Ambidexterity and the universality of finite spans

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Abstract

Pursuing the notion of ambidexterity developed by Hopkins and Lurie, we prove that the span ∞-category of finite n-truncated spaces is the free n-semiadditive ∞-category generated by a single object. Passing to presentable ∞-categories one obtains a description of the free presentable n-semiadditive ∞-category in terms of a new notion of n-commutative monoids, which can be described as spaces in which families of points parameterized by finite n-truncated spaces can be coherently summed. Such an abstract summation procedure can be used to give a formal ∞-categorical definition of the finite path integral described by Freed, Hopkins, Lurie and Teleman in the context of 1-dimensional topological field theories.

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1 Introduction

The notion of **ambidexterity**, as developed by Lurie and Hopkins in [7] in the ∞-categorical setting, is a duality phenomenon concerning diagrams $p: K \longrightarrow \mathcal{C}$

whose limit and colimit **coincide**. The simplest case where this can happen is when K is empty. In this case a colimit of K is simply an initial object of $\mathcal{C}$, and a limit of K is a final object of $\mathcal{C}$. If $\mathcal{C}$ has both an initial object $\varnothing \in \mathcal{C}$ and a final object $* \in \mathcal{C}$ then there is essentially a unique map $\varnothing \longrightarrow *$. Given that both $\varnothing$ and $*$ exist there is hence a canonical way to require that they coincide, namely, asserting that the unique map $\varnothing \longrightarrow *$ is an equivalence. In this case we say that $\mathcal{C}$ is **pointed**. An object $0 \in \mathcal{C}$ which is both initial and final is called a **zero object**.

Generalizing this property to cases where K is non-empty involves an immediate difficulty. In general, even if $p : K \longrightarrow \mathcal{C}$ admits both a limit and a colimit, there is apriori no natural choice of a map relating the two. Informally speaking, choosing a map $\operatorname{colim}(p) \longrightarrow \lim(p)$ is the same as choosing, compatibly for every two objects $x, y \in K$, a map $p(x) \longrightarrow p(y)$ in $\mathcal{C}$. The diagram p, on its part, provides such maps $p(e) : p(x) \longrightarrow p(y)$ for every $e \in \operatorname{Map}_K(x, y)$. We thus have a whole space of maps $p(e) : p(x) \longrightarrow p(y)$ at our disposal, but no a-priori way to choose a specific one naturally in both x and y.

To see how this problem might be resolved assume for a moment that $\mathcal{C}$ is pointed, i.e., admits a zero object $0 \in \mathcal{C}$, and that $\operatorname{Map}_K(x, y)$ is either empty or contractible for every $x, y \in K$ (i.e. K is equivalent to a partially ordered set, or a poset). Then for every $X, Y \in \mathcal{C}$ there is a distinguished point in $\operatorname{Map}_{\mathcal{C}}(X, Y)$, namely the essentially unique map which factors as $f : X \longrightarrow 0 \longrightarrow Y$, where $0 \in \mathcal{C}$ is a zero object. We may call this map $X \longrightarrow Y$ the **zero map**. We then obtain a choice of a map $f_{x,y} : p(x) \longrightarrow p(y)$ which is natural in both x and y: if $\operatorname{Map}_K(x, y)$ is contractible then we take $f_{x,y}$ to be $p(e)$ for the essentially unique map $e : x \longrightarrow y$, and if $\operatorname{Map}_K(x, y)$ is empty then we just take the zero map. It is then meaningful to ask whether the limits and colimit of a diagram $p : K \longrightarrow \mathcal{C}$ coincide: assuming both of them exist, we may ask whether the map $\operatorname{colim}(p) \longrightarrow \lim(p)$ we have just constructed is an equivalence.

For general posets and general pointed ∞-categories $\mathcal{C}$ the map $\operatorname{colim}(p) \longrightarrow \lim(p)$ is rarely an equivalence. For example, if $K = [1]$ then our map $p(1) \simeq \operatorname{colim}(p) \longrightarrow \lim(p) \simeq p(0)$ is the 0-map, and is hence an equivalence if and only if both $p(0)$ and $p(1)$ are zero objects. However, there is a class of posets for which this property turns out to yield something interesting, namely, the class of **finite sets**, i.e., finite posets for which the order relation is the equality. In this case we may identify $\operatorname{colim}(p) \simeq \coprod_{x \in K} p(x)$ and $\lim(p) \simeq \prod_{x \in K} p(x)$. The map

$$f_p : \coprod_{x \in K} p(x) \longrightarrow \prod_{x \in K} p(x)$$

we constructed above is then given by the "matrix" of maps $[f_{x,y}]_{x,y \in K}$, where $f_{x,y} : p(x) \longrightarrow p(y)$ is the identity if $x = y$ and the zero map if $x \neq y$. When a pointed ∞-category satisfies the property that f_p is an equivalence for every finite set K and every diagram $p : K \longrightarrow \mathcal{C}$ we say that $\mathcal{C}$ is **semiadditive**. Examples of semiadditive ∞-categories include all abelian (discrete) categories and all stable ∞-categories. For more general examples, if $\mathcal{C}$ is any ∞-category with finite products then the ∞-category $\operatorname{Mon}_{\mathbb{E}_\infty}(\mathcal{C})$ of $\mathbb{E}_\infty$-monoids in $\mathcal{C}$ is semiadditive.

In their paper [7], Lurie and Hopkins observed that the passage from pointed ∞-categories to semiadditive ones is just a first step in a more general process. Suppose, for example, that $\mathcal{C}$ is a semiadditive ∞-category. Then for every $X, Y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(X,Y)$ carries a natural structure of an $\mathbb{E}_\infty$-monoid, where the sum of two maps $f, g : X \longrightarrow Y$ is given by the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \simeq Y \coprod Y \xrightarrow{\varepsilon} Y.$$

Now suppose that K is an ∞-category whose mapping spaces are equivalent to finite sets and that $p : K \longrightarrow \mathcal{C}$ is a diagram which admits both a limit and colimit. Then we may construct a natural map $\mathrm{colim}(p) \longrightarrow \lim(p)$ by choosing, for every $x, y \in K$, the map

$$f_{x,y} = \sum_{e \in \mathrm{Map}_K(x,y)} p(e) : p(x) \longrightarrow p(y), \tag{1}$$

where the sum is taken with respect to the natural $\mathbb{E}_\infty$-monoid structure on $\mathrm{Map}_{\mathcal{C}}(X,Y)$. We may now ask if the induced map

$$f_p : \mathrm{colim}(p) \longrightarrow \lim(p), \tag{2}$$

which is often called the **norm map**, is an equivalence. When K is a finite discrete **groupoid** this happens in many interesting examples, e.g., when $\mathcal{C}$ is a $\mathbb{Q}$-linear ∞-category. When (1) is an equivalence for every finite groupoid K we say that $\mathcal{C}$ is **1-semiadditive**. This process can now be continued inductively, where at the m'th stage we consider ∞-groupoids whose homotopy groups are all finite and vanish above dimension m. In this paper we will refer to such ∞-groupoids as **finite m-truncated ∞-groupoids**. This yields a notion of an m-semiadditive ∞-category for every m. The main result of [7] identifies an interesting and subtle case where this happens: for every prime number, the associated ∞-category of $K(n)$-local spectra is m-semiadditive for every m.

The goal of this paper is form a link between the theory of ambidexterity as developed in [7] and the ∞-category of **spans** of finite m-truncated spaces. The understand the role of this ∞-category, let us consider for a moment the central role played by the ∞-category $\mathrm{Sp}_{\mathrm{fin}}$ of **finite spectra** in the theory of **stable ∞-categories**. To begin, $\mathrm{Sp}_{\mathrm{fin}}$ can be described as the free stable ∞-category generated by a single object $\mathbb{S} \in \mathrm{Sp}_{\mathrm{fin}}$. Furthermore, one can use $\mathrm{Sp}_{\mathrm{fin}}$ in order to characterize the stable ∞-categories inside the ∞-category $\mathrm{Cat}_{\mathrm{fin}}$ of all small ∞-categories with finite colimits (and right exact functors between them). Indeed, $\mathrm{Cat}_{\mathrm{fin}}$ carries a natural symmetric monoidal structure (see [6, §4.8.1]) whose unit is the the smallest full subcategory of spaces $\mathcal{S}_{\mathrm{fin}} \subseteq \mathcal{S}$ closed under finite colimits. One can then show that $\mathrm{Sp}_{\mathrm{fin}}$ is an **idempotent** object in $\mathrm{Cat}_{\mathrm{fin}}$ in the following sense: the suspension spectrum functor $\Sigma^\infty : \mathcal{S}_{\mathrm{fin}} \longrightarrow \mathrm{Sp}_{\mathrm{fin}}$ induces an equivalence $\mathrm{Sp}_{\mathrm{fin}} \simeq \mathrm{Sp}_{\mathrm{fin}} \otimes \mathcal{S}_{\mathrm{fin}} \xrightarrow{\simeq} \mathrm{Sp}_{\mathrm{fin}} \otimes \mathrm{Sp}_{\mathrm{fin}}$. The fact that $\mathrm{Sp}_{\mathrm{fin}}$ is idempotent has a remarkable consequence: it endowed $\mathrm{Sp}_{\mathrm{fin}}$ with a canonical commutative algebra structure in $\mathrm{Cat}_{\mathrm{fin}}$ such that the forgetful functor $\mathrm{Mod}_{\mathrm{Sp}_{\mathrm{fin}}}(\mathrm{Cat}_{\mathrm{fin}}) \longrightarrow \mathrm{Cat}_{\mathrm{fin}}$ is fully-faithful. From a conceptual point of view, this fact can be described as follows: given an ∞-category with finite colimits $\mathcal{C}$, the structure of being an $\mathrm{Sp}_{\mathrm{fin}}$-module is essentially unique once it exists, and can hence be considered as a **property**. One can then show that this property coincides with being stable. In other words, stable ∞-categories are exactly those $\mathcal{C} \in \mathrm{Cat}_{\mathrm{fin}}$ which admit an action of $\mathrm{Sp}_{\mathrm{fin}}$, in which case the action is essentially unique.

This double aspect of stability, as either a property or a structure, is very useful. On one hand, in a higher categorical setting structures are often difficult to construct explicitly, while properties are typically easier to define and to check. On the other hand, having a higher categorical structure available is often a very powerful tool. An equivalence between a given property and the existence of a given structure allows one to enjoy both advantages simultaneously. For example, while the property of being stable is easy to define and often to establish, once we know that a given ∞-category is stable we can use the canonically defined $\mathrm{Sp}_{\mathrm{fin}}$-module structure at our disposal. For example, it implies that any stable ∞-category is canonically enriched in spectra, and in particular its mapping spaces carry a canonical $\mathbb{E}_\infty$-group structure.

In this paper we describe a completely analogous picture for the property of m-semiadditivity. Let $\mathcal{K}_m$ be the set of equivalence classes of finite m-truncated Kan complexes. Let $\mathrm{Cat}_{\mathcal{K}_m}$ be the ∞-category of small ∞-categories which admit $\mathcal{K}_m$-indexed colimits and functors which preserve $\mathcal{K}_m$-indexed colimits between them. Then $\mathrm{Cat}_{\mathcal{K}_m}$ carries a natural symmetric monoidal structure whose unit is the ∞-category $\mathcal{S}_m$ of finite m-truncated spaces. Another object contained in $\mathrm{Cat}_{\mathcal{K}_m}$ is the ∞-category $\mathrm{Span}(\mathcal{S}_m)$ whose objects are finite m-truncated spaces and whose morphisms are given by spans (see 2 for a formal definition). Our main result can then be phrased as follows:

Theorem 1.1. $\mathrm{Span}(\mathcal{S}_m)$ *is the free m-semiadditive ∞-category generated by a single object. In other words, if $\mathcal{D}$ is an m-semiadditive ∞-category then evaluation at $* \in \mathrm{Span}(\mathcal{S}_m)$ induces an equivalence*

$$\mathrm{Fun}_{\mathcal{K}_m}(\mathrm{Span}(\mathcal{S}_m), \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}$$

where the left hand side denotes the ∞-category of functors which preserve $\mathcal{K}_m$-indexed colimits.

Furthermore, we will show in §5.1 that $\mathrm{Span}(\mathcal{S}_m)$ is in fact an **idempotent** object of $\mathrm{Cat}_{\mathcal{K}_m}$. Consequently, $\mathrm{Span}(\mathcal{S}_m)$ carries a canonical commutative algebra structure in $\mathrm{Cat}_{\mathcal{K}_m}$, and the forgetful functor $\mathrm{Mod}_{\mathrm{Span}(\mathcal{S}_m)}(\mathrm{Cat}_{\mathcal{K}_m}) \longrightarrow \mathrm{Cat}_{\mathcal{K}_m}$ is fully-faithful. The structure of a $\mathrm{Span}(\mathcal{S}_m)$-module on a given ∞-category $\mathcal{C} \in \mathrm{Cat}_{\mathcal{K}_m}$ is hence essentially a property. This property is exactly the property of being m-semiadditive.

The flexibility of switching the point of view between a property and a structure seems to be especially useful in the setting of m-semiadditivity. Indeed, while m-semiadditivity is a property (involving the coincidence of limits and colimits indexed by finite m-truncated spaces) it is quite hard to define it directly. The reason, as described above, is that in order to define the various norm maps which are required to induce the desired equivalences, one needs to use the fact that the ∞-category in question is already known to be $(m-1)$-semiadditive. Even then, describing these maps requires an elaborate inductive process (see [7, §4]). On the other hand, having a canonical $\mathrm{Span}(\mathcal{S}_{m-1})$-module structure on an $(m-1)$-semiadditive ∞-category leads to a direct and short definition of when an $(m-1)$-semiadditive ∞-category is m-semiadditive (see, e.g., Corollary 3.11).

The picture becomes even more transparent when one passes to the world of **presentable ∞-categories**. Let Pr^L denote the ∞-category of presentable ∞-categories and left functors between them. Then one has a natural symmetric monoidal functor $\mathcal{P}_{\mathcal{K}_m} : \mathrm{Cat}_{\mathcal{K}_m} \longrightarrow \mathrm{Pr}^L$ which sends $\mathcal{C} \in \mathrm{Cat}_{\mathcal{K}_m}$ to the ∞-category $\mathcal{P}_{\mathcal{K}_m}(\mathcal{C})$ of presheaves of spaces on $\mathcal{C}$ that take $\mathcal{K}_m$-indexed colimits in $\mathcal{C}$ to limits of spaces. Applying this functor to $\mathrm{Span}(\mathcal{S}_m)$ one obtains a presentable ∞-category whose objects can be described as certain **higher commutative monoids**, and which we will investigate in §5.2. Informally speaking, an object of $\mathcal{P}_{\mathcal{K}_m}(\mathcal{S}_m)$ can be described as a space X endowed with the following type of structure: for any map $f : S \longrightarrow X$ from a finite m-truncated space S to X, we have an associated point $\int_S f \in X$, which we can think of as the "continuous sum" of the family of points $\{f(s)\}_{s\in S}$. This association is of course required to satisfy various compatibility conditions. For $m = 0$ we have that f is indexed by a finite set and we obtain the structure of an $\mathbb{E}_\infty$-monoid. When $m = -1$ this is just the structure of a pointed space. Now since the functor $\mathcal{P}_{\mathcal{K}_m}$ is monoidal the ∞-category $\mathrm{Mon}_m := \mathcal{P}_{\mathcal{K}_m}(\mathcal{S}_m)$ of m-commutative monoids is idempotent as a presentable ∞-category, and the property characterizing Mon_m-modules in Pr^L is again m-semiadditivity. There is, however, an advantage for considering Mon_m in addition to $\mathrm{Span}(\mathcal{S}_m)$. Note that given an m-semiadditive ∞-category $\mathcal{C}$, the canonical action of $\mathrm{Span}(\mathcal{S}_m)$ described above is not **closed** in general, i.e., it doesn't endow $\mathcal{C}$ an enrichment in $\mathrm{Span}(\mathcal{S}_m)$. It does, however, endow $\mathcal{C}$ with an enrichment in Mon_m. In particular, mapping spaces in $\mathcal{C}$ are m-commutative monoids, and so we have a canonically defined summation over families of maps indexed by finite m-truncated spaces. This structure can be used in order to define norm maps of [7] and hence to define when an $(m-1)$-semiadditive ∞-category is m-semiadditive (in a manner analogous to the cases of $m = -1, 0, 1$ described above). Indeed, for every finite m-truncated space K, any diagram $p : K \longrightarrow \mathcal{C}$ and any $x, y \in K$ we obtain a natural map

$$f_{x,y} : \int_{e\in \mathrm{Map}_K(x,y)} p(e) : p(x) \longrightarrow p(y). \tag{3}$$

by using the $(m-1)$-commutative monoid structure of $\mathrm{Map}_{\mathcal{C}}(x, y)$ and the fact that the mapping spaces in K are $(m-1)$-truncated. The compatible collection of maps $f_{x,y}$ then induces a map

$$f_p : \mathrm{colim}(p) \longrightarrow \lim(p) \tag{4}$$

which coincide with the norm maps constructed in [7]. In particular, an $(m-1)$-semiadditive ∞-category $\mathcal{C}$ is m-semiadditive if and only if the maps f_p are equivalence for every $K \in \mathcal{K}_m$ and every $p : K \longrightarrow \mathcal{C}$.

In the final part of the paper we will explain a relation between the above results and 1-dimensional topological field theories, and more specifically to the **finite path integral** described in [2, §3]. In particular, our approach allows one to formally define this finite path integral whenever the target ∞-category is m-semiadditive. This requires a description of the free m-semiadditive ∞-category generated by an arbitrary ∞-category $\mathcal{D}$, which we establish in 5.3 using a formalism of **decorated spans**. The link with finite path integrals is then described in 5.4.

2 Preliminaries

In this paper we work in the higher categorical setting of ∞-categories as set up in [5]. In particular, an ∞-category we will always mean a simplicial set $\mathcal{C}$ which has the right lifting property with respect to inner horns. We will often refer to the vertices of $\mathcal{C}$ as objects and to edges in $\mathcal{C}$ as morphisms. In the same spirit, if $\mathcal{I}$ is an ordinary category then we will often depict maps $\mathrm{N}(\mathcal{I}) \longrightarrow \mathcal{C}$ to an ∞-category $\mathcal{C}$ in diagrammatic form, as would be the case if $\mathcal{C}$ was an ordinary category. By a **space** we will always mean a Kan simplicial set, which we will generally regard as an ∞-groupoid, i.e., a ∞-category in which every morphism is invertible. Given an ∞-category $\mathcal{C}$, we will denote by $\mathcal{C}^{\simeq}$, the maximal subgroupoid (i.e, maximal sub Kan complex) of $\mathcal{C}$.

2.1 ∞-Categories of spans

In this section we will recall the definition of the ∞-category of spans in a given ∞-category $\mathcal{C}$ with admits pullbacks. To obtain more flexibility it will be useful to consider a slightly more general case, following the approach of [1]. Recall that a functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ is called **faithful** if for every $x, y \in \mathcal{C}$ the induced map $f_{x,y} : \mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(f(x), f(y))$ is (-1)-truncated (i.e., each homotopy fiber of $f_{x,y}$ is either empty or contractible). Equivalently, f is faithful if the induced map $\mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ on homotopy categories is faithful and the square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathrm{Ho}(\mathcal{C}) & \xrightarrow{\mathrm{Ho}(f)} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

is homotopy Cartesian. In this case we will also say that $\mathcal{C}$ is a **subcategory of** $\mathcal{D}$ (and will often omit the explicit reference to f). We will say that an object or a morphism in $\mathcal{D}$ belongs to $\mathcal{C}$ if it belongs to the image of $\mathcal{C}$ up to equivalence (of objects or arrows).

Definition 2.1. Let $\mathcal{C}$ be an ∞-category. A **weak coWaldhausen** structure on $\mathcal{C}$ is a subcategory $\mathcal{C}^{\dagger} \subseteq \mathcal{C}$ which contains all the equivalences and such that any diagram

$$\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & Z \end{array}$$

in which g belongs to $\mathcal{C}^{\dagger}$ extends to a pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & X \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & Z \end{array}$$

in which g' belongs to $\mathcal{C}^{\dagger}$. In this case we will refer to the pair $(\mathcal{C}, \mathcal{C}^{\dagger})$ as a weak coWaldhausen ∞-category.

Example 2.2. For any ∞-category $\mathcal{C}$ the maximal subgroupoid $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ is a weak coWaldhausen structure on $\mathcal{C}$. If $\mathcal{C}$ admits pullbacks then $\mathcal{C}$ itself is a weak coWaldhausen structure as well. We may consider these examples as the minimal and maximal coWaldhausen structures respectively.

Given a weak coWaldhausen ∞-category $(\mathcal{C}, \mathcal{C}^{\dagger})$ we would like to define an associated ∞-category $\mathrm{Span}(\mathcal{C}, \mathcal{C}^{\dagger})$. Informally speaking, $\mathrm{Span}(\mathcal{C}, \mathcal{C}^{\dagger})$ as the ∞-category whose objects are the objects of $\mathcal{C}$ and whose morphisms are given by diagrams of the form

$$\begin{array}{ccc} & Z & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & & Y \end{array} \tag{5}$$

such that p belongs to $\mathcal{C}^{\dagger}$. We will refer to such diagrams as **spans** in $(\mathcal{C}, \mathcal{C}^{\dagger})$. A composition of two spans can be described by forming the diagram

$$\begin{array}{ccccccc} & & & P & & & \\ & & {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} & & \\ & Z & & & & V & \\ {\scriptstyle g}\swarrow & & \searrow{\scriptstyle f} & & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & & & Y & & & W \end{array}$$

in which the central square is a pullback square, and the external span is the composition of the two bottom spans. Note that since p, g belongs to $\mathcal{C}^{\dagger}$, Definition 2.1 insures that this pullback exists and $g \circ s$ belongs to $\mathcal{C}^{\dagger}$. To define $\mathrm{Span}(\mathcal{C}, \mathcal{C}^{\dagger})$ formally, it is convenient to use the **twisted arrow category** $\mathrm{Tw}(\Delta^n)$ of the n-simplex Δ^n. This ∞-category can be described explicitly as the nerve is the category whose objects are pairs $(i, j) \in [n] \times [n]$ with $i \leq j$ and such that $\mathrm{Hom}((i, j), (i', j'))$ is a singleton if $i' \leq i \leq j \leq j'$ and empty otherwise. Given a weak coWaldhausen ∞-category $(\mathcal{C}, \mathcal{C}^\dagger)$ we will say that a map $f : \mathrm{Tw}(\Delta^n)^{\mathrm{op}} \longrightarrow \mathcal{C}$ is **Cartesian** if for every $i' \le i \le j \le j' \in [n]$ the square

$$\begin{array}{ccc} f(i',j') & \longrightarrow & f(i',j) \\ \downarrow & & \downarrow \\ f(i,j') & \longrightarrow & f(i,j) \end{array}$$

is Cartesian and its vertical maps belong to $\mathcal{C}^\dagger$.

Definition 2.3 (cf.[1])**.** Let $(\mathcal{C}, \mathcal{C}^\dagger)$ be a weak coWaldhausen ∞-category. The **span ∞-category** $\mathrm{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ is the simplicial set whose set of n-simplices is the set of Cartesian maps $f : \mathrm{Tw}(\Delta^n)^{\mathrm{op}} \longrightarrow \mathcal{C}$.

By [1, §3.4-3.8] the simplicial set $\mathrm{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ is always an ∞-category. We refer the reader to loc.cit for a more detailed discussion of this construction and its properties.

Remark 2.4. Let $(\mathcal{C}, \mathcal{C}^\dagger)$ be a weak coWaldhausen ∞-category. Unwinding the definitions we see that the objects of $\mathrm{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ are the objects of $\mathcal{C}$ and the morphisms are given by spans of the form (5) such that p belongs to $\mathcal{C}^\dagger$. Furthermore, a homotopy from the span $X \longleftarrow Z \longrightarrow Y$ to the span $X \longleftarrow Z' \longrightarrow Y$ is given by an equivalence $\eta : Z \longrightarrow Z'$ over $X \times Y$. Elaborating on this argument one can identify the mapping space from X to Y in $\mathcal{C}$ with the full subgroupoid of $(\mathcal{C}_{/X \times Y})^{\sim}$ spanned by objects of the form (5).

2.2 Spans of finite truncated spaces

Definition 2.5. Let X be a space. For $n \ge 0$ we say that X is ***n*-truncated** if $\pi_i(X, x) = 0$ for every $i > n$ and every $x \in X$. We will say that X is (-1)-truncated if it is either empty of contractible and that X is (-2)-truncated if it is contractible. We will say that a map $f : X \longrightarrow Y$ is ***n*-truncated** if the homotopy fiber of f over every point of Y is n-truncated.

Definition 2.6. Let X be a space. We will say that X is **finite** if it is n-truncated for some n and all its homotopy groups/sets are finite.

The collection of weak equivalence types of finite n-truncated Kan complexes is a set. We will denote by $\mathcal{K}_n$ be a complete set of representatives of equivalence types of finite n-truncated Kan complexes.

Warning 2.7. The notion of a finite space should not be confused with the notion of a space equivalent to a simplicial set with finitely many non-degenerate simplices. We note that finite spaces in the sense of Definition 2.6 are also known in the literature as **π-finite spaces**.

Let Set_Δ denote the category of simplicial sets and let $\mathrm{Kan} \subseteq \mathrm{Set}_\Delta$ denote the full subcategory spanned by Kan simplicial sets. Then Kan is a fibrant simplicial category and its coherent nerve $\mathcal{S} := \mathrm{N}(\mathrm{Kan})$ is a model for the ∞-category of spaces. We let $\mathcal{S}_n \subseteq \mathcal{S}$ denote the full subcategory spanned by finite n-truncated spaces. It is then fairly standard to show that $\mathcal{S}_n$ has pullbacks and $\mathcal{K}_n$-indexed colimits and that Cartesian products in $\mathcal{S}_n$ preserves $\mathcal{K}_n$-indexed colimits in each variable separately (these properties are all inherited from the corresponding properties in $\mathcal{S}$). Furthermore, for each $m \leq n$ we may consider the subcategory $\mathcal{S}_{n,m} \subseteq \mathcal{S}_n$ containing all objects and whose mapping spaces are spanned by the m-truncated maps (i.e., those maps whose homotopy fibers are m-truncated). Then $(\mathcal{S}_n, \mathcal{S}_{m,n})$ is a weak coWaldhausen ∞-category, and we will denote by

$$\mathcal{S}_n^m := \mathrm{Span}(\mathcal{S}_n, \mathcal{S}_{n,m})$$

the associated span ∞-category (see §2.1). We note that $\mathcal{S}_{n,-2}$ is just the maximal subgroupoid of $\mathcal{S}_n$ and hence $\mathcal{S}_n^{-2} \simeq \mathcal{S}_n$. By [4, Theorem 1.3(iv)]) the Cartesian monoidal product on $\mathcal{S}_n$ induces a symmetric monoidal structure on $\mathcal{S}_n^m$, which is given on the level of objects by $(X, Y) \mapsto X \times Y$, and on the level of morphisms by taking levelwise Cartesian products of spans. We remark that this monoidal structure on $\mathcal{S}_n^m$ is **not** the Cartesian one.

2.3 Colimits in ∞-categories of spans

In this section we will prove some basic results concerning $\mathcal{K}_n$-indexed colimits in $\mathcal{S}_n^m$. We begin with a few basic lemmas. Given a space X and a point $x \in X$ we will denote by $i_x : * \longrightarrow X$ the map which sends the point to x.

Lemma 2.8. *Let $\mathcal{D}$ be an ∞-category which admits $\mathcal{K}_n$-indexed colimits and let $\mathcal{F} : \mathcal{S}_n \longrightarrow \mathcal{D}$ be a functor. Then $\mathcal{F}$ preserves $\mathcal{K}_n$-indexed colimits if and only if for every $X \in \mathcal{S}_n$ the collection $\{\mathcal{F}(i_x)\}_{x\in X}$ exhibits $\mathcal{F}(X)$ as the colimit of the constant X-indexed diagram with value $\mathcal{F}(*)$.*

Proof. The "only if" direction is clear since the collection of maps $\{i_x\}$ exhibits X as the colimit in $\mathcal{S}_n$ of the constant X-indexed diagram with value $*$. Now suppose that for every $X \in \mathcal{S}_n$ the collection $\{\mathcal{F}(i_x)\}_{x\in X}$ exhibits $\mathcal{F}(X)$ as the colimit of the constant X-indexed diagram with value $\mathcal{F}(*)$. Let $Y \in \mathcal{K}_n$ be a finite n-truncated space and let $\mathcal{G} : Y \longrightarrow \mathcal{S}_n$ be a Y-indexed diagram. Let $p : Z \longrightarrow Y$ be the unstraightening of $\mathcal{G}$, so that Z is the ∞-groupoid whose objects are pairs (y, z) where $y \in Y$ and $z \in \mathcal{G}(y)$. We note that in this case Z is necessarily finite and n-truncated, so that we can consider it as an object of $\mathcal{S}_n$. For every $y \in Y$ the collection of maps $\{i_{z'}\}_{z'\in\mathcal{G}(y)}$ exhibits $\mathcal{G}(y)$ as the colimit of the constant $\mathcal{G}(y)$-indexed diagram with value $*$. It follows that $\mathcal{G} \simeq p_! p^*(*)$, i.e., $\mathcal{G}$ is a left Kan extension along p of the constant Z-indexed diagram with value $*$. Since the collection of maps $\{i_z\}_{z\in Z}$ exhibits Z as the colimit of the constant Z-indexed diagram with value $*$ it follows that the collection of maps $\{\mathcal{G}(y) \longrightarrow Z\}_{y\in Y}$ exhibits Z as the colimit of $\mathcal{G}$. By our assumption for every $y \in Y$ the collection of maps $\{\mathcal{F}(i_{z'})\}_{z'\in\mathcal{G}(y)}$ exhibits $\mathcal{F}(\mathcal{G}(y))$ as the colimit in $\mathcal{D}$ of the constant $\mathcal{G}(y)$-indexed diagram with value $\mathcal{F}(*)$. It follows that $\mathcal{F} \circ \mathcal{G}$ is a left Kan extension along p of the constant diagram $Z \longrightarrow \mathcal{D}$ with value $\mathcal{F}(*)$.

Invoking our assumption again we get that the collection of maps $\{\mathcal{F}(i_z)\}_{z\in Z}$ exhibits $\mathcal{F}(Z)$ as the colimit in $\mathcal{D}$ of the constant Z-indexed diagram with value $\mathcal{F}(*)$, and so the collection of maps $\{\mathcal{F}(\mathcal{G}(y)) \longrightarrow \mathcal{F}(Z)\}_{y\in Y}$ exhibits $\mathcal{F}(Z)$ as the colimit of the diagram $\{\mathcal{F}(\mathcal{G}(y))\}_{y\in Y}$, as desired. □

Proposition 2.9. *For every $-2 \le m \le n$ the subcategory inclusion $\mathcal{S}_n \hookrightarrow \mathcal{S}_n^m$ preserves $\mathcal{K}_n$-indexed colimits.*

Proof. By Lemma 2.8 it will suffice to show that for every $X \in \mathcal{S}_n^m$, the collection of morphisms $\{i_x : * \longrightarrow X\}_{x\in X}$ exhibit X as the colimit of the constant diagram $\{*\}_{x\in X}$ in $\mathcal{S}_n^m$. Equivalently, we need to show that given any test object $Y \in \mathcal{S}_n^m$, the map

$$\mathrm{Map}_{\mathcal{S}_n^m}(X, Y) \longrightarrow \mathrm{Map}_{\mathcal{S}}(X, \mathrm{Map}_{\mathcal{S}_n^m}(*, Y))$$

determined by the collection of restriction maps $i_x \circ (-) : \mathrm{Map}_{\mathcal{S}_n^m}(X, Y) \longrightarrow \mathrm{Map}_{\mathcal{S}_n^m}(*, Y)$ is an equivalence of spaces. Let $p_X : X \times Y \longrightarrow X$ denote the projection on the first coordinate. By Remark 2.4 we may identify $\mathrm{Map}_{\mathcal{S}_n^m}(X, Y)$ with the full subgroupoid of $((\mathcal{S}_n)_{/X\times Y})^{\simeq}$ spanned by those objects $Z \longrightarrow X \times Y$ such that the composite map $Z \longrightarrow X \times Y \xrightarrow{p_X} X$ is m-truncated. Under this equivalence, the restriction map $i_x \circ (-)$ is induced by the pullback functor $i^*_{\{x\}\times Y} : (\mathcal{S}_n)_{/X\times Y} \longrightarrow (\mathcal{S}_n)_{/\{x\}\times Y}$.

Now by the straightening-unstraightening equivalence the collection of pullback functors $i^*_{\{x\}\times\{y\}} : \mathcal{S}_{/X\times Y} \longrightarrow \mathcal{S}$ induces an equivalence of ∞-categories

$$\mathrm{St} : \mathcal{S}_{/X\times Y} \xrightarrow{\simeq} \mathrm{Fun}(X \times Y, \mathcal{S}).$$

Using straightening-unstraightening again and the equivalence $\mathrm{Fun}(X \times Y, \mathcal{S}) \simeq \mathrm{Fun}(X, \mathrm{Fun}(Y, \mathcal{S}))$ we may conclude that the collection of pullback functors $i^*_{\{x\}\times Y} : \mathcal{S}_{/X\times Y} \longrightarrow \mathcal{S}_{/\{x\}\times Y}$ induces an equivalence of ∞-categories

$$\mathrm{St}_X : \mathcal{S}_{/X\times Y} \xrightarrow{\simeq} \mathrm{Fun}(X, \mathcal{S}_{/Y}).$$

and hence an equivalence on the corresponding maximal subgroupoids

$$\mathrm{St}_X^{\simeq} : (\mathcal{S}_{/X\times Y})^{\simeq} \xrightarrow{\simeq} \mathrm{Fun}(X, \mathcal{S}_{/Y})^{\simeq} \simeq \mathrm{Map}(X, (\mathcal{S}_{/Y})^{\simeq}).$$

Given an object $Z \longrightarrow X \times Y$ in $(\mathcal{S}_{/X\times Y})^{\simeq}$, the condition that the composite map $Z \longrightarrow X \times Y \xrightarrow{p_X} X$ is an m-truncated map is equivalent to the condition that the essential image of $\mathrm{St}_X^{\simeq}(Z) : X \longrightarrow (\mathcal{S}_{/Y})^{\simeq}$ is contained in $((\mathcal{S}_m)_{/Y})^{\simeq}$. Furthermore, since X is n-truncated this condition automatically implies that Z is n-truncated. Identifying $((\mathcal{S}_m)_{/Y})^{\simeq}$ with $\mathrm{Map}_{\mathcal{S}_n^m}(*, Y)$ we may then conclude that the collection of pullback functors $i^*_{\{x\}\times Y} : (\mathcal{S}_n)_{/X\times Y} \longrightarrow (\mathcal{S}_n)_{/\{x\}\times Y}$ induces an equivalence of ∞-groupoids

$$\mathrm{Map}_{\mathcal{S}_n^m}(X, Y) \xrightarrow{\simeq} \mathrm{Map}(X, \mathrm{Map}_{\mathcal{S}_n^m}(*, Y)),$$

as desired. □

Lemma 2.10. *Let $-2 \leq m \leq n$ be integers. Then any equivalence in $\mathcal{S}_n^m$ belongs to the essential image of the map $\mathcal{S}_n \hookrightarrow \mathcal{S}_n^m$.*

Proof. Given a span

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y \tag{6}$$

we may associate to it the functor $q_! p^* : \mathrm{Fun}(X, \mathcal{S}) \longrightarrow \mathrm{Fun}(Y, \mathcal{S})$, where $p^* : \mathrm{Fun}(X, \mathcal{S}) \longrightarrow \mathrm{Fun}(Z, \mathcal{S})$ is the restriction functor and $q_! : \mathrm{Fun}(Z, \mathcal{S}) \longrightarrow \mathrm{Fun}(Y, \mathcal{S})$ is given by left Kan extension. The Beck-Chevalley condition (see [7, Proposition 4.3.3]) implies that this association respects composition of spans up to homotopy. It follows that if (6) is an equivalence in $\mathcal{S}_n^m$ then the induced functor $q_! p^* : \mathrm{Fun}(X, \mathcal{S}) \longrightarrow \mathrm{Fun}(Y, \mathcal{S})$ is an equivalence of ∞-categories. Our goal is to show that in this case p must be an equivalence. Let Z_x denote the homotopy fiber of p above x, equipped with its natural map $i_{Z_x} : Z_x \longrightarrow X$. It will suffice to show that Z_x is contractible for every $x \in X$. Let $p_x : Z_x \longrightarrow *$ be the terminal map, and for each $x \in X$ let $i_x : * \longrightarrow X$ be the map which sends $*$ to x. Using the Beck-Chevalley condition again we may identify

$$Z_x \simeq \operatorname*{colim}_{Z_x} p_x^*(*) \simeq \operatorname*{colim}_{Z} (i_{Z_x})_! p_x^*(*) \simeq \operatorname*{colim}_{Z} p^*((i_x)_!(*)) \simeq \operatorname*{colim}_{Y} q_! p^*((i_x)_!(*)).$$

Note that the association $x \mapsto (i_x)_!(*)$ can be identified with the **Yoneda embedding** $\iota_X : X \longrightarrow \mathrm{Fun}(X, \mathcal{S})$. To show that Z_x is contractible it will hence suffice to show that $q_! p^*$ maps the image of the Yoneda embedding ι_X to the image of the Yoneda embedding ι_Y (indeed, $\mathrm{colim}_Y \, \iota_Y(y) = \mathrm{colim}_Y (i_y)_!(*) \simeq *$ for every $y \in Y$).

We now claim that the image of ι_X coincides with the full subcategory of $\mathrm{Fun}(X, \mathcal{S})$ spanned by **completely compact** objects (i.e., objects whose associated corepresentable functor preserves all colimits). Let $\mathcal{F} \in \mathrm{Fun}(X, \mathcal{S})$ be a completely compact object. By [5, Proposition 5.1.6.8] there exists an $x \in X$ and a retract diagram $\mathcal{F} \xrightarrow{i} \iota_X(x) \xrightarrow{r} \mathcal{F}$ in $\mathrm{Fun}(X, \mathcal{S})$. Since X is an ∞-groupoid and ι_X is fully-faithful it follows that the composition $ir : \iota_X(x) \longrightarrow \iota_X(x)$ is an equivalence. This means that r is a retract of an equivalence and hence an equivalence, so that $\mathcal{F}$ is in the essential image of ι_X. Since any equivalence of ∞-categories maps completely compact objects to completely compact objects the desired result now follows. ☐

Corollary 2.11. *Let X be a space. Then any X-indexed diagram in $\mathcal{S}_n^m$ comes from an X-indexed diagram in $\mathcal{S}_n$.*

Corollary 2.12. *For every $-2 \leq m \leq n$ the ∞-category $\mathcal{S}_n^m$ admits $\mathcal{K}_n$-indexed colimits. Furthermore, if $\mathcal{F} : \mathcal{S}_n^m \longrightarrow \mathcal{D}$ is any functor then $\mathcal{F}$ preserves $\mathcal{K}_n$-indexed colimits if and only if the composed functor $\mathcal{S}_n \hookrightarrow \mathcal{S}_n^m \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_n$-indexed colimits.*

Proof. Combine Corollary 2.11 and Proposition 2.9. ☐

Corollary 2.13. *The symmetric monoidal product* $\mathcal{S}_n^m \times \mathcal{S}_n^m \longrightarrow \mathcal{S}_n^m$ *preserves* $\mathcal{K}_n$*-indexed colimits in each variable separately.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_n \times \mathcal{S}_n & \longrightarrow & \mathcal{S}_n^m \times \mathcal{S}_n^m \\ \downarrow & & \downarrow \\ \mathcal{S}_n & \longrightarrow & \mathcal{S}_n^m \end{array}$$

Where the left vertical map is the Cartesian product. Since Cartesian products in $\mathcal{S}_n$ preserves $\mathcal{K}_n$-indexed colimits in each variable separately and the inclusion $\mathcal{S}_n \hookrightarrow \mathcal{S}_n^m$ is essentially surjective the desired result now follows from Corollary 2.11 and Proposition 2.9. □

We will denote by $\mathrm{Cat}_{\mathcal{K}_n}$ the ∞-category of small ∞-categories which admit $\mathcal{K}_n$-indexed colimits and functors which preserve $\mathcal{K}_n$-indexed colimits between them. If $\mathcal{C}, \mathcal{D}$ are ∞-categories which admit $\mathcal{K}_n$-indexed colimits then we denote by $\mathrm{Fun}_{\mathcal{K}_n}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by those functors which preserves $\mathcal{K}_n$-indexed colimits. Recall that by [6, corollary 4.8.4.1] we may endow this ∞-category with a symmetric monoidal structure $\mathrm{Cat}^{\otimes}_{\mathcal{K}_n} \longrightarrow \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ such that for $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_{\mathcal{K}_n}$ their tensor product $\mathcal{C} \otimes_{\mathcal{K}_n} \mathcal{D}$ admits a map $\mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{C} \otimes_{\mathcal{K}_n} \mathcal{D}$ from the Cartesian product and is characterized by the following universal property: for every ∞-category $\mathcal{E} \in \mathrm{Cat}_{\mathcal{K}_n}$ the restriction

$$\mathrm{Fun}_{\mathcal{K}_n}(\mathcal{C} \otimes_{\mathcal{K}_n} \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C} \times \mathcal{D})$$

is fully-faithful and its essential image is spanned by those functors $\mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ which preserve $\mathcal{K}_n$-indexed colimits in each variable separately. In particular, we may identify **commutative algebra objects** in $\mathrm{Cat}_{\mathcal{K}_m}$ with symmetric monoidal ∞-categories which admit $\mathcal{K}_m$-indexed colimits and such that the monoidal product preserves $\mathcal{K}_m$-indexed colimits in each variable separately.

Corollary 2.12 and Corollary 2.13 now imply the following:

Corollary 2.14. *The* ∞*-category* $\mathcal{S}_n^m$ *together with its symmetric monoidal structure determines a commutative algebra object in* $\mathrm{Cat}^{\otimes}_{\mathcal{K}_m}$

3 Ambidexterity and duality

Definition 3.1 (see [7, Definition 4.4.2])**.** Let $\mathcal{D}$ be an ∞-category and $-2 \le m$ an integer. Following [7], we shall say that $\mathcal{D}$ is m**-semiadditive** if $\mathcal{D}$ admits $\mathcal{K}_m$-indexed colimits and every m-truncated finite space is $\mathcal{D}$-ambidextrous in the sense of [7, Definition 4.1.11].

Informally speaking, m-semiadditive ∞-categories are ∞-categories in which $\mathcal{K}_m$-indexed colimits and limits coincide. The reason we do not recall [7, Definition 4.1.11] in full is that it requires a somewhat elaborate inductive process in order to define the maps which induce the desired equivalence. That said, if $\mathcal{D}$ is an ∞-category which admits an action of $\mathcal{S}_m^{m-1}$, then we will see below that the condition that $\mathcal{D}$ is m-semiadditive can be expressed rather succinctly (see Proposition 3.10 and Corollary 3.11). On the other hand, by the main result of this paper any $(m-1)$-semiadditive ∞-category acquires a canonical action of $\mathcal{S}_m^{m-1}$, and so this approach can be considered as an alternative way to define higher semiadditivity.

Examples 3.2.

1. An ∞-category $\mathcal{D}$ is (-1)-semiadditive if and only if it is **pointed**, i.e., if it contains an object which is both initial and final.
2. Every stable ∞-category is 0-semiadditive.
3. Let $\mathcal{D}$ be an ∞-category which admits finite products. Then the ∞-category of $\mathbb{E}_\infty$-monoid objects in $\mathcal{D}$ is 0-semiadditive (see §5.2).
4. For any prime p, the associated ∞-category of $K(n)$-local spectra is m-semiadditive for any m. This is the main result of [7].
5. For every $-2 \leq n \leq m$ the ∞-category $\mathcal{S}_n^m$ is m-semiadditive (see Corollary 3.13 below).
6. The ∞-category $\mathrm{Cat}_{\mathcal{K}_m}$ of small ∞-categories which admit $\mathcal{K}_m$-indexed colimits is m-semiadditive (see Proposition 5.25 below).
7. If $\mathcal{D}$ is m-semiadditive then $\mathcal{D}^{\mathrm{op}}$ is m-semiadditive.

Given an ∞-category $\mathcal{D}$ and a map $f : X \longrightarrow Y$ of spaces we have a restriction functor $f^* : \mathrm{Fun}(Y, \mathcal{D}) \longrightarrow \mathrm{Fun}(X, \mathcal{D})$. If $\mathcal{D}$ admits $\mathcal{K}_m$-indexed colimits and the homotopy fibers of f are finite and m-truncated then f^* admits a left adjoint $f_! : \mathrm{Fun}(X, \mathcal{D}) \longrightarrow \mathrm{Fun}(Y, \mathcal{D})$ given by left Kan extension. If in addition $\mathcal{D}$ is m-semiadditive then $f_!$ is also **right adjoint** to f^*. In this case we say that a natural transformation $u : \mathrm{Id} \Rightarrow f_! f^*$ **exhibits f as $\mathcal{D}$-ambidextrous** if it is a unit of an adjunction $f^* \dashv f_!$. In this section we fix an integer $m \geq -1$ and consider the situation where $\mathcal{D}$ is an ∞-category satisfying the following properties:

Assumption 3.3.

1. *$\mathcal{D}$ admits $\mathcal{K}_m$-indexed colimits.*
2. *$\mathcal{D}$ is $(m-1)$-semiadditive.*
3. *$\mathcal{D}$ admits a structure of a $\mathcal{S}_m^{m-1}$-module in $\mathrm{Cat}_{\mathcal{K}_m}$. In other words, there is an action of the monoidal ∞-category $\mathcal{S}_m^{m-1}$ on $\mathcal{D}$ such that the action map $\mathcal{S}_m^{m-1} \times \mathcal{D} \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits in each variable separately. Following [7], we will denote the functor $X \otimes (-)$ also by $[X] : \mathcal{D} \longrightarrow \mathcal{D}$.*

Our first goal in this section is to show that if $\mathcal{D}$ satisfies Assumption 3.3, and if $f : X \longrightarrow Y$ is an $(m-1)$-truncated map of finite m-truncated spaces then the unit transformations $u : \mathrm{Id} \Rightarrow f_! f^*$ exhibiting f as $\mathcal{D}$-ambidextrous can be written in terms of the $\mathcal{S}_m^{m-1}$ action on $\mathcal{D}$. We will use this description in order to give an explicit criteria characterization those $\mathcal{D}$ satisfying 3.3 which are also m-semiadditive (see Proposition 3.10 and Corollary 3.11).

Let $X \in \mathcal{S}_m^{m-1}$ be an object. Recall that for a point $x \in X$ we denote by $i_x : * \longrightarrow X$ the map in $\mathcal{S}_m \subseteq \mathcal{S}_m^{m-1}$ which sends $*$ to the point x. By Proposition 2.9 and Lemma 2.8 the collection of induced maps

$$(i_x)_* : D \longrightarrow [X](D)$$

exhibits $X \otimes D$ as the colimit in $\mathcal{D}$ of the constant X-indexed diagram with value D. Equivalently, if $p : X \longrightarrow *$ is the terminal map then the collection of maps $\{(i_x)_*\}$ can be considered as a natural transformation $p^*D \Rightarrow p^*[X](D)$ of (constant) functors $X \longrightarrow \mathcal{D}$ which exhibits $[X](D)$ as a left Kan extension of p^*D. We begin by considering the counit in the ordinary (non-ambidextrous) direction for the case of the terminal map $p : X \longrightarrow *$.

Lemma 3.4. *Let $\mathcal{D}$ be as in Assumption 3.3, let X be an m-truncated space and let $p : X \longrightarrow *$ be the terminal map in $\mathcal{S}_m$ (which we naturally consider as a map in $\mathcal{S}_m^{m-1}$). Then the natural transformation*

$$p_! p^* \simeq [X] \overset{[p]}{\Rightarrow} [*] \simeq \mathrm{Id}$$

exhibits $p_!$ as left adjoint to p^*.

Proof. If X is empty then $[X]$ is initial in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$, and since such a counit exists it must be homotopic to $[p]$. We may hence suppose that X is not empty. It will suffice to show that the natural transformation $T_p : p^* \Rightarrow p^*$ adjoint to $[p]$ is an equivalence. We note that to give a natural transformation $T : p^* \Rightarrow p^*$ is the same as giving an X-indexed family of natural transformations $\{T_x : \mathrm{Id} \Rightarrow \mathrm{Id}\}_{x \in X}$ from the identify $\mathrm{Id} \in \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ to itself. Furthermore, since the maps $[i_x] : [*] \Rightarrow [X]$ exhibit $[X]$ as the colimit in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$ of the constant diagram $\{[*] \simeq \mathrm{Id}\}_{x \in X}$ it follows that if $T : p_! p^* \Rightarrow \mathrm{Id}$ is a natural transformation then the adjoint natural transformation $T^{\mathrm{ad}} : p^* \Rightarrow p^*$ is given by the family $\{T \circ [i_x] : \mathrm{Id} \Rightarrow \mathrm{Id}\}_{x \in X}$. Since $p \circ i_x : * \longrightarrow *$ is an equivalence in $\mathcal{S}_m^{m-1}$ it follows that $[p] \circ [i_x] : \mathrm{Id} \to \mathrm{Id}$ is a natural equivalence for every $x \in X$ and so the natural transformation $[p]^{\mathrm{ad}} : p^* \to p^*$ is an equivalence. □

Now let X be an $(m-1)$-truncated space and let $\hat{p} : * \longrightarrow X$ be the morphism in $\mathcal{S}_m^{m-1}$ given by the span

$$* \overset{p}{\longleftarrow} X \longrightarrow X$$

Lemma 3.5. *Let $\mathcal{D}$ be as in Assumption 3.3 and let X be an $(m-1)$-truncated space. Then the natural transformation*

$$\mathrm{Id} \simeq [*] \overset{[\hat{p}]}{\Rightarrow} [X] \simeq p_! p^*$$

exhibits $p_!$ as ***right adjoint*** *to p^*. In other words, it exhibits $p : X \longrightarrow *$ as $\mathcal{D}$-ambidextrous.*

Proof. Since X is $\mathcal{D}$-ambidextrous there exists a counit $v_X : p^* p_! \Rightarrow \mathrm{Id}$ exhibiting $p_!$ as right adjoint to p^*. As in [7, §5.1] let us define the trace form $\mathrm{TrFm}_X : [X] \circ [X] \Rightarrow \mathrm{Id}$ by the composition

$$(p_! p^*)(p_! p^*) \simeq p_!(p^* p_!)p^* \xRightarrow{p_! v_X p^*} p_! p^* \xRightarrow{\phi_X} \mathrm{Id}$$

where ϕ_X is a counit exhibiting $p_!$ as left adjoint to p^*. Since $\mathcal{D}$ is assumed to be $(m-1)$-semiadditive, [7, Proposition 5.1.8] implies that the trace form exhibits $[X]$ as self dual in $\mathrm{Fun}(\mathcal{D},\mathcal{D})$. Let $u_X : \mathrm{Id} \Rightarrow p_! p^*$ be a unit which is compatible with v_X. It will then be enough to show that $[\hat{p}]$ is equivalent to u_X in the arrow category of $\mathrm{Fun}(\mathcal{D},\mathcal{D})$. Since $[X]$ is self dual it will suffice to compare the natural transformations $[X] \Rightarrow \mathrm{Id}$ which are dual to u_X and $[\hat{p}]$ respectively. In the case of u_X we observe that

$$p_! p^* \xRightarrow{(p_! p^*) u_X} (p_! p^*)(p_! p^*) \simeq p_!(p^* p_!)p^* \xRightarrow{p_! v_X p^*} p_! p^*$$

is homotopic to the identity in light of the compatibility of u_X and v_X. It hence follows that the map $[X] \Rightarrow \mathrm{Id}$ dual to u_X is the counit

$$\phi_X : p_! p^* \Rightarrow \mathrm{Id}.$$

On the other hand, the action functor $\mathcal{S}_m^{m-1} \longrightarrow \mathrm{Fun}(\mathcal{D},\mathcal{D})$ is monoidal and sends X to $[X]$. Since X is $(m-1)$-truncated it is self-dual in $\mathcal{S}_m^{m-1}$. It follows that the dual of $[\hat{p}]$ is the image of the dual of $\hat{p}$ in $\mathcal{S}_m^{m-1}$, which is given by the image in $\mathcal{S}_m^{m-1}$ of the terminal map $p : X \longrightarrow *$ of $\mathcal{S}_m$. It will hence suffice to show that $[p]$ is equivalent to ϕ_X in the arrow category of $\mathrm{Fun}(\mathcal{D},\mathcal{D})$. But this now follows from Lemma 3.4. □

Now let $f : X \longrightarrow Y$ be an arbitrary $(m-1)$-truncated map of finite m-truncated spaces. Recall that by the straightening-unstraightening construction (see[5, §2.1]) the ∞-category $\mathcal{S}_{/Y}$ is equivalent to the ∞-category $\mathrm{Fun}(Y,\mathcal{S})$ of functors from Y to spaces. In particular, the straightening of f corresponds to the functor $\mathrm{St}_f : Y \longrightarrow \mathcal{S}$ which sends $y \in Y$ to the homotopy fiber X_y of f over y. Since f is $(m-1)$-truncated every homotopy fiber X_y is $(m-1)$-truncated and we may consequently consider $\mathrm{St}(f)$ as a functor $Y \longrightarrow \mathcal{S}_{m-1}$. Let $\mathrm{St}_f^{m-1} : Y \longrightarrow \mathcal{S}_m^{m-1}$ be the composition of St_f with the inclusion $\mathcal{S}_{m-1} \subseteq \mathcal{S}_m^{m-1}$. Now the action of $\mathcal{S}_m^{m-1}$ on $\mathcal{D}$ determines an action of $\mathrm{Fun}(Y,\mathcal{S}_m^{m-1})$ on $\mathrm{Fun}(Y,\mathcal{D})$, and we will denote by

$$[X_{/Y}] : \mathrm{Fun}(Y,\mathcal{D}) \longrightarrow \mathrm{Fun}(Y,\mathcal{D})$$

the action of St_f^{m-1}, given informally by the formula

$$[X_{/Y}](\mathcal{F})(y) = \mathrm{St}_f(y)(\mathcal{F}(y)) = [X_y](\mathcal{F}(y)).$$

Now the diagonal map $\Delta : X \longrightarrow X \times_Y X$ induces a natural transformation $* \Rightarrow \mathrm{St}_f \circ f$ of functors $X \longrightarrow \mathcal{S}$ and hence a natural transformation $\Delta_* : f^* \longrightarrow f^*[X_{/Y}]$ of functors $\mathrm{Fun}(Y,\mathcal{D}) \longrightarrow \mathrm{Fun}(X,\mathcal{D})$. Given $\mathcal{F} \in \mathrm{Fun}(Y,\mathcal{D})$, the natural transformation Δ_* induces a natural transformation $\Delta_F : f^*\mathcal{F} \Rightarrow f^*[X_{/Y}](\mathcal{F})$ and by the pointwise formula for the left Kan extension we may conclude that Δ_F exhibits $[X_{/Y}](\mathcal{F})$ as the left Kan extension of $f^*\mathcal{F} : X \longrightarrow \mathcal{S}$ along f. We may then identify $[X_{/Y}] \simeq f_! f^*$ as functors $\mathrm{Fun}(Y,\mathcal{D}) \longrightarrow \mathrm{Fun}(Y,\mathcal{D})$.

Let $\mathrm{St}_{\mathrm{Id}}^{m-1} : Y \longrightarrow \mathcal{S}_m^{m-1}$ be the straightening of the identity map $Y \longrightarrow Y$ considered as the functor $Y \longrightarrow \mathcal{S}_m^{m-1}$ which sends $y \in Y$ to the object $\{y\} \in \mathcal{S}_m^{m-1}$. The collection of spans

$$\begin{array}{ccccc} & & X_y & & \\ & \swarrow^{f_y} & & \searrow & \\ \{y\} & & & & X_y \end{array}$$

then determines a natural transformation $\mathrm{St}_{\mathrm{Id}}^{m-1} \Rightarrow \mathrm{St}_f^{m-1}$ of functors $Y \longrightarrow \mathcal{S}_m^{m-1}$, and hence a natural transformation

$$[\hat{f}]_Y : \mathrm{Id} \Rightarrow [X_{/Y}]$$

of functors $\mathrm{Fun}(Y,\mathcal{D}) \longrightarrow \mathrm{Fun}(Y,\mathcal{D})$.

Lemma 3.6. *Let $\mathcal{D}$ be as in assumption 3.3. Then for every $(m-1)$-truncated map $f : X \longrightarrow Y$ in $\mathcal{S}_m$ the natural transformation*

$$\mathrm{Id} \xrightarrow{[\hat{f}]_Y} [X_{/Y}] \simeq f_! f^*$$

exhibits $f_!$ as right adjoint to f^*. In other words, it exhibits f as $\mathcal{D}$-ambidextrous.

Proof. Let $\mathcal{L}_X : X \longrightarrow \mathcal{D}$ and $\mathcal{L}_Y : Y \longrightarrow \mathcal{D}$ be two functors. We need to show that the composite map

$$\alpha : \mathrm{Map}(f^*\mathcal{L}_Y, \mathcal{L}_X) \longrightarrow \mathrm{Map}(f_! f^*\mathcal{L}_Y, f_!\mathcal{L}_X) \xrightarrow{(-)\circ[\hat{f}]_Y} \mathrm{Map}(\mathcal{L}_Y, f_!\mathcal{L}_X)$$

is an equivalence. By [7, Lemma 4.3.8] it is enough prove this for objects of the form $\mathcal{L}_Y = (i_y)_! D$ where $i_y : \{y\} \longrightarrow Y$ is the inclusion of some point $y \in Y$. For this, in turn, it will suffice to prove that for every $y \in Y$ the composed natural transformation

$$\mathrm{Id} \xRightarrow{u_y} i_y^*(i_y)_! \xRightarrow{[\hat{f}_Y]} i_y^* f_! f^*(i_y)_! \tag{7}$$

exhibits $i_y^* f_!$ as right adjoint to $f^*(i_y)_!$, where u_y is the unit of the adjunction $(i_y)_! \dashv i_y^*$. Now by construction the functor $[X_{/Y}]$ is determined pointwise, and in particular we have a natural equivalence $i_y^*[X_{/Y}] \simeq [X_y]i_y^*$ of functors $\mathrm{Fun}(Y, \mathcal{D}) \longrightarrow \mathcal{D}$. Identifying $[X_{/Y}]$ with $f_! f^*$ we see that this is just an incarnation of the fact that left Kan extensions are determined pointwise. The latter fact is best phrased via the Beck-Chevalley transformation $\tau_y : i_y^* f_! \Longrightarrow (f_y)_! i_{X_y}^*$ associated to the Cartesian square

$$\begin{array}{ccc} X_y & \xrightarrow{i_{X_y}} & X \\ {\scriptstyle f_y}\downarrow & & \downarrow{\scriptstyle f} \\ \{y\} & \xrightarrow[i_y]{} & Y \end{array} \tag{8}$$

By [7, Proposition 4.3.3] the transformation τ_y is an equivalence, asserting, in effect, that the value of the left Kan extension $f_!\mathcal{F}$ at a given point y is the colimit of $\mathcal{F}$ restricted to the homotopy fiber X_y. The equivalence $i_y^*[X_{/Y}] \simeq [X_y]i_y^*$ above is then obtained by composing the equivalences

$$i_y^* f_! f^* \xrightarrow[\simeq]{\tau_y} (f_y)_! i_{X_y}^* f^* \simeq (f_y)_! f_y^* i_y^*$$

Furthermore, by construction the natural transformation $[\hat{f}]_Y : \mathrm{Id} \Rightarrow [X_{/Y}]$ is determined pointwise as well and we have an equivalence $i_y^*[\hat{f}]_Y \simeq [\hat{f}_y]i_y^*$ of natural transformations $i_y^* \Rightarrow i_y^*[X_{/Y}]$ of functors $\mathrm{Fun}(Y, \mathcal{D}) \longrightarrow \mathcal{D}$, where $\hat{f}_y$ is as in Lemma 3.5). It now follows that we can identify the composed transformation

$$\mathrm{Id} \xrightarrow{u_y} i_y^*(i_y)_! \xrightarrow{[\hat{f}_Y]} i_y^* f_! f^*(i_y)_! \xrightarrow[\simeq]{\tau_y} (f_y)_! i_{X_y}^* f^*(i_y)_! \tag{9}$$

with the composed transformation

$$\mathrm{Id} \xrightarrow{[\hat{f}_y]} (f_y)_! f_y^* \xrightarrow{u_y} (f_y)_! f_y^* i_y^*(i_y)_! \simeq (f_y)_! i_{X_y}^* f^*(i_y)_! \tag{10}$$

Now the transpose of the square (8) also has a Beck-Chevalley transformation $\sigma_y : (i_{X_y})_! f_y^* \Longrightarrow f^*(i_y)_!$, which (see [7, Remark 4.1.2]) is given by the composition of transformations

$$(i_{X_y})_! f_y^* \xrightarrow{u_y} (i_{X_y})_! f_y^* i_y^*(i_y)_! \simeq (i_{X_y})_! i_{X_y}^* f^*(i_y)_! \xrightarrow{v_{X_y}} f^*(i_y)_!$$

Applying [7, Proposition 4.3.3] again we get that σ_y is an equivalence. Let $u_{X_y} : \mathrm{Id} \Longrightarrow i_{X_y}^*(i_{X_y})_!$ be a unit transformation compatible with the counit v_{X_y} above. Then the compatibility of u_{X_y} and v_{X_y} implies that (10) (and hence (9)) is homotopic to the composed transformation

$$\mathrm{Id} \xRightarrow{[\hat{f}_y]} (f_y)_! f_y^* \xRightarrow{u_{X_y}} (f_y)_! i_{X_y}^* (i_{X_y})_! f_y^* \xRightarrow[\simeq]{\sigma_y} (f_y)_! i_{X_y}^* f^* (i_y)_! \tag{11}$$

Comparing now (9) and (11) we have reduced to showing that

$$\mathrm{Id} \xRightarrow{[\hat{f}_y]} (f_y)_! f_y^* \xRightarrow{u_{X_y}} (f_y)_! i_{X_y}^* (i_{X_y})_! f_y^*$$

exhibits $(f_y)_! i_{X_y}^*$ as right adjoint to $(i_{X_y})_! f_y^*$. But this just follows from the fact that u_{X_y} is the unit of $(i_{X_y})_! \dashv i_{X_y}^*$ by construction and $[\hat{f}_y]$ exhibits $(f_y)_!$ as right adjoint to $(f_y)^*$ by Lemma 3.5. □

Definition 3.7. Given a map $f : X \longrightarrow Y$ in $\mathcal{S}_{m-1}$ let us denote by $\hat{f} : Y \longrightarrow X$ the morphism in $\mathcal{S}_m^{m-1}$ associated to the span

$$\begin{array}{ccc} & X & \\ {}^{f}\swarrow & & \searrow \\ Y & & X \end{array}$$

Lemma 3.8. *Let $\mathcal{D}$ be as in Assumption 3.3. Then for every $(m-1)$-truncated map $f : X \longrightarrow Y$ in $\mathcal{S}_m$ the natural transformation $[\hat{f}] : [X] \Rightarrow [Y]$ of functors $\mathcal{D} \longrightarrow \mathcal{D}$ is homotopic to the composition*

$$[Y] \simeq q_! q^* \xRightarrow{[\hat{f}]_Y} q_! f_! f^* q^* \simeq [X]$$

where $q : Y \longrightarrow *$ is the terminal map and $[\hat{f}]_Y$ is the natural transformation of Lemma 3.6.

Proof. Since the action map $\mathcal{S}_m^{m-1} \longrightarrow \mathrm{Fun}(\mathcal{D}, \mathcal{D})$ preserves $\mathcal{K}_m$-indexed colimits we may write $[\hat{f}] : [Y] \Rightarrow [X]$ as a colimit of natural transformations of the form

$$[Y] = \operatorname*{colim}_{y \in Y} [*] \xrightarrow{\mathrm{colim}_{y\in Y} [\hat{f}_y]} \operatorname*{colim}_{y \in Y} \operatorname*{colim}_{x \in X_y} [*] \simeq \operatorname*{colim}_{x \in X} [*] = [X]$$

and conclude by observing that $q_! [\hat{f}]_Y q^* \simeq \mathrm{colim}_{y \in Y} [\hat{f}_y]$ by the explicit pointwise definition of $[\hat{f}]_Y$. □

Definition 3.9. Let X be an m-truncated space. We will denote by $\mathrm{tr}_X : X \times X \longrightarrow *$ the morphism in $\mathcal{S}_m^{m-1}$ given by the span

$$\begin{array}{ccc} & X & \\ {}^{\Delta}\swarrow & & \searrow \\ X \times X & & * \end{array}$$

where $\Delta : X \longrightarrow X \times X$ is the diagonal map.

Proposition 3.10. *Let $\mathcal{D}$ be as in Assumption 3.3. Then $\mathcal{D}$ is m-semiadditive if and only if the natural transformation*

$$[\mathrm{tr}_X] : [X] \circ [X] \Rightarrow \mathrm{Id}$$

exhibits the functor $[X] : \mathcal{D} \longrightarrow \mathcal{D}$ as self-adjoint.

Proof. Let $q : X \longrightarrow *$ be the terminal map and $\pi_1, \pi_2 : X \times X \longrightarrow X$ be the two projections. Let $[\hat{\Delta}]_{X\times X} : \mathrm{Id} \longrightarrow \Delta_!\Delta^*$ be the natural transformation as in Lemma 3.6. Let $Q : X \times X \longrightarrow *$ be the terminal map, so that we have $Q = q \circ \pi_1 = q \circ \pi_2$. By Lemma 3.8 the map $[\mathrm{tr}_X] : [X \times X] \longrightarrow [*]$ is given by the composition

$$[X \times X] \simeq Q_!Q^* \xRightarrow{[\hat{\Delta}]_{X\times X}} Q_!\Delta_!\Delta^*Q^* \simeq q_!q^* \simeq [X] \xRightarrow{[q]} [*] \ .$$

Now let $v_X : q^*q_! \longrightarrow \mathrm{Id}$ be the composition

$$v_X : q^*q_! \simeq (\pi_1)_!\pi_2^* \xRightarrow{[\hat{\Delta}]_{X\times X}} (\pi_2)_!\Delta_!\Delta^*(\pi_1)^* \simeq \mathrm{Id} \ .$$

Using Lemma 3.4 we may conclude that $[\mathrm{tr}_X] : [X \times X] \longrightarrow [*]$ is equivalent in the arrow category of $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$ to the composition

$$Q_!Q^* \simeq q_!(q^*q_!)q^* \xRightarrow{q^*v_Xq_!} q^*q_! \xRightarrow{\phi_X} \mathrm{Id}$$

where ϕ_X is the usual counit exhibiting $q_!$ as left adjoint to q^*. In light of Lemma 3.6 we may now identify $[\mathrm{tr}_X]$ with the **trace form** associated to X by [7, Notation 5.1.7]. By [7, Proposition 5.1.8] we may conclude that $[\mathrm{tr}_X]$ exhibits $[X]$ as self-adjoint if and only if the natural transformation $v_X : q^*q_! \longrightarrow \mathrm{Id}$ is a counit of an adjunction. It follows that if $\mathcal{D}$ is m-semiadditive then $[\mathrm{tr}_X]$ exhibits $[X]$ as self-adjoint. To argue the other direction we note that if $[\mathrm{tr}_X]$ exhibits $[X]$ as self-adjoint then for every space $Y \in \mathcal{S}_m$ the levelwise natural transformation associated to $[\mathrm{tr}_X]$ exhibits the levelwise functor $[X] : \mathrm{Fun}(Y, \mathcal{D}) \longrightarrow \mathrm{Fun}(Y, \mathcal{D})$ as self-adjoint. It follows by [7, Proposition 5.1.8] that if $q' : X \times Y \longrightarrow Y$ is any pullback of $q : X \longrightarrow *$ then the natural transformation $v_p : (q')^*(q')_! \longrightarrow \mathrm{Id}$ is a counit of an adjunction, and hence X is $\mathcal{D}$-ambidextrous, as desired. □

Corollary 3.11. *Let $\mathcal{D}$ be as in Assumption 3.3. Then $\mathcal{D}$ is m-semiadditive if and only if the collection of natural transformations $[\hat{i}_x] : [X] \Rightarrow [*]$ exhibits $[X]$ as the limit, in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$, of the constant diagram X-indexed diagram with value $[*]$ (here the morphism $\hat{i}_x : X \longrightarrow *$ in $\mathcal{S}_m^{m-1}$ is as in Definition 3.7).*

Proof. Recall that the collection of natural transformations $[i_x] : [*] \Rightarrow [X]$ exhibits $[X]$ as the colimit of the constant X-indexed diagram with value $[*]$. By Proposition 3.10 it will suffice to show that the collection of natural transformations $[\hat{i}_x] : [X] \Rightarrow [*]$ exhibits $[X]$ as the limit in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$ of the constant diagram X-indexed diagram with value $[*]$ if and only if $[\mathrm{tr}_X] : [X] \circ [X] \Rightarrow \mathrm{Id}$ exhibits $[X]$ as self-adjoint. Let $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{D}$ be any other functor and let α be the composed map

$$\alpha_{\mathcal{G}} : \mathrm{Map}(\mathcal{G}, [X]) \longrightarrow \mathrm{Map}([X] \circ \mathcal{G}, [X] \circ [X]) \xrightarrow{[\mathrm{tr}_X] \circ (-)} \mathrm{Map}([X] \circ \mathcal{G}, \mathrm{Id}) .$$

Since colimits in functor categories are computed objectwise it follows that the natural transformations $[i_x] \circ \mathcal{G} : [*] \circ \mathcal{G} \Rightarrow [X] \circ \mathcal{G}$ exhibit $[X] \circ \mathcal{G}$ as the colimit of the constant X-indexed diagram with value $[*] \circ \mathcal{G} \simeq \mathcal{G}$. We may hence identify a map $[X] \circ \mathcal{G} \Rightarrow \mathrm{Id}$ with a collection of natural transformations $\mathcal{T}_x : \mathcal{G} \Rightarrow \mathrm{Id}$ indexed by $x \in X$. Since the map $\hat{i}_x : X \longrightarrow *$ is equivalent to the composition $X = X \times * \xrightarrow{\mathrm{Id} \times i_x} X \times X \xrightarrow{\mathrm{tr}_X} *$ we see that the map $\alpha_{\mathcal{G}}$ associates to a natural transformation $\mathcal{T} : \mathcal{G} \longrightarrow [X]$ the collection of natural transformations $[\hat{i}_x] \circ \mathcal{T} : \mathcal{G} \longrightarrow [*] \simeq \mathrm{Id}$. It hence follows that the collection of natural transformations $[\hat{i}_x] : [X] \Rightarrow [*]$ exhibits $[X]$ as the limit in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$ of the constant diagram X-indexed diagram with value $[*]$ if and only if $\alpha_{\mathcal{G}}$ is an equivalence for every $\mathcal{G}$, i.e., if and only if $[\mathrm{tr}_X] : [X] \circ [X] \Rightarrow \mathrm{Id}$ exhibits $[X]$ as self-adjoint.

□

Corollary 3.12. *Let $\mathcal{D}$ be an ∞-category which is tensored over $\mathcal{S}^m_m$, such that the action functor $\mathcal{S}^m_m \times \mathcal{D} \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits separately in each variable. Then $\mathcal{D}$ is m-semiadditive.*

Proof. Let us prove that $\mathcal{D}$ is m'-semiadditive for every $-2 \leq m' \leq m$ by induction on m'. Since every ∞-category is (-2)-semiadditive we may start our induction at $m' = -2$. Now suppose that $\mathcal{D}$ is m'-semiadditive for some $-2 \leq m' \leq m$. As above let us denote by $[X] : \mathcal{D} \longrightarrow \mathcal{D}$ the action of $X \in \mathcal{S}^{m'-1}_{m'}$. By Proposition 3.10 it will suffice to show that the morphism $[\mathrm{tr}_X] : [X] \circ [X] \Rightarrow \mathrm{Id}$ exhibits the functor $[X]$ as self-adjoint. But this follows from the fact that the action of $\mathcal{S}^{m'-1}_{m'}$ extends to an action of $\mathcal{S}^{m'}_{m'}$, and the morphism $\mathrm{tr}_X : X \times X \longrightarrow *$ exhibits X as self-dual in the monoidal ∞-category $\mathcal{S}^{m'}_{m'}$. □

Corollary 3.13. *For every $-2 \leq n \leq m$ the ∞-category $\mathcal{S}^m_n$ is m-semiadditive.*

Proof. Combine Corollary 2.14 and Corollary 3.12. □

4 The universal property of finite spans

In this section we will prove our main result, establishing a universal property for the ∞-categories $\mathcal{S}^m_n$ in terms of m-semiadditivity.

Theorem 4.1. *Let $-2 \leq m \leq n$ be integers and let $\mathcal{D}$ be an m-semiadditive ∞-category which admits $\mathcal{K}_n$-indexed colimits. Then evaluation at $* \in \mathcal{S}^m_n$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathcal{K}_n}(\mathcal{S}^m_n, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}.$$

In other words, the ∞-category $\mathcal{S}_n^m$ is the free m-semiadditive ∞-category which admits $\mathcal{K}_n$-indexed colimits, generated by $* \in \mathcal{S}_n^m$.

Our strategy is essentially a double induction on n and m. For this it will be useful to employ the following terminology:

Definition 4.2. Let $\mathcal{D}$ be an ∞-category and let $-2 \le m \le n$ be integers. We will say that $\mathcal{D}$ is (n,m)**-good** if the following conditions are satisfied:

1. $\mathcal{D}$ is m-semiadditive and admits $\mathcal{K}_n$-indexed colimits.
2. Evaluation at $*$ induces an equivalence of ∞-categories

$$\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_n^m, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}.$$

In other words, $\mathcal{D}$ is (n,m)-good if Theorem 4.1 holds for m, n and $\mathcal{D}$. We may hence phrase the induction step on n as follows: given an $(n-1,m)$-good ∞-category $\mathcal{D}$ which admits $\mathcal{K}_n$-indexed colimits, show that $\mathcal{D}$ is (n,m)-good. To establish this claim we will need to understand how to extend functors from $\mathcal{S}_{n-1}^m$ to $\mathcal{S}_n^m$ when $m \le n-1$. Note that if $f : Z \longrightarrow X$ is an m-truncated map and X is $(n-1)$-truncated then Z is $(n-1)$-truncated as well, and so the inclusion $\mathcal{S}_{n-1}^m \hookrightarrow \mathcal{S}_n^m$ is fully-faithful. The core argument for the induction step on n is the following:

Proposition 4.3. *Let $-2 \le m < n$ be integers. Let $\mathcal{D}$ be an m-semiadditive ∞-category which admits $\mathcal{K}_n$-indexed colimits and let $\mathcal{F} : \mathcal{S}_{n-1}^m \longrightarrow \mathcal{D}$ be a functor which preserves $\mathcal{K}_{n-1}$-indexed colimits. Let $\iota : \mathcal{S}_{n-1}^m \hookrightarrow \mathcal{S}_n^m$ be the fully-faithful inclusion. Then the following assertion hold:*

1. $\mathcal{F}$ admits a left Kan extension

$$\begin{array}{ccc} \mathcal{S}_{n-1}^m & \xrightarrow{\mathcal{F}} & \mathcal{D} \\ {\scriptstyle\iota}\downarrow & \nearrow_{\overline{\mathcal{F}}} & \\ \mathcal{S}_n^m & & \end{array}$$

2. An arbitrary extension $\overline{\mathcal{F}} : \mathcal{S}_n^m \longrightarrow \mathcal{D}$ of $\mathcal{F}$ is a left Kan extension if and only if $\overline{\mathcal{F}}$ preserves $\mathcal{K}_n$-indexed colimits.

Proof. For $Y \in \mathcal{S}_n^m$ let us denote by

$$\mathcal{I}_Y = \mathcal{S}_{n-1}^m \times_{\mathcal{S}_n^m} (\mathcal{S}_n^m)_{/Y}$$

the associated comma category. To prove (1), it will suffice by [5, Lemma 4.3.2.13] to show that the composed map

$$\mathcal{F}_Y : \mathcal{I}_Y \longrightarrow \mathcal{S}_{n-1}^m \longrightarrow \mathcal{D}$$

can be extended to a colimit diagram in $\mathcal{D}$ for every $Y \in \mathcal{S}^m_n$. Now an object of $\mathcal{I}_Y$ corresponds to an object $X \in \mathcal{S}^m_{n-1}$ together with a morphism $X \longrightarrow Y$ in $\mathcal{S}^m_n$, i.e., a span of the form

$$\begin{array}{ccc} & Z & \\ {}^{g}\swarrow & & \searrow^{f} \\ X & & Y \end{array} \tag{12}$$

where g is m-truncated (and hence Z is $(n-1)$-truncated). Since $\mathcal{S}^m_{n-1} \hookrightarrow \mathcal{S}^m_n$ is fully-faithful the mapping space from (X, Z, f, g) to (X', Z', f', g') in $\mathcal{I}_Y$ can be identified with the homotopy fiber of the map $\mathrm{Map}_{\mathcal{S}^m_n}(X, X') \longrightarrow \mathrm{Map}_{\mathcal{S}^m_n}(X, Y)$ over $(Z, f, g) \in \mathrm{Map}_{\mathcal{S}^m_n}(X, Y)$. Now let

$$\mathcal{J}_Y = \mathcal{S}_{n-1} \times_{\mathcal{S}_n} (\mathcal{S}_n)_{/Y}$$

be the analogue comma category for the inclusion $\mathcal{S}_{n-1} \hookrightarrow \mathcal{S}_n$. Then the inclusions $\mathcal{S}_{n-1} \hookrightarrow \mathcal{S}^m_{n-1}$ and $\mathcal{S}_n \hookrightarrow \mathcal{S}^m_n$ induce a functor $\varphi : \mathcal{J}_Y \longrightarrow \mathcal{I}_Y$, and it is not hard to check that φ is in fact fully-faithful, and its essential image consists of those objects as in (12) for which g is an equivalence. We now claim that φ is also **cofinal**. To prove this, we need to show that for every object $(X, Z, f, g) \in \mathcal{I}_Y$ as in (12), the comma category $\mathcal{J}_Y \times_{\mathcal{I}_Y} (\mathcal{I}_Y)_{(X,Z,f,g)/}$ is weakly contractible. Given an object $h : X' \longrightarrow Y$ in $\mathcal{J}_Y$, we may identify the mapping space from (X, Z, f, g) to $\varphi(X', h)$ in $\mathcal{I}_Y$ with the homotopy fiber of the map

$$h_* : \mathrm{Map}_{\mathcal{S}^m_n}(X, X') \longrightarrow \mathrm{Map}_{\mathcal{S}^m_n}(X, Y) \tag{13}$$

over the span $(Z, f, g) \in \mathrm{Map}_{\mathcal{S}^m_n}(X, Y)$. Now clearly any span of n-truncated spaces from X to X' whose composition with $h : X' \longrightarrow Y$ belongs to $\mathcal{S}^m_n$ already itself belongs to $\mathcal{S}^m_n$. We may hence identify the homotopy fiber of (13) with the homotopy fiber of the map

$$h_* : ((\mathcal{S}_n)_{/X \times X'})^{\sim} \longrightarrow ((\mathcal{S}_n)_{/X \times Y})^{\sim} \tag{14}$$

over the object $(f, g) : Z \longrightarrow X \times Y$. Finally, using the general equivalence $\mathcal{C}_{/A \times B} \simeq \mathcal{C}_{/A} \times_{\mathcal{C}} \mathcal{C}_{/B}$ we may identify the homotopy fiber of (14) with the homotopy fiber of the map

$$((\mathcal{S}_n)_{/X'})^{\sim} \longrightarrow ((\mathcal{S}_n)_{/Y})^{\sim} \tag{15}$$

over $f : Z \longrightarrow Y$. We may then conclude that the functor from $\mathcal{J}_Y$ to spaces given by $(X', h) \mapsto \mathrm{Map}_{\mathcal{I}_Y}((X, Z, f, g), \varphi(X', h))$ is corepresented by $f : Z \longrightarrow Y$ (considered as an object of $\mathcal{J}_Y$). This implies that the comma category $\mathcal{J}_Y \times_{\mathcal{I}_Y} (\mathcal{I}_Y)_{(X,Z,f,g)/}$ has an initial object and is hence weakly contractible. Since this is true for any $(X, Z, f, g) \in \mathcal{I}_Y$ it follows that φ is cofinal, as desired.

It will now suffice to show that each of the diagrams

$$\mathcal{G}_Y \stackrel{\text{def}}{=} (\mathcal{F}_Y)|_{\mathcal{J}_Y} : \mathcal{J}_Y \longrightarrow \mathcal{D}$$

can be extended to a colimit diagram. Let $\mathcal{J}'_Y = \mathcal{J}_Y \times_{\mathcal{S}_{n-1}} \{*\} \subseteq \mathcal{J}_Y$ be the full subcategory spanned by objects of the form $* \xrightarrow{h} Y$. Then $\mathcal{J}'_Y$ is an ∞-groupoid which is equivalent to the underlying space of Y, and the composed functor $\mathcal{J}'_Y \longrightarrow \mathcal{J}_Y \longrightarrow \mathcal{D}$ is constant with value $\mathcal{F}(*) \in \mathcal{D}$. Since we assumed that $\mathcal{F} : \mathcal{S}^m_{n-1} \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits it follows from Proposition 2.9 that the restriction $\mathcal{F}|_{\mathcal{S}_{n-1}} : \mathcal{S}_{n-1} \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_{n-1}$-indexed colimits and hence by Lemma 2.8 the functor $\mathcal{F}|_{\mathcal{S}_{n-1}}$ is a left Kan extension of its restriction to the object $* \in \mathcal{S}_{n-1}$. Now since the projection $\mathcal{J}_Y \longrightarrow \mathcal{S}_{n-1}$ is a right fibration (classified by the functor $X \mapsto \mathrm{Map}_{\mathcal{S}_n}(X, Y)$) it induces an equivalence $(\mathcal{J}_Y)_{/(X,h)} \longrightarrow (\mathcal{S}_{n-1})_{/X}$ for every $(X, h) \in \mathcal{J}_Y$. We may then conclude that $\mathcal{F}|_{\mathcal{J}_Y}$ is a left Kan extension of $\mathcal{F}|_{\mathcal{J}'_Y}$. Since $\mathcal{D}$ admits $\mathcal{K}_n$-indexed colimits (and $\mathcal{J}'_Y$ is a finite n-truncated Kan complex) the diagram $\mathcal{G}_Y|_{\mathcal{J}'_Y}$ admits a colimit. It then follows that the diagram $\mathcal{G}_Y : \mathcal{J}_Y \longrightarrow \mathcal{D}$ admits a colimit, as desired.

To prove (2), note that by the above considerations an arbitrary functor $\overline{\mathcal{F}} : \mathcal{S}^m_n \longrightarrow \mathcal{D}$ is a left Kan extension of $\mathcal{F}$ if and only if the extension $(\mathcal{J}'_Y)^{\triangleright} \longrightarrow \mathcal{D}$ determined by $\overline{\mathcal{F}}$ is a colimit diagram. By construction, this means that $\overline{\mathcal{F}}$ is a left Kan extension if and only if for every $Y \in \mathcal{S}^m_n$ the collection of maps $\{\overline{\mathcal{F}}(i_y)\}_{y \in Y}$ exhibits $\overline{\mathcal{F}}(Y)$ as the colimit of the constant Y-indexed diagram with value $\overline{\mathcal{F}}(*)$. It then follows from Lemma 2.8 that $\overline{\mathcal{F}}$ is a left Kan extension of $\mathcal{F}$ if and only if $\overline{\mathcal{F}}$ preserves $\mathcal{K}_n$-indexed colimits.

□

Corollary 4.4. *Let $-2 \leq m < n$ be integers and let $\mathcal{D}$ be an ∞-category which admits $\mathcal{K}_n$-indexed colimits. Then the restriction map*

$$\mathrm{Fun}_{\mathcal{K}_n}(\mathcal{S}^m_n, \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathcal{K}_{n-1}}(\mathcal{S}^m_{n-1}, \mathcal{D})$$

is an equivalence of ∞-categories.

Proof. This a direct consequence of Proposition 4.3 in light of [5, Proposition 4.3.2.15]. □

Corollary 4.5. *Let $-2 \leq m \leq n \leq n'$ be integers and let $\mathcal{D}$ be an m-semiadditive ∞-category which admits $\mathcal{K}_{n'}$-indexed colimits. Then $\mathcal{D}$ is (n', m)-good if and only if $\mathcal{D}$ is (n, m)-good.*

We shall now proceed to perform the induction step on m. For this it will be convenient to make use of the language of **marked simplicial sets**, as developed in [5]. Given an $m \geq -1$ let

$$\mathrm{Cone}_m = \mathcal{S}^m_m \coprod_{\mathcal{S}^{m-1}_m \times \Delta^{\{0\}}} \left[\mathcal{S}^{m-1}_m \times (\Delta^1)^\sharp\right]$$

be the right marked mapping cone of the inclusion $\iota : \mathcal{S}^{m-1}_m \hookrightarrow \mathcal{S}^m_m$. Let

$$\mathrm{Cone}_m \hookrightarrow \mathcal{M}^\natural \xrightarrow{p} \Delta^1$$

be a factorization of the projection $\mathrm{Cone}_m \longrightarrow (\Delta^1)^\sharp$ into a trivial cofibration follows by a fibration in the Cartesian model structure over $(\Delta^1)^\sharp$. In particular,

$p : \mathcal{M} \longrightarrow \Delta^1$ is a Cartesian fibrations and the marked edges of $\mathcal{M}^\natural$ are exactly the p-Cartesian edges. Let $\iota_0 : \mathcal{S}_m^m \hookrightarrow \mathcal{M} \times_{\Delta^1} \Delta^{\{0\}} \subseteq \mathcal{M}$ and $\iota_1 : \mathcal{S}_m^{m-1} \hookrightarrow \mathcal{M} \times_{\Delta^1} \Delta^{\{1\}} \subseteq \mathcal{M}$ be the corresponding inclusions. Then ι_0 and ι_1 exhibit $p : \mathcal{M} \longrightarrow \Delta^1$ as a correspondence from $\mathcal{S}_m^m$ to $\mathcal{S}_m^{m-1}$ which is the one associated to the functor $\iota : \mathcal{S}_m^{m-1} \hookrightarrow \mathcal{S}_m^m$.

Lemma 4.6. *Let $\mathcal{D}$ be an $(m, m-1)$-good ∞-category and let $\mathcal{F} : \mathcal{S}_m^{m-1} \longrightarrow \mathcal{D}$ be a functor which preserves $\mathcal{K}_m$-indexed colimits. If $\mathcal{D}$ is m-semiadditive then the collection of maps $\mathcal{F}(\hat{i}_x) : \mathcal{F}(X) \longrightarrow \mathcal{F}(*)$ exhibits $\mathcal{F}(X)$ as the limit in $\mathcal{D}$ of the constant X-indexed diagram with value $\mathcal{F}(*)$.*

Proof. Using the symmetric monoidal structure of $\mathcal{S}_m^{m-1}$ we may consider $\mathcal{S}_m^{m-1}$ as tensored over itself. Since the monoidal structure preserves $\mathcal{K}_m$-indexed colimits separately in each variable (see Corollary 2.13), and since $\mathcal{D}$ is $(m, m-1)$-good, we may endow $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^{m-1}, \mathcal{D}) \simeq \mathcal{D}$ with an action of $\mathcal{S}_m^{m-1}$ via precomposition. Then $\mathcal{D}$ is tensored over $\mathcal{S}_m^{m-1}$ and the action map $\mathcal{S}_m^{m-1} \times D \longrightarrow D$ preserves $\mathcal{K}_m$-indexed colimits separately in each variable. As above let us denote by $[X] : \mathcal{D} \longrightarrow \mathcal{D}$ the action of $X \in \mathcal{S}_m^{m-1}$.

By Corollary 3.11 the collection of natural transformations $[\hat{i}_x] : [X] \Rightarrow [*]$ exhibits $[X]$ as the limit, in $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$, of the constant diagram X-indexed diagram with value $[*]$. Evaluating at $\mathcal{F}(*)$ we may conclude that the collection of maps $[\hat{i}_x](\mathcal{F}(*)) : [X](\mathcal{F}(*)) \longrightarrow \mathcal{F}(*)$ exhibits $[X](\mathcal{F}(*))$ as the limit in $\mathcal{D}$ of the constant X-diagram with value $\mathcal{F}(*)$. By construction we may identify $[X](\mathcal{F}(*))$ with $\mathcal{F}(X)$ and $[\hat{i}_x](\mathcal{F}(*))$ with $\mathcal{F}(\hat{i}_x)$ and so the desired result follows. □

Proposition 4.7. *Let $\mathcal{D}$ be an ∞-category which admits $\mathcal{K}_m$-indexed limits and let $\mathcal{F} : \mathcal{S}_m^{m-1} \longrightarrow \mathcal{D}$ be a functor which satisfies the following property: for every $X \in \mathcal{S}_m^{m-1}$ the collection of maps $\mathcal{F}(\hat{i}_x) : \mathcal{F}(X) \longrightarrow \mathcal{F}(*)$ exhibits $\mathcal{F}(X)$ as the limit in $\mathcal{D}$ of the constant X-indexed diagram with value $\mathcal{F}(*)$. Then the following assertion hold:*

1. *There exists a right Kan extension*

$$\begin{array}{ccc} \mathcal{S}_m^{m-1} & \xrightarrow{\mathcal{F}} & \mathcal{D} \\ {\scriptstyle \iota_1}\downarrow & \nearrow_{\overline{\mathcal{F}}} & \\ \mathcal{M} & & \end{array}$$

2. *An extension $\overline{\mathcal{F}} : \mathcal{M} \longrightarrow \mathcal{D}$ as above is a right Kan extension if and only if $\overline{\mathcal{F}}$ maps p-Cartesian edges to equivalences in $\mathcal{D}$.*

Proof. For an object $X \in \mathcal{S}_m^m$ let us set

$$\mathcal{I}_X = \mathcal{M}_{\iota_0(X)/} \times_{\mathcal{M}} \mathcal{S}_m^{m-1}.$$

To prove (1), it will suffice by [5, Lemma 4.3.2.13] to show that the composed map

$$\mathcal{F}_X : \mathcal{I}_X \longrightarrow \mathcal{S}_m^{m-1} \longrightarrow \mathcal{D}$$

can be extended to a limit diagram in $\mathcal{D}$ for every $X \in \mathcal{S}^m_m$. Now an object of $\mathcal{I}_X$ corresponds to an object $Y \in \mathcal{S}^{m-1}_m$ and a morphism $\iota_0(X) \longrightarrow \iota_1(Y)$ in $\mathcal{M}$, or, equivalently, a morphism $X \longrightarrow \iota(Y)$ in $\mathcal{S}^m_m$, i.e., a span

$$\begin{array}{ccc} & Z & \\ {}^{g}\swarrow & & \searrow^{f} \\ X & & Y \end{array} \tag{16}$$

of finite m-truncated spaces.

Recall that we have denoted by $\mathcal{S}_{m,m-1} \subseteq \mathcal{S}_m$ the subcategory of $\mathcal{S}_m$ consisting of all objects and $(m-1)$-truncated maps between them. Then we have a commutative square

$$\begin{array}{ccc} \mathcal{S}^{\mathrm{op}}_{m,m-1} & \longrightarrow & \mathcal{S}^{\mathrm{op}}_m \\ \downarrow & & \downarrow \\ \mathcal{S}^{m-1}_m & \longrightarrow & \mathcal{S}^m_m \end{array} \tag{17}$$

where the vertical functors map are the identity on objects and send an $(m-1)$-truncated map $f : X \longrightarrow Y$ to the span $Y \xleftarrow{f} X \longrightarrow X$. Let $\mathcal{J}_X = \mathcal{S}^{\mathrm{op}}_{m,m-1} \times_{\mathcal{S}^{\mathrm{op}}_m} (\mathcal{S}^{\mathrm{op}}_m)_{X/}$ be the associated comma category. We will write the objects of $\mathcal{J}_X$ as maps $X \xleftarrow{g} Y$ of finite m-truncated spaces, or simply as pairs (Y, g). We note that morphisms from $X \xleftarrow{g} Y$ to $X \xleftarrow{g'} Y'$ are commutative triangles of the form

$$\begin{array}{ccc} Y & \xleftarrow{\quad h \quad} & Y' \\ & {}_{g}\searrow \quad \swarrow_{g'} & \\ & X & \end{array}$$

such that h is $(m-1)$-truncated. The square (17) induces a fully-faithful fuctor $\varphi : \mathcal{J}_X \hookrightarrow \mathcal{I}_X$ whose essential image consists of those objects as in (16) for which f is an equivalence. We now claim that φ is coinitial.

To prove this, we need to show that for every object $(Y, Z, f, g) \in \mathcal{I}_Y$ as in (16), the comma category $\mathcal{J}_X \times_{\mathcal{I}_X} (\mathcal{I}_X)_{/(Y,Z,f,g)}$ is weakly contractible. Given an object (Y', h) in $\mathcal{J}_X$, we may identify the mapping space from $\varphi(Y', h)$ to (Y, Z, f, g) in $\mathcal{I}_X$ with the homotopy fiber of the map

$$h_* : \mathrm{Map}_{\mathcal{S}^{m-1}_m}(Y', Y) \longrightarrow \mathrm{Map}_{\mathcal{S}^m_m}(X, Y) \tag{18}$$

over the span $(Z, f, g) \in \mathrm{Map}_{\mathcal{S}^m_m}(X, Y)$. As in the proof Proposition 4.3 we may identify these mapping spaces as $\mathrm{Map}_{\mathcal{S}^{m-1}_m}(Y', Y) \simeq ((\mathcal{S}^{\mathrm{op}}_{m,m-1})_{Y'/})^{\sim} \times_{\mathcal{S}_m} ((\mathcal{S}_m)_{/Y})^{\sim}$ and $\mathrm{Map}_{\mathcal{S}^m_m}(X, Y) \simeq ((\mathcal{S}^{\mathrm{op}}_m)_{X/})^{\sim} \times_{\mathcal{S}_m} ((\mathcal{S}_m)_{/Y})^{\sim}$ we may identify the homotopy fiber of (18) with the homotopy fiber of the map

$$h_* : ((\mathcal{S}^{\mathrm{op}}_{m,m-1})_{Y'/})^{\sim} \longrightarrow ((\mathcal{S}^{\mathrm{op}}_m)_{X/})^{\sim} \tag{19}$$

over the object $X \xleftarrow{g} Z$. We may then conclude that the functor from $\mathcal{J}_X$ to spaces given by $(Y',h) \mapsto \mathrm{Map}_{\mathcal{J}_Y}(\varphi(Y',h),(Y,Z,f,g))$ is represented in $\mathcal{J}_X$ by the object $X \xleftarrow{g} Z$. This implies that the comma category $\mathcal{J}_X \times_{\mathcal{I}_X} (\mathcal{I}_X)_{/(Y,Z,f,g)}$ has an terminal object and is hence weakly contractible. Since this is true for any $(Y,Z,f,g) \in \mathcal{I}_X$ it follows that φ is coinitial, as desired. It will hence suffice to show that each of the diagrams

$$\mathcal{G}_X \stackrel{\text{def}}{=} (\mathcal{F}_X)|_{\mathcal{J}_X} : \mathcal{J}_X \longrightarrow \mathcal{D}$$

can be extended to a limit diagram.

Let $\mathcal{J}'_X = \mathcal{J}_X \times_{\mathcal{S}_{m,m-1}} \{*\} \subseteq \mathcal{J}_Y$ be the full subcategory spanned by objects of the form $X \xleftarrow{g} *$. Then $\mathcal{J}'_X$ is an ∞-groupoid which is equivalent to the underlying space of X, and the composed functor $\mathcal{J}'_X \longrightarrow \mathcal{J}_X \longrightarrow \mathcal{D}$ is constant with value $\mathcal{F}(*) \in \mathcal{D}$. By our assumption on $\mathcal{F}$ it follows that the restricted functor $\mathcal{F}|_{\mathcal{S}^{\mathrm{op}}_{m,m-1}} : \mathcal{S}^{\mathrm{op}}_{m,m-1} \longrightarrow \mathcal{D}$ is a right Kan extension $\mathcal{F}|_{\{*\}}$. Now since the projection $\mathcal{J}_X \longrightarrow \mathcal{S}^{\mathrm{op}}_{m,m-1}$ is a left fibration it induces an equivalence $(\mathcal{J}_X)_{(Y,h)/} \longrightarrow (\mathcal{S}_{m,m-1})_{Y/}$ for every $(Y,h) \in \mathcal{J}_X$. We may then conclude that $\mathcal{F}|_{\mathcal{J}_X}$ is a right Kan extension of $\mathcal{F}|_{\mathcal{J}'_Y}$. Since $\mathcal{D}$ is m-semiadditive it admits $\mathcal{K}_m$-indexed limits and hence the diagram $\mathcal{G}_X|_{\mathcal{J}'_X}$ admits a limit. It follows that the diagram $\mathcal{G}_X : \mathcal{J}_X \longrightarrow \mathcal{D}$ admits a limit, as desired.

Let us now prove (2). Let $\overline{\mathcal{F}} : \mathcal{M} \longrightarrow \mathcal{D}$ be a map extending $\mathcal{F}$. Then for every $X \in \mathcal{S}^m_m$ the functor $\overline{\mathcal{F}}$ determines a diagram

$$\overline{\mathcal{G}}_X : \mathcal{J}^{\triangleleft}_X \longrightarrow \mathcal{D}$$

extending $\mathcal{G}_X$. By the considerations above $\overline{\mathcal{F}}$ is a right Kan extension of $\mathcal{F}$ if and only if each $\overline{\mathcal{G}}_X$ is a limit diagram. Let $\mathcal{J}''_X \subseteq \mathcal{J}_X$ denote the full subcategory spanned by those objects $X \xleftarrow{g} Y$ such that g is $(m-1)$-truncated. By the above arguments the functor $(\mathcal{G}_X)|_{\mathcal{J}''_X}$ is a right Kan extension of $(\mathcal{G}_X)|_{\mathcal{J}'_X}$, and so by [5, Proposition 4.3.2.8] we have that $\mathcal{G}_X$ is a right Kan extension of $(\mathcal{G}_X)|_{\mathcal{J}''_X}$. It follows that $\overline{\mathcal{G}}_X$ is a limit diagram if and only if $(\overline{\mathcal{G}}_X)|_{(\mathcal{J}''_X)^{\triangleleft}}$ is a limit diagram. Let $\diamond \in (\mathcal{J}''_X)^{\triangleleft}$ be the cone point. We now observe that the ∞-category $\mathcal{J}''_X$ has initial objects, namely every object of the form $X \xleftarrow{g} X$ such that g is an equivalence. It follows that $(\overline{\mathcal{G}}_X)|_{(\mathcal{J}''_X)^{\triangleleft}}$ is a limit diagram if and only if $\overline{\mathcal{G}}_X$ sends every edge connecting $\diamond$ to an initial object of $\mathcal{J}''_X$ to an equivalence in $\mathcal{D}$. To finish the proof it suffices to observe that these edges are exactly the edges which map to p-Cartesian edges by the natural map $(\mathcal{J}''_X)^{\triangleleft} \longrightarrow \mathcal{M}$, and that all p-Cartesian edges are obtained in this way.

□

Corollary 4.8. *Let $\mathcal{D}$ be an m-semiadditive ∞-category. Then the restriction map*

$$\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^m_m, \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^{m-1}_m, \mathcal{D})$$

is an equivalence of ∞-categories.

Proof. Let $p : \mathcal{M}^\natural \longrightarrow \Delta^1$ be as above and consider the marked simplicial set $\mathcal{D}^\natural = (\mathcal{D}, M)$ where M is the collection of edges which are equivalences in $\mathcal{D}$. For two marked simplicial sets $(X, M), (Y, N)$ let $\mathrm{Fun}^\flat((X, M), (Y, N)) \subseteq \mathrm{Fun}(X, Y)$ be the full sub-simplcial set spanned by those functors $X \longrightarrow Y$ which send M to N. We will denote by $\mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathcal{M}^\natural, \mathcal{D}^\natural) \subseteq \mathrm{Fun}^\flat(\mathcal{M}^\natural, \mathcal{D}^\natural)$ and by $\mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathrm{Cone}_m, \mathcal{D}^\natural) \subseteq \mathrm{Fun}^\flat(\mathrm{Cone}_m, \mathcal{D}^\natural)$ the respective full subcategories spanned by those makred functors whose restriction to $\mathcal{S}^{m-1}_m$ preserves $\mathcal{K}_m$-indexed colimits. Now consider the commutative diagram of functors categories and restriction maps

$$\begin{array}{ccc} & \mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathcal{M}^\natural, \mathcal{D}^\natural) & \\ \swarrow & & \searrow \\ \mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathrm{Cone}_m, \mathcal{D}^\natural) & \xrightarrow{\iota_1^*} & \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^{m-1}_m, \mathcal{D}) \end{array} \tag{20}$$

Since the inclusion of marked simplicial sets $\mathrm{Cone}_m \longrightarrow \mathcal{M}^\natural$ is a trivial cofibration in the Cartesian model structure over $(\Delta^1)^\sharp$ it follows that the left diagonal map is a trivial Kan fibration. On the other hand by Proposition 4.7 and [5, Proposition 4.3.2.15] the right diagonal map is a trivial Kan fibration. We may hence deduce that ι_1^* is an equivalence of ∞-categories.

Since the inclusion $\mathcal{S}^m_m \hookrightarrow \mathrm{Cone}_m$ is a pushout along the inclusion $\mathcal{S}^{m-1}_m \times \Delta^{\{0\}} \hookrightarrow \mathcal{S}^{m-1}_m \times (\Delta^1)^\sharp$ (which is itself a trivial cofibration in the **coCartesian** model structure over Δ^0) it follows that the map $i_0^* : \mathrm{Fun}^\flat(\mathrm{Cone}_m, \mathcal{D}^\natural) \longrightarrow \mathrm{Fun}(\mathcal{S}^m_m, \mathcal{D})$ is a trivial Kan fibation and that the composed functor

$$\mathrm{Fun}^\flat(\mathrm{Cone}_m, \mathcal{D}^\natural) \underset{\simeq}{\xrightarrow{i_0^*}} \mathrm{Fun}(\mathcal{S}^m_m, \mathcal{D}) \xrightarrow{\iota^*} \mathrm{Fun}(\mathcal{S}^{m-1}_m, \mathcal{D})$$

is homotopic to $i_1^* : \mathrm{Fun}^\flat(\mathrm{Cone}_m, \mathcal{D}^\natural) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{S}^{m-1}_m, \mathcal{D})$. We may consequently conclude that i_0^* induces an equivalence between $\mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathrm{Cone}_m, \mathcal{D}^\natural) \subseteq \mathrm{Fun}^\flat(\mathrm{Cone}_m, \mathcal{D}^\natural)$ and the full subcategory of $\mathrm{Fun}(\mathcal{S}^m_m, \mathcal{D})$ spanned by those functors whose restriction to $\mathcal{S}^{m-1}_m$ preserves $\mathcal{K}_m$-indexed colimits. By Corollary 2.12 these are exactly the functors $\mathcal{S}^m_m \longrightarrow \mathcal{D}$ which preserves $\mathcal{K}_m$-indexed colimits. We may finally conclude that

$$\iota^* : \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^m_m, \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^{m-1}_m, \mathcal{D})$$

is an equivalence of ∞-categories, as desired. □

Corollary 4.9. *Let $-1 \le m \le n$ integer and let $\mathcal{D}$ be an m-semiadditive ∞-category which admits $\mathcal{K}_n$-indexed colimits. If $\mathcal{D}$ is $(n, m-1)$-good then $\mathcal{D}$ is (n, m)-good.*

Proof. By Corollary 4.5 we know that $\mathcal{D}$ is (n, m)-good if and only if $\mathcal{D}$ is (m, m)-good, and that $\mathcal{D}$ is $(n, m-1)$-good if and only if $\mathcal{D}$ is $(m, m-1)$-good. The desired result now follows directly from Corollary 4.9. □

Proof of Theorem 4.1. We want to prove that if $\mathcal{D}$ is an m-semiadditive ∞-category which admits $\mathcal{K}_n$-indexed colimits then $\mathcal{D}$ is (n, m)-good. Let us consider the set $\mathcal{A} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} | a \le b\}$ as partially ordered saying that $(a, b) \le$

(c,d) if $a \le c$ and $b \le d$. We now note that for every $(-2,-2) \le (n',m') \le (n,m)$ in $\mathcal{A}$, the ∞-category $\mathcal{D}$ is m'-semiadditive and admits $\mathcal{K}_{n'}$-indexed colimits. Furthermore, $\mathcal{D}$ is tautologically $(-2,-2)$-good. It follows that there exists a pair $(-2,-2) \le (n',m') \le (n,m)$ for which $\mathcal{D}$ is (n',m') and which is maximal with respect to this property. If $n' < n$ then Corollary 4.5 implies that $\mathcal{D}$ is $(n'+1,m')$-good, contradicting the maximality of (n',m'). On the other hand, if $n' = n$ and $m' < m$ then $m' < n'$. By Corollary 4.9 we may conclude that $\mathcal{D}$ is $(n',m'+1)$-good, a contradiction again. It follows that $(n',m') = (n,m)$ and hence $\mathcal{D}$ is (n,m)-good, as desired. □

5 Applications

5.1 m-semiadditive ∞-categories as modules over spans

By Corollary 3.12, every ∞-category with $\mathcal{K}_m$-indexed colimits which carries a compatible action of $\mathcal{S}_m^m$ is m-semiadditive. On the other hand, our main theorem 4.1 implies that every m-semiadditive ∞-category $\mathcal{D}$ carries an action of $\mathcal{S}_m^m$ (by identifying $\mathcal{D}$, for example, with $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m,\mathcal{D})$). This suggests that the theory of m-semiadditive ∞-categories is strongly related to that of $\mathcal{S}_m^m$-modules in $\mathrm{Cat}_{\mathcal{K}_m}$. In this section we will make this idea more precise by proving a suitable equivalence of ∞-categories. This equivalence is strongly related to the fact that $\mathcal{S}_m^m$ is an **idempotent object** of $\mathrm{Cat}_{\mathcal{K}_m}$, a corollary we will deduce below.

Let $\mathrm{Add}_m \subseteq \mathrm{Cat}_{\mathcal{K}_m}$ denote the full subcategory spanned by m-semiadditive ∞-categories. The discussion above implies that the essential image of the forgetful functor

$$\mathcal{U} : \mathrm{Mod}_{\mathcal{S}_m^m}(\mathrm{Cat}_{\mathcal{K}_m}) \longrightarrow \mathrm{Cat}_{\mathcal{K}_m}$$

is exactly Add_m. We note that $\mathcal{U}$ admits a left adjoint given by $\mathcal{D} \mapsto \mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{D}$.

Lemma 5.1. *Let $\mathcal{C}$ be an $\mathcal{S}_m^m$-module. Then the counit map*

$$v_{\mathcal{C}} : \mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathcal{C}) \longrightarrow \mathcal{C}$$

is an equivalence of $\mathcal{S}_m^m$-modules. In particular, $\mathcal{U}$ is full-faithful.

Proof. Since $\mathcal{U}$ is conservative it will suffice to show that $\mathcal{U}(v_{\mathcal{C}})$ is an equivalence of ∞-categories. Since the composition

$$\mathcal{U}(\mathcal{C}) \overset{u_{\mathcal{U}(\mathcal{C})}}{\longrightarrow} \mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathcal{C}) \overset{\mathcal{U}(v_{\mathcal{C}})}{\longrightarrow} \mathcal{U}(\mathcal{C})$$

is homotopic to the identity we are reduced to showing that the functor

$$u_{\mathcal{U}(\mathcal{C})} : \mathcal{U}(\mathcal{C}) \longrightarrow \mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathcal{C})$$

is an equivalence. By Corollary 3.12 we have that $\mathcal{U}(\mathcal{C})$ is m-semiadditive and hence it will suffice to show that for every m-semiadditive ∞-category $\mathcal{D}$ the induced map

$$u^*_{\mathcal{U}(\mathcal{C})} : \mathrm{Fun}(\mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathcal{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{U}(\mathcal{C}), \mathcal{D})$$

is an equivalence. Identifying the functor ∞-category $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^m_m \otimes_{\mathcal{K}_m} \mathcal{U}(\mathcal{C}), \mathcal{D})$ with $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^m_m, \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathcal{C}), \mathcal{D}))$ and $u^*_{\mathcal{D}}$ with evaluation at $* \in \mathcal{S}^m_m$ it will suffice, in light of Theorem 4.1, to show that $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathcal{C}), \mathcal{D})$ is m-semiadditive. But this follows from Corollary 3.12 since $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{U}(\mathcal{C}), \mathcal{D})$ carries an action of $\mathcal{S}^m_m$ (given by pre-composing the action on $\mathcal{U}(\mathcal{C})$). □

Corollary 5.2. *The forgetful functor induces an equivalence of ∞-categories $\mathrm{Mod}_{\mathcal{S}^m_m}(\mathrm{Cat}_{\mathcal{K}_m}) \simeq \mathrm{Add}_m$.*

Corollary 5.3. *$\mathcal{S}^m_m$ is an idempotent object in $\mathrm{Cat}_{\mathcal{K}_m}$. In particular, the monoidal product map $\mathcal{S}^m_m \otimes_{\mathcal{K}_m} \mathcal{S}^m_m \xrightarrow{\simeq} \mathcal{S}^m_m$ is an equivalence.*

Corollary 5.4. *The inclusion $\mathrm{Add}_m \hookrightarrow \mathrm{Cat}_{\mathcal{K}_m}$ admits both a left adjoint, given by $\mathcal{D} \mapsto \mathcal{S}^m_m \otimes_{\mathcal{K}_m} \mathcal{D}$ and a right adjoint given by $\mathcal{D} \mapsto \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^m_m, \mathcal{D})$.*

We shall now discuss tensor products of m-semiadditive ∞-categories.

Proposition 5.5. *There exists a symmetric monoidal structure $\mathrm{Add}^{\otimes}_m \longrightarrow \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ on Add_m such that the functor $\mathrm{Cat}_\infty(\mathcal{K}_m) \longrightarrow \mathrm{Add}_m$ given by $\mathcal{D} \mapsto \mathcal{S}^m_m \otimes_{\mathcal{K}_m} \mathcal{D}$ extends to a symmetric monoidal functor. In particular, $\mathcal{S}^m_m$ is the unit of $\mathrm{Add}^{\otimes}_m$.*

Proof. Identify $\mathrm{Add}_m \simeq \mathrm{Mod}_{\mathcal{S}^m_m}(\mathrm{Cat}_{\mathcal{K}_m})$ using Corollary 5.2 and apply [6, Theorem 4.5.2.1] to the case $\mathcal{C} = \mathrm{Cat}_{\mathcal{K}_m}$ and $A = \mathcal{S}^m_m$. The assertion about $\mathcal{D} \mapsto \mathcal{S}^m_m \otimes_{\mathcal{K}_m} \mathcal{D}$ being monoidal follows from [6, Theorem 4.5.3.1]. □

Corollary 5.6. *$\mathcal{S}^m_m$ carries a canonical commutative algebra structure making it the initial object of $\mathrm{CAlg}(\mathrm{Add}_m)$.*

Let us refer to commutative algebra objects in $\mathrm{Cat}_{\mathcal{K}_m}$ as **$\mathcal{K}_m$-symmetric monoidal ∞-categories**. These can identified with ordinary symmetric monoidal ∞-categories such that the underlying ∞-category admits $\mathcal{K}_m$-indexed colimits and the monoidal product preserves $\mathcal{K}_m$-indexed colimits it each variable separately.

Proposition 5.7. *The inclusion $\mathrm{Add}^{\otimes}_m \hookrightarrow \mathrm{Cat}^{\otimes}_{\mathcal{K}_m}$ is lax monoidal. Furthermore the induced map*

$$\mathcal{R} : \mathrm{CAlg}(\mathrm{Add}_m) \longrightarrow \mathrm{CAlg}(\mathrm{Cat}_{\mathcal{K}_m})$$

is fully-faithful and its essential image is spanned by those $\mathcal{K}_m$-symmetric monoidal ∞-categories whose underlying ∞-category is m-semiadditive.

Proof. The fact that the inclusion is lax monoidal follows formally from its left adjoint $\mathcal{S}^m_m \otimes_{\mathcal{K}_m} (-)$ being monoidal. Furthermore $\mathcal{S}^m_m \otimes_{\mathcal{K}_m} (-)$ determines a left adjoint $\mathcal{L} : \mathrm{CAlg}(\mathrm{Cat}_{\mathcal{K}_m}) \longrightarrow \mathrm{CAlg}(\mathrm{Add}_m)$ to $\mathcal{R}$. Let $\mathcal{E} \subseteq \mathrm{CAlg}(\mathrm{Cat}_{\mathcal{K}_m})$ denote the full subcategory spanned by those $\mathcal{K}_m$-symmetric monoidal ∞-categories whose underlying ∞-category is m-semiadditive. Since the image of $\mathcal{R}$ is contained in $\mathcal{E}$ it now follows from Lemma 5.1 that the adjunction $\mathcal{L} \dashv \mathcal{R}$ restricts to an equivalence

$$\mathcal{E} \underset{\perp}{\overset{\simeq}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Cat}_{\mathcal{K}_m})$$

□

We hence obtain yet another universal characterization of $\mathcal{S}_m^m$:

Corollary 5.8. *The $\mathcal{K}_m$-symmetric monoidal ∞-category $\mathcal{S}_m^m$ is* ***initial*** *among those $\mathcal{K}_m$-symmetric monoidal ∞-categories whose underlying ∞-category is m-semiadditive.*

5.2 Higher commutative monoids

In the previous subsection we discussed the inclusion of Add_m inside the ∞-category of $\mathrm{Cat}_{\mathcal{K}_m}$ of ∞-categories admitting $\mathcal{K}_m$-indexed colimits. But there is also a dual story, when one embeds Add_m inside the ∞-category $\mathrm{Cat}^{\mathcal{K}_m}$ consisting of those ∞-categories which admit $\mathcal{K}_m$-indexed **limits**. Indeed, the symmetry here is complete: the operation $\mathcal{D} \mapsto \mathcal{D}^{\mathrm{op}}$ which sends an ∞-category to its opposite induces an equivalence $\mathrm{Cat}^{\mathcal{K}_m} \simeq \mathrm{Cat}_{\mathcal{K}_m}$ which maps Add_m to itself. We may hence apply any of the constructions of the previous section to ∞-categories with $\mathcal{K}_m$-indexed limits by "conjugating" it with the operation $\mathcal{D} \mapsto \mathcal{D}^{\mathrm{op}}$. From an abstract point of view this seems to yield no additional interest. However, for one of the procedures above applying this conjugation yields an interesting relation with the theory of commutative monoids, which is worthwhile spelling out.

By Corollary 5.4, if $\mathcal{D}$ is an ∞-category which admits $\mathcal{K}_m$-indexed colimits, then the restriction functor $r : \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}) \longrightarrow \mathcal{D}$ exhibits $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D})$ as the universal m-semiadditive ∞-category carrying a $\mathcal{K}_m$-colimit preserving functor to $\mathcal{D}$. In other words, any $\mathcal{K}_m$-colimit preserving functor from any other m-semiadditive ∞-category $\mathcal{C}$ factors essentially uniquely through r.

Now suppose that $\mathcal{D}$ admits $\mathcal{K}_m$-indexed **limits**. Then $\mathcal{D}^{\mathrm{op}}$ admits $\mathcal{K}_m$-indexed colimits and $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}^{\mathrm{op}})$ is the universal m-semiadditive ∞-category admitting a $\mathcal{K}_m$-colimit preserving functor to $\mathcal{D}^{\mathrm{op}}$. We now note that

$$(\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}^{\mathrm{op}}))^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathcal{K}_m}((\mathcal{S}_m^m)^{\mathrm{op}}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}),$$

where $\mathrm{Fun}^{\mathcal{K}_m}(-,-) \subseteq \mathrm{Fun}(-,-)$ denotes the full subcategory spanned by $\mathcal{K}_m$-limit preserving functors. It the follows that $\mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D})$ is the universal m-semiadditive ∞-category admitting a $\mathcal{K}_m$-limit preserving functor to $\mathcal{D}$. Our next goal is to relate the ∞-category $\mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D})$ with the theory of commutative monoid objects in $\mathcal{D}$.

Definition 5.9. Let $m \geq -1$ be an integer and let $\mathcal{D}$ be an ∞-category admitting $\mathcal{K}_m$-indexed limits. An **m-commutative monoid** in $\mathcal{D}$ is a functor $\mathcal{F} : \mathcal{S}_m^{m-1} \longrightarrow \mathcal{D}$ with the following property: for every $X \in \mathcal{S}_m^{m-1}$ the collection of maps $\mathcal{F}(\hat{i}_x) : \mathcal{F}(X) \longrightarrow \mathcal{F}(*)$ exhibits $\mathcal{F}(X)$ as the limit in $\mathcal{D}$ of the constant X-indexed diagram with value $\mathcal{F}(*)$. We will denote by $\mathrm{CMon}_m(\mathcal{D}) \subseteq \mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^{m-1}, \mathcal{D})$ the full subcategory spanned by the m-commutative monoids.

Example 5.10. If $m = -1$ then $\mathcal{S}_m^{m-1} = \mathcal{S}_{-1}^{-2} = \mathcal{S}_{-1}$ is the ∞-category of (-1)-truncated spaces and ordinary maps between them. In particular, we may identify $\mathcal{S}_{-1}$ with the category consisting of two objects $\varnothing, *$ and a unique non-identity morphism $\varnothing \longrightarrow *$. An ∞-category $\mathcal{D}$ admits $\mathcal{K}_{-1}$-indexed limits if and only if it admits a final object. A functor $\mathcal{S}_{-1} \longrightarrow \mathcal{D}$ is completely determined by the associated morphism $\mathcal{F}(\varnothing) \longrightarrow \mathcal{F}(*)$ in $\mathcal{D}$. By definition such a functor $\mathcal{F}$ is a (-1)-commutative monoid if and only if $\mathcal{F}(\varnothing)$ is a terminal object of $\mathcal{D}$. We may hence identify $\mathrm{CMon}_{-1}(\mathcal{D})$ with the full subcategory of the arrow category of $\mathcal{D}$ spanned by those arrows $A \longrightarrow B$ for which A is a final object. In particular, if we fix a particular final object $\star \in \mathcal{D}$ then we may form an equivalence $\mathrm{CMon}_{-1}(\mathcal{D}) \simeq \mathcal{D}_{\star/}$. In other words, we may identify $\mathrm{CMon}_{-1}(\mathcal{D})$ with the ∞-category of **pointed objects** $\mathcal{D}$.

Example 5.11. If $m = 0$ then we may identify $\mathcal{S}_m^{m-1} = \mathcal{S}_0^{-1}$ with the category whose objects are finite sets, and such that a morphism from a finite set A to a finite set B is a pair (C, f) where C is a subset of A and $f : C \longrightarrow B$ is a map. In particular, $\mathcal{S}_0^{-1}$ is equivalent to the nerve of a discrete category. By sending a finite set A to the pointed set $A_+ = A \coprod \{*\}$ and sending a map (C, f) to the map $f' : A_+ \longrightarrow B_+$ which restricts to f on C and sends $A \setminus C$ to the base point of B_+ we obtain an equivalence $\mathcal{S}_0^{-1} \simeq \mathcal{F}\mathrm{in}_*$, where $\mathcal{F}\mathrm{in}_*$ is the category of finite pointed sets. To say that an ∞-category $\mathcal{D}$ has $\mathcal{K}_0$-indexed limits is to say that $\mathcal{D}$ admits finite products. Unwinding the definitions we see that a functor $\mathcal{S}_0^{-1} \longrightarrow \mathcal{D}$ is a 0-commutative monoid object if and only if the corresponding functor $\mathcal{F}\mathrm{in}_* \longrightarrow \mathcal{D}$ is a commutative monoid object in the sense of [6, Definition 2.4.2.1], also known as an $\mathbb{E}_\infty$**-monoid**. When $\mathcal{D}$ is the ∞-category of spaces this notion of commutative monoids was first developed by Segal under the name **special Γ-spaces**.

Lemma 5.12. *Let $\mathcal{D}$ be an ∞-category which admits $\mathcal{K}_m$-indexed limits and let $\mathcal{F} : \mathcal{S}_m^m \longrightarrow \mathcal{D}$ be a functor. Then $\mathcal{F}$ preserves $\mathcal{K}_m$-indexed limits if and only if the restriction $\mathcal{F}|_{\mathcal{S}_m^{m-1}}$ is an m-commutative monoid object.*

Proof. Apply Corollary 2.12 and Lemma 2.8 to $(\mathcal{S}_m^m)^{\mathrm{op}} \simeq \mathcal{S}_m^m$ and $\mathcal{D}^{\mathrm{op}}$. □

Proposition 5.13. *Fix an $m \geq -1$ and let $\mathcal{D}$ be an ∞-category which admits $\mathcal{K}_m$-indexed limits. Then restriction along $\mathcal{S}_m^{m-1} \hookrightarrow \mathcal{S}_m^m$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}) \xrightarrow{\simeq} \mathrm{CMon}_n(\mathcal{D})$$

Proof. Let Cone_m be the right marked mapping cone of the natural inclusion $\iota : \mathcal{S}_m^{m-1} \hookrightarrow \mathcal{S}_m^m$ (see the discussion before Lemma 4.6) and let $\mathrm{Cone}_m \hookrightarrow \mathcal{M}^\natural \xrightarrow{p} \Delta^1$ be a factorization of the projection $\mathrm{Cone}_m \longrightarrow (\Delta^1)^\sharp$ into a trivial cofibration follows by a fibration in the Cartesian model structure over $(\Delta^1)^\sharp$. Let $\iota_0 : \mathcal{S}_m^m \hookrightarrow \mathcal{M} \times_{\Delta^1} \Delta^{\{0\}} \subseteq \mathcal{M}$ and $\iota_1 : \mathcal{S}_m^{m-1} \hookrightarrow \mathcal{M} \times_{\Delta^1} \Delta^{\{1\}} \subseteq \mathcal{M}$ be the corresponding inclusions, so that ι_0 and ι_1 exhibit $p : \mathcal{M} \longrightarrow \Delta^1$ as a correspondence from $\mathcal{S}_m^m$ to $\mathcal{S}_m^{m-1}$ which is the one associated to the functor $\iota : \mathcal{S}_m^{m-1} \hookrightarrow \mathcal{S}_m^m$.

Let $\mathrm{Fun}_0^\flat(\mathcal{M}^\natural, \mathcal{D}^\natural) \subseteq \mathrm{Fun}^\flat(\mathcal{M}^\natural, \mathcal{D}^\natural)$ and $\mathrm{Fun}_0^\flat(\mathrm{Cone}_m, \mathcal{D}^\natural) \subseteq \mathrm{Fun}^\flat(\mathrm{Cone}_m, \mathcal{D}^\natural)$ denote the respective full subcategories spanned by those marked functors whose restriction to $\mathcal{S}_m^{m-1}$ is an m-commutative monoid in $\mathcal{D}$. Since the inclusion of marked simplicial sets $\mathrm{Cone}_m \longrightarrow \mathcal{M}^\natural$ is a trivial cofibration in the Cartesian model structure over $(\Delta^1)^\sharp$ it follows that the restriction map $\mathrm{Fun}_0^\flat(\mathcal{M}^\natural, \mathcal{D}^\natural) \longrightarrow$

$\mathrm{Fun}_0^\flat(\mathrm{Cone}_m, \mathcal{D}^\natural)$ is a trivial Kan fibration, and by Proposition 4.7 and [5, Proposition 4.3.2.15] the restriction map $\mathrm{Fun}_0^\flat(\mathcal{M}^\natural, \mathcal{D}^\natural) \longrightarrow \mathrm{CMon}_m(\mathcal{D})$ is a trivial Kan fibration. We may hence deduce that the restriction map

$$\mathrm{Fun}_0^\flat(\mathrm{Cone}_m, \mathcal{D}^\natural) \xrightarrow{\simeq} \mathrm{CMon}_m(\mathcal{D})$$

is a an equivalence. On the other hand, by Lemma 5.12 the image of the restriction map $\mathrm{Fun}_0^\flat(\mathrm{Cone}_m, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{S}_m^m, \mathcal{D})$ consists of exactly those functors $\mathcal{S}_m^m \longrightarrow \mathcal{D}$ which preserves $\mathcal{K}_m$-indexed limits. Arguing as in the proof of Corollary 4.8 we may now conclude that the restriction map

$$\mathrm{Fun}^{\mathcal{K}_m}(\mathcal{S}_m^m, \mathcal{D}) \xrightarrow{\simeq} \mathrm{CMon}_n(\mathcal{D})$$

is an equivalence of ∞-categories, as desired. □

Corollary 5.14. *Let $\mathcal{D}$ be an ∞-category which admits $\mathcal{K}_m$-indexed limits. Then $\mathrm{CMon}_m(\mathcal{D})$ is m-semiadditive and the forgetful functor $\mathrm{CMon}_m(\mathcal{D}) \longrightarrow \mathcal{D}$ exhibits $\mathrm{CMon}_m(\mathcal{D})$ as universal among those m-semiadditive ∞-categories admitting a $\mathcal{K}_m$-limit preserving map to $\mathcal{D}$. In particular, $\mathcal{D}$ is m-semiadditive if and only if the forgetful functor $\mathrm{CMon}_m(\mathcal{D}) \longrightarrow \mathcal{D}$ is an equivalence.*

To get a feel for what these higher commutative monoids are, let us consider the example of the ∞-category $\mathcal{S}$ of spaces. Let $\mathcal{F} : \mathcal{S}_m^{m-1} \longrightarrow \mathcal{S}$ be an m-commutative monoid object and let us refer to $M = \mathcal{F}(*)$ as the **underlying space** of $\mathcal{F}$. We may then identify two types of morphisms in $\mathcal{S}_m^{m-1}$. The first type are morphisms of the form

$$Y \xleftarrow{\;f\;} X \xrightarrow{\;\mathrm{Id}\;} X$$

where f is $(m-1)$-truncated, and which we denote $\hat{f} : Y \longrightarrow X$ (see Definition 3.7). These morphisms help us to identify the spaces $\mathcal{F}(X)$: by definition, the collection of maps $\hat{i}_x : X \longrightarrow *$ exhibit $\mathcal{F}(X)$ as the limit of the constant X-indexed diagram with value $\mathcal{F}(*) = M$. In particular, we may identify $\mathcal{F}(X)$ with the mapping space $\mathrm{Map}_{\mathcal{S}}(X, M)$. Other morphisms of the form $\hat{f} : Y \longrightarrow X$ don't really give more information: if $f : X \longrightarrow Y$ is an $(m-1)$-truncated map then for every $x \in X$ we have $\hat{f} \circ \hat{i}_x = \hat{i}_{f(x)}$, and so the induced map

$$\hat{f}_* : \mathrm{Map}_{\mathcal{S}}(Y, M) \simeq \mathcal{F}(Y) \longrightarrow \mathcal{F}(X) \simeq \mathrm{Map}_{\mathcal{S}}(X, M)$$

is coincides with restriction along f. The second type of morphisms in $\mathcal{S}_m^{m-1}$ are the spans of the form

$$X \xleftarrow{\;\mathrm{Id}\;} X \xrightarrow{\;g\;} Y$$

where $g : X \longrightarrow Y$ is any map of finite m-truncated spaces. We can think of the associated map $g_* : \mathrm{Map}_{\mathcal{S}}(X, M) \longrightarrow \mathrm{Map}_{\mathcal{S}}(Y, M)$ as encoding the **structure** of M. Let X_y be homotopy fiber of g over $y \in Y$, equipped with its natural map $i_{X_y} : X_y \longrightarrow X$, and let $g_y : X_y \longrightarrow \{y\}$ be the constant map. Then $\hat{i}_y \circ g = g_y \circ \hat{i}_{X_y}$ and so for each $\varphi \in \mathrm{Map}_{\mathcal{S}}(X, M)$ the function $g_*(\varphi) \in \mathrm{Map}_{\mathcal{S}}(Y, M)$ maps the point y to the point $(g_y)_*(\varphi|_{X_y}) \in M$. We may hence think of the core algebraic structure of an m-commutative monoid as given by the maps $p_* : \mathrm{Map}(X, M) \longrightarrow M$ associated to constant maps $p : X \longrightarrow *$, while the other maps $g : X \longrightarrow Y$ specify various forms of compatibility. Informally speaking, the structure of being an m-commutative monoid means that for every m-truncated space X we can take an X-family $\{\varphi(x)\}_{x \in X}$ of points in M and "integrate" it to obtain a new point $\int_X \varphi := p_*(\varphi) \in M$. These operations are then required to satisfy various "Fubini-type" compatibility constraints when one is integrating over a space X which is fibered over another space Y. We note that when $m = 0$ the spaces involved are finite sets, and we obtain the usual notion of being able to sum a finite collection of points in a commutative monoid.

Examples 5.15.

1. For every space X, the ∞-groupoid $(\mathcal{S}_m \times_{\mathcal{S}} \mathcal{S}_{/X})^{\sim}$ classifying finite m-truncated spaces equipped with a map to X is naturally an m-commutative monoid. This is the free m-commutative monoid generated from X.

2. Any Q-vector space is an m-commutative monoid (in the category of $\mathbb{Q}$-vector spaces). Indeed, if X is a finite m-truncated space then the limit $\lim_X V$ of the constant X-indexed diagram on V is just the vector space of functions $f : \pi_0(X) \longrightarrow V$. To such an f we may associate the vector

$$\sum_{X_0 \in \pi_0(X)} \chi(X_0) f(X_0) \in V$$

   where $\chi(X_0) = \frac{\prod_{i \geq 0} |\pi_{2i}(X_0)|}{\prod_{i \geq 0} |\pi_{2i+1}(X_0)|}$ is the "homotopy cardinality" of X_0. This yields a structure of an m-commutative monoid on V.

3. More generally, if $\mathcal{D}$ is an m-semiadditive ∞-category then any object in $\mathcal{D}$ is carries a canonical m-commutative monoid structure and for each $X, Y \in \mathcal{D}$ the mapping space $\mathrm{Map}_{\mathcal{D}}(X, Y)$ is canonically an m-commutative monoid in spaces. For example, by the main result of [7], for any two $K(n)$-local spectra X, Y the mapping space from X to Y is an m-commutative monoid in spaces.

4. If $\mathcal{C}$ is an ∞-category which admits $\mathcal{K}_m$-indexed colimits (resp. $\mathcal{K}_m$-indexed limits) then $\mathcal{C}$ carries the coCartesian (resp. Cartesian) m-commutative monoid structure in Cat_∞, and its maximal ∞-groupoid is an m-commutative monoid in spaces.

Let us now discuss the role of m-commutative monoids in the setting of m-semiadditive **presentable** ∞-categories.

Lemma 5.16. *Let $\mathcal{D}$ be a presentable ∞-category. Then* $\mathrm{CMon}_m(\mathcal{D})$ *is presentable and the forgetful functor* $\mathrm{CMon}_m(\mathcal{D}) \longrightarrow \mathcal{D}$ *is conservative, accessible and preserves all limits.*

Proof. Let $\mathrm{CMon}_m(\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{S}_m^{m-1}, \mathcal{D})$ be the natural inclusion. Then $\mathrm{CMon}_m(\mathcal{D})$ is closed under limits in $\mathrm{Fun}(\mathcal{S}_m^{m-1}, \mathcal{D})$ and under κ-filtered colimits for any κ such that the simplicial sets in $\mathcal{K}_m$ are κ-small. Since $\mathrm{Fun}(\mathcal{S}_m^{m-1}, \mathcal{D})$ is presentable it now follows from [5, Corollary 5.5.7.3] that $\mathrm{CMon}_m(\mathcal{D})$ is presentable and the inclusion $\mathrm{CMon}_m(\mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{S}_m^{m-1}, \mathcal{D})$ is accessible. This, in turn, implies that the composition $\mathrm{CMon}_m(\mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{S}_m^{m-1}, \mathcal{D}) \xrightarrow{\mathrm{ev}_*} \mathcal{D}$ is accessible and preserves limits. Finally, to show that $\mathrm{CMon}_n(\mathcal{D}) \longrightarrow \mathcal{D}$ is conservative it is enough to note that if $f : M \longrightarrow M'$ is a map in $\mathrm{CMon}_m(\mathcal{D})$ such that $f_* : M(*) \longrightarrow M'(*)$ is an equivalence in $\mathcal{D}$ then for any $X \in \mathcal{S}_m^{m-1}$ the induced map $f_X : M(X) \simeq \lim_X M(*) \longrightarrow \lim_X M'(*)$ is an equivalence and hence f is an equivalence. ☐

When $\mathcal{D}$ is presentable, Lemma 5.16 and the adjoint functor theorem ([5]) imply that the forgetful functor $\mathrm{CMon}_n(\mathcal{D}) \longrightarrow \mathcal{D}$ admits a **left adjoint** $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{CMon}_m(\mathcal{D})$, which can be considered as the **free m-commutative monoid** functor. Given two presentable ∞-categories $\mathcal{C}, \mathcal{D}$ let us denote by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ the ∞-category of **left functors** from $\mathcal{C}$ to $\mathcal{D}$ (i.e. those functors which admit right adjoints) and by $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D})$ the ∞-category of **right functors** from $\mathcal{C}$ to $\mathcal{D}$.

Corollary 5.17. *Let $\mathcal{D}$ be a presentable ∞-category and let $\mathcal{E}$ be a presentable m-semiadditive ∞-category. Then post-composition with the forgetful functor* $\mathrm{CMon}(\mathcal{D}) \longrightarrow \mathcal{D}$ *induces an equivalence*

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathrm{CMon}(\mathcal{D})) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathcal{D}).$$

Dually pre-composition with $\mathcal{F} : \mathcal{D} \longrightarrow \mathrm{CMon}_m(\mathcal{D})$ *induces an equivalence*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{CMon}(\mathcal{D}), \mathcal{E}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{E}).$$

In particular, the functor $\mathcal{F}$ exhibits $\mathrm{CMon}(\mathcal{D})$ *as the free presentable m-semiadditive ∞-category generated from $\mathcal{D}$.*

Proof. Let us prove the first claim (the second then follows by the equivalence $\mathrm{Fun}^{\mathrm{R}}(-,-) \simeq \mathrm{Fun}^{\mathrm{L}}(-,-)$ which associates to every right functor its left adjoint). By Corollary 5.14 it will suffice to show that if $\mathcal{F} : \mathcal{E} \longrightarrow \mathrm{CMon}_n(\mathcal{D})$ is a functor that preserves $\mathcal{K}_m$-indexed limits then $\mathcal{F}$ belongs to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathcal{D})$ if and only if $\mathrm{ev}_* \circ \mathcal{F} : \mathcal{E} \longrightarrow \mathcal{D}$ belongs to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{E}, \mathrm{CMon}(\mathcal{D}))$. By the adjoint functor theorem right functors between presentable ∞-categories are exactly the limit preserving functors which are also accessible, i.e., preserve sufficiently filtered colimits. The result is now follows from Lemma 5.16 which asserts that ev_* preserves limits and sufficiently filtered colimits, and also detects them since it is conservative. ☐

Given an ∞-category $\mathcal{C}$, let $\mathcal{P}_{\mathcal{K}_m}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denote the full subcategory consisting of those presheaves which send $\mathcal{K}_m$-indexed colimits in $\mathcal{C}$ to limits of spaces. Then $\mathcal{P}_{\mathcal{K}_m}(\mathcal{C})$ is presentable and is an accessible localization of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ (choose an infinite cardinal κ such that all $\mathcal{K}_m$-colimit diagrams in $\mathcal{C}$ are κ-small and use [5, Corollary 5.5.7.3]). Let Pr^{L} denote the ∞-category of presentable ∞-categories and left functors between them. Identifying Pr^{L} as a full subcategory of cocomplete ∞-categories and colimit preserving functors and using [5, Corollary 5.3.6.10] we may conclude that the functor

$$\mathcal{P}_{\mathcal{K}_m} : \mathrm{Cat}_{\mathcal{K}_m} \longrightarrow \mathrm{Pr}^{\mathrm{L}} \tag{21}$$

is left adjoint to the forgetful functor $\mathrm{Pr}^{\mathrm{L}} \longrightarrow \mathcal{C}$. In particular, we may consider $\mathcal{P}_{\mathcal{K}_m}(\mathcal{C})$ as the free presentable ∞-category generated from $\mathcal{C}$. We hence obtain two universal characterizations of the ∞-category $\mathrm{CMon}_m(\mathcal{S})$. On the one hand, by Corollary 5.17 we may identify $\mathrm{CMon}_m(\mathcal{S})$ as the free presentable m-semiadditive ∞-category generated from the presentable ∞-category $\mathcal{S}$. On the other hand, since $\mathcal{S}_m^m \simeq (\mathcal{S}_m^m)^{\mathrm{op}}$ we may interpret Proposition 5.13 as identifying $\mathrm{CMon}_m(\mathcal{S}) \simeq \mathcal{P}_{\mathcal{K}_m}(\mathcal{S}_m^m)$ as the free presentable ∞-category generated from the $\mathcal{K}_m$-cocomplete ∞-category $\mathcal{S}_m^m$. Furthermore, by [6, Proposition 4.8.1.14] and [6, Remark 4.8.1.8] the functor (21) is symmetric monoidal (where Pr^{L} is endowed with the symmetric monoidal structure inherited from that of cocomplete ∞-categories). We may then deduce the following:

Corollary 5.18. *The ∞-category $\mathrm{CMon}_m(\mathcal{S})$ is an idempotent commutative algebra object in Pr^{L}. In particular, the monoidal product $\mathrm{CMon}_m(\mathcal{S}) \otimes \mathrm{CMon}_m(\mathcal{S}) \longrightarrow \mathrm{CMon}_m(\mathcal{S})$ is an equivalence.*

Lemma 5.19. *Let $\mathcal{D}$ be a presentable ∞-category. Then $\mathcal{D}$ carries an action of $\mathrm{CMon}_m(\mathcal{S})$ (with respect to the symmetric monoidal structure of Pr^{L}) if and only if $\mathcal{D}$ is m-semiadditive.*

Proof. By [6, Remark 4.8.1.17] the data of an action of $\mathrm{CMon}(\mathcal{S})$ on a presentable ∞-category $\mathcal{D}$ is equivalent to the data of a monoidal colimit preserving functor $\mathrm{CMon}(\mathcal{S}) \longrightarrow \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{D})$, which since (21) is monoidal, is equivalent to the data of a $\mathcal{K}_m$-colimit preserving monoidal functor $\mathcal{S}_m^m \longrightarrow \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{D})$, i.e., to an action of $\mathcal{S}_m^m$ on $\mathcal{D}$ which preserves $\mathcal{K}_m$-colimits in $\mathcal{S}_m^m$ and all colimits in $\mathcal{D}$. We now observe that any action of $\mathcal{S}_m^m$ on $\mathcal{D}$ which preserves $\mathcal{K}_m$-colimits in $\mathcal{S}_m^m$ will automatically preserve all colimits which exist in $\mathcal{D}$, since the object $X \in \mathcal{S}_m^m$ will necessarily acts as an X-indexed colimit of the identity functor. The desired result now follows from Corollary 5.2. □

Arguing as in the proof of Lemma 5.1 we may now conclude the following:

Corollary 5.20. *The forgetful functor $\mathrm{Mod}_{\mathrm{CMon}_m(\mathcal{S})}(\mathrm{Pr}^{\mathrm{L}}) \longrightarrow \mathrm{Pr}^{\mathrm{L}}$ is fully-faithful and its essential image consists of the m-semiadditive presentable ∞-categories.*

Remark 5.21. The statements of Corollary 5.20 and 5.18 are strongly related. In fact, under mild conditions on a symmetric monoidal ∞-category $\mathcal{C}$, the property of $A \in \mathrm{CAlg}(\mathcal{C})$ being idempotent is equivalent to the forgetful functor $\mathrm{Mod}_A(\mathcal{C}) \longrightarrow \mathcal{C}$ being fully-faithful. Idempotent commutative algebra objects in Pr^{L} feature in some recent investigations of T. Schlank ([9]), where they are called **modes**. Informally speaking, modes describe aspects of presentable ∞-categories which are both a property and a structures, such as being pointed (the mode of pointed spaces), being semiadditive (the mode of $\mathbb{E}_\infty$-spaces) being stable (the mode of spectra), being an $(n,1)$-category (the mode of n-truncated spaces), and more. Corollary 5.18 then adds a new infinite family of modes: the mode of m-commutative monoids in spaces for every m, which is associated to the property of being m-semiadditive.

Let us now consider the case where we replace $\mathcal{S}$ be the category Cat_∞ of ∞-categories. As above, we may informally consider an m-commutative monoid structure on an ∞-category $\mathcal{M}$ as giving us a rule for taking an X-indexed family of objects of $\mathcal{M}$ (where X is an m-truncated finite space) and producing a new object of $\mathcal{M}$. Two immediate examples come to mind: if $\mathcal{M}$ is an ∞-category admitting $\mathcal{K}_m$-indexed colimits then we may form the colimit of any X-indexed family of objects in $\mathcal{M}$. On the other hand, if $\mathcal{M}$ admits $\mathcal{K}_m$-indexed limits then we may form the limit of any such family. One might hence expect that if $\mathcal{M}$ admits $\mathcal{K}_m$-indexed colimits (resp. limits) then there should be a canonical m-commutative monoid structure on $\mathcal{M}$, which can be called the **coCartesian** (resp. **Cartesian**) m-commutative monoid structure. To show that these structures indeed exist we shall prove the following theorem:

Theorem 5.22.

1. *The forgetful functor* $\mathrm{CMon}_m(\mathrm{Cat}_{\mathcal{K}_m}) \longrightarrow \mathrm{Cat}_{\mathcal{K}_m}$ *is an equivalence. In other words, every object in* $\mathrm{Cat}_{\mathcal{K}_m}$ *admits an essentially unique m-commutative monoid structure.*

2. *The forgetful functor* $\mathrm{CMon}_m(\mathrm{Cat}^{\mathcal{K}_m}) \longrightarrow \mathrm{Cat}^{\mathcal{K}_m}$ *is an equivalence. In other words, every object in* $\mathrm{Cat}^{\mathcal{K}_m}$ *admits an essentially unique m-commutative monoid structure.*

Remark 5.23. If $\mathcal{M}$ is an ∞-category which admits $\mathcal{K}_m$-indexed colimits, then we may consider it as belonging to either Cat_∞ or $\mathrm{Cat}_{\mathcal{K}_m}$. Since the faithful inclusion $\mathrm{Cat}_{\mathcal{K}_m} \hookrightarrow \mathrm{Cat}_\infty$ preserves limits we obtain a natural map

$$\mathrm{CMon}_m(\mathrm{Cat}_{\mathcal{K}_m}) \times_{\mathrm{Cat}_{\mathcal{K}_m}} \{\mathcal{M}\} \longrightarrow \mathrm{CMon}_m(\mathrm{Cat}_\infty) \times_{\mathrm{Cat}_\infty} \{\mathcal{M}\} \qquad (22)$$

where the left hand side is contractible by Theorem 5.22, and the right hand side is an ∞-groupoid which can be considered as the space of m-commutative monoid structures on $\mathcal{M}$. The point in $\mathrm{CMon}_m(\mathrm{Cat}_\infty) \times_{\mathrm{Cat}_\infty} \{\mathcal{M}\}$ determined by (22) can be considered as identifying the coCartesian m-commutative monoid structure on $\mathcal{M}$. Similarly, if $\mathcal{M}$ admits $\mathcal{K}_m$-indexed limits then the image of the map

$$\mathrm{CMon}_m(\mathrm{Cat}^{\mathcal{K}_m}) \times_{\mathrm{Cat}^{\mathcal{K}_m}} \{\mathcal{M}\} \longrightarrow \mathrm{CMon}_m(\mathrm{Cat}_\infty) \times_{\mathrm{Cat}_\infty} \{\mathcal{M}\}$$

identifies the Cartesian m-commutative monoid structure.

Remark 5.24. The category $\mathcal{S}^m_m$ admits $\mathcal{K}_m$-indexed limits and colimits, but also carries an m-commutative monoid structure which is neither Cartesian nor coCartesian. To see this, observe that the operation $\mathcal{C} \mapsto \mathrm{Span}(\mathcal{C})$ which associates to any ∞-category $\mathcal{C}$ with finite limits its span category determines a limit preserving functor $\mathrm{Span} : \mathrm{Cat}^{\mathcal{K}_{\mathrm{fin}}} \longrightarrow \mathrm{Cat}^{\mathcal{K}_{\mathrm{fin}}}$, where $\mathcal{K}_{\mathrm{fin}}$ denotes the collection of simplicial sets with finitely many non-degenerate simplices. We then have an induced functor

$$\mathrm{Span}_* : \mathrm{CMon}_m(\mathrm{Cat}^{\mathcal{K}_{\mathrm{fin}}}) \longrightarrow \mathrm{CMon}_m(\mathrm{Cat}^{\mathcal{K}_{\mathrm{fin}}}).$$

Since the ∞-category $\mathcal{S}_m$ has both $\mathcal{K}_m$-indexed limits and $\mathcal{K}_m$-indexed colimits it carries both a Cartesian m-commutative monoid structure and a coCartesian m-commutative monoid structure. Applying the functor Span_* we obtain two m-commutative monoid structures on $\mathrm{Span}(\mathcal{S}_m) = \mathcal{S}^m_m$. The coCartesian m-commutative monoid structure of $\mathcal{S}_m$ induces an m-commutative monoid structure on $\mathcal{S}^m_m$ which is both coCartesian and Cartesian. The Cartesian m-commutative monoid structure on $\mathcal{S}_m$, however, induces a **different** m-commutative monoid structure on $\mathcal{S}^m_m$, which is neither Cartesian nor coCartesian. The restriction of this structure to $\mathcal{S}^{-1}_0$ determines a symmetric monoidal structure on $\mathcal{S}^m_m$ which is the one we've been considering throughout this paper.

As $\mathrm{Cat}^{\mathcal{K}_m} \simeq \mathrm{Cat}_{\mathcal{K}_m}$ by the functor which sends $\mathcal{C}$ to $\mathcal{C}^{\mathrm{op}}$, Theorem 5.22 will follows from Theorem 4.1 and Proposition 5.13 once we prove the following result:

Proposition 5.25. *The ∞-category $\mathrm{Cat}_{\mathcal{K}_m}$ is m-semiadditive.*

The proof of Proposition 5.25 we will be given below. Since $\mathrm{Cat}_{\mathcal{K}_m}$ has all limits it follows that $\mathrm{Cat}^{\mathrm{op}}_{\mathcal{K}_m}$ has all colimits and hence admits a canonical action of the ∞-category of spaces $\mathcal{S}$ which preserves colimits in each variable separately. Dually, $\mathrm{Cat}_{\mathcal{K}_m}$ admits an action of $\mathcal{S}^{\mathrm{op}}$ which preserves limits in each variable separately. Given a space $X \in \mathcal{S}^{\mathrm{op}}$ this latter action $[X] : \mathrm{Cat}_{\mathcal{K}_m} \longrightarrow \mathrm{Cat}_{\mathcal{K}_m}$ sends $\mathcal{M}$ to $\mathrm{Fun}(X, \mathcal{M}) = \lim_X \mathcal{M}$ and sends $f : X \longrightarrow Y$ to the restriction functor $\mathrm{Fun}(Y, \mathcal{M}) \longrightarrow \mathrm{Fun}(X, \mathcal{M})$. For our purposes we will only be interested in the action of the full subcategory $\mathcal{S}^{\mathrm{op}}_m \subseteq \mathcal{S}^{\mathrm{op}}$ on $\mathrm{Cat}_{\mathcal{K}_m}$. Given a span φ of the form

$$X \xleftarrow{\;g\;} Z \xrightarrow{\;f\;} Y \tag{23}$$

where X, Y, Z are finite m-truncated spaces we will denote by $T_\varphi : [Y] \longrightarrow [X]$ the natural transformation given by the composition

$$T_\varphi(\mathcal{M}) : \mathrm{Fun}(Y, \mathcal{M}) \xrightarrow{f^*} \mathrm{Fun}(Z, \mathcal{M}) \xrightarrow{g_!} \mathrm{Fun}(X, \mathcal{M})$$

where f^* denotes the restriction functor along f and $g_!$ denotes the left Kan extension functor, whose existence is insured by the fact that $\mathcal{M}$ has $\mathcal{K}_m$-indexed colimits. If φ is a span as in (23) then we will denote by $\hat{\varphi} : Y \longrightarrow X$ the **dual span** $Y \xleftarrow{f} Z \xrightarrow{g} X$.

Lemma 5.26. *Let* $\mathrm{tr}_X : X \times X \longrightarrow *$ *be the span of Definition 3.9. Then the natural transformation*

$$T_{\mathrm{tr}_X} : \mathrm{Id} \longrightarrow [X \times X] \simeq [X] \circ [X]$$

exhibits $[X]$ *as a self-adjoint functor. Furthermore, under this self-adjunction the natural transformation* $T_\varphi : [X] \Rightarrow [Y]$ *associated to a span* $\varphi : X \longrightarrow Y$ *is dual to the natural transformation* $T_{\hat{\varphi}} : [Y] \Rightarrow [X]$ *associated with the dual span* $\hat{\varphi}$*.*

Proof. The Beck-Chevalley condition for pullbacks and left Kan extensions (see [7, Proposition 4.3.3]) implies in particular that the association $\varphi \mapsto T_\varphi$ respects composition of spans up to homotopy. Both claims now follow from the fact that $\mathrm{tr} : X \times X \longrightarrow *$ exhibit X as self-dual in the monoidal ∞-category $\mathcal{S}^m_m$ and that under this self duality the dual morphism of φ is $\hat{\varphi}$. □

Proof of Proposition 5.25. Arguing by induction, let us assume that $\mathrm{Cat}_{\mathcal{K}_m}$ is $(m'-1)$-semiadditive for some $-1 \le m' \le m$ and show that it is in fact m'-semiadditive. By Corollary 5.2 (applied to $\mathrm{Cat}^{\mathrm{op}}_{\mathcal{K}_m}$) we may extend the $(\mathcal{S}_{m'})^{\mathrm{op}}$-action on $\mathrm{Cat}_{\mathcal{K}_m}$ described above to an $(\mathcal{S}^{m'-1}_{m'})^{\mathrm{op}}$-action which preserves $\mathcal{K}_{m'}$-indexed limits in each variable separately. Applying Lemma 3.6 and Lemma 3.8 to $\mathrm{Cat}^{\mathrm{op}}_{\mathcal{K}_m}$ we may deduce that for every morphism of the form $Y \xleftarrow{q} X$ in $\mathcal{S}^{\mathrm{op}}_{m',m'-1} \subseteq \mathcal{S}^{m'-1}_{m'}$ the induced transformation $[g](\mathcal{M}) : [X](M) \simeq \mathrm{Fun}(X, \mathcal{M}) \longrightarrow \mathrm{Fun}(Y, \mathcal{M}) \simeq [Y](M)$ is given by the formation of left Kan extensions. Applying now Lemma 5.26 we may conclude that for every $X \in \mathcal{S}^{m'-1}_{m'}$, the natural transformation $[\mathrm{tr}_X] : \mathrm{Id} \Rightarrow [X] \circ [X]$ exhibits $[X]$ as a self-adjoint functor. By (the dual version of) Proposition 3.10 the ∞-category $\mathrm{Cat}_{\mathcal{K}_m}$ is m'-semiadditive, as desired. □

Remark 5.27. Proposition 5.25 implies in particular that if X is an m-truncated space and $\mathcal{M} \in \mathrm{Cat}_{\mathcal{K}_m}$ is an ∞-category admitting $\mathcal{K}_m$-indexed colimits then $\mathrm{Fun}(X, \mathcal{M}) \simeq \lim_X \mathcal{M}$ is also a model for the **colimit** of the constant X-indexed diagram with value $\mathcal{M}$. Using Lemma 5.26 we can make this claim more precise: for any $\mathcal{M} \in \mathrm{Cat}_{\mathcal{K}_m}$ and $X \in \mathcal{S}_m$, the collection of left Kan extension functors $(i_x)_! : \mathcal{M} \longrightarrow \mathrm{Fun}(X, \mathcal{M})$ exhibits $\mathrm{Fun}(X, \mathcal{M})$ as the colimit of the constant X-indexed diagram with value $\mathcal{M}$.

5.3 Decorated spans

Theorem 4.1 identifies $\mathcal{S}^m_m$ as the free m-semiadditive ∞-category generated by a single object. In this section we will show how to bootstrap Theorem 4.1 in order to obtain a description of the free m-semiadditive ∞-category generated by an arbitrary small ∞-category $\mathcal{C}$.

Let $q : \mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}_m$ be a Cartesian fibration classifying the functor $X \mapsto \mathrm{Fun}(X, \mathcal{C})$. We may informally describe objects in $\mathcal{S}_m(\mathcal{C})$ as pairs (X, f) where X is a finite m-truncated space and $f : X \longrightarrow C$ is a functor. A map $(X, f) \longrightarrow (Y, g)$ in $\mathcal{S}_m(\mathcal{C})$ can be described in these terms as a pair (φ, T) where $\varphi : X \longrightarrow Y$ is a map of spaces and $T : f \Rightarrow g \circ \varphi$ is a natural transformation, i.e., a map in $\mathrm{Fun}(X, \mathcal{C})$. In particular, a morphism (φ, T) corresponds to a q-Cartesian edge of $\mathcal{S}_m(\mathcal{C})$ if and only T is an equivalence in $\mathrm{Fun}(X, \mathcal{C})$. Now since $\mathcal{S}_m$ admits pullbacks it follows that $\mathcal{S}_m(\mathcal{C})$ admits pullbacks of diagram of the form $p : \Delta^1 \coprod_{\Delta^{\{1\}}} \Delta^1 \longrightarrow \mathcal{S}_m(\mathcal{C})$ such that $p|\Delta^1$ is q-Cartesian. Let $\mathcal{S}_m^{\mathrm{coc}}(\mathcal{C}) \subseteq \mathcal{S}_m(\mathcal{C})$ denote the subcategory containing all objects and whose mapping spaces are the subspaces spanned by q-Cartesian edges. Then $\mathcal{S}_m^{\mathrm{coc}}$ determines a weak coWaldhausen structure on $\mathcal{C}$ (see §2.1) and we may consider the associated span ∞-category

$$\mathcal{S}_m^m(\mathcal{C}) \coloneqq \mathrm{Span}(\mathcal{S}_m(\mathcal{C}), \mathcal{S}_m^{\mathrm{coc}}(\mathcal{C})).$$

By Remark 2.4 we may identify the objects of $\mathcal{S}_m^m(\mathcal{C})$ with the objects of $\mathcal{S}_m(\mathcal{C})$ and the mapping space in $\mathcal{S}_m^m(\mathcal{C})$ from (X, f) to (Y, g) with the classifying space of spans

$$\begin{array}{ccccc} & & (Z,h) & & \\ & \swarrow^{(\varphi,T)} & & \searrow^{(\psi,S)} & \\ (X,f) & & & & (Y,g) \end{array} \tag{24}$$

such that (φ, T) is q-Cartesian (i.e., such that T is an equivalence in $\mathrm{Fun}(Z, \mathcal{C})$).

The fiber of the Cartesian fibration $\mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}_m$ over $* \in \mathcal{S}_m$ is equivalent to $\mathrm{Fun}(*, \mathcal{C}) \simeq \mathcal{C}$ and we may fix an equivalence $\mathcal{C} \xrightarrow{\simeq} \mathcal{S}_m(\mathcal{C}) \times_{\mathcal{S}_m} \{*\}$. We will denote by $\iota : C \longrightarrow \mathcal{S}_m(\mathcal{C})$ the composition of this equivalence with the inclusion of the fiber over $* \in \mathcal{S}_m$. Let

$$U \subseteq \mathcal{S}_m(\mathcal{C})^{\Delta^1} \times_{\mathcal{S}_m(\mathcal{C})^{\Delta^{\{1\}}}} \mathcal{C}$$

be the full subcategory spanned by those arrows $(X, f) \longrightarrow \iota(C)$ which are π-Cartesian. We may informally describe objects of U as tuples (X, f, C, T) where T is a natural equivalence from $f : X \longrightarrow \mathcal{C}$ to the constant functor $\overline{C} : X \longrightarrow \mathcal{C}$ with value C. Since $\mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}_m$ is a Cartesian fibration and $*$ is terminal in $\mathcal{S}_m$ it follows that the maps $U \xrightarrow{\simeq} \mathcal{S}_m^{\Delta^1} \times_{\mathcal{S}_m^{\Delta^{\{1\}}}} \mathcal{C} \xrightarrow{\simeq} \mathcal{S}_m \times \mathcal{C}$ are trivial Kan fibrations (whose composition can informally be described as sending (X, f, C, T) to (X, C)) and hence there exists an essentially unique section $s : \mathcal{S}_m \times \mathcal{C} \longrightarrow U$. We will denote by

$$\iota' : \mathcal{S}_m \times \mathcal{C} \longrightarrow \mathcal{S}_m(\mathcal{C})$$

the composition of s with the projection $U \longrightarrow \mathcal{S}_m(\mathcal{C})$. We may informally describe ι' by the formula $\iota'(X, C) = (X, \overline{C})$, where $\overline{C} : X \longrightarrow \mathcal{C}$ denotes the constant functor with value C.

Recall that we denote by $\mathcal{C}^{\sim} \subseteq \mathcal{C}$ the maximal subgroupoid of $\mathcal{C}$. Since $\mathcal{S}_m$ admits pullbacks it follows that $\mathcal{S}_m \times \mathcal{C}$ admits pullbacks of diagram of the form

$p : \Delta^1 \coprod_{\Delta^{\{1\}}} \Delta^1 \longrightarrow \mathcal{S}_m \times \mathcal{C}$ such that $p|\Delta^1$ belongs to $\mathcal{S}_m \times \mathcal{C}^{\sim}$. Since the functor $\iota' : \mathcal{S}_m \times \mathcal{C} \longrightarrow \mathcal{S}_m(\mathcal{C})$ maps $\mathcal{S}_m \times \mathcal{C}^{\sim}$ to $\mathcal{S}_m^{\mathrm{coc}}(\mathcal{C})$ we obtain an induced functor of span ∞-categories

$$\iota'' : \mathcal{S}_m^m \times \mathcal{C} \simeq \mathcal{S}_m^m \times \mathrm{Span}(\mathcal{C}, \mathcal{C}^{\sim}) \simeq \mathrm{Span}(\mathcal{S}_m \times \mathcal{C}, \mathcal{S}_m \times \mathcal{C}^{\sim}) \longrightarrow \mathrm{Span}(\mathcal{S}_m(\mathcal{C}), \mathcal{S}_m^{\mathrm{coc}}(\mathcal{C})) = \mathcal{S}_m^m(\mathcal{C}).$$

We may informally describe the functor $\iota'' : \mathcal{S}_m^m \times \mathcal{C} \longrightarrow \mathcal{S}_m^m(\mathcal{C})$ as the functor which sends the object (X, C) to the object $(X, \overline{C})$ and a pair $(X \longleftarrow Z \longrightarrow Y, f : C \longrightarrow D)$ of a morphism in $\mathcal{S}_m^m$ and a morphism in $\mathcal{C}$ to the span

$$\begin{array}{ccc} & (Z, \overline{C}) & \\ {}^{(\varphi, \mathrm{Id})}\swarrow & & \searrow^{(\psi, f)} \\ (X, \overline{C}) & & (Y, i_D) \end{array}$$

Our goal in this section is to prove the following characterization of the above constructions in terms of suitable universal properties:

Theorem 5.28.

1. *The functor $\iota : \mathcal{C} \longrightarrow \mathcal{S}_m(\mathcal{C})$ exhibits $\mathcal{S}_m(\mathcal{C})$ as the free ∞-category with $\mathcal{K}_m$-indexed colimits generated from $\mathcal{C}$.*
2. *The composed functor $\mathcal{C} \longrightarrow \mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}_m^m(\mathcal{C})$ exhibits $\mathcal{S}_m^m(\mathcal{C})$ as the free m-semiadditive ∞-category generated from $\mathcal{C}$.*
3. *The functor $\mathcal{S}_m^m \times \mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}_m^m(\mathcal{C})$ exhibits $\mathcal{S}_m^m(\mathcal{C})$ as the tensor product $\mathcal{S}_m^m \otimes_{\mathcal{K}_m} \mathcal{S}_m(\mathcal{C})$ in $\mathrm{Cat}_{\mathcal{K}_m}$.*

The rest of this section is devoted to the proof of Theorem 5.28, which is covered by Corollaries 5.32, 5.35 and 5.37. We begin with the following general lemma about colimits in Cartesian fibrations.

Lemma 5.29. *Let K be a Kan complex and let $\overline{p} : K^{\triangleright} \longrightarrow \mathcal{C}$ be a diagram taking values in an ∞-category $\mathcal{C}$. Let $\pi : \mathcal{D} \longrightarrow \mathcal{C}$ be a Cartesian fibration classified by a functor $\chi : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_{\infty}$ which sends $\overline{p}$ to a limit diagram in Cat_{∞}. Then a lift $\overline{q} : K^{\triangleright} \longrightarrow \mathcal{D}$ of $\overline{p}$ is a π-colimit diagram in $\mathcal{D}$ if and only if $\overline{q}$ sends every morphism in $K^{\triangleright}$ to a π-Cartesian edge.*

Proof. Let $p = \overline{p}|_K$ and $q = \overline{q}|_K$ and consider the induced map $\pi_* : \mathcal{D}_{q/} \longrightarrow \mathcal{C}_{p/}$. By definition, $\overline{q}$ is a π-colimit diagram if and only if the object $\overline{q} \in \mathcal{D}_{q/}$ is π_*-initial. By (the dual of) [5, Proposition 2.4.3.2] the map π_* is a Cartesian fibration, and hence by [5, Corollary 4.3.1.16] we have that $\overline{q}$ is π_*-initial if and only if it is initial when considered as an object of $\mathcal{D}_{q/} \times_{\mathcal{C}_{p/}} \{\overline{p}\}$. Using the natural equivalence (see [5, §4.2.1])

$$\mathcal{D}_{q/} \times_{\mathcal{C}_{p/}} \{\overline{p}\} \simeq \mathcal{D}^{q/} \times_{\mathcal{C}^{p/}} \{\overline{p}\} \simeq \mathrm{Fun}(K^{\triangleright}, \mathcal{D}) \times_{\mathrm{Fun}(K, \mathcal{D}) \times \mathrm{Fun}(K^{\triangleright}, \mathcal{C})} \{(q, \overline{p})\} \quad (25)$$

it will suffice to show that $\overline{q}: K^{\triangleright} \longrightarrow \mathcal{D}$ is initial when considered as an object of the RHS of (25) if and only it sends all edges to π-Cartesian edges. Let $L = (K \times \Delta^1)^{\triangleright}$ and let $L_1, L_2 \subseteq L$ be the subsimplicial sets given by

$$L_1 := (K \times \Delta^{\{1\}})^{\triangleright} \coprod_{K \times \Delta^{\{1\}}} K \times \Delta^1 \hookrightarrow (K \times \Delta^1)^{\triangleright} \hookleftarrow (K \times \Delta^{\{0\}})^{\triangleright} := L_2 \tag{26}$$

Let $\overline{L}$ be the marked simplicial set whose underlying simplicial set is L and the marked edges are those which are contained in $(K \times \Delta^{\{1\}})^{\triangleright}$. Similarly, let $\overline{L}_1$ and $\overline{L}_2$ be the marked simplicial sets whose underlying simplicial sets are L_1 and L_2 respectively and whose markings are inherited from L. In particular, $\overline{L}_2 = L_2^{\flat} \cong K^{\triangleright}$. We now claim that the inclusions $\overline{L}_1 \hookrightarrow \overline{L}$ and $\overline{L}_2 \hookrightarrow \overline{L}$ are marked anodyne. For $\overline{L}_1$ this follows from the fact that $L_1 \hookrightarrow L$ is inner anodyne by [5, Lemma 2.1.2.3] and all the marked edges of L are contained in L_1. For $\overline{L}_2$ we can write the inclusion $K \times \Delta^{\{0\}} \hookrightarrow K \times \Delta^1$ as a transfinite composition of pushouts along $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 0$, yielding a description of $\overline{L}_2 \hookrightarrow \overline{L}$ as a transfinite composition of pushouts along marked maps of the form $(\Lambda^{n+1}_{n+1}, (\Lambda^{n+1}_{n+1})_1 \cap \Delta^{\{n,n+1\}}) \hookrightarrow (\Delta^{n+1}, \Delta^{\{n,n+1\}})$ which are marked anodyne by definition. Let $\mathcal{D}^{\natural}$ be the marked simplicial set whose underlying simplicial set is $\mathcal{D}$ and the marked edges are the π-Cartesian edges. Then $\mathcal{D}^{\natural}$ is fibrant in the Cartesian model structure over $\mathcal{C}$ and so we obtain a zig-zag of trivial Kan fibrations

$$\begin{array}{c} \mathrm{Fun}^{\flat}(\overline{L}_1, \mathcal{D}^{\natural}) \times_{\mathrm{Fun}(K \times \Delta^{\{0\}}, \mathcal{D}) \times \mathrm{Fun}(L_1, \mathcal{C})} \{(q, \overline{p}'_1)\} \\ \uparrow \simeq \\ \mathrm{Fun}^{\flat}(\overline{L}, \mathcal{D}^{\natural}) \times_{\mathrm{Fun}(K \times \Delta^{\{0\}}, \mathcal{D}) \times \mathrm{Fun}(L, \mathcal{C})} \{(q, \overline{p}')\} \\ \downarrow \simeq \\ \mathrm{Fun}^{\flat}(\overline{L}_2, \mathcal{D}^{\natural}) \times_{\mathrm{Fun}(K \times \Delta^{\{0\}}, \mathcal{D}) \times \mathrm{Fun}(L_2, \mathcal{C})} \{(q, \overline{p}'_2)\} \end{array} \tag{27}$$

where $\overline{p}' : L \longrightarrow \mathcal{C}$ is the composition of $\overline{p}$ with the projection $L \longrightarrow K^{\triangleright}$ and $\overline{p}'_i = \overline{p}'|_{\overline{L}_i}$. Let $\overline{r} : \overline{L}_1 \longrightarrow \mathcal{D}^{\natural}$ be an object which corresponds to $\overline{q} : L_2 \longrightarrow \mathcal{D}$ under the zig-zag of equivalences (27). We now observe that if a map $\overline{L} \longrightarrow \mathcal{D}^{\natural}$ sends all edges in $\overline{L}_2$ to π-Cartesian edges then it sends all edges in $\overline{L}$ to π-Cartesian edges. It then follows that $\overline{q}$ sends all edges to π-Cartesian edges if and only if $\overline{r}$ sends all edges to π-Cartesian edges. To finish the proof it will hence suffice to show that $\overline{r}$ is initial in $\mathrm{Fun}^{\flat}(\overline{L}_1, \mathcal{D}^{\natural}) \times_{\mathrm{Fun}(K \times \Delta^{\{0\}}, \mathcal{D}) \times \mathrm{Fun}(L_1, \mathcal{C})} \{(q, \overline{p}'_1)\}$ if and only if it sends all edges to π-Cartesian edges.

We now invoke our assumption that the functor $\chi : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_{\infty}$ maps $\overline{p}$ to a limit diagram in Cat_{∞}. By [5, Proposition 3.3.3.1] and using the fact that K is a Kan complex we may conclude that the projection

$$\mathrm{Fun}^{\flat}(\overline{L}_1, \mathcal{D}^{\natural}) \times_{\mathrm{Fun}(K \times \Delta^{\{0\}}, \mathcal{D}) \times \mathrm{Fun}(L_1, \mathcal{C})} \{(q, \overline{p}'_1)\} \xrightarrow{\simeq}$$

$$\mathrm{Fun}(K \times \Delta^1, \mathcal{D}) \times_{\mathrm{Fun}(K \times \Delta^{\{0\}}, \mathcal{D}) \times \mathrm{Fun}(K \times \Delta^1, \mathcal{C})} \{(q, p')\}$$

is a weak equivalence, where $p' : K \times \Delta^1 \longrightarrow \mathcal{C}$ is the composition of p with the projection $K \times \Delta^1 \longrightarrow K$. We now observe that $\overline{r}|_{K\times\Delta^1}$ is initial in

$$\mathrm{Fun}(K\times\Delta^1,\mathcal{D})\times_{\mathrm{Fun}(K\times\Delta^{\{0\}},\mathcal{D})\times\mathrm{Fun}(K\times\Delta^1,\mathcal{C})}\{(q,p')\} \simeq (\mathrm{Fun}(K,\mathcal{D})\times_{\mathrm{Fun}(K,\mathcal{C})}\{p\})_{q/},$$

if and only if the morphism in $\mathrm{Fun}(K,\mathcal{D}) \times_{\mathrm{Fun}(K,\mathcal{C})} \{p\}$ determined by $\overline{r}|_{K\times\Delta^1}$ is an equivalence, and so the desired result follows. □

For an object $(X, f) \in \mathcal{S}_m(\mathcal{C})$ and a point $x \in X$, let us denote by $i_x : (\{x\}, f(x)) \longrightarrow (X, f)$ the corresponding morphism in $\mathcal{S}_m(\mathcal{C})$.

Lemma 5.30.

1. *The ∞-category $\mathcal{S}_m(\mathcal{C})$ admits $\mathcal{K}_m$-indexed colimits. Furthermore, if $\overline{p} : K^{\triangleright} \longrightarrow \mathcal{S}_m(\mathcal{C})$ is a cone diagram with $K \in \mathcal{K}_m$ then $\overline{p}$ is a colimit diagram if and only if $\pi \circ \overline{p} : K^{\triangleright} \longrightarrow \mathcal{S}_m$ is a colimit diagram and $\overline{p}$ sends every morphism in $K^{\triangleright}$ to a π-Cartesian edge.*

2. *For any ∞-category $\mathcal{D}$ with $\mathcal{K}_m$-indexed colimits, an arbitrary functor $\mathcal{F} : \mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits if and only if for every $(X, f) \in \mathcal{S}_m(\mathcal{C})$ the collection of maps $\mathcal{F}(i_x) : \mathcal{F}(x, f(x)) \longrightarrow \mathcal{F}(X, f)$ exhibits $\mathcal{F}(X, f)$ as the colimit of the diagram $\{\mathcal{F}(x, f(x))\}_{x\in X}$.*

Proof. Let us first prove (1). By definition, the Cartesian fibration $\pi : \mathcal{S}_m(\mathcal{C})$ is classified by the functor $\mathcal{F}_C : \mathcal{S}_m^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ given by $\mathcal{F}_{\mathcal{C}}(X) = \mathrm{Fun}(X, \mathcal{C})$. Since the inclusion $\mathcal{S}_m \hookrightarrow \mathcal{S}$ preserves $\mathcal{K}_m$-indexed colimits and the inclusion $\mathcal{S} \longrightarrow \mathrm{Cat}_\infty$ preserves all colimits it follows that $\mathcal{F}_{\mathcal{C}}$ sends $\mathcal{K}_m$-indexed colimit diagrams to limit diagrams in Cat_∞. Now let K be a finite m-truncated Kan complex, let $q : K \longrightarrow \mathcal{S}_m(\mathcal{C})$ be a diagram and let $p = \pi \circ q : K \longrightarrow \mathcal{S}_m$. Since $\mathcal{S}_m$ admits $\mathcal{K}_m$-indexed colimits we may extend p to a colimit diagram $\overline{p} : K^{\triangleright} \longrightarrow \mathcal{C}$. Since $\mathcal{F}_{\mathcal{C}} \circ \overline{p}^{\mathrm{op}} : (K^{\mathrm{op}})^{\triangleleft} \longrightarrow \mathrm{Cat}_\infty$ is a limit diagram and K is a Kan complex we may use [5, Proposition 3.3.3.1] to deduce the existence of a dotted lift

$$\begin{array}{ccc} K & \xrightarrow{\ q\ } & \mathcal{S}_m(\mathcal{C}) \\ \downarrow & \overset{\overline{q}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ K^{\triangleright} & \xrightarrow{\ \overline{p}\ } & \mathcal{S}_m \end{array}$$

such that $\overline{q}$ sends all edges in $K^{\triangleright}$ to π-Cartesian edges. By Lemma 5.29 we may conclude that $\overline{q}$ is a π-colimit diagram, and since $\overline{p}$ is a colimit diagram it follows that $\overline{q}$ is also a colimit diagram in $\mathcal{S}_m(\mathcal{C})$. Finally, by uniqueness of colimits this construction covers all colimit of $\mathcal{K}_n$-indexed diagram, and so the characterization of colimits given in (1) follows.

We shall now prove (2). The "only if" direction is clear since the collection of maps $\{i_x\}$ exhibits (X, f) as the colimit in $\mathcal{S}_n(\mathcal{C})$ of the diagram $\{(\{x\}, f(x))\}$ by the characterization of colimits diagram given in (1). Now suppose that for every $X \in \mathcal{S}_n(\mathcal{C})$ the collection $\{\mathcal{F}(i_x)\}_{x\in X}$ exhibits $\mathcal{F}(X, f)$ as the colimit of the diagram $\{(\{x\}, f(x))\}$. Let $Y \in \mathcal{K}_n$ be a finite n-truncated space and let $\mathcal{G} : Y \longrightarrow \mathcal{S}_n(\mathcal{C})$ be a Y-indexed diagram, and for each $y \in Y$ let us write $\mathcal{G}(y) = (Z_y, h_y)$ where Z_y is an m-truncated space and $h_y : Z_y \longrightarrow \mathcal{C}$ is a functor. By (1) we may identify the colimit of $\mathcal{G}$ in $\mathcal{S}_m(\mathcal{C})$ with the pair (Z, h) where Z is the total space of the left fibration $p : Z \longrightarrow Y$ classified by $y \mapsto Z_y$ and $h : Z \longrightarrow \mathcal{C}$ is the essentially unique functor such that $h|_{Z_y} = h_y$.

By our assumption for every $y \in Y$ the collection of maps $\{\mathcal{F}(i_{z'})\}_{z' \in Z_y}$ exhibits $\mathcal{F}(\mathcal{G}(y)) = \mathcal{F}(Z_y, h|_{Z_y})$ as the colimit in $\mathcal{D}$ of the Z_y-indexed diagram $\{\mathcal{F}(\{z'\}, h(z'))\}_{z' \in Z_y}$. It follows that $\mathcal{F} \circ \mathcal{G}$ is a left Kan extension along $p : Z \longrightarrow Y$ of the Z-indexed diagram $\{\mathcal{F}(\{z\}, h(z))\}_{z \in Z}$. Invoking the assumption again we get that the collection of maps $\{\mathcal{F}(i_z)\}_{z \in Z}$ exhibits $\mathcal{F}(Z, h)$ as the colimit in $\mathcal{D}$ of the Z-indexed diagram $\{\mathcal{F}(\{z\}, h(z))\}_{z \in Z}$, and so the collection of maps $\{\mathcal{F}(\mathcal{G}(y)) \longrightarrow \mathcal{F}(Z)\}_{y \in Y}$ exhibits $\mathcal{F}(Z)$ as the colimit of the diagram $\{\mathcal{F}(\mathcal{G}(y))\}_{y \in Y}$, as desired.

□

Now let $(X, f) \in \mathcal{S}_m(\mathcal{C})$ be an object. The forgetful functor $\pi : \mathcal{S}_m(\mathcal{C})_{/(X,f)} \longrightarrow (\mathcal{S}_m)_{/X}$ is a Cartesian fibration classifying the functor $(Y \xrightarrow{\varphi} X) \mapsto \mathrm{Fun}(Y, \mathcal{C})_{/f \circ \varphi}$. Since all the fibers of π have terminal objects we may choose an essentially unique Cartesian section $s : (\mathcal{S}_m)_{/X} \longrightarrow \mathcal{S}_m(\mathcal{C})_{/(X,f)}$ such that $s(Y, \varphi)$ is terminal in the fiber over (Y, φ) for every $\varphi : Y \longrightarrow X$ in $(\mathcal{S}_m)_{/X}$. We may then form the diagram of ∞-categories

$$\begin{array}{ccc} V & \longrightarrow & (\mathcal{S}_m)_{/X} \\ \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\ \mathcal{C} \times_{\mathcal{S}_m(\mathcal{C})} \left(\mathcal{S}_m(\mathcal{C})_{/(X,f)}\right) & \longrightarrow & \mathcal{S}_m(\mathcal{C})_{/(X,f)} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{S}_m(\mathcal{C}) \end{array}$$

where V is chosen so that the top square is Cartesian (and the bottom square is Cartesian by construction as well). It then follows that the external rectangle is Cartesian. Using this we may then identify objects of V with pairs $(C, \varphi : Y \longrightarrow X, \eta)$ where η is an equivalence from $(*, C)$ to $(Y, f \circ \varphi)$ in $\mathcal{S}_m(\mathcal{C})$. In particular, $Y \simeq *$ must be contractible and the data of η is just an equivalence from $f(\varphi(Y)) = f(\varphi(*))$ to C in $\mathcal{C}$. We may hence identify V with the ∞-groupoid X, in which case the map

$$s' : X \longrightarrow \mathcal{C} \times_{\mathcal{S}_m(\mathcal{C})} \left(\mathcal{S}_m(\mathcal{C})_{/(X,f)}\right)$$

can be informally described by the formula

$$s'(x) = (f(x), (\{x\}, f(x))) \in \mathcal{C} \times_{\mathcal{S}_m(\mathcal{C})} \left(\mathcal{S}_m(\mathcal{C})_{/(X,f)}\right)$$

Lemma 5.31. *The map s' is cofinal.*

Proof. The map s' is a base change of the map s, which is cofinal since it is a terminal section of a Cartesian fibration. $\square$

Corollary 5.32. *The inclusion $\iota : \mathcal{C} \longrightarrow \mathcal{S}_m(\mathcal{C})$ exhibits $\mathcal{S}_m(\mathcal{C})$ as the ∞-category obtained from $\mathcal{C}$ by freely adding $\mathcal{K}_m$-indexed colimits. In particular, if $\mathcal{D}$ is an ∞-category with $\mathcal{K}_m$-indexed colimits then restriction along ι induces an equivalence of ∞-categories*

$$\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}_m(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

Proof. By Lemma 5.30(1) we know that $\mathcal{S}_m(\mathcal{C})$ has $\mathcal{K}_m$-indexed colimits. Now suppose that $\mathcal{D}$ is an ∞-category that admits $\mathcal{K}_m$-indexed colimits and let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. Since $\mathcal{D}$ admits colimits indexed by X for every $X \in \mathcal{S}_m$ Lemma 5.31 and [5, Lemma 4.3.2.13] together imply that any functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ admits a left Kan extension $\overline{\mathcal{F}} : \mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{D}$, and that an arbitrary functor $\overline{\mathcal{F}} : \mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{D}$ extending $\mathcal{F}$ is a left Kan extension of $\mathcal{F}$ if and only if for every $(X, f) \in \mathcal{S}_m(\mathcal{C})$ the collection of maps $\{\overline{\mathcal{F}}(i_x) : \overline{\mathcal{F}}(\{x\}, f(x)) = \mathcal{F}(f(x)) \longrightarrow \overline{\mathcal{F}}(X, f)\}$ exhibit $\overline{\mathcal{F}}(X, f)$ as the colimit of the diagram $\{\mathcal{F}(f(x))\}_{x \in X}$. By Lemma 5.30(2) the latter condition is equivalent to the condition that $\overline{\mathcal{F}}$ preserves $\mathcal{K}_m$-indexed colimits. The desired result now follows from the uniqueness of left Kan extensions (see [5, Proposition 4.3.2.15]). $\square$

We now address the universal property of $\mathcal{S}^m_m(\mathcal{C})$ as described in the second claim of Theorem 5.28. We begin with the following Lemma:

Lemma 5.33.

1. *The ∞-category $\mathcal{S}^m_m(\mathcal{C})$ admits $\mathcal{K}_m$-indexed colimits and the inclusion $\mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}^m_m(\mathcal{C})$ preserves $\mathcal{K}_m$-indexed colimits. Furthermore, every $\mathcal{K}_m$-indexed diagram in $\mathcal{S}^m_m(\mathcal{C})$ comes from a $\mathcal{K}_m$-indexed diagram in $\mathcal{S}_m(\mathcal{C})$.*
2. *For any ∞-category $\mathcal{D}$ with $\mathcal{K}_m$-indexed colimits, an arbitrary functor $\mathcal{F} : \mathcal{S}^m_m(\mathcal{C}) \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits if and only if for every $(X, f) \in \mathcal{S}^m_m(\mathcal{C})$ the collection of maps $\{\mathcal{F}(i_x) : \mathcal{F}(x, f(x)) \longrightarrow \mathcal{F}(X, f)\}$ exhibit $\mathcal{F}(X, f)$ as the colimit of the diagram $\{\mathcal{F}(x, f(x))\}$.*
3. *The functor $\iota'' : \mathcal{S}^m_m \times \mathcal{C} \longrightarrow \mathcal{S}^m_m(\mathcal{C})$ preserves $\mathcal{K}_m$-indexed colimits in the $\mathcal{S}^m_m$ variable.*

Proof. Let us begin with Claim (1). We first claim that every equivalence in $\mathcal{S}^m_m(\mathcal{C})$ is in the image of the map $\mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}^m_m(\mathcal{C})$. Indeed, a morphism in $\mathcal{S}^m_m(\mathcal{C})$ is given by a span

$$\begin{array}{ccccc} & & (Z, h) & & \\ & \overset{(\varphi, T)}{\swarrow} & & \overset{(\psi, S)}{\searrow} & \\ (X, f) & & & & (Y, g) \end{array} \tag{28}$$

such that T is an equivalence in $\mathrm{Fun}(Z, \mathcal{C})$. If (28) is an equivalence then its image in $\mathcal{S}_m^m$ is an equivalence which means by Lemma 2.10 that $\varphi : Z \longrightarrow X$ is an equivalence in $\mathcal{S}_m$ and hence that $(\varphi, T) : (Z, h) \longrightarrow (X, f)$ is an equivalence in $\mathcal{S}_m(\mathcal{C})$. In this case the span (28) is essentially equivalent to an honest map, i.e., is in the image of $\mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}_m^m(\mathcal{C})$. Since the inclusion $\mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}_m^m(\mathcal{C})$ is faithful it follows that any $\mathcal{K}_m$-indexed diagram in $\mathcal{S}_m^m(\mathcal{C})$ is the image of an essentially unique $\mathcal{K}_m$-indexed diagram in $\mathcal{S}_m(\mathcal{C})$. It will hence suffice to prove that the map $\mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}_m^m(\mathcal{C})$ preserves $\mathcal{K}_m$-indexed colimits.

By Lemma 5.30(2) it will suffice to show that for every $(X, f) \in \mathcal{S}_m^m(\mathcal{C})$, the collection of maps $f_x : (\{x\}, f(x)) \longrightarrow (X, f)$ exhibit (X, f) as the colimit of the X-indexed diagram $\{(\{x\}, f(x))\}_{x \in X}$ in $\mathcal{S}_m^m(\mathcal{C})$. In other words, we need to show that the data of a span of the form (28) such that $T : h \xrightarrow{\simeq} f \circ \varphi$ is an equivalence in $\mathrm{Fun}(Z, \mathcal{C})$ is equivalent to the data of an X-indexed family of spans

$$\begin{array}{ccc} & (Z_x, h|_{Z_x}) & \\ {}^{(\varphi|_{Z_x}, T|_{Z_x})}\swarrow & & \searrow^{(\psi|_{Z_x}, S|_{Z_x})} \\ (\{x\}, f(x)) & & (Y, g) \end{array} \tag{29}$$

where Z_x denotes the homotopy fiber of $\varphi : Z \longrightarrow X$ over $x \in X$. But this is now a consequence of the fact that the collection of fiber functors $i_x^* : \mathcal{S}_{/X} \longrightarrow \mathcal{S}$ identifies $\mathcal{S}_{/X}$ with $\mathrm{Fun}(X, \mathcal{S})$ and for every $Z \longrightarrow X$ the collection of maps $i_x^* Z \longrightarrow Z$ exhibits Z as the homotopy colimit of the X-indexed family $\{Z_x\}_{x \in X}$. Claim (2) is now a direct consequence of the above and Lemma 5.30(2).

Let us now prove Claim (3). We have a commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathcal{S}_m \times \mathcal{C} & \xrightarrow{\iota'} & \mathcal{S}_m(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{S}_m^m \times \mathcal{C} & \xrightarrow{\iota''} & \mathcal{S}_m^m(\mathcal{C}) \end{array}$$

where the vertical maps are faithful. Let $C \in \mathcal{C}$ be an object, let K be a finite m-truncated Kan complex and let $p : K \longrightarrow \mathcal{S}_m^m \times \{C\}$ be a diagram. By Lemma 2.10 the diagram p is the image of an essentially unique diagram $p' : K \longrightarrow \mathcal{S}_m \times \{C\}$, and the inclusion $\mathcal{S}_m \times \{C\} \longrightarrow \mathcal{S}_m^m \times \{C\}$ preserves $\mathcal{K}_m$-indexed colimits. By Claim (2) above it will suffice to show that the top horizontal map preserves $\mathcal{K}_m$-indexed colimits, which is clear in light of the characterization of colimit cones in $\mathcal{S}_m(\mathcal{C})$ given in 5.30(1). □

Since $\mathcal{S}_m(\mathcal{C})$ admits $\mathcal{K}_m$-indexed colimits it carries a canonical action of $\mathcal{S}_m$, given informally by the formula $X \otimes (Y, g) = \mathrm{colim}_{x \in X}(Y, g) = (X \times Y, g \circ p_Y)$, where $p_Y : X \times Y \longrightarrow Y$ is the projection on the second factor. By Lemma 5.30(1) we get that if $\varphi : X \longrightarrow X'$ is a map of finite m-truncated Kan complexes then the induced edge $X \otimes (Y, g) \longrightarrow X' \otimes (Y, g)$ is π-Cartesian in $\mathcal{S}_m(\mathcal{C})$. On the other hand, for a fixed space X and a π-Cartesian map $(Y, g) \longrightarrow (Z, h)$ the induced map $X \otimes (Y,g) \longrightarrow X \otimes (Z,h)$ is again π-Cartesian. It then follows that the action of $\mathcal{S}_m$ on $\mathcal{S}_m(\mathcal{C})$ induces an action of $\mathrm{Span}(\mathcal{S}_m)$ on $\mathrm{Span}(\mathcal{S}_m(\mathcal{C}), \mathcal{S}_m^{\mathrm{coc}}(\mathcal{C}))$. Furthermore, by Lemma 5.33 and Lemma 2.10 this action preserves $\mathcal{K}_m$-indexed colimits in each variable separately. By Corollary 3.12 we now get that $\mathcal{S}_m^m(\mathcal{C})$ is **m-semiadditive**.

Let us now consider the left marked mapping cone

$$\mathrm{Cone}_m(\mathcal{C}) = \left[\mathcal{S}_m^m \times \mathcal{C} \times (\Delta^1)^\sharp\right] \coprod_{\mathcal{S}_m^m \times \mathcal{C} \times \Delta^{\{1\}}} \mathcal{S}_m^m(\mathcal{C})$$

of the inclusion $\iota'' : \mathcal{S}_m^m \times \mathcal{C} \hookrightarrow \mathcal{S}_m^m(\mathcal{C})$. Let $\mathrm{Cone}_m \hookrightarrow \mathcal{M}^\natural \xrightarrow{p} \Delta^1$ be a factorization of the projection $\mathrm{Cone}_m \longrightarrow (\Delta^1)^\sharp$ into a trivial cofibration follows by a fibration in the coCartesian model structure over $(\Delta^1)^\sharp$. In particular, $p : \mathcal{M} \longrightarrow \Delta^1$ is a coCartesian fibration and the marked edges of $\mathcal{M}^\natural$ are exactly the p-coCartesian edges. Let $\iota_0 : \mathcal{S}_m^m \times \mathcal{C} \hookrightarrow \mathcal{M} \times_{\Delta^1} \Delta^{\{0\}} \subseteq \mathcal{M}$ and $\iota_1 : \mathcal{S}_m^m(\mathcal{C}) \hookrightarrow \mathcal{M} \times_{\Delta^1} \Delta^{\{1\}} \subseteq \mathcal{M}$ be the corresponding inclusions. Then ι_0 and ι_1 exhibit $p : \mathcal{M} \longrightarrow \Delta^1$ as a correspondence from $\mathcal{S}_m^m \times \mathcal{C}$ to $\mathcal{S}_m^m(\mathcal{C})$ which is the one associated to the functor $\iota'' : \mathcal{S}_m^m \times \mathcal{C} \hookrightarrow \mathcal{S}_m^m(\mathcal{C})$.

Proposition 5.34. *Let $\mathcal{D}$ be an m-semiadditive ∞-category and let $\mathcal{F} : \mathcal{S}_m^m \times \mathcal{C} \longrightarrow \mathcal{D}$ be a functor which preserves $\mathcal{K}_m$-indexed colimits in the $\mathcal{S}_m^m$ variable. Then the following holds:*

1. *$\mathcal{F}$ admits a left Kan extension*

$$\begin{array}{ccc} \mathcal{S}_m^m \times \mathcal{C} & \xrightarrow{\mathcal{F}} & \mathcal{D} \\ {\scriptstyle \iota_0}\downarrow & \nearrow_{\overline{\mathcal{F}}} & \\ \mathcal{M} & & \end{array}$$

2. *An arbitrary functor $\overline{\mathcal{F}} : \mathcal{M} \longrightarrow \mathcal{D}$ extending $\mathcal{F}$ is a left Kan extension if and only if $\overline{\mathcal{F}}$ maps p-coCartesian edges in $\mathcal{M}$ to equivalences in $\mathcal{D}$ and $\overline{\mathcal{F}} \circ \iota_1 : \mathcal{S}_m^m(\mathcal{C}) \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits.*

Proof. For $(Y,g) \in \mathcal{S}_m^m(\mathcal{C})$ let us denote by

$$\mathcal{I}_{(Y,g)} = \mathcal{S}_m^m(\mathcal{C}) \times_{\mathcal{S}_m^m \times \mathcal{C}} (\mathcal{S}_m^m \times \mathcal{C})_{/(Y,g)}.$$

To prove (1), it will suffice by [5, Lemma 4.3.2.13] to show that the composed map

$$\mathcal{F}_{(Y,g)} : \mathcal{I}_{(Y,g)} \longrightarrow \mathcal{S}_m^m \times \mathcal{C} \longrightarrow \mathcal{D}$$

can be extended to a colimit diagram in $\mathcal{D}$ for every $(Y,g) \in \mathcal{S}_m^m(\mathcal{C})$. Now an object of $\mathcal{I}_{(Y,g)}$ corresponds to an object $(X,C) \in \mathcal{S}_m^m \times \mathcal{C}$ and a morphism $(X,\overline{C}) \longrightarrow (Y,g)$ in $\mathcal{S}_m^m(\mathcal{C})$, i.e., a span

$$\begin{array}{ccc} & (Z,h) & \\ {\scriptstyle (\varphi,T)}\swarrow & & \searrow{\scriptstyle (\psi,S)} \\ (X,\overline{C}) & & (Y,g) \end{array} \tag{30}$$

where $T : h \longrightarrow \overline{C} \circ \varphi$ is an equivalence in $\mathrm{Fun}(Z, \mathcal{C})$. Let $\mathcal{J}_{(Y,g)} = (\mathcal{S}_m \times C) \times_{\mathcal{S}_m(\mathcal{C})} (\mathcal{S}_m(\mathcal{C}))_{/(Y,g)}$ be the comma category over (Y, g) associated to the inclusion $\iota' : \mathcal{S}_m \times \mathcal{C} \longrightarrow \mathcal{S}_m(\mathcal{C})$. Then the faithful maps $\mathcal{S}_m \times \mathcal{C} \hookrightarrow \mathcal{S}_m^m \times \mathcal{C}$ and $\mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}_m^m(\mathcal{C})$ induce a fully-faithful inclusion $\mathcal{J}_{(Y,g)} \hookrightarrow \mathcal{I}_{(Y,g)}$ whose essential image consists of those objects as in (30) for which $\psi : Z \longrightarrow Y$ is an equivalence. We now claim that ρ is cofinal.

Consider an object $P \in \mathcal{I}_{(Y,g)}$ of the form (30). We need to show that the comma ∞-category $\mathcal{J}_{(Y,g)} \times_{\mathcal{I}_{(Y,g)}} (\mathcal{I}_{(Y,g)})_{P/}$ is weakly contractible. Given an object $(\psi', S') : (X', \overline{C'}) \longrightarrow (Y, g)$ of $\mathcal{J}_{(Y,g)}$ the mapping space from P to $\varphi(X', \overline{C'}, \psi', S')$ in $\mathcal{I}_{(Y,g)}$ is given by the homotopy fiber of the map

$$\mathrm{Map}_{\mathcal{S}_m^m \times \mathcal{C}}(X \times C, X' \times C') \longrightarrow \mathrm{Map}_{\mathcal{S}_m^m(\mathcal{C})}((X, \overline{C}), (Y, g)) \tag{31}$$

over the map determined by P. In light of Remark 2.4 we may identify the homotopy fiber of 31 with the homotopy fiber of the map

$$((\mathcal{S}_m)_{/X})^{\sim} \times_{\mathcal{S}_m} ((\mathcal{S}_m)_{/X'})^{\sim} \times \mathrm{Map}_{\mathcal{C}}(C, C') \longrightarrow (\mathcal{S}_m^{\mathrm{coc}}(\mathcal{C})_{/(X,\overline{C})})^{\sim} \times_{\mathcal{S}_m(\mathcal{C})} (\mathcal{S}_m(C)_{/(Y,g)})^{\sim} \tag{32}$$

over the object corresponding to P. Now since the map $((\mathcal{S}_m)_{/X})^{\sim} \longrightarrow (\mathcal{S}_m^{\mathrm{coc}}(\mathcal{C})_{/(X,\overline{C})})^{\sim}$ is an equivalence we may identify the homotopy fiber of 32 with the homotopy fiber of the map

$$\mathrm{Map}_{\mathcal{S}_m}(Z, X') \times \mathrm{Map}_{\mathcal{C}}(C, C') \longrightarrow \mathrm{Map}_{\mathcal{S}_m(\mathcal{C})}((Z, \overline{C}), (Y, g)) \tag{33}$$

over the point $(\psi, S) \in \mathrm{Map}_{\mathcal{S}_m(\mathcal{C})}((Z, \overline{C}), (Y, g))$. Unwinding the definitions we recover that the map 33 sends a pair $(\psi' : Z \longrightarrow X', \alpha : C \longrightarrow C')$ to the composition

$$(Z, \overline{C}) \overset{(\psi', \alpha')}{\longrightarrow} (X', \overline{C'}) \overset{(\psi', S')}{\longrightarrow} (Y, g).$$

We may then conclude that the functor $\mathcal{J}_{(Y,g)} \longrightarrow \mathcal{S}$ defined by $(X', \overline{C'}, \psi', S') \mapsto \mathrm{Map}_{\mathcal{I}_{(Y,g)}}(P, \varphi(X', \overline{C'}, \psi', S'))$ is corepresented in $\mathcal{J}_{(Y,g)}$ by the object $(\psi, S) : (Z, \overline{C}) \longrightarrow (Y, g)$. It then follows that $\mathcal{J}_{(Y,g)} \times_{\mathcal{I}_{(Y,g)}} (\mathcal{I}_{(Y,g)})_{P/}$ has an initial object and is hence weakly contractible. This means that $\rho : \mathcal{J}_{(Y,g)} \hookrightarrow \mathcal{I}_{(Y,g)}$ is cofinal, as desired.

It will now suffice to show that each of the diagrams

$$\mathcal{G}_{(Y,g)} := (\mathcal{F}_{(Y,g)})|_{\mathcal{J}_{(Y,g)}} : \mathcal{J}_{(Y,g)} \longrightarrow \mathcal{D}$$

can be extended to a colimit diagram. Let $\mathcal{J}'_{(Y,g)} = \mathcal{J}_{(Y,g)} \times_{\mathcal{S}_n} \{*\} \subseteq \mathcal{J}_Y$ be the full subcategory spanned by objects of the form $(\psi, S) : (*, \overline{C}) \longrightarrow (Y, g)$. Since we assumed that $\mathcal{F} : \mathcal{S}_m^m \times \mathcal{C} \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits in the first coordinate it follows from Proposition 2.9 that the restriction $\mathcal{F}|_{\mathcal{S}_m \times \mathcal{C}} : \mathcal{S}_m \times \mathcal{C} \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits in the first coordinate and by combining Lemma 2.8 with Lemma 5.31 we may conclude that the functor $\mathcal{F}|_{\mathcal{S}_m \times \mathcal{C}}$ is a left Kan extension of its restriction to $\{*\} \times \mathcal{C} \in \mathcal{S}_m$. Now since the projection $\mathcal{J}_Y \longrightarrow \mathcal{S}_m \times \mathcal{C}$ is a right fibration it induces an equivalence $(\mathcal{J}_{(Y,g)})_{/(X', C', \psi', S')} \longrightarrow$

$(\mathcal{S}_m \times \mathcal{C})_{/(X',C')}$ for every $(X', C', \psi', S') \in \mathcal{J}_{(Y,g)}$. We may then conclude that $\mathcal{F}|_{\mathcal{J}_{(Y,g)}}$ is a left Kan extension of $\mathcal{F}|_{\mathcal{J}'_{(Y,g)}}$. Since $\mathcal{D}$ admits $\mathcal{K}_m$-indexed colimits and $\mathcal{J}'_{(Y,g)}$ is contains a finite m-truncated Kan complex as a cofinal subcategory by Lemma 5.31 the diagram $\mathcal{G}_Y|_{\mathcal{J}'_{(Y,g)}}$ admits a colimit. It then follows that the diagram $\mathcal{G}_Y : \mathcal{J}_{(Y,g)} \longrightarrow \mathcal{D}$ admits a colimit, as desired.

To prove (2), we begin by noting that by the above considerations, an arbitrary extension $\overline{\mathcal{F}} : \mathcal{M} \longrightarrow \mathcal{D}$ is a left Kan extension if and only if for every (Y, g) the diagram

$$(\mathcal{J}'_{(Y,g)})^{\triangleright} \longrightarrow \mathcal{D}$$

determined by $\overline{\mathcal{F}}$ is a colimit diagram. By Lemma 5.31 the functor $Y \longrightarrow \mathcal{J}'_{(Y,g)}$ sending $y \in Y$ to the object $(\{y\}, g(y)) \longrightarrow (Y, g)$ is cofinal and so $\overline{\mathcal{F}}$ is a left Kan extension of $\mathcal{F}$ if and only if for every (Y, g) the diagram

$$\overline{\mathcal{G}}_{(Y,g)} : Y^{\triangleright} \longrightarrow \mathcal{D} \tag{34}$$

determined by $\overline{\mathcal{F}}$ is a colimit diagram. Now by Lemma 2.8 and Lemma 2.12 we know that for each $Y \in \mathcal{S}^m_m$ the collection of maps $\iota_y : \{y\} \longrightarrow Y$ exhibits Y as the colimit in $\mathcal{S}^m_m$ of the constant Y-diagram with value $*$. Since $\mathcal{F} : \mathcal{S}^m_m \times \mathcal{C} \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits in the first variable it follows that each $(Y, C) \in \mathcal{S}^m_m \times \mathcal{C}$ the collection of maps $\mathcal{F}(\iota_y, \mathrm{Id}_C) : \mathcal{F}(\{y\}, C) \longrightarrow \mathcal{F}(Y, C)$ exhibit $\mathcal{F}(Y, C)$ as the colimit of the diagram $\{\mathcal{F}(\{y\}, C)\}_{y \in Y}$. This means that $\overline{\mathcal{G}}_{(Y,\overline{C})}$ is a colimit diagram if and only if $\overline{\mathcal{F}}$ maps every p-coCartesian edge in $\mathcal{M}$ of the form $(Y, C) \longrightarrow (Y, \overline{C})$ (covering the map $0 \longrightarrow 1$ of Δ^1) to an equivalence in $\mathcal{D}$. Since all the other p-coCartesian edges of $\mathcal{M}$ are equivalences we may conclude that $\overline{\mathcal{F}}$ maps π-coCartesian edges to equivalences if and only if the diagrams $\overline{\mathcal{G}}_{(Y,\overline{C})}$ are colimit diagrams for every $Y \in \mathcal{S}^m_m$ and $C \in \mathcal{C}$. On the other hand, when these two equivalent conditions hold for $Y = *$ and all $C \in \mathcal{C}$ then the condition that $\overline{\mathcal{G}}_{(Y,g)}$ is a colimit diagram is equivalent by Lemma 5.33(2) to the condition that $\overline{\mathcal{F}} \circ \iota_1 : \mathcal{S}^m_m(\mathcal{C}) \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits. We may hence conclude that $\overline{\mathcal{F}}$ is a left Kan extension of $\overline{\mathcal{F}}$ if and only if it maps all p-coCartesian edges of $\mathcal{M}$ to equivalences in $\mathcal{D}$ and $\overline{\mathcal{F}} \circ \iota_1 : \mathcal{S}^m_m(\mathcal{C}) \longrightarrow \mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits. □

Given an ∞-category $\mathcal{D}$ admitting $\mathcal{K}_m$-indexed colimits, let us denote by $\mathrm{Fun}_{\mathcal{K}_m/C}(\mathcal{S}^m_m \times \mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{S}^m_m \times \mathcal{C}, \mathcal{D})$ the full subcategory spanned by those functors which preserves $\mathcal{K}_m$-indexed colimits in the $\mathcal{S}^m_m$ variable.

Corollary 5.35. *Let $\mathcal{D}$ be an m-semiadditive ∞-category. Then restriction along $\iota'' : \mathcal{S}^m_m \times \mathcal{C} \hookrightarrow \mathcal{S}^m_m(\mathcal{C})$ induces an equivalence of ∞-categories:*

$$\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^m_m(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{K}_m/C}(\mathcal{S}^m_m \times \mathcal{C}, \mathcal{D}).$$

Proof. Let $p : \mathcal{M}^{\natural} \longrightarrow \Delta^1$ be as above and consider the marked simplicial set $\mathcal{D}^{\natural} = (\mathcal{D}, M)$ where M is the collection of edges which are equivalences in $\mathcal{D}$. Let $\mathrm{Fun}^{\flat}_{\mathcal{K}_m}(\mathcal{M}^{\natural}, \mathcal{D}^{\natural}) \subseteq \mathrm{Fun}^{\flat}(\mathcal{M}^{\natural}, \mathcal{D}^{\natural})$ and $\mathrm{Fun}^{\flat}_{\mathcal{K}_m}(\mathrm{Cone}_m(\mathcal{C}), \mathcal{D}^{\natural}) \subseteq \mathrm{Fun}^{\flat}(\mathrm{Cone}_m(\mathcal{C}), \mathcal{D}^{\natural})$

be the respective full subcategories spanned by those makred functors whose restriction to $\mathcal{S}^m_m \times \mathcal{C}$ preserves $\mathcal{K}_m$-indexed colimits in the left variable and whose restriction to $\mathcal{S}^m_m(\mathcal{C})$ preserves $\mathcal{K}_m$-indexed colimits. Since the map $\mathrm{Cone}_m(\mathcal{C}) \longrightarrow \mathcal{M}^\natural$ is marked anodyne it follows that the restriction map $\mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathcal{M}^\natural, \mathcal{D}^\natural) \longrightarrow \mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathrm{Cone}_m(\mathcal{C}), \mathcal{D}^\natural)$ is an equivalence and by Proposition 5.34 the restriction map $\mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathcal{M}^\natural, \mathcal{D}^\natural) \longrightarrow \mathrm{Fun}_{\mathcal{K}_m/C}(\mathcal{S}^m_m \times \mathcal{C}, \mathcal{D})$ is an equivalence. We may hence deduce that the restriction map

$$\iota_0^* : \mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathrm{Cone}_m(\mathcal{C}), \mathcal{D}^\natural) \longrightarrow \mathrm{Fun}_{\mathcal{K}_m/C}(\mathcal{S}^m_m \times \mathcal{C}, \mathcal{D})$$

is an equivalence.

Now since the inclusion $\mathcal{S}^m_m \times C \hookrightarrow \mathrm{Cone}_m(\mathcal{C})$ is a pushout along the inclusion $\mathcal{S}^m_m \times \times \mathcal{C} \times \Delta^{\{1\}} \hookrightarrow \mathcal{S}^m_m \times \mathcal{C} \times (\Delta^1)^\sharp$ (which is itself a trivial cofibration in the **Cartesian** model structure over Δ^0) it follows that the map $i_1^* : \mathrm{Fun}^\flat(\mathrm{Cone}_m(\mathcal{C}), \mathcal{D}^\natural) \longrightarrow \mathrm{Fun}(\mathcal{S}^m_m(\mathcal{C}), \mathcal{D})$ is a trivial Kan fibation and that the composed functor

$$\mathrm{Fun}^\flat(\mathrm{Cone}_m(\mathcal{C}), \mathcal{D}^\natural) \xrightarrow[\simeq]{i_1^*} \mathrm{Fun}(\mathcal{S}^m_m(\mathcal{C}), \mathcal{D}) \xrightarrow{(\iota'')^*} \mathrm{Fun}(\mathcal{S}^m_m \times \mathcal{C}, \mathcal{D})$$

is homotopic to $i_0^* : \mathrm{Fun}^\flat(\mathrm{Cone}_m(\mathcal{C}), \mathcal{D}^\natural) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{S}^m_m \times \mathcal{C}, \mathcal{D})$. We may consequently conclude that i_1^* induces an equivalence between $\mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathrm{Cone}_m(\mathcal{C}), \mathcal{D}^\natural)$ and the full subcategory of $\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^m_m(\mathcal{C}), \mathcal{D})$ spanned by those functors whose restriction to $\mathcal{S}^m_m \times \mathcal{C}$ which preserves $\mathcal{K}_m$-indexed colimits in the left variable. By Lemma 5.33(3) the latter condition is automatic and hence the restriction map $\iota_1^* : \mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathrm{Cone}_m(\mathcal{C}), \mathcal{D}^\natural) \longrightarrow \mathrm{Fun}^\flat_{\mathcal{K}_m}(\mathcal{S}^m_m(\mathcal{C}), \mathcal{D}^\natural)$ is an equivalence. We may then conclude that

$$(\iota'')^* : \mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^m_m(\mathcal{C}), \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathcal{K}_m/C}(\mathcal{S}^m_m \times \mathcal{C}, \mathcal{D})$$

is an equivalence of ∞-categories, as desired. □

Corollary 5.36. *Let $\mathcal{D}$ be an m-semiadditive ∞-category. Then restriction along the inclusion $\{*\} \times \mathcal{C} \hookrightarrow \mathcal{S}^m_m(\mathcal{C})$ induces an equivalence of ∞-categories:*

$$\mathrm{Fun}_{\mathcal{K}_m}(\mathcal{S}^m_m(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Proof. Combine Corollary 5.35 and Theorem 4.1. □

Corollary 5.37. *The functor $\mathcal{S}^m_m \times \mathcal{S}_m(\mathcal{C}) \longrightarrow \mathcal{S}^m_m(\mathcal{C})$ exhibits $\mathcal{S}^m_m(\mathcal{C})$ as the tensor product $\mathcal{S}^m_m \otimes \mathcal{S}_m(\mathcal{C})$ in $\mathrm{Cat}_{\mathcal{K}_m}$.*

Proof. Combine Corollary 5.35 and Corollary 5.32. □

Remark 5.38. For $m \leq n$ one may also consider the subcategory $\mathcal{S}^m_n(\mathcal{C}) \subseteq \mathcal{S}^n_n(\mathcal{C})$ containing all objects and whose mapping spaces are spanned by those morphisms as in (24) for which φ is m-truncated. A similar argument then shows that $\mathcal{S}^m_n(\mathcal{C})$ is the free m-semiadditive ∞-category with $\mathcal{K}_n$-indexed colimits generated from $\mathcal{C}$.

5.4 Higher semiadditivity and topological field theories

In this section we will discuss a relation between the results of this paper and 1-dimensional topological field theories, and more specifically, with the notion of **finite path integrals** as described in [2, §3]. We first discuss the universal constructions of §5.3 in the presence of a symmetric monoidal structure. Recall that by [5, Proposition 4.8.1.10] the free-forgetful adjunction $\mathrm{Cat}_\infty \dashv \mathrm{Cat}_{\mathcal{K}_m}$ induces an adjunction $\mathrm{CAlg}(\mathrm{Cat}_\infty) \dashv \mathrm{CAlg}(\mathrm{Cat}_{\mathcal{K}_m})$ on commutative algebra objects which is compatible with the free-forgetful adjunction. In particular, if $\mathcal{D}^\otimes \in \mathrm{CAlg}(\mathrm{Cat}_\infty)$ is a symmetric monoidal ∞-category then the ∞-category $\mathcal{S}_m(\mathcal{D})$ (which, by Corollary 5.32, is the image of $\mathcal{D}$ in $\mathrm{Cat}_{\mathcal{K}_m}$ under the free functor $\mathrm{Cat}_\infty \longrightarrow \mathrm{Cat}_{\mathcal{K}_m}$) carries a canonical symmetric monoidal structure, under which it can be identified with the image of $\mathcal{D}^\otimes$ under the left adjoint $\mathrm{CAlg}(\mathrm{Cat}_\infty) \longrightarrow \mathrm{CAlg}(\mathrm{Cat}_{\mathcal{K}_m})$. Since the tensor product on $\mathcal{S}_m(\mathcal{C})$ preserves $\mathcal{K}_m$-colimits in each variable separately the characterization of colimits given in Lemma 5.30 yields an explicit formula for the monoidal product as $(X, f) \otimes (Y, g) = (X \times Y, f \otimes g)$, where $f \otimes g : X \times Y \longrightarrow \mathcal{D}$ is the map $(f \otimes g)(x, y) = f(x) \otimes g(y)$. We also note that by the above the unit map $\mathcal{D} \longrightarrow \mathcal{S}_m(\mathcal{D})$ is symmetric monoidal, and if $\mathcal{D}$ already has $\mathcal{K}_m$-indexed colimits and its monoidal structure commutes with $\mathcal{K}_m$-indexed in each variable separately then the counit map $\mathcal{S}_m(\mathcal{D}) \longrightarrow \mathcal{D}$ is symmetric monoidal as well.

Corollary 5.37 tells us that we have a similar phenomenon with $\mathcal{S}^m_m(\mathcal{D})$: indeed, by Proposition 5.5 the ∞-category $\mathcal{S}^m_m(\mathcal{D})$ inherits a canonical commutative algebra structure in $\mathrm{Add}_m \simeq \mathrm{Mod}_{\mathcal{S}^m_m}(\mathrm{Cat}_{\mathcal{K}_m})$ under which it can be identified with the image of $\mathcal{S}_m(\mathcal{D})^\otimes \in \mathrm{CAlg}(\mathrm{Cat}_{\mathcal{K}_m})$ under the left functor $\mathrm{Cat}_{\mathcal{K}_m} \longrightarrow \mathrm{Add}_m$. Combined with the above considerations we may further identify $\mathcal{S}^m_m(\mathcal{C})^\otimes \in \mathrm{CAlg}(\mathrm{Add}_m)$ with the image of $\mathcal{D}$ under the left functor $\mathrm{CAlg}(\mathrm{Cat}_\infty) \longrightarrow \mathrm{CAlg}(\mathrm{Add}_m)$. In explicit terms, $\mathcal{S}^m_m(\mathcal{D})$ carries a symmetric monoidal structure which preserves $\mathcal{K}_m$-indexed colimits in each variable separately and the unit map $\mathcal{D} \longrightarrow \mathcal{S}^m_m(\mathcal{D})$ extends to a symmetric monoidal functor. Furthermore, if $\mathcal{D}$ is already m-semiadditive and its symmetric monoidal structure commutes with $\mathcal{K}_m$-indexed colimits in each variable separately then the counit map $\mathcal{S}^m_m(\mathcal{D}) \longrightarrow \mathcal{D}$ is symmetric monoidal as well.

The following lemma appears to be well-known, but we could not find a reference. Note that, while the lemma is phrased for $\mathcal{S}^m_m(\mathcal{D})$, it has nothing to do with the finiteness or truncation of the spaces in $\mathcal{S}^m_m$. In particular, the analogous claim holds if one replaces $\mathcal{S}^m_m(\mathcal{D})$ by the analogous ∞-category of decorated spans between arbitrary spaces.

Lemma 5.39. *Let $\mathcal{D}$ be a symmetric monoidal ∞-category. Let $(X, f) \in \mathcal{S}^m_m(\mathcal{D})$ be such that $f(x)$ is dualizable in $\mathcal{D}$ for every $x \in X$. Then (X, f) is dualizable in $\mathcal{S}^m_m(\mathcal{D})$.*

Proof. Let $\mathcal{D}^{\mathrm{dl}} \subseteq \mathcal{D}$ be the full subcategory spanned by dualizable objects and let $(\mathcal{D}^{\mathrm{dl}})^\simeq \subseteq \mathcal{D}^{\mathrm{dl}}$ be the maximal subgroupoid of $\mathcal{D}^{\mathrm{dl}}$. Let $\mathrm{Bord}_1^\otimes$ be the 1-dimensional framed cobordism ∞-category. By the 1-dimensional cobordism hypothesis ([8],[3]), evaluation at the positively 1-framed point $*_+ \in \mathrm{Bord}_1$ induces an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{\otimes}, \mathcal{D}^{\otimes}) \xrightarrow{\simeq} (\mathcal{D}^{\mathrm{dl}})^{\sim}. \tag{35}$$

Now let (X, f) be an object of $\mathcal{S}_m^m(\mathcal{D})$ such that $f(x)$ is dualizable for every $x \in X$. Then the map $f : X \longrightarrow \mathcal{D}$ factors through a map $f' : X \longrightarrow (\mathcal{D}^{\mathrm{dl}})^{\sim}$. By the equivalence (35) we may lift f to a map $f' : X \longrightarrow \mathrm{Fun}^{\otimes}(\mathrm{Bord}_1^{\otimes}, \mathcal{D}^{\otimes})$. Evaluation at the negatively 1-framed point $*_- \in \mathrm{Bord}_1$ now yields map $\hat{f} : X \longrightarrow \mathcal{D}$. Furthermore, for every $x \in X$, evaluation at the evaluation bordism $\mathrm{ev} : *_+ \coprod *_- \longrightarrow \varnothing$ induces a map $f'(x, \mathrm{ev}) : f(x) \otimes \hat{f}(x) \longrightarrow 1_{\mathcal{D}}$ exhibiting $\hat{f}(x)$ as dual to $f(x)$. Allowing x to vary we obtain a natural transformation $f'(\mathrm{ev}) : (f \otimes \hat{f}) \circ \Delta \Rightarrow \overline{1_{\mathcal{D}}}$ of functors $X \longrightarrow \mathcal{D}$, where $\Delta : X \longrightarrow X \times X$ is the diagonal map. Similarly, we may evaluate at the coevaluation cobordism $\mathrm{coev} : \varnothing \longrightarrow *_- \coprod *_+$ and obtain a natural transformation $f'(\mathrm{coev}) : \overline{1_{\mathcal{D}}} \Rightarrow (\hat{f} \otimes f) \circ \Delta$, which, for each x, determines a compatible coevaluation map $1 \longrightarrow \hat{f}(x) \otimes f(x)$.

Now let $q : X \longrightarrow *$ denote the constant map and consider the morphisms $\mathrm{ev}_{(X,f)} : (X, f) \otimes (X, \hat{f}) = (X \times X, f \otimes \hat{f}) \longrightarrow *$ and $* \longrightarrow (X \times X, \hat{f} \otimes f)$ in $\mathcal{S}_m^m(\mathcal{C})$ given by the spans

$$(X \times X, f \otimes \hat{f}) \xleftarrow{(\Delta, \mathrm{Id})} (X, (f \otimes \hat{f}) \circ \Delta) \xrightarrow{(q, f'(\mathrm{ev}))} (*, 1_{\mathcal{D}}) \tag{36}$$

and

$$(*, 1_{\mathcal{D}}) \xleftarrow{(q, \mathrm{Id})} (X, \overline{1_{\mathcal{D}}}) \xrightarrow{(\Delta, f'(\mathrm{coev}))} (X \times X, \hat{f} \otimes f) \tag{37}$$

It is now straightforward to check that the morphisms (36) and (37) satisfy the evaluation-coevaluation identities and hence exhibit (X, f) and $(X, \hat{f})$ as dual to each other. □

Let us now explain the relation of the above construction with the notion of finite path integrals as described in [2]. Given a family $f : X \longrightarrow \mathcal{D}$ of dualizable objects in $\mathcal{D}$ (e.g., a family of invertible objects), one obtains, as described in Lemma 5.39, a dualizable object (X, f) of the decorated span ∞-category $\mathcal{S}_m^m(\mathcal{D})$. By the cobordism hypothesis this object determines a 1-dimensional topological field theory $Z_f : \mathrm{Bord}_1 \longrightarrow \mathcal{S}_m^m(\mathcal{D})$ which sends the point to (X, f). The term **quantization** is sometimes used to describe a procedure in which the topological field theory Z_f can be "integrated" into a topological field theory taking values in $\mathcal{D}$ (see, e.g., [10]). This can often be achieved, at various levels of rigor, but performing some kind of a **path integral**.

Such a path integral is described informally in [2] in the setting of **finite groupoids** (i.e. $m = 1$) and where the target ∞-category $\mathcal{D}$ is the category of vector spaces over the complex numbers. More generally, the authors of [2] work with an n-categorical version of the span construction and consider n-dimensional topological field theories. However, as the paper [2] is expository in nature, it discusses these ideas somewhat informally, leaving many assertions without a formal proof or a precise formulation. In a recent paper [11], F. Trova suggests to use the formalism of Nakayama categories in order to give a formal definition of quantization in the setting of finite groupoids and 1-dimensional field theories, when the target is an ordinary category satisfying suitable conditions. We shall now explain how the results of this paper can be used to give a formal definition of quantization when the target is an m-semiadditive ∞-category and finite groupoids are generalized to finite m-truncated spaces.

Let $Z_f : \mathrm{Bord}_1 \longrightarrow \mathcal{S}^m_m(\mathcal{D})$ be the topological field theory determined by a diagram $f : X \longrightarrow \mathcal{D}$ of dualizable objects in $\mathcal{D}$. Suppose that $\mathcal{D}$ is m-semiadditive and that the monoidal structure on $\mathcal{D}$ preserves $\mathcal{K}_m$-indexed colimits in each variable separately. Then we may consider the **counit map** $\nu_{\mathcal{D}} : \mathcal{S}^m_m(\mathcal{D}) \longrightarrow \mathcal{D}$ associated to the free-forgetful adjunction $\mathrm{CAlg}(\mathrm{Cat}_\infty) \dashv \mathrm{CAlg}(\mathrm{Cat}_{\mathcal{K}_m})$. This counit map is a symmetric monoidal functor, and we may consequently post-compose the topological field theory Z_f with ν_D to obtain a topological field theory $\overline{Z}_f : \mathrm{Bord}_1 \longrightarrow \mathcal{D}$. The association $Z_f \mapsto \overline{Z}_f$ can be considered as a **quantization procedure**, and by comparing values on the point it must be compatible with the approach of [2]. We note that one may write the counit map $\nu_D : \mathcal{S}^m_m(\mathcal{D}) \longrightarrow \mathcal{D}$ explicitly by using the formation of colimits and the formal summation of $\mathcal{K}_m$-families of maps in $\mathcal{D}$ via its canonical enrichment in m-commutative moniods established in §5.2. The resulting formulas can then be considered as explicit forms of path integrals. We may also summarize this process with the following corollary:

Corollary 5.40. *Let $\mathcal{D}$ be an m-semiadditive ∞-category equipped with a symmetric monoidal structure which preserves $\mathcal{K}_m$-indexed colimits in each variable separately. Then the collection of dualizable objects in $\mathcal{D}$ is closed under $\mathcal{K}_m$-indexed colimits. Furthermore, if X is finite m-truncated space and $f : X \longrightarrow \mathcal{D}$ is a diagram of dualizable objects in $\mathcal{D}$, then the 1-dimensional topological field theory $\mathrm{Bord}_1 \longrightarrow \mathcal{D}$ determined by the dualizable object $\mathrm{colim}(f)$ is the quantization of the topological field theory $\mathrm{Bord}_1 \longrightarrow \mathcal{S}^m_m(\mathcal{C})$ determined by the dualizable object $(X, f) \in \mathcal{S}^m_m(\mathcal{C})$.*

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