The classical obstructions to rational and integral points
Outline

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- The relative étale shape
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- Future directions
The local obstruction

\[ X(\mathbb{A}_K) = \emptyset \implies X(K) = \emptyset \]
The Classical Obstructions

Number Fields

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  - Pairing with elements in the Brauer group
  \[ X(\mathbb{A}_K) \times H^2_{\text{ét}}(X, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z} \]
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- A finer obstruction set

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The Classical Obstructions

Number Fields

- $Y \rightarrow X$ - a torsor under an algebraic group $G/K$

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- Intersecting over various families of algebraic groups gives various obstruction sets -
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- D. Harari (02): for $X$ smooth projective one has $X^{\text{con}}(\mathbb{A}) = X^{\text{Br}}(\mathbb{A})$
The Classical Obstructions

Number Fields

- The étale-Brauer obstruction

Defined by A. Skorobogatov in 1999. Uses the Brauer-Manin obstruction applied to finite torsors. Obtain a finer obstruction set $X_{\text{fin}}, Br(A_k) = \emptyset \implies X(K) = \emptyset$. Stronger than all previously known obstructions. A. Skorobogatov (09), C. Demarch (09): for $X$ smooth projective one has $X_{\text{fin}}, Br(A) = X_{\text{desc}}(A)$.

Not a complete obstruction - counter example constructed by B. Poonen in 2008.
The Classical Obstructions

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The Relative Étale Shape and Obstructions to Rational Points
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Integral Points

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The Classical Obstructions

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  - \( \mathcal{O}_{K,S} \) - the ring of \( S \)-integers of a number field \( K \)

Study \( S \)-integral points, i.e. sections of the form

\[ X \rightarrow \text{Spec}(\mathcal{O}_{K,S}) \]

Intersect obstruction sets with \( S \)-integral adelic points:

\[ X_{\text{Br}}(A_K, S) = X_{\text{Br}}(A) \cap X(A_K, S) \]

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of étale fundamental groups

- A $k$-rational point of $X$ induces a section

$$\pi_1^{\text{ét}}(X) \xrightarrow{\text{section}} \text{Gal}(\overline{k}/k)$$
The Étale Homotopy Type

- $X$ - a scheme

Classical Shape theory - Associate to a site $C$ a pro-object $\vert C \vert = \{ X_\alpha \}$ in the homotopy category of topological spaces $H_n(C, F) \sim \lim_{\alpha} H_n(\vert C \vert, F)$ for constant sheaves $F$. 

The comparison theorem: for $X/\mathcal{C}$ we have $\text{Ét}(X) \sim \text{the pro-finite completion of } X(\mathcal{C})$. 

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The Relative Étale Shape and Obstructions to Rational Points
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$$f : \mathbb{A}^1_{/\mathbb{C}} \rightarrow \mathbb{A}^1_{/\mathbb{C}}$$

with $f(x) = x^2$
The Relative Étale Homotopy Type

- A relative situation $X \longrightarrow S$: wish to study sections

\[
\begin{array}{ccc}
X & \longrightarrow & S \\
\downarrow & & \downarrow \\
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- Solution: relative étale homotopy type $\hat{\text{Et}}_{/S}(X)$ - a pro-object in the homotopy category of “sheaves of spaces” on $S$
What is an (étale) sheaf of spaces on $S$?
The Relative Étale Homotopy Type

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- $S = \text{Spec}(k)$ when $k$ is a general field - a space with an action of $\text{Gal}(\overline{k}/k)$
- General case - formulate via Quillen’s notion of a model category (Jardin, Joyal and others)
Section Obstructions in Algebraic Topology

$E \to B$ - a Serre fibration of topological spaces with fiber $F$

$B$ is a CW complex $\Rightarrow$ can study sections inductively on skeletons.

In $n$'th step face an obstruction in $H_{n+1}(B, \pi_n(F))$.

If all obstructions vanish - a spectral sequence $H_s(B, \pi_t(F)) \Rightarrow \pi_{t-s}(\text{Sec}(f))$.
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The Relative Homotopy Obstruction

General Base Schemes

$\xrightarrow{} X \rightarrow S$ - a scheme over a base scheme $S$
The Relative Homotopy Obstruction

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The Relative Homotopy Obstruction

General Base Schemes

- $X \longrightarrow S$ - a scheme over a base scheme $S$
- $\hat{\text{Et}}/S(X)$ - an inverse family of sheaves of spaces $\{F_\alpha\}_{\alpha \in I}$ on $S$
- For each $F_\alpha$ obtain obstructions to the existence of a (homotopy) global section that live in $H^{n+1}_{\text{ét}}(S, \pi_n(F_\alpha))$
The Relative Homotopy Obstruction
General Base Schemes

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- For each $\mathcal{F}_\alpha$ obtain obstructions to the existence of a (homotopy) global section that live in $H^{n+1}_{\text{ét}}(S, \pi_n(\mathcal{F}_\alpha))$
- If all obstructions vanish - a spectral sequence

$$H^s_{\text{ét}}(S, \pi_t(\mathcal{F}_\alpha)) \Rightarrow \pi_{t-s}(h\text{Sec}(S, \mathcal{F}_\alpha))$$
A section $X \to S$ gives a compatible choice of (homotopy) global sections $s_\alpha \in \mathcal{F}_\alpha(S)$. In particular one obtains a map $X(S) \to X(hS)$ defined as $\lim_\alpha \pi_0(hSec(S, F_\alpha))$. Can use the obstructions above to show $X(hS) = \emptyset$ and hence $X(S) = \emptyset$. If $X(hS) \neq \emptyset$ still use to classify $S$-points of $X$. 
A section $X \hookrightarrow S$ gives a compatible choice of (homotopy) global sections $s_\alpha \in \mathcal{F}_\alpha(S)$.

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Can use the obstructions above to show \( X(hS) = \emptyset \) and hence \( X(S) = \emptyset \).

If \( X(hS) \neq \emptyset \Rightarrow \) can still use to classify \( S \)-points of \( X \).
Generalized proper base change theorem:

\[ \{(\mathcal{F}_\alpha)_s\} \cong \mathbf{Et}_k(X_s) \]

for each closed point \( s : \text{Spec}(k) \hookrightarrow S \).
The Relative Homotopy Obstruction

Proper Base Change

- Generalized proper base change theorem:

\[ \{ (F_\alpha)_s \} \cong \hat{\text{Et}}/k(X_s) \]

for each closed point \( s : \text{Spec}(k) \hookrightarrow S \)

- Allows one to predict the homotopy type of the fibers of \( \hat{\text{Et}}/S(X) \)
Generalized proper base change theorem:

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for each closed point \( s : \text{Spec}(k) \hookrightarrow S \).

- Allows one to predict the homotopy type of the fibers of \( \acute{\text{E}}t_{/S}(X) \).

- Example: if \( S = \text{Spec}(k) \) for a field \( k \) then the underlying pro homotopy type of \( \acute{\text{E}}t_{/k}(X) \) is \( \acute{\text{E}}t_{/\overline{k}}(X \otimes_k \overline{k}) \) (but we have an additional structure of a \( \Gamma_k \)-action).
The Relative Homotopy Obstruction
For Fields

- $S = \text{Spec}(k)$ for a field $k$
The Relative Homotopy Obstruction

For Fields

- $S = \text{Spec}(k)$ for a field $k$
- First Obstruction to $X(hS) \neq \emptyset$ being non-empty is

\[ o_1 \in H^2_{\text{Gal}}(k, \pi^\text{ét}_1(\bar{X})) \]
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For Fields

- $S = \text{Spec}(k)$ for a field $k$
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- Exactly Grothendieck’s section obstruction

  $$1 \longrightarrow \pi_1^{\text{ét}}(\bar{X}) \longrightarrow \pi_1^{\text{ét}}(X) \longrightarrow \text{Gal}(\bar{k}/k) \longrightarrow 1$$
Let $X \rightarrow S$ be a $\mathbb{G}_m$-torsor
The Relative Homotopy Obstruction

Examples

- Let $X \rightarrow S$ be a $\mathbb{G}_m$-torsor
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- Obtain a sheaf $\mathcal{F}_n \in \hat{\text{Et}}_S(X)$ with connected fibers such that

$$\pi_1(\mathcal{F}_n) \cong \mu_n$$

and no higher homotopy groups
The Relative Homotopy Obstruction

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- Let $X \rightarrow S$ be a $\mathbb{G}_m$-torsor
- $n$ - coprime to the characteristics of all closed points in $S$
- Obtain a sheaf $\mathcal{F}_n \in \acute{E}t_S(X)$ with connected fibers such that
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  and no higher homotopy groups
- Obtain an obstruction element in $H^2_{\acute{E}t}(S, \mu_n)$
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  \[ \pi_1(\mathcal{F}_n) \cong \mu_n \]
  and no higher homotopy groups
- Obtain an obstruction element in $H^2_{\text{ét}}(S, \mu_n)$
- Can be shown to match the image of the element $c \in H^1(S, \mathbb{G}_m)$ classifying $X$
Let \( \text{char}(k) = 0 \) and \( X/k \) given by
\[
\sum_{i=0}^{n} a_i x_i^2 = 1 \quad \text{with} \quad 0 \neq a_i \in k
\]
Let \( \text{char}(k) = 0 \) and \( X/k \) given by
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\]

\( \hat{\text{Et}}_k(X) \cong \) pro-finite completion of \( n \)-sphere
(with some \( \Gamma_K \)-action) \( \Rightarrow \hat{\text{Et}}_k(X) \) contains
the space \( K(\mathbb{Z}/2, n) \)
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Examples

- Let $\text{char}(k) = 0$ and $X/k$ given by
  $\sum_{i=0}^{n} a_i x_i^2 = 1$ with $0 \neq a_i \in k$

- $\mathcal{E}t_{/k}(X) \cong$ pro-finite completion of $n$-sphere
  (with some $\Gamma_K$-action) $\Rightarrow \mathcal{E}t_{/k}(X)$ contains
  the space $K(\mathbb{Z}/2, n)$

- Obtain an obstruction element in Galois cohomology $H^{n+1}(\Gamma_k, \mathbb{Z}/2)$
The Relative Homotopy Obstruction

Examples

- Let $\text{char}(k) = 0$ and $X/k$ given by $\sum_{i=0}^{n} a_i x_i^2 = 1$ with $0 \neq a_i \in k$
- $\hat{\text{Et}}_k(X) \cong \text{pro-finite completion of } n\text{-sphere (with some } \Gamma_K\text{-action)} \Rightarrow \hat{\text{Et}}_k(X)$ contains the space $K(\mathbb{Z}/2, n)$
- Obtain an obstruction element in Galois cohomology $H^{n+1}(\Gamma_k, \mathbb{Z}/2)$
- Can be shown to equal the cup product $\bigcup_{i=0}^{n} [a_i]$ where $[a_i] \in H^1(\Gamma_k, \mathbb{Z}/2) \cong k^*/(k^*)^2$ is the class of $a_i$
The Relative Homotopy Obstruction

Examples - The Affine Line

\[
\mathbb{A}^1 \longrightarrow \text{Spec}(\mathbb{Z})
\]
The Relative Homotopy Obstruction

Examples - The Affine Line

- $\mathbb{A}^1 \to \text{Spec}(\mathbb{Z})$

- For every $p$ the fiber $(\mathbb{A}_1)_p = \text{Spec}(\mathbb{F}_p[t])$ has big fundamental group - many Artin-Schrier extensions
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Examples - The Affine Line

- $\mathbb{A}^1 \to \text{Spec}(\mathbb{Z})$
- For every $p$ the fiber $(\mathbb{A}_1)_p = \text{Spec}(\mathbb{F}_p[t])$ has big fundamental group - many Artin-Schrier extensions
- E.g. the extension $y^p + y = t$ translates to a sheaf of spaces $\mathcal{F}_p \in \check{\text{Et}}_{/\text{Spec}(\mathbb{Z})}(\mathbb{A}^1)$ with connected fibers such that

$$\pi_1(\mathcal{F}_p) = (\iota_p)_*\mathbb{Z}/p$$

and higher homotopy groups vanish
The Relative Homotopy Obstruction

Examples - The Affine Line (cont.)

\[ \pi_0 \left( \text{hSec}(S, \mathcal{F}_p) \right) \cong H^1_{\text{ét}}(\text{Spec}(\mathbb{Z}), \pi_1(\mathcal{F}_p)) \cong \mathbb{Z}/p \]
\( \pi_0(\text{hSec}(S, \mathcal{F}_p)) \cong H_1^{\text{ét}}(\text{Spec}(\mathbb{Z}), \pi_1(\mathcal{F}_p)) \cong \mathbb{Z}/p \)

- The resulting map

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The Relative Homotopy Obstruction

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- \( \Rightarrow \) the map \( \mathbb{A}^1(\text{Spec}(\mathbb{Z})) \longrightarrow \mathbb{A}^1(\text{hSpec}(\mathbb{Z})) \) is injective
The Relative Homotopy Obstruction

Examples - The Affine Line (cont.)

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\[ \Rightarrow \text{ the map } \mathbb{A}^1(\text{Spec}(\mathbb{Z})) \longrightarrow \mathbb{A}^1(h\text{Spec}(\mathbb{Z})) \text{ is injective} \]

\[ \Rightarrow \text{ the map } X(\text{Spec}(\mathbb{Z})) \longrightarrow X(h\text{Spec}(\mathbb{Z})) \text{ is injective for every affine scheme} \]

\[ X \longrightarrow \text{Spec}(\mathbb{Z}) \]
The Relative Homotopy Obstruction
The Local Global Principle

- $S = \text{Spec}(K)$ for a number field $K$ with absolute Galois group $\Gamma_K$
The Relative Homotopy Obstruction

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The Relative Homotopy Obstruction
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Obtain a commutative diagram of sets

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\begin{array}{c}
X(K) \rightarrow X(hK) \\
\downarrow \quad \downarrow \\
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Rightarrow a new obstruction set

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⇒ a new obstruction set

\[X^h(\mathbb{A}) \subseteq X(\mathbb{A})\]

by taking adelic points whose corresponding homotopy fixed points are rational.
Theorem (H., S. 2010)

For $X$ smooth and geometrically connected one has

$$X^h(\mathbb{A}) = X^{\text{fin}, \text{Br}}(\mathbb{A})$$
Given an augmented functor $F : \text{Ho}(\text{Top}) \longrightarrow \text{Ho}(\text{Top})$ one can construct a new (weaker) obstruction set $X^F(\mathbb{A})$ by replacing $\acute{\text{E}}t_K(X)$ with $F(\acute{\text{E}}t_K(X))$. 

Examples (H. S. 2010):

- For $F = P_1$ the first Postnikov piece functor recover finite descent: $X^F(A) = X^{\text{fin}}(A)$
- For $F = \mathbb{Z}$ the free abelian group functor recover the Brauer-Manin obstruction: $X^F(A) = X^{\text{Br}}(A)$
- For $F = P_1 \circ \mathbb{Z}$ recover finite abelian descent: $X^F(A) = X^{\text{fin-ab}}(A)$
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Applications

- Let $K$ be number field and assume all varieties are smooth and geometrically connected.
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\end{array}
\]
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- Higher dimensional fields
Thank you for listening!