) \xrightarrow{\psi}$$
$$\pi_0(f^*E, (x, e)) \longrightarrow \pi_0(X, x) \times \pi_0(E, e) \longrightarrow *$$

where

$$\psi(\alpha, \beta) = p_*(\beta)^{-1} f_*(\alpha) \in \pi_1(B, b)$$
$$\psi(\gamma) = [(x, \gamma(e))] \in \pi_0(f^*E, (x, e))$$

and the other maps are the obvious maps induced by p_X and p_E. In order to show exactness we need to check the following claims:

1. The map
$$(p_{X*}, p_{E*}) : \pi_1(f^*E, (x, e)) \longrightarrow \pi_1(X, x) \times \pi_1(E, e)$$
is injective. This just follows from the fact that p_{X*} is injective - a fact that follows from the fact that p_X is a covering map.

2. For $\alpha \in \pi_1(X, x), \beta \in \pi_1(E, e)$ we have that $f_*(\alpha) = p_*(\beta)$ if and only if there exists a $\gamma \in \pi_1(f^*E, (x, e))$ such that $p_{X*}(\gamma) = \alpha$ and $p_{X*}(\gamma) = \alpha$. The if part is clear because $f_* \circ p_{X*} = p_* \circ p_{E*}$.

   In order to show the only if part assume that $f_*(\alpha) = p_*(\beta)$. This means that
$$\alpha(x, e) = (x, f_*\alpha(e)) = (x, p_*\beta(e)) = (x, e)$$
   and so there exists a γ such that $p_{E*}(\gamma) = \alpha$. Define $\beta' = p_{E*}\gamma$. Then
$$p_*\beta' = p_*p_{E*}\gamma = f_*p_{E*}\gamma = f_*\alpha = p_*\beta$$
   But p_* is injective and so $\beta' = \beta$. Hence
$$(p_{X*}\gamma, p_{E*}\gamma) = (\alpha, \beta)$$
   which is what we wanted.

3. For $\gamma \in \pi_1(B, b)$ we need to show that $[(x, \gamma(e))] = [(x, e)]$ if and only if there exist $\alpha \in \pi_1(X, x), \beta \in \pi_1(E, e)$ such that
$$\gamma = (p_*\beta)^{-1} f_*\alpha \in \pi_1(B, b)$$
   For this we note that $[(x, \gamma(e))] = [(x, e)]$ if and only if there exists an $\alpha \in \pi_1(X, x)$ such that
$$\alpha(x, e) = (x, f_*\alpha(e))$$
   But $(x, \gamma(e)) = (x, f_*\alpha(e))$ so this is equivalent to
$$\gamma(e) = f_*\alpha(e)$$
   Since the stabilizer of e is exactly the image of p_* we see that this is equivalent to there being a $\beta \in \pi_1(E, e)$ such that
$$\gamma = (p_*\beta)^{-1} f_*\alpha$$

4. The map $\pi_0(f^*E, (x, e)) \longrightarrow \pi_0(X, x) \times \pi_0(E, e)$ is surjective. This is equivalent that the fact that every connected components of $X \times E$ contains a point of f^*E. Let $(x, e) \in X \times E$ be a point. Since B is path connected there is a path in B from $f(x)$ to $p(b)$ lifting this path to E we can construct a path in $X \times E$ from (x, e) to (x, e') such that $p(e') = f(x)$, i.e. such that $(x, e') \in f^*E$.

Note that if $E \longrightarrow B$ is the universal covering then we get an exact sequence
$$\{1\} \longrightarrow \pi_1(f^*E, (x, e)) \longrightarrow \pi_1(X, x) \xrightarrow{\varphi} \pi_1(B, b) \xrightarrow{\psi}$$
$$\pi_0(f^*E, (x, e)) \longrightarrow \pi_0(X, x) \longrightarrow *$$

Corollary 2. *If $E \longrightarrow B$ is the universal covering then*

1. *$p_{X*}(\pi_1(f^*E,e))$ is the kernel of f_* (and in particular a normal subgroup).*

2. *If X is path connected and f_* is injective then f^*E is just a disjoint union of fundamental coverings of X.*

3. *If X is path-connected then by varying the base point e one can also identify the set of connected components of f^*E with the set of cosets of $f_*(\pi_1(X,x))$ in $\pi_1(B,b)$ (something we also refer to as the* ***cokernel*** *of f_*).*

2 The Universal Covering of M_g

Consider the surface with g handles M_g. We want to show that it's universal covering is contractible. We will show this for $g = 2$ but this proof can be easily generalized. In order to do so we will construct M_2 in the following way: we start with two toruses T_1, T_2. We then cut out two small discs, one from each of the T_i's. We denote the resulting spaces by T'_1, T'_2. Each of these spaces has a boundary which is a circle. Hence we can take a small collar $S^1 \times I$ and glue it to $T'_1 \coprod T'_2$ along it's boundary $\partial C = S^1 \times \{0,1\}$. The resulting space is just M_2. We will use the following notations for the natural inclusions:

$$\iota_1 : T'_1 \hookrightarrow M_2$$

$$\iota_2 : T'_2 \hookrightarrow M_2$$

$$\iota_3 : C \hookrightarrow M_2$$

$$\iota_4 : \partial C \hookrightarrow M_2$$

Further more we will denote by $T''_1 = M_2 \setminus T'_2$ and $T''_2 = M_2 \setminus T'_1$. In particular T'_1, T'_2 are closed subsets of M_2 and T''_1 and T''_2 as open neighborhoods of them which deformation retract to them.

Now let $E \longrightarrow M_2$ be the universal covering. Then we can partition the universal covering according to this construction of M_2 as follows:

$$E = \left[\iota_1^*E \coprod \iota_2^*E\right] \coprod_{\iota_4^*E} \iota_3^*E$$

Let $x \in C$ be a point. Since both $T''_1 \sim T'_1$ and $T''_2 \sim T'_2$ deformation retract to a wedge of 2 circles we get that

$$\pi_1(T''_1, x) \cong < a, b >$$

$$\pi_1(T''_2, x) \cong < c, d >$$

$$\pi_1(C, x) \cong < e >$$

The inclusion of C in T''_1 maps the element e to $aba^{-1}b^{-1}$ and the inclusion in T''_2 maps e to $cdc^{-1}d^{-1}$. Now from Van-Kampen's theorem applied to the open covering

$$M_2 = T''_1 \cup T''_2$$

we get that we can identify $\pi_1(M_2, x)$ with

$$\pi_1(M_2, x) \cong < a, b > *_{<e>} < c, d > \cong < a, b, c, d > /aba^{-1}a = cdc^{-1}d^{-1}$$

In particular since the inclusion $< e > \hookrightarrow < a, b >$ and $< e > \hookrightarrow < c, d >$ are injective we get that all the ι_j's induce injective homomorphisms on the fundamental group.

By Corollary 2 we see that all the $\iota_j^* E$'s are (infinite) disjoint unions of simply connected spaces. In particular $\iota_1^* E$ and $\iota_2^* E$ are equivalent to a disjoint unions of infinite graphs (the fundamental covering of a wedge of 2 circles), $\iota_3^* E$ is just a disjoint union of infinite strips $\mathbb{R} \times I$ and $\iota_4^* E$ is an infinite union of $\mathbb{R} \times \{0, 1\}$'s.

Now note that $\mathbb{R} \times I$ deformation retracts to its subspace $A \subseteq \mathbb{R} \times I$ given by

$$A = \mathbb{R} \times \{0, 1\} \bigcup 0 \times I$$

This means that $\iota_3^* E$ deformation retracts to a subspace $B \subseteq \iota_3^* E$ which is a disjoint union of infinitely many copies of A. Let $B' \subseteq B$ be the parts coming from $0 \times I \subseteq A$. Then B' is an infinite collection of segments. We denote by $\partial B' \subseteq B'$ the points coming from $0 \times \{0, 1\} \subseteq A$.

Since B contains $\iota_4^* E$ we can do this deformation retract on all of E and get that

$$E \simeq \left[\iota_1^* E \coprod \iota_2^* E\right] \coprod_{\iota_4^* E} B \cong \left[\iota_1^* E \coprod \iota_2^* E\right] \coprod_{\partial B'} B'$$

Since each component of $[\iota_1^* E \coprod \iota_2^* E]$ deformation retracts to a graph we see that E in it's entirety is homotopy equivalent to one huge graph. Since E is the universal covering of M_2 it is simply connected (which means in particular path-connected). But any simply connected graph if contractible and we're done.

Basic Notions in Algebraic Topology 7

Yonatan Harpaz

1 Serre Fibrations and Fiber Bundles

Definition 1. A **Serre fibration** is a map $p : E \longrightarrow B$ such that for every $n \geq 0$ and every diagram

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{g} & E \\ \downarrow & & \downarrow p \\ I^{n+1} & \xrightarrow{f} & B \end{array}$$

there exists a lift $\widetilde{f} : I^{n+1} \longrightarrow E$ making the diagram

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{g} & E \\ \downarrow & \nearrow^{\widetilde{f}} & \downarrow \\ I^{n+1} & \xrightarrow{f} & B \end{array}$$

commutative.

In this situation we refer to B as the **base space** and to E as the **total space**. Now let $p : (E, e) \longrightarrow (B, b)$ be a pointed Serre fibration (i.e. a Serre fibration which happens to send e to b). We call the subspace $F = p^{-1}(B)$ the **fiber** of p and let $\iota : F \hookrightarrow E$ be the inclusion. We also like to say that

$$F \xrightarrow{\iota} E \xrightarrow{p} B$$

is a (Serre) fibration sequence.

The important feature of Serre fibrations for us is that in this situation one can use the lifting property above in order to define homomorphisms

$$\partial_n : \pi_n(B, b) \longrightarrow \pi_{n-1}(F, e)$$

for every $n \geq 1$ (for $n = 1$ this is a map of pointed sets) which fit in a long (=infinite) exact sequence of pointed sets (most of whom are abelian groups):

$$... \longrightarrow \pi_n(F, e) \xrightarrow{\iota_*} \pi_n(E, e) \xrightarrow{p_*} \pi_n(B, b) \xrightarrow{\partial_n}$$

$$\pi_{n-1}(F, e) \xrightarrow{\iota_*} \pi_{n-1}(E, e) \xrightarrow{p_*} \pi_{n-1}(B, b) \longrightarrow ...$$

This means in particular that if we can represent a space X as a total space of some fibration sequence $F \longrightarrow X \longrightarrow B$ such that the homotopy groups of both B and F are better understood then we can use this long exact sequence in order to learn more about the homotopy groups of X.

One way to obtain Serre fibrations is through the notion of **fiber bundles**.

Definition 2. A **fiber bundle** with fiber F is a map $p : E \longrightarrow B$ such that for every $b \in B$ there is a neighborhood $b \in U$ and a homeomorphism

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\simeq} & U \times F \\ & {\scriptstyle p}\searrow \qquad \swarrow & \\ & U & \end{array}$$

making the diagram above commutative (the left vertical map is the natural projection $U \times F \longrightarrow U$).

Theorem 3. *Every fiber bundle $p : E \longrightarrow B$ is a Serre fibration.*

Proof. Consider a diagram

$$\begin{CD} I^n \times \{0\} @>g>> E \\ @VVV @VVpV \\ I^n \times I @>f>> B \end{CD}$$

Let $B = \bigcup_{\alpha \in I} U_\alpha$ be an open covering of B such that for every α we have a homeomorphism

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\simeq} & U_\alpha \times F \\ & {\scriptstyle p}\searrow \qquad \swarrow & \\ & U & \end{array}$$

From Lebesgue's number theorem we can partition the cube I^n to very little cubes C_i for $i = 1, ..., m$ and the segment I to very little segments S_j for $j = 1, ..., k$ such that $f(C_i \times S_j)$ is completely contained in $U_{\alpha_{i,j}}$ for some $\alpha_{i,j} \in I$. Note that it is enough to define a lift on $I^n \times S_1$ because then one can repeat this process until the lift is defined on all of $I^n \times I$.

Now let $0 \le l \le m-1$ and suppose that we have defined the lift $\widetilde{f_l}$ on $C_i \times S_1$ for $i = 1, ..., l$. We wish to define the lift on $C_{l+1} \times S_1$. Let

$$A = (C_{l+1} \times S_1) \cap \left(\cup_{i=1}^{l} C_i \times S_1\right)$$

Then A is a (possibly empty) union of faces of the cube $C_{l+1} \times S_1$ which doesn't include the faces $C_1 \times \{0\}$ and $C_{l+1} \times \{1\}$. This means that $A' = A \cup (C_{l+1} \times \{0\}$ is a non-empty union of faces of $C_{l+1} \times S_1$ which is **not** the entire boundary of $C_{l+1} \times S_1$. Note that in that case A' is a **retract** of $C_{l+1} \times S_1$.

Now we have the problem of extending a lift $\widetilde{f}_l$ from A' to all of $C_{l+1} \times S_1$. Since

$$p^{-1}(U_{\alpha_{l+1,1}}) = U_{\alpha_{l+1,1}} \times F$$

we see that $\widetilde{f}_l|_{A'}$ is just given by a map $Id \times \varphi : A' \longrightarrow U_{\alpha_{l+1,1}} \times F$. Since A' is a retract of $C_{l+1} \times S_1$ we can extend φ to a map from $C_{l+1} \times S_1 \longrightarrow F$ which gives us an extension of $\widetilde{f}_l$ from A' to all of $C_{l+1} \times S_1$. This finishes the proof. $\square$

We will apply this theorem to the **Hopf** map $S^3 \longrightarrow S^2$. This map is obtained as follows: First identify S^2 with $\mathbb{C}P^1$. Then we have the natural projection

$$p : \mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{C}P^1 \cong S^2$$

Now embed S^3 in $\mathbb{C}^2$ as all pairs (z, w) such that $|z|^2 + |w|^2 = 1$ and let h be the restriction of p to S^3.

The Hopf map is a fiber bundle. The fiber over every point is homeomorphic to S^1. Hence we get a long exact sequence in homotopy groups

$$... \longrightarrow \pi_n(S^1) \longrightarrow \pi_n(S^3) \longrightarrow \pi_n(S^2) \longrightarrow \pi_{n-1}(S^1) \longrightarrow ...$$

For $n \geq 3$ we get that $\pi_n(S^1) = \pi_{n-1}(S^1) = 0$ (because it's higher homotopy groups are like those of its universal cover $\mathbb{R}$). Hence for $n \geq 3$ we have $\pi_n(S^2) = \pi_n(S^3)$. In particular

$$\pi_3(S^2) \cong \mathbb{Z}$$

and is generated by the Hopf map.

Basic Notions in Algebraic Topology 8

Yonatan Harpaz

1 Simplicial Sets and their Homology Groups

Recall the standard simplex Δ^n defined by

$$\Delta^n = \left\{ \sum_i a_i e_i \in \mathbb{R}^{n+1} \middle| \sum_i a_i = 0 \right\}$$

where $\{e_i\} \in \mathbb{R}^{n+1}$ is the standard basis. We will denote by $[n]$ the set $\{0, ..., n\}$. We say that a map $\varphi : \Delta^n \longrightarrow \Delta^k$ is **simplicial** if there exists an **order preserving** map $f : [n] \longrightarrow [k]$ (i.e. $i \leq j \Longrightarrow f(i) \leq f(j)$) such that

$$\varphi \left(\sum_i a_i e_i \right) = \sum_i a_i e_{f(i)}$$

Note that the composition of two simplicial maps is again a simplicial map (because the composition of two weak order preserving maps is weak order preserving). Let Δ be the subcategory of the category of topological spaces whose objects are the standard simplices and morphisms simplicial maps. Since every simplicial map comes from a unique order preserving map, we can identify Δ with the category whose objects are the sets $[n]$ for $n = 0, 1, ...$ and morphisms weak order preserving maps.

Definition 1. A simplicial set is a functor from Δ^{op} to the category $\mathcal{S}ets$ of sets. Given a simplicial set S we will denote the object $S(\Delta^n)$ simply by S_n. We will also sometimes denote a simplicial set S by $S_\bullet$. We refer to elements in S_n as the set of n**-simplices** of S.

Recall that a functor from Δ^{op} to any category C is the same as a contravariant functor from Δ to C. A **map of simplicial sets** is just a natural transformation of functors. We denote by $\mathcal{S}ets_\Delta$ the category of simplicial sets and maps of simplicial sets.

We will now define the homology group of a simplicial set $S_\bullet$. For each n and $0 \leq i \leq n$ let $l_i^n : [n-1] \longrightarrow [n]$ denote the order preserving map

$$l_i^n(x) = \begin{cases} x & x < i \\ x+1 & x \geq i \end{cases}$$

Since S is a functor from Δ^{op} to $\mathcal{S}ets$ we get maps

$$d_i^n \stackrel{def}{=} S(l_i^n) : S_n \longrightarrow S_{n-1}$$

which we denote by d_i^n. We now compose S with the functor $A \mapsto \mathbb{Z}A$ from sets to abelian groups (which takes a set A to the abelian group freely generated from the elements of A). We denote $\mathbb{Z}S_n$ by C_n. We then get homomorphisms

$$(d_i^n)^* : CS_n \longrightarrow C_{n-1}$$

Note the homomorphisms of abelian groups can be **summed**. We use this to define

$$\partial_n = \sum_{i=0}^{n} (-1)^n (d_i^n)^*$$

We get a sequence of groups and homomorphisms:

$$... \longrightarrow C_n \stackrel{\partial_n}{\longrightarrow} C_{n-1} \stackrel{\partial_{n-1}}{\longrightarrow} C_{n-2} \longrightarrow ...$$

it can be shown that $\partial_{n-1} \circ \partial_n = 0$. Hence the image of ∂_n is contained in the kernel of ∂_{n-1}. We define

$$H_n(S_\bullet) = \ker(\partial_{n-1})/\mathrm{Im}\,(\partial_n)$$

we call elements of $\ker(\partial_n)$ n**-cycles** and elements in $\mathrm{Im}\,(\partial_{n+1})$ n**-boundaries**.

Note that for every n the homology group $H_n(S_\bullet)$ is functorial in $S_\bullet$, i.e. we can extend it to a functor which for every map of simplicial set $f : T_\bullet \longrightarrow S_\bullet$ gives a homomorphism of groups $f_* : H_n(S_\bullet) \longrightarrow H_n(T_\bullet)$.

Given a topological space X one can create a simplicial set $\mathrm{Sing}(X)$ in the following way: define $\mathrm{Sing}(X)_n$ to be the set of all continuous map $f : \Delta^n \longrightarrow X$ (as a set, without any topology). For every simplicial map $l : \Delta^n \longrightarrow \Delta^n$ define the map $\mathrm{Sing}(X)_k \longrightarrow \mathrm{Sing}(X)_n$ by

$$f \mapsto f \circ l$$

We **define** the homology groups

$$H_n(X) \stackrel{def}{=} H_n(\mathrm{Sing}(X))$$

1.1 Realization of Simplicial Sets

Given a simplicial set $S_\bullet$ we define the **realization** of $S_\bullet$ to be the topological space given by

$$|S_\bullet| = \coprod_n S_n \times \Delta^n / \sim$$

where the equivalence relation $\sim$ on the set $\coprod_n S_n \times \Delta^n$ is defined as follows: we say that $(\alpha, y) \in S_k \times \Delta^k$ is equivalent to $(\beta, x) \in S_n \times \Delta^n$ if there exists a simplicial map $f : \Delta^n \longrightarrow \Delta^k$ such that $f(x) = y$ and $f^*(\alpha) = \beta$ (recall that S

is a functor on Δ^{op}). We then take the equivalence relation generated by this relation.

In order to compute this realization we need some additional terminology: for $i \in \{0, ..., n\}$ let $m_i^n : [n+1] \longrightarrow [n]$ be given by

$$m_i^n(x) = \begin{cases} x & x \leq i \\ x-1 & x > i \end{cases}$$

and for each simplicial set $S_\bullet$ let $s_i^n : S_n \longrightarrow S_{n+1}$ be the maps induced by the m_i^n's. We then note that the equivalence relation above is generated by the following identifications: for each $\alpha \in S_n, x \in \Delta^{n-1}$ we identify

$$(*) \;\; (\alpha, l_i^n(x)) \sim (d_i^n(\alpha), x), \forall \alpha \in S_n, x \in \Delta^{n-1}$$

$$(**) \;\; (s_i^n(\beta), x) \sim (\beta, m_i^n(x)), \forall \beta \in S_{n-1}, x \in \Delta^n$$

We say that an element $\alpha \in S_n$ is **degenerate** if there exists a $\beta \in S_{n-1}$ and an $i \in \{0, ..., n-1\}$ such that $\alpha = s_i^n(\beta)$.

Theorem 2. *The realization $|S_\bullet|$ has a structure of a CW complex.*

Proof. Let $|S|_n$ be the image of $\coprod_{i=0}^n S_n \times \Delta^n$ in $|S|$. Clearly S_0 is discrete (i.e. a bunch of points). We claim that S_{n+1} is obtained from S_n by attaching n-cells. Let $\alpha \in S_{n+1}$ be an element and $P \subseteq |S|_{n+1}$ be the union of $|S|_n$ and the image of $\{\alpha\} \times \Delta^{n+1}$. For each i we have a map $\{d_i^n(\alpha)\} \times \Delta^{n-1} \longrightarrow |S|_{n-1}$. Going over all $i \in \{0, ..., n\}$ we get a map $\varphi : \partial \Delta^{n+1} \longrightarrow |S|_n$.

Since we have the identifications $(*)$ in $|S|$ we see that we get a surjective map from the pushout

$$|S|_n \cup_\varphi \Delta^{n+1} \longrightarrow P$$

If α is not degenerate then these are all the identifications involving α and lower simplices. Hence in that case P is homeomorphic to the pushout $|S|_n \cup_\varphi \Delta^{n+1}$ and we are done.

If α is degenerate then there exists a β and an $i \in \{0, ..., n\}$ such that $\alpha = s_i^n(\beta)$. Then P is obtained from $|S|_n \cup_\varphi \Delta^{n+1}$ by collapsing Δ^{n+1} onto one of it's faces. It is not hard to show that if two points on the boundary of Δ^{n+1} are identified by this collapsing, then they are already identified in $|S|_n$. This means that the new cell didn't change anything, so we are good as well. $\square$

This motivates the following definition:

Definition 3. Let $S_\bullet$ be a simplicial set. The **dimension** of $S_\bullet$ is the smallest number n such that for all $m > n$ all the elements of S_m are degenerate. If no such n exists we say that the dimension is ∞.

Corollary 4. *Let $S_\bullet$ be a simplicial set of dimension n. Then*

$$|S_\bullet| = \coprod_{i=0}^n S_i \times \Delta^i / \sim$$

where the equivalence relation is restricted only the the simplices of dimension $\leq n$. In particular $|S|$ carries a structure of an n-dimensional CW complex.

Given a map of simplicial sets $f : S_\bullet \longrightarrow T_\bullet$ we have a natural map $f_* : |S_\bullet| \longrightarrow |T_\bullet|$. This observation leads to the fact that the realization procedure is in fact a functor from simplicial sets to spaces. Note that if we compose the Sing functor after the realization functor we get a functor from simplicial sets to simplicial sets

$$S_\bullet \mapsto \mathrm{Sing}(|S_\bullet|)$$

For each simplicial set we have a natural map from $S_\bullet$ to $\mathrm{Sing}(|S_\bullet|)$ (because every element of S_n gives a map form Δ^n to $|S_\bullet|$, or an element of $\mathrm{Sing}(|S_\bullet|)_n$). Formally these maps are really a **natural transformation** from the identity functor to the functor $S_\bullet \mapsto \mathrm{Sing}(|S_\bullet|)$. This natural transformation induces homomorphisms

$$H_n(S_\bullet) \longrightarrow H_n(\mathrm{Sing}(|S_\bullet|))$$

Theorem 5. *This homomorphism is an isomorphism.*

We will prove this theorem later in the course. In particular we get

Corollary 6. *Let S be a simplicial sets. Then the homologies of S are isomorphic to the homologies of $|S|$.*

Hence we will be able to compute the homology groups of a space by representing it as a realization of a relatively small and simple simplicial set.

2 Computing homologies of simplicial sets

Let S be a simplicial set and $T_n \subseteq S_n$ the subsets of degenerate simplices. Note that the T_n's don't form a sub simplicial set. However, if we define

$$D_n = \mathbb{Z}T_n \subseteq \mathbb{Z}S_n = C_n$$

and $\partial_n : C_n \longrightarrow C_{n-1}$ we do have the following result:

Lemma 7. *If $\sigma \in D_n$ is an element then $\partial\sigma \in D_{n-1}$.*

Proof. It is enough to show that for every $\alpha \in T_n$ we have $\partial\alpha \in D_{n-1}$. Since α is degenerate there exists a $\beta \in S_{n-1}$ and a $k \in \{0, ..., n-1\}$ such that $\alpha = s_k^{n-1}(\beta)$. Then we note the following relations:

1. If $0 \leq j \leq k-1$ then

$$d_j^n(s_k^{n-1}(\beta)) = s_{k-1}^{n-2}(d_j^n(\beta))$$

2. If $k \leq j \leq k+1$ we have

$$d_j^n(s_k^{n-1}(\beta)) = \beta$$

3. if $k+2 \leq j \leq n+1$ we have

$$d_j^n(s_k^{n-1}(\alpha)) = s_k^{n-2}(d_{j-1}^{n-1}(\beta))$$

This means that

$$\partial\alpha = \sum_{j=0}^{n}(-1)^j d_j^n(\alpha) = \sum_{j=0}^{n}(-1)^j d_j^n(s_k^{n-1}(\beta)) =$$

$$\sum_{j=0}^{k-1}(-1)^j s_{k-1}^{n-2}(d_j^{n-1}(\beta) + (-1)^k\beta + (-1)^{k+1}\beta + \sum_{j=k+2}^{n} s_k^{n-2}(d_{j-1}^{n-1}(\beta) =$$

$$\sum_{j=0}^{k-1}(-1)^j s_{k-1}^{n-2}(d_j^{n-1}(\beta) + \sum_{j=k+2}^{n} s_k^{n-2}(d_{j-1}^{n-1}(\beta) \in D_{n-1}$$

and we are done. □

Let $\overline{C}_n = C_n/D_n$. By the above lemma we get that ∂_n induces a well defined map

$$\overline{\partial}_n : \overline{C}_n \longrightarrow \overline{C}_{n-1}$$

It is immediate that $\overline{\partial_n} \circ \overline{\partial}_{n+1} = 0$. We define the **non-degenerate** homology groups of S to be

$$H_n^{nd}(S) = \ker(\overline{\partial}_n)/\mathrm{Im}\,(\overline{\partial}_{n+1})$$

The following lemma will be proven in later classes:

Lemma 8. *The quotient map $C_n \longrightarrow \overline{C}_n$ induces an **isomorphism**:*

$$H_n(S) \xrightarrow{\simeq} H_n^{nd}(S)$$

This lemma means that if we have a simplicial set with only finitely many non-degenerate simplices then we can compute it's homologies by hand.

Examples

1. For each n we can construct a simplicial set $S_\bullet$ whose realization if homeomorphic to the n-sphere. This is done as follows: identify the n-sphere with the space X obtained by taking the n-simplex Δ^n and collapsing all its boundary $\partial\Delta^n$ to a point. Let $x_0 = [\partial\Delta^n] \in X$ be the collapse point.

   Consider the quotient map $q : \Delta^n \longrightarrow X$. We will now construct a simplicial set $S_\bullet$ whose realization is X. We define S_k to be the set of all maps $\varphi : \Delta^k \longrightarrow X$ which are either constant with image x_0 or of the form $q \circ f$ where $f : \Delta^k \longrightarrow \Delta^n$ is a simplicial map. For each simplicial map $g : \Delta^m \longrightarrow \Delta^k$ we define $g^* : S_k \longrightarrow S_m$ by

   $$g^*\varphi = \varphi \circ g$$

   Note that if φ satisfies the condition above then so does $\varphi \circ g$, so everything is well defined. We need to show that the realization of S is homeomorphic to X. In order to do that we go back to the proof of Theorem 2 and see that we can realize $|S|$ as a CW complex by looking only at the non-degenerate simplices. Note that S has only two non-degenerate simplices:

one 0-simplex $\alpha \in S_0$ and one n-simplex $\beta \in S_n$. The faces of β all degenerate to α so we get that the gluing map of $\{\alpha\} \times \Delta^n$ is constant. Hence we get exactly the sphere X.

Let us now use this in order to compute the homologies of the sphere: the groups $\overline{C}_n$ look like this:

$$... \longrightarrow 0 \longrightarrow 0 \longrightarrow ... \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_n} 0 \longrightarrow ...0 \xrightarrow{\partial_1} \mathbb{Z}$$

where the $\mathbb{Z}$'s sit at the n'th and 0'th place. Hence for all $n > 1$ we immediately get

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & k \neq 0, n \end{cases}$$

for $n = 1$ we get that the two $\mathbb{Z}$'s are consecutive:

$$... \longrightarrow 0 \longrightarrow 0 \longrightarrow ... \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}$$

so we need to compute $\partial_1 : \overline{C}_1 \longrightarrow \overline{C}_0$. Since the two vertices of the non-degenerate 1-simplex $\beta \in S_1$ are glued to the same (and only) 0-simplex in S_0 we get that

$$\partial_1(\beta) = d_0^1(\beta) - d_1^1(\beta) = 0$$

and so the kernel of ∂_1 is everything and it's image is trivial. This means that for $n = 1$ we have (similarly to the higher n's):

$$H_k(S^1) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & k \neq 0, 1 \end{cases}$$

It is not hard to complete the picture and see that for S^0 we have

$$H_k(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

2. Let us now compute the homologies of the 2-torus $\mathbb{T}^2$. The the standard CW structure of $\mathbb{T}^2$ and divide the 2-cell into two triangles by adding an extra diagonal 1-simplex. We get a structure with three 1-cells and two 2-cells:

These 2-cells happen to look like triangles. This observation can be translated to there being maps $f_1, f_2 : \Delta^2 \longrightarrow \mathbb{T}^2$ such that the restriction of f_i to the interior of Δ^2 gives a homeomorphism to the interior of the i'th 2-cell. Further more for each of the three one cells we have maps $e_1, e_2, e_3 : \Delta^1 \longrightarrow \mathbb{T}^2$ and the restriction of f_i to any of the edges of the triangle gives one of the e_i's. Of course there is also a map $v : \Delta^0 \longrightarrow \mathbb{T}^2$ whose image of the vertex in $(\mathbb{T}^2)_0$.

We then define a simplicial set $S_\bullet$ by defining S_k to be the set of all maps $\varphi : \Delta^k \longrightarrow \mathbb{T}^2$ which are of the form $f_i \circ g, e_i \circ g$ or $v \circ g$ for some simplicial map g. As in the previous example it is not hard to show that the realization $|S|$ is homeomorphic to $\mathbb{T}^2$.

Looking at the groups $\overline{C}_n$ we get the following chain:

$$... \xrightarrow{\partial_4} 0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

There is a unique 0-simplex all the 1-simplices start and end with it. Hence ∂_1 is the 0-map. It is left to compute ∂_2. Call the 1-simplices a, b, c where a, b are the familiar ones and c is the one we added when partitioning the 2-cell. Let U and L denote the two 2-simplices (U for upper and L for lower). Then we see that

$$\partial_2(U) = a + b - c$$

$$\partial_2(L) = a + b - c$$

Hence the kernel of ∂_2 is generated by $U - L$ and is isomorphic to $\mathbb{Z}$. The cokernel of ∂_2 is the abelian group generated by a, b, c modulu the relation $a + b - c = 0$, or $c = a + b$. Clearly this group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. To conclude we get that

$$H_2(\mathbb{T}^2) \cong \mathbb{Z}$$

$$H_1(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Basic Notions in Algebraic Topology 9

Yonatan Harpaz

1 Barycentric Subdivision

Consider the standard simplex

$$\Delta^n = \left\{ (a_0, ..., a_n) \in \mathbb{R}^{n+1} \middle| a_i \geq 0, \sum_i a_i = 1 \right\}$$

Let Σ_{n+1} be the permutation group on the set $\{0, ..., n\}$. For every permutation $\rho \in \Sigma_{n+1}$ consider the subset

$$\Delta^\rho = \left\{ (a_0, ..., a_n) \in \Delta^n \middle| a_{\rho(0)} \leq a_{\rho(1)} \leq ... \leq a_{\rho(n)} \right\} \subseteq \Delta^n$$

We claim that each one of the Δ^ρ's is actually homeomorphic to Δ^n. In order to construct the homeomorphism let us define for every vector $(x_0, ..., x_n) \in \mathbb{R}^{n+1}$ and permutation $\rho \in \Sigma_{n+1}$ the vector

$$\rho(x_0, ..., x_n) = (x_{\rho^{-1}(0)}, x_{\rho^{-1}(1)}, ..., x_{\rho^{-1}(n)})$$

Then the homeomorphism $\tau_\rho : \Delta^n \longrightarrow \Delta^\rho$ is given by

$$\tau_\rho(b_0, ..., b_n) = \rho\left(\frac{b_0}{n+1}, \frac{b_0}{n+1} + \frac{b_1}{n}, \frac{b_0}{n+1} + \frac{b_1}{n} + \frac{b_2}{n-1}, ..., \sum_{i=0}^{n} \frac{b_i}{n+1-i} \right)$$

In particular the volume of each Δ^ρ is exactly $\frac{1}{(n+1)!}$ times the volume of the standard simplex.

Each two of the Δ^ρ's intersect along a common face (of some dimension) and their union is all of Δ^n. In fact they form a **triangulation** of Δ^n. It is called the **barycentric triangulation** of Δ^n.

For example when $n = 0$ it is the trivial triangulation. When $n = 1$ the barycentric triangulation corresponds to partitioning the segment into two halves.

For $n = 2$ and $n = 3$ the barycentric triangulation are shown in the illustrations above.

We package the data of this triangulation by an element $B_n \in C_n(\Delta^n)$ (where $C_n(\Delta^n)$ denotes the singular chain complex of Δ^n) given by:

$$B_n = \sum_{\rho \in \Sigma_{n+1}} \mathrm{sign}(\rho)\tau_\rho \in C_n(\Delta^n)$$

There is an inductive way to describe this triangulation. Let $A \subseteq \mathbb{R}^n$ be a convex set and $b \in A$ a point. Let $C_n(A)$ be the singular chain complex of A. For every $\varphi : \Delta^k \longrightarrow A$ let $\varphi^+ : \Delta^{k+1} \longrightarrow \Delta^n$ be the map given by

$$\varphi^+(b_0, ..., b_{k+1}) = (1 - b_0)\varphi\left(\frac{b_1}{1 - b_0}, ..., \frac{b_{k+1}}{1 - b_0}\right) + b_0 p$$

for $b_0 < 1$ and completed by $\varphi^+(1, 0, ..., 0) = p$. This is the singular chain analogue of the construction given in question 2 of exercise 8.

Now the construction $\varphi \mapsto \varphi^+$ induces a map of sets $S_n(A) \longrightarrow S_{n+1}(A)$ which induces a homomorphism of groups

$$h_n^p : C_n(A) \longrightarrow C_{n+1}(A)$$

These maps give a chain homotopy from the identity to the chain version of the constant map at p.

Recalling that each Δ^n has a standard embedding as a convex set in $\mathbb{R}^{n+1}$ we can use the map h^p on the singular chain complex of $A = \Delta^n$. In particular it is not hard to see that if $p \in \Delta^n$ is the center of mass point

$$p = \left(\frac{1}{n+1}, ..., \frac{1}{n+1}\right) \in \Delta^n$$

and $l_j^n : \Delta^{n-1} \hookrightarrow \Delta^n$ is the embedding of the j'th face (i.e. the simplicial map corresponding to the order preserving map $[n-1] \hookrightarrow [n]$ whose image is $[n] \setminus \{j\}$) then you will show in the exercise that:

$$B_n = \sum_{j=0}^{n} (-1)^j h_{n-1}^p ({l_j^n}_* (B_{n-1}))$$

Note that this recursion rule completely determines the element B_n provided we define $B_0 \in C_0(\Delta^0)$ to be the element corresponding to the identity $\Delta^0 \longrightarrow \Delta^0$.

Now let X be any topological space. Then we define a map of chain complexes $B^X : C_\bullet(X) \longrightarrow C_\bullet(X)$ by setting

$$B_n^X(\sigma) = \sigma_*(B_n) \in C_n(X)$$

for every $\sigma \in S_n(X)$ and extending linearly to all of $C_n(X)$. It is not hard to see that the maps B_n^X respect the differential and so fit together to form a map of chain complexes.

Basic Notions in Algebraic Topology 10

Yonatan Harpaz

1 The Homologies of a CW complex

Let us see what the process of adding the n-cells to the $(n-1)$-skeleton of a CW complex does to its homologies. Let X be a connected finite CW complex. Then its 1-skeleton X_1 is homotopy equivalent to a wedge of circles $X_1 = \bigvee_{i=1}^{n} S^1$. Note that for each i there exists a $U_i \subseteq X_1$ which deformation retracts to the i'th circle and such that $U_i \cap U_j \simeq *$ for $i \neq j$. Hence from the Mayer-Vietors sequence we get that

$$H_k(X_1) \cong \oplus_{i=1}^{n} H_i(S^1)$$

for every $i > 0$. Since $H_i(S^1) = \mathbb{Z}$ for $i = 1$ and $H_i(S^1) = 0$ for $i > 1$ we know all homologies of X_1 (note that X_1 is connected so $H_0(X_1) \cong \mathbb{Z}$).

Now let $n \geq 2$ and consider the passage from X_{n-1} to X_n. Suppose X_n is obtained from X_{n-1} be adding a single n-cell.

Then we can construct X_n from X_{n-1} via a pushout of the form

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ X_{n-1} & \longrightarrow & X_n \end{array}$$

If X_{n-1} was connected then so it X_n so we don't bother much with H_0.

Now for the higher homologies we need to replace the above pushout diagram with a pushout diagram of open sets. Let $U \subseteq X_n$ be the interior of D^n and $p \in U$ a point. Let $V = X_n \setminus \{p\}$. Then the above pushout square is equivalent to the pushout square with

$$\begin{array}{ccc} V \cap U & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X_n \end{array}$$

and so we can apply to it the Mayer Vietoris sequence. Recall that we assume $n \geq 2$ and so the map $H_0(S^{n-1}) \longrightarrow H_0(D^n)$ is an isomorphism. This means that the map $H_0(S^{n-1}) \longrightarrow H_0(X_{n-1}) \oplus H_0(D^n)$ is in particular injective and so we can "cut" our Mayer Vietoris sequence at $k = 1$. Noting further more that $H_i(D^n) = 0$ for $i > 0$ we get the segment

$$... \longrightarrow H_{k+1}(X_{n-1}) \longrightarrow H_{k+1}(X_n) \longrightarrow H_k\left(S^{n-1}\right) \longrightarrow$$

$$H_k(X_{n-1}) \longrightarrow H_k(X_n) \longrightarrow ... \longrightarrow H_1(X_n) \longrightarrow 0$$

Now first of all for $k > 0$ and $k \neq n, n-1$ we get from the segments

$$... \longrightarrow H_k\left(S^{n-1}\right) \longrightarrow H_k(X_{n-1}) \longrightarrow H_k(X_n) \longrightarrow H_{k-1}\left(S^{n-1}\right) \longrightarrow ...$$

or

$$... \longrightarrow H_1\left(S^{n-1}\right) \longrightarrow H_1(X_{n-1}) \longrightarrow H_1(X_n) \longrightarrow 0$$

that $H_k(X_n) \cong H_k(X_{n-1})$ (because $H_i(S^{n-1}) = 0$ for $i \neq 0, n-1$). From this we already draw far reaching conclusions:

Corollary 1. *If X is a connected n-dimensional CW complex then $H_k(X) = 0$ for $k > n$ and $H_k(X) \cong H_k(X_{k+1})$ for $k < n$.*

Proof. The first skeleton X_1 is equivalent to a wedge of circles and so doesn't have homologies beyond dimension 1. Adding k-cells doesn't change the homologies above dimension k. Hence an n-dimensional connected CW complex doesn't have homologies above dimension n. The second conclusion is also an immediate consequence of the same observation. □

In order to understand the behavior of H_n and H_{n-1} we observe the segment

$$0 \longrightarrow H_n(X_{n-1}) \longrightarrow H_n(X_n) \longrightarrow H_{n-1}\left(S^{n-1}\right) \longrightarrow H_{n-1}(X_{n-1}) \longrightarrow H_{n-1}(X_n) \longrightarrow 0$$

Since $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ we can write this as

$$0 \longrightarrow H_n(X_{n-1}) \longrightarrow H_n(X_n) \longrightarrow \mathbb{Z} \longrightarrow H_{n-1}(X_{n-1}) \longrightarrow H_{n-1}(X_n) \longrightarrow 0$$

Hence we see that the $n-1$ homology of X_n is a quotient of the $n-1$ homology of X_{n-1} obtained by killing the image of the map

$$H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(X_{n-1})$$

The n-homology of X_n is simply the kernel of that map.

Examples:

1. Let us calculate the homologies of the surface of genus $g \geq 1$. The 1-skeleton of this surface is a wedge of $2g$ circles

$$(M_g)_1 = S^1_{a_1} \vee S^1_{b_1} \vee S^1_{a_2} \vee S^1_{b_2} \vee ... \vee S^1_{a_g} \vee S^1_{b_g}$$

and we have one 2-cell. Hence the sequence above becomes

$$0 \longrightarrow H_2(M_g) \longrightarrow H_1(S^1) \longrightarrow H_1((M_g)_1)) \longrightarrow H_1(M_g) \longrightarrow 0$$

So all we need to do is to understand what the gluing map

$$S^1 \longrightarrow S^1_{a_1} \vee S^1_{b_1} \vee S^1_{a_2} \vee S^1_{b_2} \vee ... \vee S^1_{a_g} \vee S^1_{b_g}$$

induces on H_1.

The gluing map is given by the word

$$\prod_i a_i b_i a_i^{-1} b_i^{-1}$$

whose Hurewitz image is

$$\sum_i a_i + b_i - a_i - b_i = 0$$

Hence the gluing map sends the generator of $H_1(S^1)$ to 0. Hence from the sequence above we get

$$0 \longrightarrow H_2(M_g) \cong H_1(S^1) \cong \mathbb{Z}$$

$$H_1(M_g) \cong\!\!\longrightarrow H_1((M_g)_1))$$

2. Let us do the same for $\mathbb{R}P^2$. The 1-skeleton is $(\mathbb{R}P^2)_1 \cong S^1$ and we have one 2-cell. Hence the sequence above becomes

$$0 \longrightarrow H_2(\mathbb{R}P^2) \longrightarrow H_1(S^1) \longrightarrow H_1((\mathbb{R}P^2)_1)) \longrightarrow H_1(\mathbb{R}P^2) \longrightarrow 0$$

The gluing map
$$S^1 \longrightarrow (\mathbb{R}P^2)_1 \cong S^1$$
is just the multiplication by 2 map. This map induces multiplication by 2 on H_1 and so we get the sequence

$$0 \longrightarrow H_2(\mathbb{R}P^2) \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow H_1(\mathbb{R}P^2) \longrightarrow 0$$

and so
$$H_2(\mathbb{R}P^2) = 0$$
$$H_1(\mathbb{R}P^2) = \mathbb{Z}/2$$

The fact that $H_2(\mathbb{R}P^2) = 0$, while $H_2(M_g) \cong \mathbb{Z}$ is connected to a geometric difference: the surface M_g is orientable and $\mathbb{R}P^2$ isn't. We will learn about orientability later in the course.

2 Pushouts in the Category of Simplicial Sets

Consider a diagram

$$\begin{array}{ccc} S_\bullet & \xrightarrow{\varphi} & T_\bullet \\ \downarrow{\scriptstyle\psi} & & \\ R_\bullet & & \end{array}$$

of simplicial sets. We claim that there exists a pushout P for this diagram which is given by

$$P_n = T_n \coprod_{\varphi_n(\sigma)\sim\psi_n(\sigma),\sigma\in S_n} R_n$$

Note that for every simplicial map $f : \Delta^n \longrightarrow \Delta^k$ we have a commutative diagram

$$\begin{array}{ccccc} R_k & \xleftarrow[\psi_k]{} & S_k & \xrightarrow{\varphi_k} & T_k \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ R_n & \xleftarrow[\psi_n]{} & S_n & \xrightarrow{\varphi_n} & T_k \end{array}$$

which induces a map

$$P_k \longrightarrow P_n$$

Hence $P_\bullet$ admits a structure of a simplicial set. It is not hard to see that $P_\bullet$ is indeed the pushout because it realizes the pushout of sets at each level.

Now consider our standard functor from simplicial sets to chain complexes which associates to a simplicial set $S_\bullet$ the chain complex $C_\bullet(S_\bullet)$ given by

$$C_n(S_\bullet) = \mathbb{Z}S_\bullet$$

$$\partial_n = \sum_i (-1)^i d_i^n$$

Now the natural map

$$C_\bullet(T_\bullet) \oplus_{C_\bullet(S)} C_\bullet(R_\bullet) \longrightarrow C_\bullet(P)$$

is easily seen to be an isomorphism of chain complexes, i.e. the functor $C_\bullet$ respects pushouts.

Hence we get a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(S_\bullet) \longrightarrow C_\bullet(T_\bullet) \oplus C_\bullet(R_\bullet) \longrightarrow C_\bullet(P) \longrightarrow 0$$

Applying the snake lemma we get a long exact sequence

$$... \longrightarrow H_n(S_\bullet) \longrightarrow H_n(T_\bullet) \oplus H_n(R_\bullet) \longrightarrow H_n(P_\bullet) \longrightarrow$$

$$\longrightarrow H_{n-1}(S_\bullet) \longrightarrow H_{n-1}(T_\bullet) \oplus H_{n-1}(R_\bullet) \longrightarrow H_{n-1}(P_\bullet) \longrightarrow ... \longrightarrow H_0(P_\bullet) \longrightarrow 0$$

We call this the **simplicial Mayer-Vietoris sequence**.

The following lemma is an easy exercise:

Lemma 2. *The realization functor $S_\bullet \longrightarrow |S_\bullet|$ respects pushouts, i.e. the natural map*

$$|T_\bullet| \coprod_{|S_\bullet|} |R_\bullet| \longrightarrow |P_\bullet|$$

is a homeomorphism.

3 The Equivalence of Simplicial and Singular Homology

Theorem 3. *Let $S_\bullet$ be a simplicial set with finitely many non-degenerate simplices. Then the natural map*

$$H_i(S_\bullet) \longrightarrow H_i(|S_\bullet|)$$

is an isomorphism for every i.

Proof. We will prove by induction on the dimension of S (i.e. the maximal dimension of a non-degenerate simplex appearing in S). For 0-dimensional simplicial sets. Such simplicial sets look like this, there exists a set A such that $S_n = A$ for every n and every simplicial map $g : \Delta^n \longrightarrow \Delta^k$ induces the identity $A \longrightarrow A$. In this case the realization $|S|$ is just A as a discrete space. In this case we get that

$$H_0(S_\bullet) \xrightarrow{\simeq} H_0(|S_\bullet|) \cong \mathbb{Z}A$$

and

$$H_n(S_\bullet) \cong H_n(|S_\bullet|) \cong 0$$

Now let $n \geq 1$ and suppose we have proved the theorem for all finite simplicial sets of dimension $< n$. Every finite n-dimensional simplicial set is obtained from its $(n-1)$-dimensional skeleton by a sequence of gluings of Δ_n's along $\partial\Delta_n$. This means that the induction step will be complete once we prove the following lemma:

Lemma 4. *Suppose that S is an n-dimensional simplicial set for which theorem 3 is true. Suppose that S' is obtained from S by adding a single n-simplex. Then the theorem is true for S' as well.*

Proof. Let $T^n_\bullet$ be the simplicial set represented by Δ^n, i.e.

$$T^n_k = \mathrm{Hom}_\Delta(\Delta^k, \Delta^n)$$

We saw in exercise 8 that $|T^n_\bullet| \cong \Delta^n$. Let $R^n_\bullet \subseteq T^n_\bullet$ be the simplicial subset corresponding to the boundary $\partial\Delta^n$, i.e. R^n_k is the set of all simplicial maps $\Delta^k \longrightarrow \Delta^n$ whose image is contained in $\partial\Delta^n \subseteq \Delta^n$. Then $|R^n_\bullet| \cong \partial\Delta^n$.

We have a pushout square of simplicial sets

$$\begin{array}{ccc} R_\bullet & \longrightarrow & S \\ \downarrow & & \downarrow \\ T_\bullet & \longrightarrow & S' \end{array}$$

From the hypothesis of the lemma we know that the natural maps

$$H_i(S_\bullet) \longrightarrow H_i(|S_\bullet|)$$

are isomorphisms for every i. Since $R^n_\bullet$ is $(n-1)$-dimensional we know from the induction hypothesis that

$$H_i(R^n_\bullet) \longrightarrow H_i(|R^n_\bullet|) = H_i(\partial\Delta^n)$$

For every i. From question 2 in exercise 8 we have that

$$H_i(T^n_\bullet) \longrightarrow H_i(|T^n_\bullet|) = H_i(\Delta^n)$$

is also an isomorphism for every i.

We now want to use the Mayer Vietoris sequences. Note that the images of the maps $|S| \longrightarrow |S'|$ and $|T^n_\bullet| \longrightarrow |S'|$ are two sub CW complexes, whose intersection is the image of $|R^n_\bullet|$. Then there exists open subsets $U, V \subseteq |S'|$ such that U deformation retracts to the image of $|S_\bullet|$, V deformation retracts to the image of $|T^n_\bullet|$ and $U \cap V$ deformation retracts to the intersection of $|R^n_\bullet|$.

Then we have two short exact sequence of chain complexes and compatible maps between them (i.e. everything commutes):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_\bullet(R_\bullet) & \longrightarrow & C_\bullet(S_\bullet) \oplus C_\bullet(T^n_\bullet) & \longrightarrow & C_\bullet(S'_\bullet) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_\bullet(\partial\Delta^n) & \longrightarrow & C_\bullet(|S|) \oplus C_\bullet(\Delta^n) & \longrightarrow & C_\bullet(V) \oplus_{C_\bullet(V\cap U)} C_\bullet(U) & \longrightarrow & 0
\end{array}$$

This gives us a sequence of compatible maps between the two long exact sequences:

$$\begin{array}{ccccccccccccc}
... & \longrightarrow & H_i(R^n_\bullet) & \longrightarrow & H_i(S_\bullet) \oplus H_i(T^n_\bullet) & \longrightarrow & H_i(S'_\bullet) & \longrightarrow & H_{i-1}(R^n_\bullet) & \longrightarrow & H_{i-1}(S_\bullet) \oplus H_{i-1}(T^n_\bullet) & \longrightarrow & \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \downarrow \simeq & & \\
... & \longrightarrow & H_i(\partial\Delta_n) & \longrightarrow & H_i(|S_\bullet|) \oplus H_i(\Delta_n) & \longrightarrow & H_i(|S'_\bullet|) & \longrightarrow & H_{i-1}(\partial\Delta_n) & \longrightarrow & H_{i-1}(|S_\bullet|) \oplus H_{i-1}(\Delta_n) & \longrightarrow &
\end{array}$$

All the vertical maps besides those going from $H_i(S'_\bullet)$ to $H_i(|S'_\bullet|)$ are isomorphisms. Hence we get for every i a map between one exact sequence of 5 groups to another exact sequence of 5 groups which is an isomorphism everywhere except for the middle group. We claim that in this situation it has to be an isomorphism for the middle group as well. This is called **the five lemma** :

Lemma 5. *Let*

$$\begin{array}{ccccccccc}
A & \xrightarrow{f_A} & B & \xrightarrow{f_B} & C & \xrightarrow{f_C} & D & \xrightarrow{f_D} & E \\
\downarrow g_A & & \downarrow g_B & & \downarrow g_C & & \downarrow g_D & & \downarrow g_E \\
A' & \xrightarrow{f'_A} & B' & \xrightarrow{f'_B} & C' & \xrightarrow{f'_C} & D' & \xrightarrow{f'_D} & E'
\end{array}$$

be a commutative diagram of abelian groups such that the rows are exact and the maps g_A, g_B, g_D, g_E are isomorphisms. Then g_C is an isomorphism.

Proof. Suppose that $c \in \ker(g_C)$. Then $g_D(f_C(c)) = f'_C(g_C(c)) = 0$ and since g_D is injective $f_C(c) = 0$. This means that there exists a $b \in B$ such that $f_B(b) = c$. Let $b' = g_B(b)$. Then $f'_B(b') = g_C(f_B(b)) = g_C(c) = 0$ so there exists an $a' \in A$ such that $f'_A(a') = b'$. Since g_A is surjective there exists an $a \in A$ such that $g_A(a) = a'$.

We claim that $f_A(a) = b$. The reason is that $g_B(f_A(a)) = f'_A(g_A(a)) = f'_A(a') = b' = g_B(b)$ and g_B is injective. Hence $c = f_B(f_A(a)) = 0$. This shows that g_C is injective.

Let us now show that g_C is surjective. Let $c' \in C'$ be an element and $d' = f'_C(c')$. Since g_D is surjective there exists a $d \in D$ such that $g_D(d) = d'$. Then $g_E(f_D(d)) = f'_D(g_D(d)) = f'_D(d') = 0$ (because d' is the image of c') and since g_E is injective we get that $f_D(d) = 0$. This means that there exists a $c \in C$ such that $f_C(c) = d$.

Comparing $g_C(c)$ with c' we see that both of them satisfy

$$f'_C(g_C(c)) = g_D(f_C(c)) = d' = f'_C(c')$$

and so $g_C(c) - c'$ is in the kernel of f'_C, i.e. in the image of f'_B. Hence there exists a $b' \in B'$ such that $f'_B(b') = g_C(c) - c'$ or $c' = g_C(c) + f'_B(b')$.

Since g_B is surjective there exists a $b \in B$ such that $g_B(b) = b'$. Then $g_C(f_B(b)) = f'_B(g_B(b)) = f'_B(b')$ and so

$$g_C(c + f_B(b)) = g_C(c) + f'_B(b') = c'$$

Hence c' has a pre-image in C and we are done. $\square$

This finishes the proof of the lemma. $\square$

This finishes the proof of the theorem. $\square$

This finishes the notes.

Basic Notions in Algebraic Topology 11

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1 Cellular Homology

In the last TA session we saw that we can use the Mayer-Vietoris sequence in order to analyze the homology of a CW complex. Our approach applied the Mayer-Vietoris sequence to each cell we glued separately. Hence it might be a bit cumbersome to apply it to a CW complex with more than 2-3 cells. It turns out that our approach has a more sophisticated version, called **cellular homology**.

Recall the relative homology groups defined in exercise 10: Let X be a topological space and $A \subseteq X$ a subspace. Let $C_\bullet(X, A)$ be the chain complex given by

$$C(X, A)_\bullet = C_n(X)/C_n(A)$$

with the boundary maps $\partial_n : C_n(X, A) \longrightarrow C_{n-1}(X, A)$ induced from the boundary maps of $C_\bullet(X)$. The homology groups $H_n(C_\bullet(X, A))$ of this complex are called the **relative homology groups** of the pair (X, A) and are denoted by $H_n(X, A)$. Since we have a short exact sequence

$$0 \longrightarrow C_\bullet(A) \longrightarrow C_\bullet(X) \longrightarrow C_\bullet(X, A) \longrightarrow 0$$

of chain complexes and so from the snake lemma a long exact sequence:

$$... \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(X, A) \longrightarrow ...$$

In exercise 10 you will show that

$$H_n(X, A) \cong \widetilde{H}_0(X \cup_A CA)$$

Lemma 1. *Let (X, A) be a pair satisfying the homotopy extension property. Then the cone $C \cup_A CA$ is homotopy equivalent to the quotient space X/A.*

Proof. Exercise. □

Now let X be a CW complex with a finite number of cells in each dimension and let X_k be the k'th skeleton. Consider the long exact sequence

$$... \longrightarrow H_n(X_{k-1}) \xrightarrow{i_*^k} H_n(X_k) \xrightarrow{p_*^k} H_n(X_k, X_{k-1}) \xrightarrow{\delta_*^k}$$

$$H_{n-1}(X_{k-1}) \xrightarrow{i_*^k} H_{n-1}(X_k) \xrightarrow{p_*^k} H_{n-1}(X_k, X_{k-1}) \longrightarrow ...$$

Let I_k be the set indexing the k-cells of X, i.e. we have a pushout square

$$\begin{array}{ccc} \coprod_{\alpha \in I_k} S^{k-1} & \longrightarrow & \coprod_{\alpha \in I_k} D^k \\ \downarrow & & \downarrow \\ X_{k-1} & \longrightarrow & X_k \end{array}$$

This means that $X_k/X_{k-1} \cong \bigvee_\alpha S^k$. Since (X_k, X_{k-1}) satisfy the homotopy extension property we have that

$$H_n(X_k, X_{k-1}) \cong \widetilde{H}_n(X_k/X_{k-1}) \cong \oplus_{\alpha \in I} \widetilde{H}_n(S^k)$$

and in particular $H_n(X_n, X_{n-1}) \cong \mathbb{Z}I_n$.

Let us try to make this isomorphism more explicit: for each $\alpha \in I_n$ we have a gluing map $\varphi_\alpha : S^{n-1} \longrightarrow X_{n-1}$ and a corresponding extension

$$\widetilde{\varphi}_\alpha : D^n \longrightarrow X_n$$

Choose some homeomorphism $T : \Delta^n \longrightarrow D^n$ which sends $\partial\Delta^n$ to ∂D^n. Then for each n-cell $\alpha \in I_n$ consider the singular n-simplex $\sigma : \Delta^n \longrightarrow X_n$ given by $\sigma_\alpha = \widetilde{\varphi}_\alpha \circ T$. The corresponding element in $C_n(X_n)$ is not a cycle, but it's boundary is contained in X_{n-1}. This means that its image $p(\sigma_\alpha) \in C_n(X_n, X_{n-1})$ is a cycle. The element

$$[p(\sigma_\alpha)] \in H_n(X_n, X_{n-1})$$

corresponds to the generator α in $\mathbb{Z}I_n$.

Now let

$$\partial_n : H_n(X_n, X_{n-1}) \longrightarrow H_{n-1}(X_{n-1}, X_{n-2})$$

be the composition of the maps

$$H_n(X_n, X_{n-1}) \xrightarrow{\delta_n^n} H_{n-1}(X_{n-1}) \xrightarrow{p_*^{n-1}} H_{n-1}(X_{n-1}, X_{n-2})$$

where the first one is taken from the long exact sequence of the pair (X_n, X_{n-1}) and the second from the long exact sequence of the pair (X_{n-1}, X_{n-2}). Define

$$C_n^{\text{cel}}(X) = H_n(X_n, X_{n-1})$$

$$\partial_n = p_*^{n-1} \circ \delta_n^n : C_n^{\text{cel}}(X) \longrightarrow C_{n-1}^{\text{cel}}(X)$$

Lemma 2. *The composition $\partial_{n-1} \circ \partial_n$ equals the 0 map.*

Proof.

$$\partial_{n-1} \circ \partial_n = (p_*^{n-2} \circ \delta_{n-1}^{n-1}) \circ (p_*^{n-1} \circ \delta_n^n) = p^{n-2} \circ (\delta_{n-1}^{n-1} \circ p_*^{n-1}) \circ \delta_n^n$$

but $\delta_{n-1}^{n-1} \circ p_*^{n-1} = 0$ because these are two consecutive maps in the long exact sequence of the pair (X_{n-1}, X_{n-2}) so get that

$$\partial_{n-1} \circ \partial_n = 0$$

□

This means that the groups $C_n^{\text{cel}}(X)$ fit together to form a **chain complex**. This chain complex is called the **cellular chain complex** of X. It's homology groups of called the **cellular homology groups** of X and are denoted by $H_n^{\text{cel}}(X)$. Our main result is the following:

Theorem 3. *We have a canonical isomorphism*

$$H_n^{\text{cel}}(X) \cong H_n(X)$$

for every n and every CW complex X.

Proof.

Lemma 4. *The kernel of the map $\partial_n : C_n^{\text{cel}}(X) \longrightarrow C_{n-1}^{\text{cel}}(X)$ is naturally isomorphic to $H_n(X_n)$.*

Proof. In the previous TA session we saw that adding an n-cell doesn't change the homology groups below dimension $n-1$. Let $n \geq 1$. Then in particular $H_{n-1}(X_{n-2}) = 0$ and so the map $p_*^n : H_{n-1}(X_{n-1}) \longrightarrow H_{n-1}(X_{n-1}, X_{n-2})$ is injective. This means that the kernel of $\partial_n : C_n^{\text{cel}}(X) \longrightarrow C_{n-1}^{\text{cel}}(X)$ is equal to the kernel of

$$\delta_n : H_n(X_n, X_{n-1}) \longrightarrow H_{n-1}(X_{n-1})$$

This kernel is equal to the cokernel of the map

$$H_n(X_{n-1}) \longrightarrow H_n(X_n)$$

but since $H_n(X_{n-1}) = 0$ this kernel is just $H_n(X_n)$. □

The natural map $H_n(X_n) \longrightarrow H_n(X)$ gives us a map

$$T : \ker(\partial_n) \longrightarrow H_n(X)$$

In the last TA session we also saw that the map $H_n(X_n) \longrightarrow H_n(X_{n+1})$ is surjective. Since $H_n(X) \cong H_n(X_{n+1})$ our map T is surjective.

Now the kernel of T consists of exactly those $\alpha \in H_n(X_n)$ which maps to 0 in $H_n(X_{n+1})$. Then kernel of this map is equal to the image of the map

$$H_{n+1}(X_{n+1}, X_n) \longrightarrow H_n(X_n)$$

which is exactly the image of ∂_n. Hence T induces an isomorphism

$$H_n^{\text{cel}}(X) \longrightarrow H_n(X)$$

□

In order to use this tool in computations we need an idea of how to compute the boundary map $\partial_n : H_n(X_n, X_{n-1}) \longrightarrow H_{n-1}(X_{n-1}, X_{n-2})$. It is enough to understand what are the images under ∂_n of the generators $[p(\sigma_\alpha)]$ described above. Now the boundary of Δ^n is mapped under σ_α to X_{n-1} and so we can consider $\partial_n \sigma_\alpha$ as an $(n-1)$-cycle of $C_{n-1}(X_{n-1})$. By definition we then have

$$\delta_n([p(\sigma_\alpha)]) = [\partial_n \sigma_\alpha] \in H_{n-1}(X_{n-1})$$

But note that

$$[\partial_n \sigma_\alpha] = \varphi_{\alpha *}(g)$$

where $g \in H_{n-1}(S^{n-1})$ is given by the class of the cycle ∂T considered as a cycle in $\partial D^n = S^{n-1}$. This element is a generator of $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, so this gives a nice description of $\delta_n([p(\sigma_\alpha)])$. Then

$$\partial_n([p(\sigma_\alpha)]) = p_*(\varphi_{\alpha *}(g))$$

where $p_* : H_{n-1}(X_{n-1}) \longrightarrow H_{n-1}(X_{n-1}, X_{n-2})$.

Example:

Identify S^3 with the subset

$$S^3 = \{(z, w) \in \mathbb{C}^2 | |z|^2 + |w|^2 = 1\}$$

Let $n \geq 3$ and let $\omega \in \mathbb{C}$ be a primitive n'th root of unity. Let X be the quotient space of S^3 under the action of $\mathbb{Z}/n$ where the generator $g \in \mathbb{Z}/n$ acts as

$$g(z, w) = (\omega z, \omega w)$$

and let $q : S^3 \longrightarrow X$ be the quotient map.

We wish to compute the homology groups of X. We can find a CW structure on X as follows: Let $D \subseteq S^3$ be the subset given by

$$K = \left\{ \left(z, \sqrt{1 - |z|^2}\,\mathrm{cis}\,(\theta)\right) \middle| |z| \leq 1, |\theta| \leq \frac{\pi i}{n} \right\}$$

Note that K is homeomorphic to the 3-ball $D^3 = \{a, b, c | a^2 + b^2 + c^2 \leq 1\}$ via the homeomorphism

$$\varphi(a, b, c) = \left(a + bi, \sqrt{1 - a^2 - b^2} \cdot \mathrm{cis}\left(\frac{\pi i}{n} \frac{c}{\sqrt{1 - a^2 - b^2}}\right)\right)$$

(by this definition of φ we mean implicitly that when $a^2 + b^2 = 1$ then $\varphi(a, b, c) = (a + bi, 0)$).

Now the interior of K is given by

$$\mathrm{Int}(K) = \left\{ \left(z, \sqrt{1 - |z|^2}\,\mathrm{cis}(\theta)\right) \in K \middle| |z| < 1, \theta \in \left(-\frac{2\pi i}{n}, \frac{2\pi i}{n}\right) \right\}$$

Note that no two distinct points of $\mathrm{Int}(K)$ are identified by the action of $\mathbb{Z}/n$. Further more every point in S^3 is equivalent under the action to a point in K. In this case we say that K is a **fundamental domain** for the action of $\mathbb{Z}/n$ on S^3. What this gives us is that we can compute X by taking K and identifying points on the boundary ∂K which are identified by $\mathbb{Z}/n$.

The boundary of K is given by

$$\partial K = \left\{ \left(z, \sqrt{1-|z|^2}\,\mathrm{cis}\left(\frac{2\pi i}{n}\right)\right) \middle| \, |z| \leq 1 \right\} \cup \left\{ \left(z, \sqrt{1-|z|^2}\,\mathrm{cis}\left(-\frac{2\pi i}{n}\right)\right) \middle| \, |z| \leq 1 \right\}$$

$$\stackrel{def}{=} K_+ \cup K_-$$

Our identification of K with D^3 identifies K_+ and K_- with the two hemispheres of $\partial D^3 = S^2$. The action of $\mathbb{Z}/n$ on ∂K identifies the point

$$\left(z, \sqrt{1-|z|^2}\,\mathrm{cis}\left(-\frac{2\pi i}{n}\right)\right) \in K_-$$

with the point

$$\left(\omega z, \sqrt{1-|\omega z|^2}\,\mathrm{cis}\left(\frac{2\pi i}{n}\right)\right) \in K_+$$

Let $E = K_+ \cap K_-$ be the equator. It is given by points of the form $(z, 0)$ such that $|z| = 1$. Note that each such point $(z, 0) \in E$ is identified with $(w^i z, 0) \in K_+ \cap K_-$ for every $i = 0, ..., n-1$. Note that E is a circle and $E/\sim$ is also a circle. The map $E \longrightarrow E/\sim$ is an n-fold cover, and in particular a map of degree n.

Hence we see that $\partial K/\sim$ is a 2-dimensional CW complex whose 1-skeleton is $E/\sim$ and with a single two cell K_+ attached via the gluing map

$$\partial K_+ = E \longrightarrow E/\sim$$

which is an n-fold map $S^1 \longrightarrow S^1$. Finally the space $K/\sim$ is obtained from $\partial K/\sim$ by gluing a single 3-cell K attached via the gluing map

$$\partial K \longrightarrow \partial K/\sim = K_+/\sim$$

Now according to cellular homology we can compute the homologies of X from the chain complex

$$... \longrightarrow 0 \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \longrightarrow 0$$

The map ∂_1 is 0 (both edges of the 1-cell are glued to the same vertex). From our construction it is clear that ∂_2 is multiplication by n (because an n-fold cover of S^1 by S^1 is a map of degree n and so induces multiplication by n on H_1).

The only slight mystery is ∂_3. However it is completely constrained - since $\ker(\partial_2) = 0$ the image of ∂_3 must be 0. Hence we get that

$$H_1(X) \cong \mathbb{Z}/n$$

$$H_2(X) \cong \mathbb{Z}$$

and

$$H_3(X) = 0$$

Basic Notions in Algebraic Topology 13

Yonatan Harpaz

In this TA session we are going to talk about the functors $\mathrm{Tor}(-,-)$ and $\mathrm{Ext}(-,-)$ in the category of abelian groups. These notions are useful in analyzing the behavior of homology with coefficients. The theory we are going to discuss is part of a field called **homological algebra** which is an indispensable tool in the algebraic topologist toolkit.

1 The Functors Tensor and Hom

Let $\mathcal{A}b$ denote the category of abelian groups. The starting point of this discussion are the the tensor product operation $(A, B) \mapsto A \otimes B$ and the Hom operation $(A, B) \mapsto \mathrm{Hom}(A, B)$. Let us recall the definition of tensor product:

Definition 1. Let A, B be abelian group. We define $A \otimes B$ to be the abelian group generated by expressions of the form $a \otimes b$ for $a \in A, b \in B$ modulu the relations

$$a_1 \otimes b + a_2 \otimes b = a_3 \otimes b$$

whenever $a_1 + a_2 = a_3$ and

$$a \otimes b_1 + a \otimes b_2 = a \otimes b_2$$

whenever $b_1 + b_2 = b_3$.

Examples:

1. $A \otimes \mathbb{Z}$ is isomorphic to A. The map $a \mapsto a \otimes 1$ is an isomorphism.
2. $A \otimes \mathbb{Z}/n$ is isomorphic to the group A/nA.

   Proof. We have a homomorphism

   $$A \longrightarrow A \otimes \mathbb{Z}/n$$

   sending a to $a \otimes 1$. Now

   $$(na) \otimes 1 = n(a \otimes 1) = a \otimes n = a \otimes 0 = 0$$

   so this homomorphism induces a well defined homomorphism $A/nA \longrightarrow A \otimes \mathbb{Z}/n$. This homomorphism has an inverse sending $a \otimes r$ to the image of ra in A/nA. Hence it is an isomorphism. □

3.
$$(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$$

The obvious map $(A \otimes C) \oplus (B \otimes C) \longrightarrow (A \oplus B) \otimes C$ is clearly surjective. It is a simple exercise to show that it is also injective.

Note that these operations take two abelian groups and return a single abelian group. Formally we can think of $\otimes$ as a functor from $\mathcal{A}b \times \mathcal{A}b$ to $\mathcal{A}b$ where $\mathcal{A}b \times \mathcal{A}b$ is the category whose objects are pairs of groups (A, B) and morphisms from (A, B) to (C, D) are pairs of homomorphisms $T : A \longrightarrow C, S : B \longrightarrow D$. We also say that $\otimes$ is a bi-functor or that $\otimes$ is functorial in both coordinates.

As for $\mathrm{Hom}(-,-)$ we can think of it as a functor from $\mathcal{A}b^{op} \times \mathcal{A}b$ to $\mathcal{A}b$ where $\mathcal{A}b^{op}$ is the opposite category of abelian groups. An alternative terminology is to say that $\mathrm{Hom}(A, B)$ is contra-variant in the first coordinate and covariant in the second.

We will not worry to much about these formalities because we are going to fix a group D and consider the functors

$$A \mapsto A \otimes D$$

and

$$A \mapsto \mathrm{Hom}(A, D)$$

The first is a covariant functors (or just a functor from $\mathcal{A}b$ to $\mathcal{A}b$) and the second is contra-variant (or a functor from $\mathcal{A}b^{op}$ to $\mathcal{A}b$). We are going to be concerned with the behavior of these functor under short exact sequences, i.e. we will want to know what happens if we take a short exact sequence of abelian groups and hit it with it.

1.1 The Functor $- \otimes D$

We will start by analyzing the functor $- \otimes D$. Surprisingly, it will be useful to start our discussion by **generalizing** this functor from abelian groups to chain complexes (which also plays an important role in defining the product structure on cohomology):

Definition 2. Let $C_\bullet, D_\bullet$ be two chain complexes. We define their tensor product chain complex $C_\bullet \otimes D_\bullet$ by

$$(C \otimes D)_n = \oplus_{i+j=n} C_i \otimes D_j$$

The boundary map is given as follows: if $x \in C_i, y \in D_j$ such that $i + j = n$ then

$$\partial_n(x \otimes y) = (\partial_i x) \otimes y + (-1)^i x \otimes (\partial_j y)$$

It is immediate to compute that $\partial_{n+1} \circ \partial_n = 0$ so this is indeed a chain complex.

We will also need the following notion:

Definition 3. Let A be an abelian group. We say that a chain complex $A_\bullet$ is a free resolution of A if $A_\bullet$ is free and

$$H_i(A_\bullet) = \begin{cases} A & i = 0 \\ 0 & i > 0 \end{cases}$$

Note that every abelian group A has a free resolution: choose some set of generators and take A_0 to be the free group on these generators. Then we have a surjective map $A_0 \twoheadrightarrow A$ whose kernel is some (free) subgroup $A_1 \hookrightarrow A_0$ (because any subgroup of a free abelian group is free. For a proof of this claim see the wikipedia entry "free abelian group"). Hence

$$... \longrightarrow 0 \longrightarrow A_1 \longrightarrow A_0$$

is a complex whose 0'th homology is A and all the other homologies vanish. Note that in fact every abelian group has a free resolution of length 2. Now in question 1 in exercise 12 you prove that every two free chain complexes with isomorphic homologies are chain homotopy equivalent. In particular this implies that every two free resolutions are homotopy equivalent.

Now let A, D be two abelian groups and $A_\bullet, D_\bullet$ two respective free resolutions. Consider the chain complex $A_\bullet \otimes D_\bullet$. Note that if $A'_\bullet, D'_\bullet$ were two other free resolutions then $A_\bullet \simeq A'_\bullet$ and $D_\bullet \simeq D'_\bullet$ which implies that $A_\bullet \otimes D_\bullet \simeq A'_\bullet \otimes D'_\bullet$ (exercise). Hence the chain homotopy type of $A_\bullet \otimes D_\bullet$ depends only on A and D. This makes it interesting to wonder about the homology groups of the complex $A_\bullet \otimes D_\bullet$. These homology groups are denoted by

$$\text{Tor}_i(A, D) \stackrel{def}{=} H_i(A_\bullet \otimes D_\bullet)$$

They are called the Tor groups of A and D.

Remark 4. If $f : A \longrightarrow B$ is a homomorphism and $A_\bullet, B_\bullet$ are two respective free resolutions then there exists a (unique up to chain homotopy) map $\widetilde{f} : A_\bullet \longrightarrow B_\bullet$ which induces f on H_0. This makes it possible to define $\text{Tor}_i(A, B)$ as functors from $\mathcal{A}b \times \mathcal{A}b$ to $\mathcal{A}b$.

For a fixed D, the functors $\text{Tor}_i(-, D)$ are called the left derived functors of $- \otimes D$.

Remark 5. Since $A_\bullet \otimes D_\bullet \cong D_\bullet \otimes A_\bullet$ we see that $\text{Tor}_i(A, D)$ is symmetric in A and D.

Theorem 6.

$$\text{Tor}_0(A, D) \cong A \otimes D$$

and

$$\text{Tor}_i(A, D) = 0$$

for $i \geq 2$.

Proof. Since we can compute this using any free resolution we want why not take the standard free resolution: we let $A_0 = \oplus_{a \in A} \mathbb{Z} < a >$ be the free abelian group generated by the elements of A (considered as a set) and $A_1 \subseteq A_0$ the (free) subgroup generated by relations of the form $a_1 + a2 - a_3$ when ever $a_1 + a_2 = a_3$. It is clear that $A_0/A_1 \cong A$ so $A_1 \longrightarrow A_0$ is free resolution of length 2 of A. Similarly we take $D_1 \longrightarrow D_0$ be the standard free resolution of D.

We obtain the complex:

$$... \longrightarrow 0 \longrightarrow A_1 \otimes D_1 \xrightarrow{\partial_2} [A_0 \otimes D_1] \oplus [A_1 \otimes D_0] \longrightarrow A_0 \otimes D_0 \longrightarrow 0$$

Note that ∂_2 is injective and so the homologies of this complex vanish in dimension 2 and up.

We can identify $A_0 \otimes D_0$ with the free abelian group generated by symbols of the form $a \otimes d$ for $a \in A, d \in D$. Then the elements coming from $[A_0 \otimes D_1] \oplus [A_1 \otimes D_0]$ are exactly the standard relations defining $A \otimes D$. Hence we get that the 0'th homology of this complex is isomorphic to $A \otimes D$. This finishes the proof of the theorem. □

What is the role of the Tor groups? They will help us study the behavior of the functor $- \times D$ under short exact sequence. Consider a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

when we tensor with D we get the sequence

$$0 \longrightarrow A \otimes D \longrightarrow B \otimes D \longrightarrow C \otimes D \longrightarrow 0$$

and we ask our selves if this sequence is exact. Clearly there are some easy situations in which it is exact:

1. If $D = \mathbb{Z}I$ is free with generator set I then it is easy to see that

   $$0 \longrightarrow \oplus_{i \in I} A \longrightarrow \oplus_{i \in I} B \longrightarrow \oplus_{i \in I} C \longrightarrow 0$$

   is exact. We can generalize this observation quite easily to chain complexes: if

   $$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0$$

   is a short exact sequence of chain complexes and $D_\bullet$ is free then

   $$0 \longrightarrow A_\bullet \otimes D_\bullet \longrightarrow B_\bullet \otimes D_\bullet \longrightarrow C_\bullet \otimes D_\bullet \longrightarrow 0$$

   is exact.

2. If $B \cong C \oplus A$ and i, p are the natural inclusion and projections respectively then

   $$B \otimes D \cong (C \otimes D) \oplus (A \otimes D)$$

   so the sequence remains exact after applying to it the functor $- \otimes D$ for any D. In this situation we say that the short exact sequence **splits**. Note that if C is free then any short exact sequence ending with C splits (exercise).

Now let us analyze the general case. We first use the observation above in order to show that one can compute $\mathrm{Tor}_i(A, D)$ by resolving just one of A, D:

Lemma 7. *Let $A_\bullet, D_\bullet$ be free resolutions of A, D respectively. Let $\overline{A}$ be the complex which is A at the 0'th place and 0 elsewhere, and similarly $\overline{D}$. Then the natural maps*

$$A_\bullet \otimes D_\bullet \longrightarrow \overline{A}_\bullet \otimes D_\bullet$$

$$A_\bullet \otimes D_\bullet \longrightarrow A_\bullet \otimes \overline{D}$$

induce isomorphism on homology groups.

Proof. The claims are symmetric so we just prove the first. Let $\underline{D}$ be the kernel of the map $D_\bullet \longrightarrow \overline{D}_\bullet$. Then $\underline{D}_\bullet$ is given by

$$... \longrightarrow 0 \longrightarrow D_1 \longrightarrow D_1$$

We have a short exact sequence of chain complexes

$$0 \longrightarrow \underline{D} \longrightarrow D_\bullet \longrightarrow \overline{D}_\bullet \longrightarrow 0$$

Let us now tensor this short exact sequence with the free complex $A_\bullet$. We will get a short exact sequence

$$0 \longrightarrow \underline{D} \otimes A_\bullet \longrightarrow D_\bullet \otimes A_\bullet \longrightarrow \overline{D}_\bullet \otimes A_\bullet \longrightarrow 0$$

From the snake lemma we see that in order to show that the map $D_\bullet \otimes A_\bullet \longrightarrow \overline{D}_\bullet \otimes A_\bullet$ induces an isomorphism on homology groups it is enough to show that all the homologies of $\underline{D} \otimes A_\bullet$ vanish. But $\underline{D}$ is free without homologies and so chain homotopic to the 0 complex. Hence $\underline{D} \otimes A_\bullet$ is chain homotopic to the 0 complex and it's homologies vanish. It is also possible to verify that the homologies of $\underline{D} \otimes A$ vanish directly (just write it down and you will see). This finishes the proof of the lemma. $\square$

Let us now return to analyzing the behavior of $- \otimes D$ under short exact sequences. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of abelian groups and $D_\bullet$ a free resolution of D. Since D is free the sequence

$$0 \longrightarrow \overline{A}_\bullet \otimes D_\bullet \longrightarrow \overline{B}_\bullet \otimes D_\bullet \longrightarrow \overline{C}_\bullet \otimes D_\bullet \longrightarrow 0$$

of chain complexes is exact. Using Lemma 7 and the snake lemma we get a long exact sequence

$$... \longrightarrow \mathrm{Tor}_2(C, D) \longrightarrow \mathrm{Tor}_1(A, D) \longrightarrow \mathrm{Tor}_1(B, D) \longrightarrow \mathrm{Tor}_1(C, D) \longrightarrow$$

$$A \otimes D \longrightarrow B \otimes D \longrightarrow C \otimes D \longrightarrow 0$$

Since all $\mathrm{Tor}_i(-, -) = 0$ for $i \geq 2$ we can write this as

$$0 \longrightarrow \mathrm{Tor}_1(A, D) \longrightarrow \mathrm{Tor}_1(B, D) \longrightarrow \mathrm{Tor}_1(C, D) \longrightarrow$$

$$A \otimes D \longrightarrow B \otimes D \longrightarrow C \otimes D \longrightarrow 0$$

We can summarize this analysis as follows:

1. If
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$
is a short exact sequence then
$$A \otimes D \longrightarrow B \otimes D \longrightarrow C \otimes D \longrightarrow 0$$
is exact. We say that $- \otimes D$ is exact from right.

2. One can use the functor $\mathrm{Tor}_1(-,-)$ in order to measure the non-exactness of $- \otimes D$ from the left. This is manifested in the fact that the map
$$\mathrm{Tor}_1(C, D) \longrightarrow \ker(A \otimes D \longrightarrow B \otimes D)$$
is surjective.

3. The functor $- \otimes D$ will preserve short exact sequences (a.k.a - $- \otimes D$ is **exact**) if and only if $\mathrm{Tor}(A, D) = 0$ for any group A. We know that this happens for example when D is free so we conclude that $\mathrm{Tor}_i(A, D) = 0$ when D is free and A is any group.

   By a Zorn lemma argument one can show that if $\mathrm{Tor}_1(\mathbb{Z}/n, D) = 0$ for every n then $\mathrm{Tor}_1(A, D) = 0$ for every A. We will calculate in a minute that $\mathrm{Tor}_1(\mathbb{Z}/n, D) = D[n]$, and so $- \otimes D$ is exact if and only if D is torsion free. This is one of the reasons that Tor is called Tor.

Let us calculate indeed that $\mathrm{Tor}(\mathbb{Z}/n, D) = 0$. Consider the short exact sequence
$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$
Tensoring with D we get an exact sequence
$$0 \longrightarrow \mathrm{Tor}_1(\mathbb{Z}, D) \longrightarrow \mathrm{Tor}_1(\mathbb{Z}, D) \longrightarrow \mathrm{Tor}_1(\mathbb{Z}/n, D) \longrightarrow$$
$$\mathbb{Z} \otimes D \longrightarrow \mathbb{Z} \otimes D \longrightarrow \mathbb{Z}/n \otimes D \longrightarrow 0$$
which since $\mathrm{Tor}_i(\mathbb{Z}, D) = \mathrm{Tor}_i(D, \mathbb{Z}) = 0$ becomes
$$\mathrm{Tor}_1(\mathbb{Z}/n, D) \longrightarrow D \xrightarrow{n} D \longrightarrow \mathbb{Z}/n \otimes D \longrightarrow 0$$
and so $\mathrm{Tor}_1(\mathbb{Z}/n, D) \cong D[n]$.

1.2 The functor $\mathrm{Ext}^i(-,-)$

Let D be an abelian group. Suppose we take a short exact sequence
$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$
and hit it with the functor $\mathrm{Hom}(-, D)$. We get a sequence
$$0 \longrightarrow \mathrm{Hom}(C, D) \xrightarrow{p^*} \mathrm{Hom}(B, D) \xrightarrow{i^*} \mathrm{Hom}(A, D) \longrightarrow 0$$

and we ask whether it is still exact. Clearly since the map $B \longrightarrow C$ is injective we see that p^* is injective: if $T : C \longrightarrow D$ is a homomorphism such that $T(p(b)) = 0$ for every b then $T(c) = 0$ for every $c \in C$ so $T = 0$. It is also not hard to see that this sequence is also exact in the middle: if a homomorphism $T : B \longrightarrow D$ becomes the zero homomorphism when restricted to A then it induces a well defined homomorphism $\widetilde{T} : C \longrightarrow D$. Then $T = \widetilde{T} \circ p = p^*\widetilde{T}$ so T is in the image of p^*.

On the end, however, we will see that our sequence is **not** exact, because the map

$$\mathrm{Hom}(B, D) \longrightarrow \mathrm{Hom}(A, D)$$

will in general not be surjective. For example, suppose that $D = A$. Then the identity $I \in \mathrm{Hom}(A, A)$ extends to a homomorphism $B \longrightarrow A$ if and only if the sequence **splits**. Hence if the sequence doesn't split (for example $A \cong B \cong \mathbb{Z}$ and i is multiplication by 2) then $\mathrm{Hom}(B, A) \longrightarrow \mathrm{Hom}(A, A)$ will not be surjective. Note that if the sequence splits then it will remain exact after applying the functor $\mathrm{Hom}(-, D)$ for any D.

We say that the contravariant functor $\mathrm{Hom}(-, D)$ is exact from the left, or left exact.

Now suppose we hit a short exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

with the functor $\mathrm{Hom}(A, -)$. It is not hard to see that we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, C) \longrightarrow \mathrm{Hom}(A, D)$$

but that the homomorphism $\mathrm{Hom}(A, C) \to \mathrm{Hom}(A, D)$ will not be surjective. We say that the functor $\mathrm{Hom}(A, -)$ is exact from the left.

Now in order to understand both the functors $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, D)$ and in particular the understand their gap from being exact, we will generalize the $\mathrm{Hom}(-, -)$ functor to complexes. This time, however, we're going to need to work with **cochain** complexes as well.

Definition 8. Let $A_\bullet$ be a chain complex and $D^\bullet$ a cochain complex. We define a **cochain** complex $\mathrm{Hom}(A_\bullet, D^\bullet)$ as follows:

$$\mathrm{Hom}(A_\bullet, D^\bullet)^n = \prod_{i+j=n} \mathrm{Hom}(A_i, D^j)$$

with the differential given by

$$\partial f = \partial_D \circ f + (-1)^n f \circ \partial_A$$

Now let A, D be two abelian groups. We would like to apply this generalization of the Hom functor to some resolutions of A and D. However it turns out that the concept of free resolution we have encountered before will not suffice. We need to add to it the concept of an **injective resolution**, which is a dual notion to the notion of free resolution.

In order to understand how to dualize the notion of free resolution we first need to formulate it in a different way.

Definition 9. We say that a group A is **projective** if for every diagram

$$\begin{array}{ccc} & & C \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & D \end{array}$$

such that g is **surjective** there exists a lift $\widetilde{f} : A \longrightarrow C$.

Remark 10. Another way to formulate the same property is to say that if $g : C \longrightarrow D$ is surjective then the induced map $\mathrm{Hom}(A, C) \longrightarrow \mathrm{Hom}(A, D)$ is surjective. This is equivalent to saying that the functor $\mathrm{Hom}(A, -)$ is exact.

Lemma 11. *An abelian group is projective if and only if it is free.*

Proof. It is clear that if A is free then it is projective (construct $\widetilde{f}$ on each generator independently). Now if A is projective and A_0 is a free group which admits a surjective homomorphism $g : A_0 \longrightarrow A$ then from projectivity one can lift the identity $If : A \longrightarrow A$ to a map $s : A \longrightarrow A_0$. Since $g \circ s = Id$ we get that s is injective and so A can be embedded in a free group. Since every subgroup of a free group is free we get the desired result. $\square$

Now the definition of projectivity has a natural dual notion:

Definition 12. We say that an abelian group D is **injective** if for every diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & D \\ \downarrow{\scriptstyle g} & & \\ B & & \end{array}$$

such that g is **injective** there exists an extension $\overline{f} : B \longrightarrow D$.

Remark 13. Another way to formulate the same property is to say that if $g : A \longrightarrow B$ is injective then the induced map $\mathrm{Hom}(B, D) \longrightarrow \mathrm{Hom}(A, D)$ is surjective. This is equivalent to saying that the functor $\mathrm{Hom}(-, D)$ is exact.

Theorem 14. *An abelian group D is injective if and only if it is a* ***divisible group****, i.e. if for each $x \in D$ and $0 \neq n \in \mathbb{Z}$ there exists a $y \in D$ such that $ny = x$.*

Proof. Sketch of proof: first one shows that D is divisible if and only if for every diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & D \\ \downarrow{\scriptstyle g} & & \\ B & & \end{array}$$

such that g is **injective** and $B/g(A)$ is **cyclic** there exists an extension $\overline{f} : B \longrightarrow D$.

This already gives that injectivity implies divisibilty. Now let D be a divisible group and consider a general diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & D \\ \downarrow{\scriptstyle g} & & \\ B & & \end{array}$$

such that g is injective. By Zorm's lemma there exists a maximal subgroup $g(A) \subseteq A_{\max} \subseteq B$ such that f extends to $\overline{f} : A_{\max} \longrightarrow D$. We need to show that $A_{\max} = B$. Otherwise there would exists an element $x \in B$ which is not in $A_{\max}$. Let $A' \subseteq B$ be the subgroup generated by $A_{\max}$ and x. Then $A'/A_{\max}$ is cyclic (generated by the image of x) and and so $\overline{f}$ extends to A' (because D is divisible) in contradiction to the maximality of $A_{\max}$.

□

Definition 15. We say that a cochain complex $D^{\bullet}$ is an **injective resolution** if a group D if all the groups D^n are injective and

$$H^i = \begin{cases} D & i = 0 \\ 0 & i > 0 \end{cases}$$

Lemma 16. *Every group D has an injective resolution.*

Proof. Sketch of proof: by formally adding solutions to equations of the form $ny = x$ we can embed D in a divisible group D^0. The quotient $D^1 = D^0/D$ is then divisible as well, and we get an injective resolution

$$D^0 \longrightarrow D^1 \longrightarrow 0 \longrightarrow 0 \longrightarrow ...$$

□

Lemma 17. *Every two injective resolutions of D are homotopy equivalent.*

Proof. Left for the reader. Just dualize the proof you gave for question 1 in exercise 12. □

Now let A, D be two groups. Let $A_{\bullet}$ be a projective (i.e. free) resolution of A and $D^{\bullet}$ an injective resolution of D. We define

$$\mathrm{Ext}^i(A, D) = H^i(\mathrm{Hom}(A_{\bullet}, D^{\bullet}))$$

Remark 18.

1. As before this definition does not depends on the choice of resolutions $A_{\bullet}, D^{\bullet}$. Further more one can show that $\mathrm{Ext}^i(A, D)$ is functorial in D and contraviariantly functorial in A.

2. As in the definition of Tor_i one can show that

$$H^i(\mathrm{Hom}(A_\bullet, D^\bullet)) \cong H^i(\mathrm{Hom}(A_\bullet, D)) \cong H^i(\mathrm{Hom}(A, D^\bullet))$$

so it is enough to resolve just one of the groups (although for A we must use a projective resolution and for D an injective one). The proof still works because in this case $\mathrm{Hom}(A_\bullet, -)$ and $\mathrm{Hom}(-, D^\bullet)$ are exact functors.

3. Since we can choose $A_\bullet$ to be a resolution of length 2 we get that

$$\mathrm{Ext}^i(A, D) = 0$$

if $i \geq 2$.

4. Since $\mathrm{Hom}(-, D)$ is exact from the left it preserves kernels and so

$$\mathrm{Ext}^0(A, D) = H^0(\mathrm{Hom}(A_\bullet, D)) = \ker(\mathrm{Hom}(A_0, D) \longrightarrow \mathrm{Hom}(A_1, D)) =$$

$$\mathrm{Hom}(A, D)$$

Theorem 19. *Let D be a fixed abelian group and*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

a short exact sequence of abelian groups. Then we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, D) \longrightarrow \mathrm{Hom}(B, D) \longrightarrow \mathrm{Hom}(A, D) \longrightarrow$$
$$\mathrm{Ext}^1(C, D) \longrightarrow \mathrm{Ext}^1(B, D) \longrightarrow \mathrm{Ext}^1(A, D) \longrightarrow 0$$

Proof. Choose an injective resolution $D^\bullet$ of D. Then the sequence

$$0 \longrightarrow \mathrm{Hom}(C, D^\bullet) \longrightarrow \mathrm{Hom}(B, D^\bullet) \longrightarrow \mathrm{Hom}(A, D^\bullet) \longrightarrow 0$$

is exact (because each D^n is injective) and so we can apply the snake lemma (in its cohomological version) and get the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, D) \longrightarrow \mathrm{Hom}(B, D) \longrightarrow \mathrm{Hom}(A, D) \longrightarrow \mathrm{Ext}^1(C, D) \longrightarrow$$

$$\mathrm{Ext}^1(B, D) \longrightarrow \mathrm{Ext}^1(A, D) \longrightarrow \mathrm{Ext}^2(C, D) \longrightarrow ...$$

Since $\mathrm{Ext}^i(-, -) = 0$ for $i \geq 2$ we get the desired result. □

Theorem 20. *Let A be a fixed abelian group and*

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

a short exact sequence of abelian groups. Then we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, C) \longrightarrow \mathrm{Hom}(A, D) \longrightarrow$$
$$\mathrm{Ext}^1(A, B) \longrightarrow \mathrm{Ext}^1(A, C) \longrightarrow \mathrm{Ext}^1(A, D) \longrightarrow 0$$

Proof. Choose a projective (i.e. free) resolution $A_\bullet$ of A. Then the sequence

$$0 \longrightarrow \mathrm{Hom}(A_\bullet, B) \longrightarrow \mathrm{Hom}(A_\bullet, C) \longrightarrow \mathrm{Hom}(A_\bullet, D) \longrightarrow 0$$

is exact (because each A_n is projective) and so we can apply the snake lemma and get the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, C) \longrightarrow \mathrm{Hom}(A, D) \longrightarrow \mathrm{Ext}^1(A, B) \longrightarrow$$

$$\mathrm{Ext}^1(A, C) \longrightarrow \mathrm{Ext}^1(A, D) \longrightarrow \mathrm{Ext}^2(A, B) \longrightarrow ...$$

Since $\mathrm{Ext}^i(-,-) = 0$ for $i \geq 2$ we get the desired result. □

2 Universal Coefficients Theorems

Theorem 21 (The universal coefficients theorem for homology). *Let $C_\bullet$ be a free chain complex and A an abelian group. Then there exists a natural short exact sequence*

$$0 \longrightarrow H_n(C_\bullet) \otimes A \longrightarrow H_n(C_\bullet \otimes A) \longrightarrow \mathrm{Tor}_1(H_{n-1}(C_\bullet), A) \longrightarrow 0$$

which splits (though not naturally).

In particular if X is a topological space and A an abelian group then we have a natural exact sequence

$$0 \longrightarrow H_n(X) \otimes A \longrightarrow H_n(X, A) \longrightarrow \mathrm{Tor}_1(H_{n-1}(X), A) \longrightarrow 0$$

which splits (though not naturally).

Proof. In question 1 of exercise 12 you prove that if $C_\bullet$ is a **free** chain complex with boundary maps $\partial_n : C_n \longrightarrow C_{n-1}$ then there exists a (non-unique) isomorphism

$$C_\bullet \cong \oplus_n A^n_\bullet$$

where $A^n_\bullet$ is the 2-fold complex

$$... \longrightarrow 0 \longrightarrow \mathrm{Im}(\partial_{n+1}) \stackrel{\iota}{\hookrightarrow} \ker(\partial_n) \longrightarrow 0 \longrightarrow ... \longrightarrow 0$$

so that $H_n(A^n_\bullet) \cong H_n(C_\bullet)$ and $H_k(A^n_\bullet) = 0$ for $k \neq n$. Let us write this a bit differently as follows: define the **suspension** $\Sigma D_\bullet$ of a chain complex $D_\bullet$ by $\Sigma D_n = D_{n-1}$. Note that $H_n(\Sigma D_\bullet) \cong H_{n-1}(D_\bullet)$ for every n. Then one can write

$$C_\bullet \cong \oplus_n \Sigma^n B^n_\bullet$$

where $B^n_\bullet$ is the complex

$$... \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{Im}(\partial_{n+1}) \stackrel{\iota}{\hookrightarrow} \ker(\partial_n)$$

This means that

$$H_k(C_\bullet \otimes A) \cong \oplus_n H_k(\Sigma^n B^n_\bullet \otimes A) = \oplus_n H_{k-n}(B^n_\bullet \otimes A)$$

Note that $B^n_\bullet$ is a free resolution of $H_0(B^n_\bullet) \cong H_n(C_\bullet)$ and so

$$H_0(B^n_\bullet \otimes A) \cong H_n(C) \otimes A$$

$$H_1(B^n_\bullet \otimes A) \cong \mathrm{Tor}_1(H_n(C), A)$$

and

$$H_i(B^n_\bullet \otimes A) = 0$$

for $i > 1$. This means that

$$H_k(C_\bullet \otimes A) \cong H_0(B^k_\bullet \otimes A) \oplus H_1(B^{k-1}_\bullet, A) \cong [H_k(C) \otimes A] \oplus \mathrm{Tor}_1(H_{k-1}(C_\bullet), A)$$

Now this isomorphism is not unique because we could have chosen many different isomorphisms

$$C_\bullet \cong \oplus_n A^n_\bullet$$

Note however that for the inclusions $\iota_k : A^n_k \hookrightarrow C_k$ there is a natural candidate for ι_n (the natural inclusion $\ker(\partial_k) \subseteq C_k$) but not for ι_{k+1}. On the other hand if we are considering projections $p_k : C_k \longrightarrow A^n_k$ then there is a natural candidate for p_{k+1} but not for p_k. This results in the fact that out of the isomorphism above, the short exact sequence

$$0 \longrightarrow H_n(C_\bullet) \otimes A \longrightarrow H_n(C_\bullet \otimes A) \longrightarrow \mathrm{Tor}_1(H_{n-1}(C_\bullet), A) \longrightarrow 0$$

is natural, but the choice of splitting is not natural.

□

Theorem 22 (The universal coefficients theorem for cohomology). *Let $C_\bullet$ be a free chain complex and A an abelian group. Then there exists a natural short exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(C_\bullet), A) \longrightarrow H^n(\mathrm{Hom}(C, A)) \longrightarrow \mathrm{Hom}(H_n(C_\bullet), A) \longrightarrow 0$$

which splits unnaturally.

In particular if X is a topological space and A an abelian group then there exists a natural exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(X), A) \longrightarrow H^n(X, A) \longrightarrow \mathrm{Hom}(H_n(X), A) \longrightarrow 0$$

which splits unnaturally.

Proof. Let $C_\bullet$ be a free chain complex. As in the proof of the universal coefficients theorem for homology we write

$$C_\bullet \cong \oplus_n \Sigma^n B^n_\bullet$$

where $B^n_\bullet$ is the complex

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{Im}(\partial_{n+1}) \overset{\iota}{\hookrightarrow} \ker(\partial_n)$$

This means that

$$H^k(\mathrm{Hom}(C_\bullet, A)) \cong \prod_n H^k(\mathrm{Hom}(\Sigma^n B^n_\bullet, A)) = \prod_n H^{k-n}(\mathrm{Hom}(B^n_\bullet, A))$$

As $B^n_\bullet$ are free resolutions of the groups $H_0(B^n_\bullet) \cong H_n(C_\bullet)$ we get that

$$H^0(\mathrm{Hom}(B^n_\bullet, A)) \cong \mathrm{Hom}(H_n(C), A)$$

$$H^1(\mathrm{Hom}(B^n_\bullet, A)) \cong \mathrm{Ext}^1(H_n(C), A)$$

and

$$H_i(\mathrm{Hom}(B^n_\bullet, A)) = 0$$

for $i > 1$. This means that

$$H^k(\mathrm{Hom}(C_\bullet, A)) \cong H^0(\mathrm{Hom}(B^k_\bullet, A)) \times H_1(\mathrm{Hom}(B^{k-1}_\bullet, A)) \cong \mathrm{Hom}(H^k(C), A) \times \mathrm{Ext}^1(H_{k-1}(C_\bullet), A)$$

Now this isomorphism is not unique because we could have chosen many different isomorphisms

$$C_\bullet \cong \oplus_n A^n_\bullet$$

However from the same considerations as above we get that the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(C_\bullet), A) \longrightarrow H^n(\mathrm{Hom}(C, A)) \longrightarrow \mathrm{Hom}(H_n(C_\bullet), A) \longrightarrow 0$$

is natural, but the choice of splitting is not natural. □

Basic Notions in Algebraic Topology 13

Yonatan Harpaz

1 Alexander's Duality

In this section we will show a simplicial version of Alexander's Duality. It is less general then the one you've learned in class, but it gives a rather concrete and combinatorial way of understanding the concept.

1.1 Posets and Barycentric Subdivision

Let us recall the concept of barycenric subdivision. Let X be a simplicial complex with vertex set $V = \{v_0, ..., v_n\}$ and simplices $\mathcal{F} \subseteq P(V)$. In the barycentric subdivision $B(X)$ of X we put a vertex in the middle of each simplex. Hence the vertices of $B(X)$ are in bijection with the set $\mathcal{F}$ of simplices of X.

What are the simplices of $B(X)$? By identifying the vertex set with $\mathcal{F}$ we see that the simplices of $B(X)$ correspond to sets $\{\sigma^{n_1}, \sigma^{n_2}, ..., \sigma^{n_k}\} \subseteq \mathcal{F}$ such that each σ^{n_i} is a face of (i.e. contained in) $\sigma^{n_{i+1}}$.

To make this more convenient recall the notion of a **poset**. A poset (initials for partially ordered set) is a set X of elements together with a relation $a < b$ which is transitive and non-reflexive. The strict inclusion relation on $\mathcal{F}$ is a partial order making $\mathcal{F}$ into a poset. Hence we will denote it from now on by $<$.

Let X be a poset. We call a subset $S \subseteq X$ a **chain** if every two elements $x, y \in S$ are **comparable**, i.e. either $x < y$ or $y < x$. Note that the set of chains in X is closed to taking subsets and contains the empty set. Hence we almost have a structure of a simplicial complex. Note that we don't have a natural numbering on our vertices so we need to ask how we choose the orientations on all the simplices.

Note that each chain has a natural **order** to its vertices induced from $<$. In fact we have two choices we can make here. We can think of a chain as a descending sequence

$$x_1 > x_2 > ...x_k$$

or an ascending sequence

$$y_1 < y_2 < ...y_k$$

This we give us two possible simplicial complex structures on X. The realizations of these two structures are homeomorphic but differ as abstract simplicial complexes).

Note that this means that whenever we come across a poset in mathematics we have a natural topological space in the background, which is quite cool.

To conclude, the simplicial complex $B(X)$ is just the simplicial complex associated to the poset $\mathcal{F}$ with respect to the inclusion order. With $B(X)$ we will work with option one (descending sequences). We will denote by $\widehat{B}(X)$ the simplicial complex obtained by working with ascending sequences.

We have a natural inclusion of complexes $C^{\Delta}_{\bullet}(X) \hookrightarrow C^{\Delta}_{\bullet}(B(X))$ which sends

$$\sigma^k \mapsto \sum \pm(\sigma^k > \sigma^{k-1} > ... > \sigma^0)$$

where the sum is taken over all chains $\sigma^k > \sigma^{k-1} > ... > \sigma^0$ which start with σ^k and descend one dimension at a time until they reach a vertex (hence they are chains of length $k+1$ which are k-simplices of $B(X)$).

The sign represents the possible gap between the orientation of $\sigma^k > \sigma^{k-1} > ... > \sigma^0$ and the orientation of σ^k itself. This gap can be measured as follows. Recall the original vertex set V. Each σ^i in this sequence has a vertex $v_{j_i} \in V$ which σ^{i-1} doesn't have (in the end σ^0 has a single vertex v_{j_0}). Hence we get an order

$$v_{j_k}, v_{j_{k-1}}, ..., v_{j_0}$$

on the vertex set of σ^k. The sign is just the sign of this permutation (measuring with respect to the natural order on these vertices coming from the values of $j_k, ..., j_0$).

The inclusion $C^{\Delta}_{\bullet}(X) \hookrightarrow C^{\Delta}_{\bullet}(B(X))$ induces isomorphisms on homologies.

1.2 Simplicial Alexander's Duality

Let X be a simplicial complex on the vertex set $V = \{v_0, ..., v_n\}$ and simplices set $\mathcal{F} \subseteq P(V)$ such that $V \notin \mathcal{F}$. Hence X can be considered as a subcomplex of $\partial\Delta_n$ which is the simplicial complex on V whose simplices $\mathcal{G}$ are all the non-empty subsets besides V itself.

Define the dual complex $\widehat{X}$ to be the simplicial complex on V with simplices set $\widehat{\mathcal{F}}$ defined by the rule that $S \subseteq V$ is in $\widehat{\mathcal{F}}$ if the complement $V \setminus S$ is **not** in $\mathcal{F}$. Note that $\widehat{\mathcal{F}}$ is closed under takingk subsets and so defines some simplicial complex on V. Also note that V itself is not in $\widehat{\mathcal{F}}$ so we can consider $\widehat{X}$ as a subcomplex of S^{n-1}.

We claim that $\widehat{X}$ is actually homotopy equivalent to the complement of X in S^{n-1}. We saw in previous exercises that if we take the full subcomplex on some subset of the vertexes $A \subseteq V$ then we could obtain a model for the complement by considering the full subcomplex on the complement vertex set $V \setminus A$.

In order to use this result we first switch to the barycentric subdivision. Let $B(S^{n-1}), B(X), B(\widehat{X})$ denote the barycentric subdivisions of S^{n-1}, X and $\widehat{X}$ respectively and $\widehat{B}(S^{n-1})$ the dual barycentric subdivision of S^{n-1}.

$B(S^{n-1})$ has $\mathcal{G}$ as a vertex set (and chains on $\mathcal{G}$ as simplices) and $B(X)$ becomes the full subcomplex of $B(S^{n-1})$ on the set $\mathcal{F} \subset \mathcal{G}$. Similarly $\widehat{B}(\widehat{X}) \subseteq B(\widehat{X})$ becomes the full subcomplex on $\widehat{\mathcal{F}} \subseteq \mathcal{G}$.

Let $X^c \subseteq \widehat{B}(S^{n-1})$ denote the full subcomplex on the vertex set $\mathcal{G} \setminus \mathcal{F}$. By the exercise we know that the realization of X^c is homotopy equivalent to the complement of X in S^{n-1}.

There is a bijection between the set $\widehat{\mathcal{F}}$ and $\mathcal{G} \setminus \mathcal{F}$ which is obtained by sending $S \subseteq V$ to $V \setminus S$. This bijection is order reversing and so induces an isomorphism of simplicial complexes

$$\widehat{B}(\widehat{X}) \xrightarrow{\simeq} X^c$$

Hence we get that the realization of $B(\widehat{X})$ is homotopy equivalent to the complement of X in S^{n-1}.

We now want to prove Alexander's duality in this case. Consider the chain complex $C^{\Delta}_{\bullet}(S^{n-1})$. We have a natural bilinear map (also called **pairing**):

$$(-,-) : C^{\Delta}_k(S^{n-1}) \times C^{\Delta}_{n-1-k}(S^{n-1}) \longrightarrow \mathbb{Z}$$

given by

$$(\sigma^k, \sigma^{n-1-k}) = \begin{cases} \pm 1 & \sigma^k \cup \sigma^{n-1-k} = V \\ 0 & \sigma^k \cup \sigma^{n-1-k} \neq V \end{cases}$$

where the sign ± 1 is determined as follows. If $\sigma^k = \{v_{i_0}, ..., v_{i_k}\}$ and $\sigma^{n-1-k} = \{v_{i_{k+1}}, ..., v_{i_n}\}$ (where $i_0, ..., i_k$ and $i_{k+1}, ..., i_n$ are well ordered) then we take the sign above to be the sign of the permutation corresponding to the (somewhat weird looking) ordering

$$v_{i_n}, ..., v_{k+1}, v_0, ..., v_{i_k}$$

For example, if $V = \{v_0, v_1, v_2\}$ then $(\{v_0, v_1\}, \{v_2\}) = \text{sign}(v_2, v_0, v_1) = 1$, $(\{v_0\}, \{v_1, v_2\}) = \text{sign}(v_2, v_1, v_0) = -1$ and $(\{v_0, v_1\}, \{v_1\}) = 0$.

This pairing respects the differentials ∂_k in the sense that if $\sigma^k \in C^{\Delta}_k, \sigma^{n-k} \in C^{\Delta}_{n-k}$ then

$$(\partial_k \sigma^k, \sigma^{n-k}) = (\sigma^k, \partial_{n-k} \sigma^{n-k})$$

This is a rather simple sign chasing argument. Now this pairing is clearly non-degenerate (in the appropriate basis it is represented by the identity matrix). Hence if we introduce some subgroup $C^{\Delta}_k(X) \subseteq C^{\Delta}_k(S^{n-1})$ we will get a non-degenerate pairing between the quotient $C^{\Delta}_k(S^{n-1})/C^{\Delta}_k(X)$ and the subgroup of $C^{\Delta}_{n-1-k}(S^{n-1})$ which is orthogonal to $C^{\Delta}_k(X)$ (under this pairing).

This subgroup is just the subgroup generated by all simplices $\sigma^{n-1-k} \in C^{\Delta}_{n-1-k}(S^{n-1})$ whose complement is **not** in X, i.e. this is exactly the subgroup

$$C^{\Delta}_{n-1-k}(\widehat{X}) \subseteq C^{\Delta}_{n-1-k}(S^{n-1})$$

Hence we get a non-degenerate pairing

$$(-,-) : C^{\Delta}_k(S^{n-1}, X) \times C^{\Delta}_{n-1-k}(\widehat{X}) \longrightarrow \mathbb{Z}$$

satisfying the same property

$$(\partial_k \sigma^k, \sigma^{n-k}) = (-1)^k (\sigma^k, \partial_{n-k} \sigma^{n-k})$$

with respect to the differentials.

Hence it induces an isomorphism

$$\varphi : C_k^{\Delta}(S^{n-1}, X) \longrightarrow C_{\Delta}^{n-1-k}(\widehat{X})$$

From the property above we see that this isomorphism identifies ∂_k with ∂^*_{n-k}. Hence for each $k = 0, ..., n-1$ it induces an isomorphism

$$H_k^{\Delta}(S^{n-1}, X) \cong H_{\Delta}^{n-1-k}(\widehat{X})$$

from the long exact homology sequence for the pair (S^{n-1}, X) we get that for $0 \leq k < n-2$ we have an isomorphism

$$\widetilde{H}_k^{\Delta}(X) \cong H_{\Delta}^{n-2-k}(\widehat{X})$$

As for $k = n-2$ we it can be shown that we can form map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{n-1}^{\Delta}(S^{n-1}) & \longrightarrow & H_{n-1}^{\Delta}(S^{n-1}, X) & \longrightarrow & H_{n-2}^{\Delta}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_{\Delta}^{0}(\widehat{X}) & \longrightarrow & \widetilde{H}_{\Delta}^{0}(\widehat{X}) & \longrightarrow & 0 \end{array}$$

where the first two vertical rows are isomorphisms, and hence also the third.