Arizona winter school - some solutions

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1 The Brauer group of the generic affine quadric

Let k be an algebraically closed field of characteristic $\neq 2$ and let $F = k(b, c, d)$ be the function field over k in three variables. Let U be the affine conic over F given by the equation

$$x^2 + by^2 + cz^2 + d = 0 \tag{1}$$

Our goal is to prove that $\mathrm{Br}(U)/\mathrm{Br}(F) = 0$. Let $\overline{F}$ be an algebraic closure of F and $\overline{U} = U \otimes_F \overline{F}$ the corresponding base change. One may then compute that $\mathrm{Pic}(\overline{U}) \cong \mathbb{Z}$ and the action of $\mathrm{Gal}(\overline{F}/F)$ is given by the quadratic character $\mathrm{Gal}(\overline{F}/F) \longrightarrow \{-1, 1\}$ associated with the quadratic extension $K = F(\sqrt{bcd})$. In particular $H^1(F, \mathrm{Pic}(\overline{U})) \cong \mathbb{Z}/2$.

Now let $E = F(\sqrt{-b})$ and let $U_E = U \otimes_F E$ be the corresponding base change. Then $H^1(E, \mathrm{Pic}(\overline{U_E})) \cong \mathbb{Z}/2$ as well and since U_E has a point defined over E one may use the method of [CTX09] to write down an explicit element in $\mathrm{Br}(U_E)$ corresponding to the non-trivial element in $H^1(E, \mathrm{Pic}(\overline{U_E}))$. This element is

$$(x + \sqrt{-b}y, -cd) \in \mathrm{Br}(U_E)$$

We may then conclude that

$$\mathrm{Br}(U_E)/\mathrm{Br}(E) \cong \mathbb{Z}/2$$

Let $G = \mathrm{Gal}(E/F) \cong \mathbb{Z}/2$. Let us now use the Hocshield-Serre spectral sequence

$$H^p(G, H^q(U_E, \mathbb{G}_m)) \Rightarrow H^{p+q}(U, \mathbb{G}_m)$$

associated with the field extension E/F to compute the Brauer group of U_F. Since $\overline{U}$ has no non-constant invertible functions we have $H^0(U_E, \mathbb{G}_m) = E^*$ and since the action of $\mathrm{Gal}(\overline{E}/E)$ on $\mathrm{Pic}(\overline{U})$ is via a non-trivial quadratic character we have $H^1(U_E, \mathbb{G}_m) = H^0(E, \mathrm{Pic}(\overline{U})) = 0$. It follows that the quotient $H^2(U, \mathbb{G}_m)/H^2(G, E^*)$ injects into $H^2(U_E, \mathbb{G}_m)^G$. We note that the elements of $H^2(U, \mathbb{G}_m)$ that come from $H^2(G, E^*)$ are constant, i.e., they come from $H^2(F, \mathbb{G}_m)$ (more precisely, they come from the kernel of the map $H^2(F, \mathbb{G}_m) \longrightarrow H^2(E, \mathbb{G}_m)$). It will hence suffice to show that $H^2(U_E, \mathbb{G}_m)^G \subseteq H^2(E, \mathbb{G}_m)^G$. Now recall that we have a short exact sequence

$$0 \longrightarrow H^2(E, \mathbb{G}_m) \longrightarrow H^2(U_E, \mathbb{G}_m) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

and a corresponding exact sequence

$$0 \longrightarrow H^2(E,\mathbb{G}_m)^G \longrightarrow (H^2(U_E,\mathbb{G}_m))^G \longrightarrow \mathbb{Z}/2 \xrightarrow{\partial} H^1(G,H^2(E,\mathbb{G}_m)).$$

In order to finish the proof we need to show that the non-trivial element $1 \in \mathbb{Z}/2$ is mapped by ∂ to a non-trivial element in $H^1(G,H^2(E,\mathbb{G}_m))$. Recall that the group $H^1(G,H^2(E,\mathbb{G}_m))$ can be identified with the middle homology of the complex

$$H^2(E,\mathbb{G}_m) \xrightarrow{x-\sigma(x)} H^2(E,\mathbb{G}_m) \xrightarrow{x+\sigma(x)} H^2(E,\mathbb{G}_m)$$

Now recall that the quaternion algebra $A = (x+\sqrt{b}y, -cd)$ represents an element in $H^2(U_E,\mathbb{G}_m)$ whose image in $\mathbb{Z}/2$ is 1. We can then compute that $\partial(1) \in H^1(G,H^2(E,\mathbb{G}_m))$ is represented by

$$A - \sigma(A) = (x^2+by^2,-cd) = (-cz^2-d,-cd) = (-c,-d) \in H^2(E,\mathbb{G}_m)$$

It is left to show that the element $(-c,-d) \in H^2(E,\mathbb{G}_m)$ cannot be written in the form $x-\sigma(x)$ for some $x \in H^2(E,\mathbb{G}_m)$. Indeed, suppose that x was such that

$$x - \sigma(x) = (-c,-d) \tag{2}$$

an element. Let us consider E as the field of functions of $\mathrm{spec}(k[\sqrt{b},c,d]) \cong \mathbb{A}^3$. Let $P \subseteq \mathrm{spec}(k[\sqrt{b},c,d])$ be the plane given by $\sqrt{b}=0$. By subtracting from x an Azumaya algebra of the form $(\sqrt{b},f(c,d))_n$ (which is invariant under G since -1 is an n'th power in k for every n) we may assume that x is unramified at P. Since $(-c,-d)$ is also unramified we may restrict the equation 2 to P. Since P is G-invariant and the induced action of G on P is trivial we get that

$$(-c,-d)|_P = x|_P - \sigma(x)|_P = 0$$

which is a contradiction, since the element $(-c,-d)|_P$ is non-trivial in $H^2(k(P),\mathbb{G}_m) = \mathrm{Br}(k(c,d))$.

2 Brauer-Manin obstruction for rational points on singular cubic 3-folds

Let k be a number field. Let $K/k, L/k$ be two linearly disjoint cubic cyclic extensions of k and let $G = \mathrm{Gal}(KL/k) \cong \mathrm{Gal}(K/k) \times \mathrm{Gal}(L/k)$. We fix generators $\sigma \in \mathrm{Gal}(K/k)$ and $\tau \in \mathrm{Gal}(L/k)$. By abuse of notation we will also denote by σ the element $(\sigma,1)$ of $\mathrm{Gal}(LK/k)$ and similarly for τ. Let $V = \mathrm{R}_{K/k}\mathbb{A}^1_K \times \mathrm{R}_{L/k}\mathbb{A}^1_L$ be the product of the corresponding restriction of scalars. We will use x as a coordinate on $\mathrm{R}_{K/k}\mathbb{A}^1_K$ and y as a coordinates on $\mathrm{R}_{L/k}\mathbb{A}^1_L$. Let $a \in K, b \in L$ be two non-zero elements. Consider the cubic 3-fold $X \subseteq \mathbb{P}(V)$ given by the pair of equations

$$\begin{gathered} \mathrm{N}_{K/k}(x) = \mathrm{N}_{L/k}(y) \\ \mathrm{Tr}(ax) = \mathrm{Tr}(by) \end{gathered} \tag{3}$$

Note that X has exactly nine singular points (over $\overline{k}$) and G acts simply transitively on them (in particular, no singular point is rational). Let $X^0 \subseteq X$ be the smooth locus. We will now calculate the Brauer group of X^0 and show that the Brauer-Manin obstruction may indeed cause a non-trivial obstruction to the Hasse principle on X.

The coordinate x can be considered as a regular function $\mathrm{R}_{K/k}\,\mathbb{A}^1_K \longrightarrow \mathbb{A}^1_K$ **defined over** K and similarly for y. We hence have six rational functions on X defined over the field KL, given by $x, \sigma(x), \sigma^2(x), y, \tau(y), \tau^2(y)$. Let $D_{i,j}$ be the 0-locus of $\sigma^i(x) = \tau^j(y) = 0$. Then each $D_{i,j}$ is a 2-plane lying inside X, defined over KL, and the group G acts on the $D_{i,j}$'s simply and transitively. Let $H \subseteq X$ be a hyperplane section defined over k. According to [**?**] the geometric Picard group $\mathrm{Pic}(\overline{X^0})$ is generated by the classes of H and $D_{i,j}$ modulu the relations

$$\sum_{j=0}^{2} D_{i,j} = H$$

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We observe that $\mathrm{Pic}(\overline{X^0})$ is a free abelian group of rank 5. Let $N \subseteq G$ be the cyclic subgroup generated by the element $\sigma\tau \in G$ and let $P_N \subseteq \mathrm{Pic}(\overline{X^0})$ be the subgroup fixed by N.

Lemma 2.1. *P_N is a subgroup of rank 3 generated freely by $[H], L, M$ where $L = [D_{0,1}] - [D_{1,0}]$ and $M = [D_{0,0}] + [D_{0,1}] + [D_{1,1}]$.*

Proof. It is straightforward to verify that $[H], L, M \subseteq P_N$. Let $\overline{L}, \overline{M}$ and $\overline{[D_{i,j}]}$ denote the images of L, M and $[D_{i,j}]$ respectively in the quotient group $\mathrm{Pic}(\overline{X^0})/H$. We observe that $\mathrm{Pic}(\overline{X^0})/H$ has rank 4 and is freely generated by $\overline{[D_{0,0}]}, \overline{[D_{0,1}]}, \overline{[D_{1,0}]}, \overline{[D_{1,1}]}$. To finish the proof it will be enough to show that $\overline{L}, \overline{M}$ freely generate $\left(\mathrm{Pic}(\overline{X^0})/H\right)^N$. Now the action of $\sigma\tau$ on $\mathrm{Pic}(\overline{X^0})/H$ is given by

$$\sigma\tau(\overline{[D_{0,0}]}) = \overline{[D_{1,1}]}$$

$$\sigma\tau(\overline{[D_{0,1}]}) = -\overline{[D_{1,0}]} - \overline{[D_{1,1}]}$$

$$\sigma\tau(\overline{[D_{1,0}]}) = -\overline{[D_{0,1}]} - \overline{[D_{1,1}]}$$

$$\sigma\tau(\overline{[D_{1,1}]}) = \overline{[D_{0,0}]} + \overline{[D_{0,1}]} + \overline{[D_{1,0}]} + \overline{[D_{1,1}]}$$

Let $x = \sum_{i,j\in\{0,1\}} a_{i,j}\overline{[D_{i,j}]}$ be an element which is fixed by $\sigma\tau$. Then $a_{0,0} = a_{1,1}$ and so by subtracting a multiple of $\overline{M}$ we may assume that x is of the form $x = a_{1,0}\overline{[D_{1,0}]} + a_{0,1}\overline{[D_{0,1}]}$. But then $a_{1,0} = -a_{0,1}$ and so x is a multiple of $\overline{L}$. It is hence clear that $\overline{L}, \overline{M}$ freely generate $(\mathrm{Pic}(\overline{X^0})/H)^N$. □

Lemma 2.2. *Let C be a cyclic subgroup of order 3 and let V be a C-module which, as an abelian group, is freely generated by two elements $v, u \in V$. Suppose that a generator $g \in C$ acts on V by $g(v) = u$ and $g(u) = -v - u$. Then $H^1(C, V) \cong \mathbb{Z}/3$ and a generating 1-cocycle sends g to v.*

Proof. We may equivariantly embed V inside $\mathbb{Z}C$ be sending v to $g - 1$ and u to $g^2 - g$ and obtain a short exact sequence

$$0 \longrightarrow V \longrightarrow \mathbb{Z}C \xrightarrow{d} \mathbb{Z} \longrightarrow 0.$$

where $d(a + bg + cg^2) = a + b + c$. Since $H^1(C, \mathbb{Z}C) = 0$ we may identify $H^1(C, V)$ with the cokernel of the induced map $d_* : H^0(C, \mathbb{Z}C) \longrightarrow \mathbb{Z}$. This cokernel is clearly cyclic of order 3. Direct examination verifies the assertion on the generating 1-cocycle. □

Lemma 2.3. *$H^1(N, \mathrm{Pic}(\overline{X^0})) = 0$ and $H^1(G, \mathrm{Pic}(\overline{X^0})) \cong H^1(G/N, P_N) \cong \mathbb{Z}/3$. A generator for $H^1(G/N, P_N)$ is given by a 1-cocycle which sends a generator of G/N to L.*

Proof. Consider the short exact sequence

$$0 \longrightarrow P_N \longrightarrow \mathrm{Pic}(\overline{X^0}) \longrightarrow \mathrm{Pic}(\overline{X^0})/P_N \longrightarrow 0$$

Then P_N is torsion free with trivial N-action and so $H^1(N, P_N) = 0$, which means that $H^1(N, \mathrm{Pic}(\overline{X^0}))$ injects into $H^1(N, \mathrm{Pic}(\overline{X^0})/P_N)$. Let $A, B \in \mathrm{Pic}(\overline{X^0})/P_N$ be the images of $[D_{0,0}]$ and $[D_{1,0}]$ respectively. Then it is straightforward to verify that $\mathrm{Pic}(\overline{X^0})/P_N$ is freely generated by A, B. Let $\theta \in G/N$ be the image of σ (or τ^{-1}). Then θ sends A to B and B to $-A - B$. By Lemma 2.2 we get that $H^1(N, \mathrm{Pic}(\overline{X^0})/P_N) \cong \mathbb{Z}/3$ and a generating 1-cocycle u sends θ to A. Let $\alpha \in H^1(N, \mathrm{Pic}(\overline{X^0})/P_N)$ be the generating class. Then the image of α in $H^2(N, P_N) \cong P_N/3P_N$ is given by the class $[H] \in P_N/3P_N$. Since H is not divisible by 3 in P_N this class is non-trivial and hence α does not lift to $H^1(N, \mathrm{Pic}(\overline{X^0})$. This implies that $H^1(N, \mathrm{Pic}(\overline{X^0})) = 0$. By the Hocshield-Serre spectral sequence we now have $H^1(G, \mathrm{Pic}(\overline{X^0})) \cong H^1(G/N, P_N)$. It is left to compute $H^1(G/N, P_N)$.

Let $\langle [H] \rangle \subseteq P_N$ denote the cyclic subgroup generated by H. Consider the short exact sequence

$$0 \longrightarrow \langle [H] \rangle \longrightarrow P_N \longrightarrow P_N / \langle [H] \rangle \longrightarrow 0$$

Since $H^0(G/N, P_N / \langle [H] \rangle) = 0$ we see that $H^1(G/N, P_N)$ injects into $H^1(G/N, P_N / \langle [H] \rangle)$. Let $\overline{L}, \overline{M} \in P_N / \langle [H] \rangle$ denote the images of L and M respectively. Then $P_N / \langle [H] \rangle$ is freely generated by $\overline{L}, \overline{M}$ and the generator $\theta \in G/N$ acts on them by

$$\theta(\overline{L}) = \overline{M}$$
$$\theta(\overline{M}) = -\overline{L} - \overline{M}$$

and so by Lemma 2.2 we get that $H^1(G/N, P_N / \langle [H] \rangle) \cong \mathbb{Z}/3$ and a generating 1-cocycle sends θ to $\overline{L}$. Since $L + \theta(L) + \theta^2(L) = 0$ we see that this 1-cocycle lifts to $H^1(G/N, P_N)$. □

Our next goal is to find an explicit Azumaya algebra corresponding to a non-trivial element in $H^1(G, \mathrm{Pic}(\overline{X^0}))$. Let $E \subseteq KL$ be the fixed field of $N \subseteq G$. Recall the element $L \in \mathrm{Pic}(\overline{X^0})$ which is fixed by N. Then L is defined over E as a **divisor class**. Let $\alpha \in H^1(G/N, P_N)$ be the class whose 1-cocycle sends $\theta \in G/N$ to L. As we saw above α is a genrator for $H^1(G/N, P_N) \cong H^1(G, P_N)$. However, in order to produce an Azumaya algebra from α and L we must first represent L by a divisor which is itself defined over E. Note that the divisor $D_{1,0} - D_{0,1}$ used to define L is not invariant, as a divisor, under the action of N. In general not every Galois invariant divisor class can be represented by an invariant divisor. However, we will show that in this case the obstruction to do so vanishes. Instead of following the abstract root let us take a concrete approach. Consider the rational function

$$f = \frac{\sigma(x)}{\tau(y)}$$

on X. Then

$$\mathrm{div}(f) = \sum_i D_{i,1} - \sum_j D_{1,j} = [D_{0,1} - D_{1,0}] - [D_{1,2} - D_{2,1}] = [D_{0,1} - D_{1,0}] - \sigma\tau\,[D_{0,1} - D_{1,0}]$$

Since

$$f \sigma\tau(f)(\sigma\tau)^2(f) = \frac{\mathrm{N}_{K/k}(x)}{\mathrm{N}_{L/k}(y)} = 1$$

we get from Hilbert 90 that there exists a rational function g on X, defined over KL, such that

$$\frac{\sigma\tau(g)}{g} = f \tag{4}$$

To be even more explicit, one may choose the function

$$g = \frac{\tau(y)\tau^2(y)}{\sigma(x)} + \tau(y) + x$$

Let $D_g = D_{0,1} - D_{1,0} + \mathrm{div}(g)$. Then D_g is a divisor which is invariant under $\sigma\tau$ and is linearly equivalent to $D_{0,1} - D_{1,0}$. We may now use D_g to produce an explicit Azumaya algebra. By 4 we have

$$\sigma(g) = \tau^2(g)\tau^2(f)$$

$$\sigma^2(g) = \frac{\tau(g)}{\sigma^2(f)}$$

and so

$$N_{LK/L}(g) = N_{LK/K}(g)\frac{\tau^2(f)}{\sigma^2(f)} = N_{LK/K}(g)\frac{\sigma(x)}{y}\frac{\tau(y)}{x}$$

which means that

$$F \stackrel{\mathrm{def}}{=} \frac{y}{\tau(y)} N_{LK/L}(g) = \frac{\sigma(x)}{x} N_{LK/K}(g)$$

is a function which is invariant under both σ and τ, i.e., F is a rational function on X defined over k. Considering D_g as a divisor defined over E we have

$$\operatorname{div}(F) = D_g + \theta(D_g) + \theta^2(D_g)$$

where $\theta \in G/N$ is the image of σ (or τ^{-1}). We may hence form the unramified Azumaya algebra

$$A = (E/k, F) \in \operatorname{Br}(X)$$

This is the element corresponding to the generator of $H^1(G, \operatorname{Pic}(\overline{X^0}))$.

Warning 2.4. *In general one may apply the above procedure to any N-invariant divisor D' (in place of D_g), and obtain some element in $\operatorname{Br}(X)$. In [SB12, Proposition 4.2] the author claims that a generator for $\operatorname{Br}(X)$ is obtained by applying the above procedure to the N-invariant divisor $D_{0,0} + D_{1,1} + D_{2,2} - H$. However, when applied to this element the procedure produces the trivial element $0 \in \operatorname{Br}(X)$.*

The following lemmas will help to compute the evaluation of A on adelic points. Given a place v of k we will denote $K_v = K \otimes_k k_v$, $L_v = L \otimes_k k_v$, $E = E \otimes_k k_v$ and $(LK)_v = LK \otimes_k k_v$.

Lemma 2.5. *The restriction of A to L can be identified with the quaternion algebra*

$$\left(KL/L, \frac{y}{\tau(y)}\right)$$

Proof. This follows from the fact that $EL \cong KL$ and so $N_{KL/L}(g)$ becomes a norm from EL. □

Corollary 2.6. *Let v be a place of k which splits in L but not in K. Let $y \mapsto (y^0, y^1, y^2)$ be an isomorphism of étale algebras $L_v \cong k_v \times k_v \times k_v$ such that the Galois group of L/k acts by cyclic permutations on the right hand side. Then for any point $(x, (y^0, y^1, y^2)) \in X(k_v)$ at least two of the y^i's are non-zero and the restriction of the quaternion algebra A to k_v can be identified, up to a sign, with the gluing of the quaternion algebras*

$$\left(K_{v_2}/k_{v_2}, \frac{y^1}{y^0}\right) = \left(K_{v_2}/k_{v_2}, \frac{y^2}{y^1}\right) = \left(K_{v_2}/k_{v_2}, \frac{y^0}{y^2}\right)$$

Proof. Let $(x, (y^0, y^1, y^2)) \in X(k_v)$ be a point such that at least one of the y^i's is 0. Then we have $\mathrm{N}_{K_v/k_v}(x)\,\mathrm{N}_{L_v/k_v}(y) = 0$ and since K/k is non-trivial it follows that $x = 0$. This means that at least one of the y^i's must be non-zero. Let $(b^0, b^1, b^2) \in k_v \times k_v \times k_v$ be the components of b. Since $b \neq 0$ it follows that each b^i is non-zero. The equation

$$0 = \operatorname{Tr}(ax) = \operatorname{Tr}(by) = \sum b^i y^i$$

implies that at most one of the y^i's can vanish, as desired. The explicit description of A now follows from Lemma 2.5. □

Lemma 2.7. *Let v be a place in k and let $(x_v, y_v) \in X(k_v)$ be a local point. If there exists a $z_v \in (KL)_v$ such that $\mathrm{N}_{(KL)_v/K_v}(z_v) = x_v$ and $\mathrm{N}_{(KL)_v/L_v}(z_v) = y_v$. Then $A(x_v, y_v) = 0$.*

Proof. Let $w_v = \frac{\sigma(z_v)}{\tau(z_v)} g(x_v, y_v)$. We claim that w_v belongs to E. Indeed, using 4 we obtain

$$\sigma\tau(w_v) = \frac{\sigma^2(\tau(z_v))}{\tau^2(\sigma(z_v))} \frac{\sigma(x_v)}{\tau(y_v)} g(x_v, y_v) = \frac{\tau(\sigma(z_v))\sigma(z_v)}{\sigma(\tau(z_v))\tau(z_v)} g(x_v, y_v) = w_v$$

On the other hand

$$\mathrm{N}_{E_v/k_v}(w_v) = \mathrm{N}_{(LK)_v/L_v}\left(\frac{\sigma(z_v)}{\tau(z_v)} g(x_v, y_v)\right) = \frac{y_v}{\tau(y_v)} \mathrm{N}_{(LK)_v/L_v}(g(x_v, y_v)) = F(x_v, y_v)$$

and so $A(x_v, y_v) = (E_v/k_v, F(x_v, y_v))_v = 0$. $\square$

Corollary 2.8. *Let v be a place of k such that $K \otimes_k k_v$ is isomorphic (over k_v) to $L \otimes_k k_v$. Then $X(k_v) \neq \emptyset$ and for any $(x_v, y_v) \in X$ one has $A(x_v, y_v) = 0$.*

Proof. Choosing an isomorphism $K_v \longrightarrow L_v$ we may assume that $K_v = L_v$. To see that $X(k_v) \neq \emptyset$ it is enough to take any $x \neq 0 \in K_v$ such that $\mathrm{Tr}(ax) = \mathrm{Tr}(bx)$ and then (x, x) is a k_v-point on X. We will prove that the assumption of Lemma 2.7 is satisfies for any point $(x, y) \in X(k_v)$. If v splits in K (and hence also in L) then the claim is immediate. Otherwise, we may assume that $x, y \neq 0$. Let $T : K_v \otimes_{k_v} K_v \xrightarrow{\cong} K_v \times K_v \times K_v$ be an isomorphism such that the Galois group of the first component acts by cyclic shifts and the Galois group of the second component acts by cyclic shifts follows by its coordinate-wise action. Unwinding the definitions, what we need to do is to find three elements $a, b, c \in K_v$ such that $abc = x$ and $a\sigma(b)\sigma^2(c) = y$. Since $\frac{x}{y} \in K_v$ is an element of norm 1 we know by Hilbert 90 that there exists a $z \in K_v$ such that $\frac{z}{\sigma(z)} = \frac{x}{y}$. We may then set $a = \frac{x}{z}$, $b = z$ and $c = 1$. $\square$

Lemma 2.9. *Let v be a place of residue characteristic $\neq 3$ such that both K and L are unramified and such that a, b are units at any place over v. Then for any $(x_v, y_v) \in X$ one has $A(x_v, y_v) = 0$.*

Proof. If K_v is isomorphic to L_v over k_v than the claim follows from Corollary 2.8. If K_v is not isomorphic to L_v then our assumptions imply that at least one of K_v, L_v splits then we may assume without loss of generality that L_v splits, and so we are in the situation of Corollary 2.6. If K_v splits as well the desired result follows. Otherwise, let $(x_v, (y_v^0, y_v^1, y_v^2)) \in X(k_v)$ be a point. We may assume that x_v is integral in K_v and each y_v^i is integral in k_v, and that at least one of x_v, y_v^0, y_v^1, y_v^2 is a unit. Let $(b^0, b^1, b^2) \in k_v \times k_v \times k_v$ be the element corresponding to b. Since b is a unit at v it follows that each b^i is a unit. The same argument as in the proof of Corollary 2.6 now shows that at most one of the y^i's has a positive valuation. Let $i \neq j$ be such that y_v^i, y_v^j are units. Then by Corollary 2.6

$$A\left(x_v, y_v^0, y_v^1, y_v^2\right) = \pm\left(K_v/k_v, \frac{y_v^i}{y_v^j}\right) = 0$$

$\square$

We now come to out explicit example where the element $A \in \mathrm{Br}(X)$ yields a non-trivial obstruction to the Hasse principal.

Example 2.10. Let $\omega \in \overline{\mathbb{Q}}$ be a primitive cubic root of unity and set $k = \mathbb{Q}(w)$. Let $p \in \mathbb{Z}$ be an odd prime which is equal to 5 mod 9. Set $K = k(\sqrt[3]{2p^2})$ and $L = k(\sqrt[3]{p})$ and let $a = 1, b = \sqrt[3]{p}$. Let X be the associated cubic 3-fold given by 3. Then $X(k) = \emptyset$ but $X(\mathbb{A}_k) \neq \emptyset$.

Proof. Let v_2, v_3, v_p be the unique places of k which lie above $2, 3$ and p respectively. To see that X has a local point at v_2 observe that p is a cube in k_{v_2} and the split étale algebra $L_{v_2} \cong k_{v_2} \times k_{v_2} \times k_{v_2}$ must have a non-zero element $y \in L_{v_2}$ such that $\mathrm{N}_{L_{v_2}/k_{v_2}}(y) = 0$ and $\mathrm{Tr}(by) = 0$. Then $(0, y)$ determines a k_{v_2}-point of X. Now since $p = 5$ mod 9 it follows that $k_{v_3}(\sqrt[3]{2p^2}) \cong k_{v_3}(\sqrt[3]{p})$ and so by Corollary 2.8 it follows that X has a local point at v_3. Finally, since $p-1$ is not divisible by 3 it follows that 2 (as any other integer coprime to p) is a cube at $\mathbb{Q}_p$ and hence $k_{v_p}(\sqrt[3]{2p^2}) \cong k_{v_p}(\sqrt[3]{p^2}) \cong k_{v_p}(\sqrt[3]{p^2})$. By Corollary 2.8 it follows that X has a local point at v_p. To show that X has local points at all other places consider the subvariety $Y \subseteq X$ given by the additional equation

$$\mathrm{Tr}(x) = \mathrm{Tr}(by) = 0$$

Then Y is a smooth cubic surface and has good reduction at every place $w \neq v_2, v_3, v_p$. By the Hasse-Weil bounds this implies that Y has a local point at $w \neq v_2, v_3, v_p$ as soon as the residue field of $w \neq v_2, v_3$ has at least 7 elements. Since 5 is inert in k this holds for all places $w \neq v_2, v_3, v_p$.

It is left to show that $X(k) = \emptyset$. By Corollary 2.6, Corollary 2.8 and Lemma 2.9 it will be enough to show that $A(x, y) \neq 0$ for every $(x, y) \in X(k_{v_2})$.

Let $\overline{\omega}$ be the reduction of ω mod 2 so that we can identify the residue field $\mathbb{F}_{v_2}$ with $\mathbb{F}_2[\overline{\omega}]$. Let $\alpha \in k_{v_2}$ be a cube root of p. Then we have an isomorphism of étale algebras

$$L_{v_2} \cong k_{v_2} \times k_{v_2} \times k_{v_2}$$

sending $b = \sqrt[3]{p}$ to $(\alpha, \omega\alpha, \omega^2\alpha)$. Under this isomorphism we may identify a point $(x, y) \in X(k_{v_2})$ with a tuple $(x, (y^0, y^1, y^2))$ such that $x \in K_{v_2}$, $y^i \in k_{v_2}$ and the equations

$$\begin{aligned} \mathrm{N}_{K_{v_2}/k_{v_2}}(x) &= y^0 y^1 y^2 \\ \mathrm{Tr}(x) = \mathrm{Tr}(b) &= \alpha y^0 + \omega\alpha y^1 + \omega^2 \alpha y^2 \end{aligned} \tag{5}$$

hold. Furthermore, by Corollary 2.8 the restriction of the quaternion algebra A to k_{v_2} may be identified, up to a sign, with the gluing of

$$\left(K_{v_2}/k_{v_2}, \frac{y^1}{y^0}\right) = \left(K_{v_2}/k_{v_2}, \frac{y^2}{y^1}\right) = \left(K_{v_2}/k_{v_2}, \frac{y^0}{y^2}\right)$$

Now let $(x, (y^0, y^1, y^2))$ be a k_{v_2}-point. Let w_2 be the unique place of K_{v_2} lying above v_2. We may assume without loss of generality that $\mathrm{val}_{w_2}(x) \geq 0$ and

$\mathrm{val}_{v_2}(y^i) \geq 0$ for every i. Furthermore, at least one of x, y^0, y^1, y^2 must be a unit. We now distinguish between two cases. If $\mathrm{val}_{v_2}(y^0 y^1 y^2) > 0$ then by 5 it follows that $\mathrm{val}_{w_2}(x) > 0$ and so $\mathrm{val}_{v_2}(\mathrm{Tr}(x)) = 0$. Since $\mathrm{val}_{w_2}(x) > 0$ we see that at least one of the y^i must be a unit, and from the second equation of 5 we get that exactly one of the y^i is a non-unit. Furthermore, if $i \neq j$ are such that y^i, y^j are units the the second equation of 5 implies that

$$\frac{y^i}{y^j} = \omega^{\pm 1}$$

We then get that

$$A(x, (y^0, y^1, y^2)) = \pm \left(K_{v_2}/k_{v_2}, \frac{y^i}{y^j} \right) = \pm \left(K_{v_2}/k_{v_2}, \omega \right) \neq 0$$

since $\overline{\omega} \in \mathbb{F}_{v_2}$ is not a cube. Let us now consider the case where all the y^i's are units. Let $\overline{y^i} \in \mathbb{F}_{v_2}$ be the reduction of y^i. Let us assume by contradiction that $A(x, (y^0, y^1, y^2)) = 0$. Then we would get that $\frac{\overline{y^i}}{\overline{y^j}}$ is a cube root in $\mathbb{F}_{v_2}$ and hence equal to $1 \in \mathbb{F}_{v_2}$. Let $z \in \mathbb{F}_{v_2}$ be such that $\overline{y^i} = z$ for every i, let $\overline{\alpha} \in \mathbb{F}_{v_2}$ be the reduction of α and let $\overline{x} \in \mathbb{F}_{w_2} = \mathbb{F}_{v_2}$ be the reduction of x. Since K_{v_2}/k_{v_2} is purely ramified the reduction of $\mathrm{Tr}(x)$ mod v_2 can be identified with $3\overline{x} = \overline{x}$. We then get that

$$\overline{x} = \overline{\alpha}\overline{y^0} + \omega\overline{\alpha}\overline{y^1} + \omega^2\overline{\alpha}\overline{y^2} = z\overline{\alpha}(1 + \omega + \omega^2) = 0$$

and so $\mathrm{val}_{w_2}(x) > 0$, yielding a contradiction. It hence follows that $A(x, (y^0, y^1, y^2)) \neq 0$ for every k_{v_2}-point of X and so $X(k) = \emptyset$ as desired □

Remark 2.11. The equations describing Example 2.10 can be written more explicitly as

$$x_0^3 + 2p^2 x_1^3 + 4p^4 x_2^3 - 6p^2 x_0 x_1 x_2 = y_0^3 + p y_1^3 + p^2 y_2^3 - 3p y_0 y_1 y_2$$

$$x_0 = y_2$$

By restricting to the subvariety $Y \subseteq X$ given by $x_0 = y_2 = 0$ we obtain a smooth cubic surface given by the equation

$$2p^2 x_1^3 + 4p^4 x_2^3 = y_0^3 + p y_1^3$$

which after a simple variable change becomes the surface

$$2px^3 + 4y^3 + p^2 z^3 + w^3 = 0.$$

This surface is actually defined over $\mathbb{Q}$, and it is straightforward to check that it has points everywhere locally. We hence obtain a family of cubic surfaces over $\mathbb{Q}$ which are counter-examples to the Hasse principle. This family is a particular subfamiliy of the family [CTKS87, §III.2].

References

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