The descent-fibration method for integral points - Bristol lecture

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1 Introduction

A prominent problem in number theory and arithmetic geometry is to understand the set of integral solutions to equations with integral coefficients. In modern terms one usually fixes a number field k (i.e., a finite extension of the field $\mathbb{Q}$ of rational numbers), a finite set S of places of k, and a scheme $\mathcal{X}$ of finite type over the ring $\mathcal{O}_S \subseteq k$ of **S-integers**. To make things more concrete, let us take $k = \mathbb{Q}$, in which case we can identify S with a finite set of prime numbers (although technically we always include the infinite places in S), and the ring of S-integers $\mathbb{Z}_S$ is the ring of numbers of the form $\frac{a}{b}$ where b is only divisible by primes in S. To make things even more concrete, one may also imagine that $\mathcal{X}$ is affine, i.e., it is defined by a finite set of equations with coefficients in $\mathbb{Z}_S$. We will denote by the letter X the base change of $\mathcal{X}$ from $\mathbb{Z}_S$ to $\mathbb{Q}$, i.e., the variety over $\mathbb{Q}$ given by the same equations as $\mathcal{X}$. In this talk, we will attempt to say something intelligent about S-integral points on the affine scheme $\mathcal{X} \subseteq \mathbb{A}^4$ given by an equation of the form

$$at^2x^2 + bs^2x^2 + ct^2y^2 + ds^2y^2 = 1 \tag{1}$$

where $a, b, c, d \in \mathbb{Z}_S$ are S-integers such that $abcd(ad - bc) \neq 0$. However, we first wish to give a proper context for this question.

When studying S-integral points, one often begins by considering the set of S-integral **adelic points**

$$\mathcal{X}(\mathbb{A}) \stackrel{\text{def}}{=} \prod_{p \in S} X(\mathbb{Q}_p) \times \prod_{p \notin S} \mathcal{X}(\mathbb{Z}_p).$$

If $\mathcal{X}(\mathbb{A}) = \emptyset$ one may immediately deduce that $\mathcal{X}$ has no S-integral points. When a family of schemes satisfies the implication $\mathcal{X}(\mathbb{A}) \neq \emptyset \Rightarrow \mathcal{X}(\mathbb{Z}_S) \neq \emptyset$ we say that the family satisfies the S-integral **Hasse principle**. The name is due to the Hasse-Minkowski theorem which asserts the analogous claim for rational points when X is a quadratic hypersurface. For example, if $q(x_1, ..., x_n)$ is an **indefinite** quadratic form in $n \geq 4$ variables then the equation $q(x_1, ..., x_n) = a$

satisfies the S-integral Hasse principle (Eichler, Kneser). The same holds for certain forms of higher degree, if the number of variables is large enough (circle method).

In general, it can certainly happen that $\mathcal{X}(\mathbb{A}) \neq \emptyset$ but $\mathcal{X}(\mathbb{Z}_S)$ is still empty. One way to account for this phenomenon is given by the integral version of the **Brauer-Manin obstruction**, introduced in [CTX09]. This is done by constructing a natural subset $\mathcal{X}(\mathbb{A})^{\mathrm{Br}} \subseteq \mathcal{X}(\mathbb{A})$, which can be defined informally as those S-integral adelic points which satisfy a certain coherence condition relating the various primes, and based on the notion of global reciprocity arising from class field theory.

When $\mathcal{X}(\mathbb{A})^{\mathrm{Br}} = \emptyset$ one says that there is a Brauer-Manin obstruction to the existence of S-integral points. A central question in the study of integral points is then the following:

question 1.1. *Given a family $\mathcal{F}$ of varieties does the property $\mathcal{X}(\mathbb{A})^{\mathrm{Br}} \neq \emptyset$ implies that $\mathcal{X}$ has an S-integral point for every $\mathcal{X} \in \mathcal{F}$?*

When the answer to Question 1.1 is yes one says that the Brauer-Manin obstruction is the only obstruction to the existence of S-integral points for the family $\mathcal{F}$. In their paper, Colliot-Thélène and Xu showed that if X is a homogeneous space under a simply-connected linear algebraic group G with connected geometric stabilizers, and G satisfies a certain non-compactness condition over S, then the Brauer-Manin obstruction is the only obstruction to the existence of S-integral points on $\mathcal{X}$. Similar results hold when X is a principal homogeneous space of an algebraic group of multiplicative type (Wei, Xu). On the other hand, there are several known types of counter-examples, i.e., families for which the answer to Question 1.1 is negative. One way to construct such counter-example is to consider varieties which are not simply-connected. In this case, one can sometimes refine the Brauer-Manin obstruction by applying it to various unramified coverings of X (Colliot-Thèléne, Wittenberg). Other types of counter-examples occur when X lacks a sufficient supply of local S-points "at infinity". For example, let $r < p < q \in \mathbb{Z}$ be odd positive non-square integers such that the quaternion algebra $(r, -p)$ is nowhere ramified (e.g. $r, p = 13, 17$). Then the affine scheme $\mathcal{X}$ given by the equation

$$x^2 + py^2 + qz^2 + pqw^2 = r$$

has a real point and integral points everywhere locally. However, since r is not a square and $p, q > r$ there is no global integral point on $\mathcal{X}$. This violation of the integral Hasse principal is not explained by the Brauer-Manin obstruction. We claim that the problem here arises from the fact that X lacks real points at infinity (note that in this case the set S contains only the real place). To see this, we may consider the smooth compacitification $\overline{X} \subseteq \mathbb{P}^4$ of X given by

$$x^2 + py^2 + qz^2 + pqw^2 = rv^2$$

The complement $D = \overline{X} \setminus X$ is determined, inside X, by the condition $v = 0$, and can hence be identified with the surface inside $\mathbb{P}^3$ given by

$$x^2 + py^2 + qz^2 + pqw^2 = 0$$

Since $p, q > 0$ we see that D does not have any real points. Another way to see this phenomenon (in this case) is that the space of real points of X is compact. As a result, there can a-priori only be finitely many integral points, even though all S-integral adelic points satisfy "global reciprocity".

We hence see that the behavior of Question 1.1 for integral points is quite subtle. In order to obtain a better understanding of it it is important to have good tools to establish the existence of integral points, when possible. For this let us recall the standard techniques from the world of **rational points** which are used to establish existence (under suitable conditions):

1. Method relying on special structure and special circumstances. Examples include homogeneous spaces (Borovoi), varieties with special configurations of singular points, varieties with a 0-dimensional moduli, such as del Pezzo surfaces of degree ≥ 5 (Manin, CT), and others.

2. Analytic methods, such as modern variants of the circle method, sieve methods and others. Such methods typically apply to varieties defined by equations whose degrees are small compared to the dimension of the variety (for smooth proper varieties this can be expressed intrinsically by saying that anti-canonical class is "big"). For example, smooth cubic hypersurfaces of dimension n have a rational point if $n \geq 9$ (Heath-Brown) and satisfy the Hasse principle if $n \geq 8$ (Hooley).

3. The descent method. When X satisfies certain geometric conditions and $X(\mathbb{A})^{\mathrm{Br}} \neq \emptyset$, one may construct a torus torsor $Y \longrightarrow X$, known as the **universal torsor**, such that $Y(\mathbb{A}) \neq \emptyset$. Furthermore the geometry of Y is often simpler than that of X. One may then reduce question 1.1 for X to the question of the Hasse principle for Y, and attack the latter using one of the methods above. The most well known example of this is CT, SD and Sansuc's proof that the Brauer-Manin obstruction is the only for Châtlet surfaces.

4. The fibration method. This method can sometimes be applied when X is equipped with a dominant morphism $\pi : X \longrightarrow B$ such that the arithmetic of B and the generic fiber of π is simple enough. In a typical application the base B is the projective line $\mathbb{P}^1$, the generic fiber of π is geometrically integral, and one is interested in reducing Question 1.1 for X to Question 1.1 for the smooth fibers of π. This can be achieved in the following circumstances:

    (a) When all the fibers of π but one are split (CT, Sansuc, Swinnerton-Dyer, Skorobogatov, Harari).

    (b) Under Schinzel's hypothesis, when all the fibers of π split in an abelian extension of the base field and the Brauer groups of all the fibers is trivial (CT, Sansuc, Swinnerton-Dyer, Serre, Skorobogatov).

    (c) When all the non-split fibers are defined over the base field (H, Wittenberg).

5. The descent-fibration method. This method first appeared in Swinnerton-Dyer's paper [SD95], where it was applied to diagonal Del-Pezzo surfaces of degree 4. It was later expanded and generalized to deal with semi-diagonal Del-Pezzo surfaces of degree 4 ([BSD01],[CT01],[Wit07]), diagonal Del-Pezzo surfaces of degree 3 ([SD01]), Kummer surfaces ([SDS05], [HS]) and more general elliptic fibrations ([CT01],[Wit07]). In all these papers one is trying to establish the existence of rational points on a variety X. In order to apply the method one exploits a suitable geometric structure on X in order to reduced the problem to the construction of rational points on a suitable fibered variety $Y \longrightarrow \mathbb{P}^1$ whose fibers are torsors under a family $A \longrightarrow \mathbb{P}^1$ of abelian varieties. The first step is to apply the fibration method above in order to find a $t \in \mathbb{P}^1(k)$ such that the fiber Y_t has points everywhere locally (this part typically uses the vanishing of the Brauer-Manin obstruction, and often requires Schinzel's hypothesis). The second step then consists of modifying t until the Tate-Shafarevich group $Ш^1(A_t)$ (or a suitable part of it) vanishes, implying the existence of a k-rational point on Y_t. This part usually assumes, in additional to a possible Schinzel hypothesis, the finiteness of the Tate-Shafarevich group for all relevant abelian varieties, and crucially relies on the properties of the Cassels-Tate pairing. The descent-fibration method is currently the only method powerful enough to prove (though often conditionally) the existence of rational points on families of varieties which includes non-rationally connected varieties, such as K3 surfaces.

In principle, methods (1)-(3) above can often be generalized to the context of integral points, occasionally under addition conditions (such as conditions regarding real points at infinity, etc.). However, for the descent-fibration method no such generalization is known. As a consequence, there is currently no class of non-proper varieties for which question 1.1 is known to have a positive answer, except for classes which include only homogeneous spaces or only varieties with very big (log) anti-canonical class. In particular, question 1.1 is not known to have a positive answer for any class of varieties containing log K3 surfaces (or, more generally, log Calabi-Yau varieties, i.e., those whose log anti-canonical class vanishes). In this talk I will describe an approach for adapting the descent-fibration method to the world of integral points.

The key point is that when working with integral points, one should replace torsors under abelian varieties with **torsors under algebraic tori**. Let us return to our example $\mathcal{X} \subseteq \mathbb{A}^4$ given by

$$at^2x^2 + bs^2x^2 + ct^2y^2 + ds^2y^2 = 1 \tag{2}$$

Let us set $f(t, s) = at^2 + bs^2$ and $g(t, s) = ct^2 + ds^2$. We can then write our equation as

$$f(t, s)x^2 + g(t, s)y^2 = 1$$

If we fix a pair (t_0, s_0), the equation becomes an affine conic

$$ax^2 + by^2 = 1$$

with $a = f(t_0, s_0)$ and $b = g(t_0, s_0)$. Furthermore, if a, b are coprime then this conic is a torsor under the norm 1 torus $\mathcal{T}_{(t,s)}$ given by the equation

$$x^2 - dy^2 = 1$$

where $d = -ab$. We may hence consider $\mathcal{X}$ has a two parameter family of torsors under tori. The homogeneous quadratic dependence on t, s suggests that we should actually think of this as a one parameter family. Indeed, there is a natural action of $\mathbb{G}_m$ on $\mathcal{X}$ given by

$$(t, s, x, y) \mapsto (ut, us, u^{-1}x, u^{-1}y)$$

The quotient $\mathcal{Y} = \mathcal{X}/\mathbb{G}_m$ is a smooth 2-dimensional scheme over $\mathbb{Z}_S$ and we obtain a Cartesian square of $\mathbb{Z}_S$-schemes

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{q} & \mathcal{Y} \\ \downarrow & & \downarrow \pi \\ \mathbb{A}^2_S \setminus \{(0,0)\} & \longrightarrow & \mathbb{P}^1_S \end{array}$$

where the bottom horizontal map is the standard projection. In particular, given an S-integral point $(t, s) \in \mathbb{A}^2_S \setminus \{(0, 0)\}$ we may identify the fiber $\mathcal{X}_{(t,s)}$ with the fiber $\mathcal{Y}_{(t:s)}$. The surface $\mathcal{Y}$ is now a (one dimensional) pencil of torsors under algebraic tori. The varieties $\mathcal{X}$ and $\mathcal{Y}$ are very similar. They are both log Calabi-Yau, and for any field k the map $X(k) \longrightarrow Y(k)$ is surjective. Furthermore, one can show that Question 1.1 is equivalent for $\mathcal{X}$ and $\mathcal{Y}$.

The first part of the descent-fibration method is the fibration method. Applied to the case of either $\mathcal{Y}$ or $\mathcal{X}$ we may obtain a pair (t_0, s_0) such that the associated $\mathcal{T}_{(t_0,s_0)}$-torsor

$$ax^2 + by^2 = 1 \tag{3}$$

has S-integral points everywhere locally. In general, this does not imply that 5 has an S-integral points. However, like with elliptic curves or abelian varieties, there exists a finite abelian group, that we will denote here by $Ш^1(\mathcal{T}_{(t_0,s_0)}) = Ш^1(\mathcal{T}_{(t_0,s_0)}, S(t_0, s_0))$ (where $S(t_0, s_0)$ is suitable set of places depending on (t_0, s_0)), which classifies exactly such everywhere soluble torsors modulu the soluble ones (where soluble here means soluble by an $S(t_0, s_0)$-integral element). Furthermore, our torsors 3 are not arbitrary torsors of $\mathcal{T}_{t,s}$. They are in fact classified by a 2-torsion element of $Ш^1(\mathcal{T}_{(t_0,s_0)})$. Following the descent-fibration strategy, one would like to modify the fiber (t_0, s_0) into another one, (t_1, s_1), such that the 2-torsion part $Ш^1(\mathcal{T}_{(t_1,s_1)})[2]$ vanishes. To achieve this, one may consider the natural perfect pairing

$$Ш^1(\mathcal{T}_{(t,s)}) \times Ш^2(\widehat{\mathcal{T}}_{(t,s)}) \longrightarrow \mathbb{Q}/\mathbb{Z} \tag{4}$$

where $\widehat{\mathcal{T}}_{(t,s)}$ is the character group of $\mathcal{T}_{(t,s)}$, considered as an étale sheaf over $\mathrm{spec}(\mathbb{Z}_{S(t,s)})$. Hence if we can find a fiber (t_1, s_1) such that $Ш^2(\widehat{\mathcal{T}}_{(t_1,s_1)})[2] =$

0 then we would be able to conclude that $Ш^1(\mathcal{T}_{(t_1,s_1)})[2] = 0$, and hence that our fiber $\mathcal{X}_{(t_1,s_1)}$ has an S-integral point. In order to be able to control $Ш^1(\mathcal{T}_{(t_1,s_1)})[2]$ we have developed a suitable 2-**descent formalism**, analogous to that of elliptic curves an abelian varieties, in which one describes a certain **Selmer group** $\mathrm{Sel}(\widehat{\mathcal{T}}_{(t,s)})$ that is mapped surjectively onto $Ш^2(\widehat{\mathcal{T}}_{(t,s)})[2]$. The kernel of this mapping can be explicitly determined, and so given the group $\mathrm{Sel}(\widehat{\mathcal{T}}_{(t,s)})$ one may deduce the size of $Ш^2(\widehat{\mathcal{T}}_{(t,s)})[2]$. In particular, one wishes to reduce the size of $\mathrm{Sel}(\widehat{\mathcal{T}}_{(t,s)})$ until it will coincide with the above said kernel. This requires a delicate analysis of the dependence of $\mathrm{Sel}(\widehat{\mathcal{T}}_{(t,s)})$ on (t,s), which is very similar to the analysis of the fiber dependence of the Selmer group in a pencil of elliptic curves, done in the classical application of the descent fibration method.

Let us now state the main assumptions that are needed in order to apply the method.

Assumption 1.2.

1. *The homogeneous Schinzel's hypothesis holds for the pair* f, g.

2. *The classes of* -1, $-\frac{a}{b}$, $-\frac{c}{d}$, $\frac{ad-bc}{a}$ *and* $-\frac{ad-bc}{c}$ *are linearly independent in* $H^1(k, \mathbb{Z}/2)$*. In particular,* k *does not contain a square root of* -1.

3. *There exists a place* $v_\infty \in S_0$ *and a point* $(t_{v_\infty}, s_{v_\infty}, x_{v_\infty}, y_{v_\infty}) \in \mathcal{X}(k_{v_\infty})$ *such that* $-f(t_{v_\infty}, s_{v_\infty})g(t_{v_\infty}, s_{v_\infty})$ *is a square in* k_{v_∞} *(note that despite the notation* v_∞ *is not assumed to be an infinite place, just a place of* S_0*).*

4. *For every place* $v \notin S_0$ *there exists a* $(t_v, s_v) \in \mathcal{O}_v$ *such that* $\mathrm{val}_v(f(t_v, s_v)g(t_v, s_v)) \leq 1$.

Remark 1.3.

1. Condition 1.2(2) is analogous in some sense to Condition (D) of [CTSSD98b]. Similarly to the case there, one can also show here that condition 1.2(2) implies that the 2-torsion of $\mathrm{Br}(X)$ is contained in $\mathrm{Br}^{\mathrm{vert}}(X)$.

2. Condition 1.2(3) guarantees that S_0-integral points are no obstructed at ∞ (see [Ha] for a discussion of this notion). For example, if $k = \mathbb{Q}$ and $S_0 = \{\infty\}$ then this condition is equivalent to saying that a, b, c, d do not all have the same sign. Note that if $k = \mathbb{Q}$, $S_0 = \{\infty\}$ and all the a, b, c, d have the same sign then an S_0-integral point exists if and only if one of a, b, c, d is equal to 1. In general, if condition 1.2(2) is not satisfied, then there exists a finite procedure for determining whether or not an S_0-integral point exists.

3. Condition 1.2(4) seems to be a biproduct of the method of proof. It is possible that this condition can be removed if the 2-descent procedure (and the arithmetic duality result it relies on) will be extended to more general types of tori.

We are now ready to state our main theorems:

Theorem 1.4. *Let k be a number field and S_0 a finite set of places of k containing all the archimedean places and all the places above 2. Let $a, b, c, d \in \mathcal{O}_{S_0}$ be elements such that $abcd(ad - bc) \neq 0$ and such that conditions 1.2(1)-(4) are satisfied. If there exists an S_0-integral adelic point on $\mathcal{X}$ (resp. $\mathcal{Y}$) which is orthogonal to the vertical Brauer group then there exists an S_0-integral point on $\mathcal{X}$ (resp. $\mathcal{Y}$).*

2 Some more details, if there is time

Let S_0 be a finite set of places of $\mathbb{Q}$ containing the real place, and let $d \in \mathbb{Z}_{S_0}$ be a non-zero S_0-integer. For every divisor $a|d$ we may consider the affine scheme $\mathcal{Z}^a_{S_0}$ given by the equation

$$ax^2 + by^2 = 1 \tag{5}$$

where $b = -\frac{d}{a}$. We are interested in the solubility in $\mathbb{Z}_{S_0}$ of 5. For this we see that a necessary condition is that a and b will be coprime outside S_0. Our goal in this section is formulate a sufficient condition for equations of the form 5 to satisfy the S_0-integral Hasse principle (see Proposition 2.12 below).

Let $K = \mathbb{Q}(\sqrt{d})$ and consider the subring

$$\mathcal{O}_d = \{a + b\sqrt{d} | a, b \in \mathcal{O}_{S_0}\} \subseteq K$$

Let T_0 denote the set of places of K lying above S_0. We note that in general $\mathcal{O}_d$ might fail to coincide with the subring of T_0-integral elements in K, but for our purposes we may assume that it does. Let $I_a \subseteq \mathcal{O}_d$ be the $\mathcal{O}_d$-ideal generated by a and $\sqrt{d}$. The association $(x, y) \mapsto ax + \sqrt{d}y$ identifies the set of S_0-integral points of $\mathcal{X}$ with the set of elements in I_a whose norm is a. We note that I_a is an ideal of norm a (i.e., $\mathcal{O}_d/I_a \cong \mathbb{Z}/a$), and hence we may consider the scheme $\mathcal{Z}^{a,b}_{S_0}$ as parameterizing generators for I_a whose norm is exactly a. Let $\mathcal{T}^d_{S_0}$ denote the algebraic group given the equation

$$x^2 - dy^2 = 1$$

We may identify the S-integral points of $\mathcal{T}^d_{S_0}$ with the set of units in $\mathcal{O}_d$ whose norm is 1 (in which case the group operation is given by multiplication in $\mathcal{O}_d$). We have a natural action of the algebraic group $\mathcal{T}^d_{S_0}$ on the scheme $\mathcal{Z}^{a,b}_{S_0}$ corresponding to multiplying a generator by a unit. This action exhibits $\mathcal{Z}^a_{S_0}$ as a **torsor** under $\mathcal{T}_{S_0}$.

We now note that $\mathcal{Z}^a_{S_0}$ is not an arbitrary torsor of $\mathcal{T}_{S_0}$. Since the norm of I_a is a and since I_a^2 contains a it follows that $I_a^2 = (a)$. We may define a map

$$q : \mathcal{Z}^a_{S_0} \longrightarrow \mathcal{T}_{S_0}$$

which sends a generator x of I_a of norm a to the element $\frac{x^2}{a}$ whose norm is 1. The action of $\mathcal{T}_{S_0}$ on $\mathcal{Z}^a_{S_0}$ is compatible with the action of $\mathcal{T}_{S_0}$ on itself via the multiplication-by-2 map $\mathcal{T}_{S_0} \xrightarrow{2} \mathcal{T}_{S_0}$. We will say that q is a map of $\mathcal{T}_{S_0}$-torsors covering the map $\mathcal{T}_{S_0} \xrightarrow{2} \mathcal{T}_{S_0}$.

The analogy one should keep in mind is that of abelian varieties. Let A be an abelian varity over k and let $\widehat{A}$ be the dual abelian variety. Let X be an A-torsor equipped with a map of A-torsors $q : X \longrightarrow A$ covering the map $A \xrightarrow{2} A$. Then q itself is a torsor under $A[2]$, and determines an element in $\alpha \in H^1(k, A[2])$. If X has points everywhere locally, then α belongs to the Selmer group $\mathrm{Sel}_2(A)$. The torsor X has a rational point if and only if image of α in $\text{Ш}^1(A)[2]$ is trivial. Now we have a the Cassels-Tate pairing

$$\text{Ш}^1(A) \times \text{Ш}^1(\widehat{A}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

which is a perfect pairing if $\text{Ш}^1(A)$ is **finite**. In this case, the vanishing of the 2-torsion subgroup $\text{Ш}^1(\widehat{A})[2]$ would imply the vanishing of the 2-torsion subgroup $\text{Ш}^1(A)[2]$, and hence the solubility of X. In order to determine whether $\text{Ш}^1(\widehat{A})[2]$ vanishes, one may perform the classical **2-descent** process on $\widehat{A}$: one may use local computations in order to compute the finite group $\mathrm{Sel}_2(\widehat{A})$. If, for example, $\mathrm{Sel}_2(\widehat{A})$ consists only if the elements which comes from $A[2]$ under the boundary map associated to the Kummer sequence, then one may conclude that $\text{Ш}^1(\widehat{A})[2]$ is zero and hence X has a rational point.

Remark 2.1. The procedure described above only has a chance to work if A is not isomorphic to $\widehat{A}$ over k, otherwise we will find α itself in $\mathrm{Sel}(\widehat{A}) \cong \mathrm{Sel}(A)$, which will normally not come from a 2-torsion point. However, if A admits a principal polarization coming from a divisor in $\mathrm{Pic}(A)$ then $\text{Ш}^1(A)$ will carry an alternating self-pairing which is non-degenerate if $\text{Ш}^1(A)$ is finite. In this case the 2-torsion subgroup $\text{Ш}^1(A)[2]$ must have an even 2-rank. Then, if we know that $\mathrm{Sel}_2(A)$ is generated by α and the image of $A[2]$ we may conclude that $\dim_2 \text{Ш}^1(A)[2] \leq 1$ and hence that $\text{Ш}^1(A)[2] = 0$ as desired. This idea is a core point in Swinnerton-Dyer's method, where the abelian varieties in question are most often elliptic curves.

In order to perform our 2-descent process we will need to enlarge our set S_0 so that $\mathcal{T}_{S_0}$ will become an algebraic torus. Let S be the union of S_0 with all the places which ramify in K and the prime 2, and let T be the set of places of K which lie above S. Let $\mathcal{O}_{T_1}$ denote the ring of T_1-integers in K. Let $\mathcal{Z}^a_S$ and $\mathcal{T}_S$ be the base changes of $\mathcal{Z}^a_S$ and $\mathcal{T}_S$ from $\mathbb{Z}_{S_0}$ to $\mathbb{Z}_S$. We note that $\mathcal{T}_S$ becomes isomorphic to $\mathbb{G}_m$ after base changing from $\mathbb{Z}_S$ to $\mathcal{O}_{T_1}$, and $\mathcal{O}_{T_1}/\mathbb{Z}_S$ is an étale extension of rings. This means that $\mathcal{T}_S$ is an **algebraic torus** over $\mathbb{Z}_S$. Furthermore, $\mathcal{Z}^a_S$ is now a **torsor** under the torus $\mathcal{T}_S$. We will denote by $\widehat{\mathcal{T}}_S$ be the character group of $\mathcal{T}_S$ considered as an étale sheaf over $\mathrm{spec}(\mathcal{O}_S)$.

In order to proceed with our analysis we will now enforce the following assumption:

Assumption 2.2. *The ring $\mathcal{O}_d$ coincides with the ring of T_0-integers in K.*

Remark 2.3. Condition 2.2 implies, in particular, that d is square-free outside S_0.

Let us begin by checking that this base change did not lose any information, solubility wise:

Lemma 2.4. *Assume condition 2.2 and suppose in addition that either $2 \in S_0$ or that $d = 2d'$ with d' odd. If $\mathcal{Z}^a_S$ has an S-integral point then $\mathcal{Z}^a_{S_0}$ has an S_0-integral point.*

Proof. □

We will use the notation $H^i(\mathbb{Z}_S, \mathcal{F})$ to denote étale cohomology of $\mathrm{spec}(\mathbb{Z}_S)$ with coefficients in the sheaf $\mathcal{F}$.

Definition 2.5.

1. We will denote by $Ш^1(\mathcal{T}_S) \subseteq H^1(\mathbb{Z}_S, \mathcal{T}_S)$ the kernel of the map

$$H^1(\mathbb{Z}_S, \mathcal{T}_S) \longrightarrow \prod_{p \in S} H^1(\mathbb{Q}_p, \mathcal{T}_S \otimes_{\mathbb{Z}_S} \mathbb{Q}_p).$$

2. We will denote by $Ш^2(\widehat{\mathcal{T}}_S) \subseteq H^2(\mathbb{Z}_S, \widehat{\mathcal{T}}_S)$ the kernel of the map

$$H^2(\mathcal{O}_S, \widehat{\mathcal{T}}_S) \longrightarrow \prod_{p \in S} H^2(\mathbb{Q}_p, \widehat{\mathcal{T}}_S \otimes_{\mathbb{Z}_S} \mathbb{Q}_p).$$

Since $\mathcal{T}_S$ is an algebraic torus we may apply [Mil, Theorem 4.6(a), 4.7] and deduce that the groups $Ш^1(\mathcal{T}_S)$ and $Ш^2(\widehat{\mathcal{T}}_S)$ are finite and that the cup product in étale cohomology with compact support induces a perfect pairing

$$Ш^1(\mathcal{T}_S) \times Ш^2(\widehat{\mathcal{T}}_S) \longrightarrow \mathbb{Q}/\mathbb{Z} \tag{6}$$

Since 2 is invertible in $\mathbb{Z}_S$ the multiplication by 2 map $\mathcal{T}_S \longrightarrow \mathcal{T}_S$ is surjective when considered as a map of étale sheaves on $\mathrm{spec}(\mathbb{Z}_S)$. We hence obtain a short exact sequence of étale sheaves

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathcal{T}_S \xrightarrow{2} \mathcal{T}_S \longrightarrow 0.$$

We define the **Selmer group** $\mathrm{Sel}(\mathcal{T}_S) \subseteq H^1(\mathcal{O}_S, \mathbb{Z}/2)$ be the subgroup consisting of all elements whose image in $H^1(\mathbb{Z}_S, \mathcal{T}_S)$ belongs to $Ш^1(\mathcal{T}_S)$. We hence obtain a short exact sequence

$$0 \longrightarrow \mathcal{T}_S(\mathbb{Z}_S)_2 \longrightarrow \mathrm{Sel}(\mathcal{T}_S) \longrightarrow Ш^1(\mathcal{T}_S)[2] \longrightarrow 0$$

where $\mathcal{T}_S(\mathbb{Z}_S)_2$ denotes the cokernel of the map $\mathcal{T}_S(\mathbb{Z}_S) \xrightarrow{2} \mathcal{T}_S(\mathbb{Z}_S)$. Similarly, we have a short exact sequence of étale sheaves

$$0 \longrightarrow \widehat{\mathcal{T}}_S \xrightarrow{2} \widehat{\mathcal{T}}_S \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

and we define the **dual Selmer group** $\mathrm{Sel}(\widehat{\mathcal{T}}_S) \subseteq H^1(\mathbb{Z}_S, \mathbb{Z}/2)$ to be the subgroup consisting of all elements whose image in $H^2(\mathbb{Z}_S, \widehat{\mathcal{T}}_S)$ belongs to $Ш^2(\widehat{\mathcal{T}}_S)$. The dual Selmer group then sits in a short exact sequence of the form

$$0 \longrightarrow H^1(\mathbb{Z}_S, \widehat{\mathcal{T}}_S)_2 \longrightarrow \mathrm{Sel}(\widehat{\mathcal{T}}_S) \longrightarrow Ш^2(\widehat{\mathcal{T}}_S)[2] \longrightarrow 0$$

Remark 2.6. By Dirichlet's unit theorem the 2-rank of $\mathcal{T}_S(\mathcal{O}_S)_2$ is equal to the number of places of S which split in K plus 1. On the dual side, a direct computation shows that the 2-rank of $H^1(\mathbb{Z}_S, \widehat{\mathcal{T}}_S)_2$ is 1. This is in contrast to the elliptic curve case where the analogous group is the Mordell-Weil group whose rank is hard to determine in general.

The map $H^1(\mathbb{Z}_S, \mathbb{Z}/2) = \mathbb{Z}_S^*/(\mathbb{Z}_S^*)^2 \longrightarrow H^1(\mathcal{T}_S)$ can be described explicitly as follows. To each $a \in \mathbb{Z}_S^*$ we may associate the $\mathcal{T}_S$-torsor $\mathcal{Z}_S^a$ given by the equation

$$ax^2 - \frac{d}{a}y^2 = 1.$$

By the association $(x, y) \mapsto ax + \sqrt{d}y$ we may identify S-integral points of $\mathcal{Z}_S^a$ with T-units in K whose norm is a. Since both d and a are S-units we observe that each such torsor has an $\mathcal{O}_v$-point for every $v \notin S$. We hence see that the class of a belongs to $\mathrm{Sel}(\mathcal{T}_S)$ if and only if a is everywhere locally a norm from K. The image of $\mathcal{T}_S(\mathcal{O}_S)_2 \longrightarrow \mathrm{Sel}(\mathcal{T}_S)$ consists of those elements whose image in $H^1(\mathcal{O}_S, \mathcal{T}_S)$ is 0. These are the classes represented by S-units a which are norm of T-units.

On the dual side, we may consider the short exact sequence

$$0 \longrightarrow \widehat{\mathcal{T}}_S \longrightarrow \widehat{\mathcal{T}}_S \otimes \mathbb{Q} \longrightarrow \widehat{\mathcal{T}}_S \otimes (\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

Since $\widehat{\mathcal{T}}_S \otimes \mathbb{Q}$ is a uniquely divisible sheaf we get an identification

$$H^2(\mathbb{Z}_S, \widehat{\mathcal{T}}_S) \cong H^1(\mathbb{Z}_S, \widehat{\mathcal{T}}_S \otimes (\mathbb{Q}/\mathbb{Z}))$$

By the Hocshiled-Serre spectral the latter may be identified with the kernel of the corestriction map

$$\mathrm{Cores} : H^1(\mathcal{O}_T, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(\mathbb{Z}_S, \mathbb{Q}/\mathbb{Z})$$

The map $H^1(\mathbb{Z}_S, \mathbb{Z}/2) \longrightarrow H^2(\mathbb{Z}_S, \widehat{\mathcal{T}}_S)$ can then be identified with the composite

$$H^1(\mathbb{Z}_S, \mathbb{Z}/2) \xrightarrow{\mathrm{res}} H^1(\mathcal{O}_T, \mathbb{Z}/2) \longrightarrow H^1(\mathcal{O}_T, \mathbb{Q}/\mathbb{Z})$$

Indeed, since K/k is a quadratic extension the corestriction of the restriction of any quadratic extension is 0. The group $Ш^2(\widehat{\mathcal{T}}_S)$ is then the group which classifies everywhere unramified cyclic extensions of K which split over T and whose corestriction to $\mathcal{O}_S$ vanishes. Finally, the 2-torsion part $Ш^2(\widehat{\mathcal{T}}_S)[2]$ is the group classifying everywhere unramified quadratic extensions of K, splitting over T, whose corestriction to k vanishes. Using the fact that S contains all the places with residue characteristic 2 we then obtain the following explicit description of $\mathrm{Sel}(\widehat{\mathcal{T}})$:

Corollary 2.7. *Let $a \in \mathbb{Z}_S^*$ be an element. Then $[a] \in \mathrm{Sel}(\widehat{\mathcal{T}}_S)$ if and only if every place in T splits in $K(\sqrt{a})$.*

Corollary 2.8. *The kernel of the map $\mathrm{Sel}(\widehat{\mathcal{T}}_S) \longrightarrow Ш^2(\widehat{\mathcal{T}}_S)$ has rank 1 and is generated by the class $[d] \in \mathrm{Sel}(\widehat{\mathcal{T}}_S)$.*

Remark 2.9. In might not be immediately clear why the map

$$\mathrm{Sel}(\widehat{\mathcal{T}}) \longrightarrow Ш^2(\widehat{\mathcal{T}}_S)[2]$$

is surjective. Elements in the latter are generally represented by elements $\alpha \in K$, which are squares in K_w for every $w \in T$, whose valuations are even outside T and such that $\mathrm{N}_{K/k}(\alpha) = b^2$ for some $b \in \mathbb{Z}_S^*$. By Hilbert 90 we see that in this case there exists a $\beta \in K$ such that $\frac{\beta}{\sigma(\beta)} = \frac{\alpha}{b}$ and so $\alpha = \frac{b\beta^2}{\mathrm{N}_{K/k}(\beta)}$. In particular, α is equivalent mod squares to an element $a = \frac{b}{\mathrm{N}_{K/k}(\beta)}$ which comes from k. Since α has even valuations outside T-unit it follows that a must have even valuations outside S (recall that S contains all the ramified places of K). Since S contains a set of generators for the class group we get that a is equivalent mod squares to an S-unit.

Our proposed method of 2-descent can be considered as way to calculate the 2-ranks of $\mathrm{Sel}(\mathcal{T}_S)$ and $\mathrm{Sel}(\widehat{\mathcal{T}}_S)$. When $k = \mathbb{Q}$ our method can be considered as a repackaging of Gauss' classical genus theory for computing the 2-torsion of the class groups of quadratic fields.

The notation introduced in the next few paragraphs follows the analogous notation of [CTSSD98b] and [CT01]. For each $p \in S$, let V_p and V^p denote two copies of $H^1(\mathbb{Q}_p, \mathbb{Z}/2) \cong \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$, considered as $\mathbb{F}_2$-vector spaces. We will also denote $V_S = \oplus_{p\in S} V_p$ and $V^S = \oplus_{p\in S} V^p$. By taking the sum of the Hilbert symbol pairings

$$\langle , \rangle_p : V_p \times V^p \longrightarrow \mathbb{Z}/2 \tag{7}$$

we obtain a non-degenerate pairing

$$\langle , \rangle_S : V_S \times V^S \longrightarrow \mathbb{Z}/2 \tag{8}$$

Let I_S and I^S be two copies of the group $\mathbb{Z}_S^*/\mathbb{Z}_S^* \cong H^1(\mathcal{O}_S, \mathbb{Z}/2)$. From our assumptions on S it follows that the inclusions $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ induces embeddings

$$I_S \hookrightarrow V_S$$
$$I^S \hookrightarrow V^S$$

and that the dimensions of I_S, I^S is exactly half the dimensions of V_S, V^S respectively. Furthermore, I_S is the orthogonal complement of I^S with respect to 8 and vice versa.

For each $p \in S$ let $W^p \subseteq V^p$ be the subspace generated by the class $[d]$ and we let $W_p \subseteq V_p$ be the orthogonal complement of W^p with respect to 7. We will denote by $W_S = \oplus_{p\in S} W_v$ and $W^S = \oplus_{p\in S} W^v$. By construction we see that W_S is the orthogonal complement of W^S with respect to 8 and vice versa.

Now consider the induced pairings

$$I_S \times W^S \longrightarrow \mathbb{Z}/2 \tag{9}$$

and

$$W_S \times I^S \longrightarrow \mathbb{Z}/2 \tag{10}$$

We then have the following straightforward lemma (compare [CT01]).

Lemma 2.10. *The Selmer group* $\mathrm{Sel}(\mathcal{T}_S)$ *can be identified with each of the following groups:*

1. *The intersection* $I_S \cap W_S$.
2. *The left kernel of 9.*
3. *The left kernel of 10.*

Similarly, the dual Selmer group $\mathrm{Sel}(\widehat{\mathcal{T}}_S)$ *can be identified with each of the following groups:*

1. *The intersection* $I^S \cap W^S$.
2. *The right kernel of 9.*
3. *The right kernel of 10.*

Proof. □

Remark 2.11. Let $S_{\mathrm{split}} \subseteq S_0$ denote the places of S_0 which split in K (recall that by construction all the places of $S \setminus S_0$ do not split in K). From our assumptions on S in follows that $\dim(I^S) = |S|$ and it is straightforward to verify that $\dim(W_S) = |S| - |S_{\mathrm{split}}|$. Since the vector $(d, d, ..., d) \in W_S$ is always in the left kernel of 10 we see that the pairing 10 is never of full rank. However, the rank of 10 may achieve is maximal value of $|S| - |S_{\mathrm{split}}| - 1$, in which case $\mathrm{Sel}(\widehat{\mathcal{T}})$ is generated by $[d]$, $\mathrm{Sel}(\mathcal{T})$ has dimension $|S_{\mathrm{split}}| + 1$ and $Ш^1(\mathcal{T}_S)[2] = Ш^2(\widehat{\mathcal{T}})[2] = 0$. In particular, in such a case every S-unit which is a norm everywhere locally is a norm of a T-unit.

We are now ready to prove our main result of this section, applying the above 2-descent formalism to obtain sufficient condition for the solubility in $\mathcal{O}_{S_0}$ of our equation of interest:

Proposition 2.12. *Assume condition 2.2 and suppose in addition that either* $2 \in S_0$ *or that* $d = 2d'$ *with* d' *odd. If* $\mathrm{Sel}(\widehat{\mathcal{T}})$ *is generated by* $[d]$ *then for every* $a|d$ *the torsor* $\mathcal{Z}^a_{S_0}$ *(see 5) satisfies the* S_0*-integral Hasse principle.*

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