The Relative Étale Shape and Obstructions to Rational Points

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Outline

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- The classical obstructions to rational and integral points

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- Future directions

The Classical Obstructions

Number Fields

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  - A finer obstruction set

$$X^{\mathrm{Br}}(\mathbb{A}_K) = \emptyset \Longrightarrow X(K) = \emptyset$$

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  - Not a complete obstruction - counter example constructed by B. Poonen in 2008

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  - Intersect obstruction sets with S-integral adelic points:

$$\mathcal{X}^{\mathrm{Br}}(\mathbb{A}_{K,S}) = X^{\mathrm{Br}}(\mathbb{A}) \cap \mathcal{X}(\mathbb{A}_{K,S})$$
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  - A k-rational point of X induces a section

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- $\acute{E}t(X) = |X_{ét}|$ - known as the **étale homotopy type** (Artin and Mazur, 1969)
- The comparison theorem: for $X/\mathbb{C}$ we have $\acute{E}t(X) \cong$ the pro-finite completion of $X(\mathbb{C})$

The Relative Étale Homotopy Type

- A relative situation $X \longrightarrow S$: wish to study sections

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- Solution: relative étale homotopy type $\acute{E}t_{/S}(X)$ - a pro-object in the homotopy category of "sheaves of spaces" on S

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  - General case - formulate via Quillen's notion of a **model category** (Jardin, Joyal and others)

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$$H^s(B, \pi_t(F)) \Rightarrow \pi_{t-s}(\mathrm{Sec}(f))$$

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The Relative Homotopy Obstruction

General Base Schemes

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- Can use the obstructions above to show $X(hS) = \emptyset$ and hence $X(S) = \emptyset$
- If $X(hS) \neq \emptyset \Rightarrow$ can still use to classify S-points of X

- Generalized proper base change theorem:

$$\{(\mathcal{F}_\alpha)_s\} \cong \acute{E}t_{/k}(X_s)$$

for each closed point $s : \mathrm{Spec}(k) \hookrightarrow S$

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Proper Base Change

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- Allows one to predict the homotopy type of the fibers of $\acute{E}t_{/S}(X)$
- Example: if $S = \mathrm{Spec}(k)$ for a field k then the underlying pro homotopy type of $\acute{E}t_{/k}(X)$ is $\acute{E}t_{/\overline{k}}(X \otimes_k \overline{k})$ (but we have an additional structure of a Γ_k-action)

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- Exactly Grothendieck's section obstruction

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- Obtain an obstruction element in $H^2_{\acute{e}t}(S, \mu_n)$
- Can be shown to match the image of the element $c \in H^1(S, \mathbb{G}_m)$ classifying X

- Let $\mathrm{char}(k) = 0$ and X/k given by $\sum_{i=0}^{n} a_i x_i^2 = 1$ with $0 \neq a_i \in k$

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- Can be shown to equal the cup product $\bigcup_{i=0}^{n} [a_i]$ where $[a_i] \in H^1(\Gamma_k, \mathbb{Z}/2) \cong k^*/(k^*)^2$ is the class of a_i

- $\mathbb{A}^1 \longrightarrow \mathrm{Spec}(\mathbb{Z})$

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Examples - The Affine Line

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- For every p the fiber $(\mathbb{A}_1)_p = \mathrm{Spec}(\mathbb{F}_p[t])$ has big fundamental group - many Artin-Schrier extensions
- E.g. the extension $y^p + y = t$ translates to a sheaf of spaces $\mathcal{F}_p \in \acute{E}t_{/\mathrm{Spec}(\mathbb{Z})}(\mathbb{A}^1)$ with connected fibers such that

$$\pi_1(\mathcal{F}_p) = (\iota_p)_* \mathbb{Z}/p$$

and higher homotopy groups vanish

- $\pi_0\left(\mathrm{hSec}(S, \mathcal{F}_p)\right) \cong H^1_{ét}(\mathrm{Spec}(\mathbb{Z}), \pi_1(\mathcal{F}_p)) \cong \mathbb{Z}/p$

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Examples - The Affine Line (cont.)

- $\pi_0(\mathrm{hSec}(S, \mathcal{F}_p)) \cong H^1_{\acute{e}t}(\mathrm{Spec}(\mathbb{Z}), \pi_1(\mathcal{F}_p)) \cong \mathbb{Z}/p$
- The resulting map

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The Relative Homotopy Obstruction

Examples - The Affine Line (cont.)

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The Relative Homotopy Obstruction

The Local Global Principle

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- $\Rightarrow$ a new obstruction set

$$X^h(\mathbb{A}) \subseteq X(\mathbb{A})$$

by taking adelic points whose corresponding homotopy fixed points are rational

Theorem (H., S. 2010)

For X smooth and geometrically connected one has

$$X^h(\mathbb{A}) = X^{\mathrm{fin,Br}}(\mathbb{A})$$

- Given an augmented functor $F : \mathrm{Ho}(\mathrm{Top}) \longrightarrow \mathrm{Ho}(\mathrm{Top})$ one can construct a new (weaker) obstruction set $X^F(\mathbb{A})$ by replacing $\acute{E}t_{/K}(X)$ with $F(\acute{E}t_{/K}(X))$

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  - For $F = P_1 \circ \mathbb{Z}$ recover finite abelian descent: $X^F(\mathbb{A}) = X^{\text{fin-ab}}(\mathbb{A})$

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Thank you for listening!