(CO)CARTESIAN FIBRATIONS

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Introduction. Let X be a nice topological space. A classical result in algebraic topology asserts an equivalence between the category of **covering spaces** $Y \longrightarrow X$ and the category of functors $\Pi_1(X) \longrightarrow \mathrm{Set}$, where $\Pi_1(X)$ is the **fundamental groupoid** of X, i.e., the groupoid whose objects are the points in X and whose morphisms are homotopy classes of paths. From a modern perspective this result is just the tip of the iceberg. Let us replace the concept of a nice topological spaces by the equivalent one of a **Kan complex**, which we may consider in particular as an ∞-category all of whose morphisms are invertible, i.e., an **∞-groupoid**. From this point of view the fundamental gruopoid can be identified with the **homotopy category** of X. The classical classification of covering spaces can now be promoted to a much more comprehensive result, stating that the concept of a Kan fibration $p : Y \longrightarrow X$ is essentially equivalent to that of a functor $X \longrightarrow \mathcal{S}$ to the ∞-category of spaces, where we associate to each point $x \in X$ the fiber $p^{-1}(x) \in \mathcal{S}$.

A similar phenomenon occurs in classical category theory. Given a category $\mathcal{C}$ and a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$, one may assemble the various categories $\{\mathcal{F}(A)\}_{A \in \mathcal{C}}$ into a global category $\int_{\mathcal{C}} \mathcal{F}$. The objects of $\int_{\mathcal{C}} \mathcal{F}$ are pairs (A, X) where A is an object of $\mathcal{C}$ and X is an object of $\mathcal{F}(A)$. A morphism from (A, X) to (B, Y) is a pair (f, φ), where $f : A \longrightarrow B$ is a morphism in $\mathcal{C}$ and $\varphi : f_!(X) \longrightarrow Y$ is a morphism in $\mathcal{F}(B)$, where we denoted by $f_! : \mathcal{F}(A) \longrightarrow \mathcal{F}(B)$ the functor associated to f by $\mathcal{F}$. If $(f, \varphi) : (A, X) \longrightarrow (B, Y)$ and $(g, \psi) : (B, Y) \longrightarrow (C, Z)$ are morphisms then the composition $(\varphi, f) \circ (\psi, g) = (\eta, f \circ g)$, where η is the composed map $g_! f_! X \xrightarrow{g_! \varphi} g_! Y \xrightarrow{\psi} Z$. The category $\int_{\mathcal{C}} \mathcal{F}$ is known as the **Grothendieck construction** of $\mathcal{F}$, and carries a natural functor $\int_{\mathcal{C}} \mathcal{F} \longrightarrow \mathcal{C}$ given by $(A, X) \mapsto A$. A fundamental insight of Grothendieck is that under suitable conditions one can also go the other way and associate to a category over $\mathcal{C}$ a functor from $\mathcal{C}$ to Cat. Let us begin by observing that some of the morphisms in $\int_{\mathcal{C}} \mathcal{F}$ are more special then others. These are the morphisms (f, φ) where $\varphi : f_!(X) \longrightarrow X'$ is an isomorphism. We claim that these morphisms can be characterized intrinsically in a way that depends only on $\int_{\mathcal{C}} \mathcal{F}$ as an abstract category together with the projection to $\mathcal{C}$.

Definition 1. Let $\pi : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor. We will say that a map $\phi : X \longrightarrow Y$ in $\mathcal{D}$ is **π-coCartesian** if it has the **unique relative extension property**, that is, if for every map $\psi : X \longrightarrow Z$ in $\mathcal{D}$ and every dotted extension

$$\begin{tikzcd} \pi(X) \arrow[r, "\pi\psi"] \arrow[d, "\pi\phi"'] & \pi(Z) \\ \pi(Y) \arrow[ur, dashed, "\rho"'] & \end{tikzcd}$$

in the projection down to $\mathcal{C}$, there exists a unique dotted extension

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\phi"'] & Z \\ Y \arrow[ur, dashed, "\varepsilon"'] & \end{tikzcd}$$

in $\mathcal{D}$ such that $\pi\varepsilon = \rho$.

Exercises.

(1) If $\mathcal{C} = *$ then a morphism in $\mathcal{D}$ is π-coCatesian if and only if it is an isomorphism.
(2) If we take $\mathcal{D} = \int_{\mathcal{C}} \mathcal{F}$ with the projection $\pi : \int_{\mathcal{C}} \mathcal{F} \longrightarrow \mathcal{C}$ as above, then a morphism $(f, \varphi) : (A, X) \longrightarrow (B, Y)$ is π-coCartesian if and only if $\varphi : f_! X \longrightarrow Y$ is an isomorphism in $\mathcal{F}(B)$.

Grothendieck's idea was that functors of the form $\int_{\mathcal{C}} \mathcal{F} \longrightarrow \mathcal{C}$ can be intrinsically characterized as those functors which have a sufficient supply of π-coCartesian maps. More precisely, we have the following definition:

Definition 2. Let $\pi : \mathcal{D} \longrightarrow \mathcal{C}$ be a functor. We will say that π is a **coCartesian fibration** if for every morphism $f : A \longrightarrow B$ in $\mathcal{C}$ and every object $X \in \mathcal{D}$ such that $\pi(X) = A$, there exists a π-coCatersian morphism $\phi : X \longrightarrow Y$ such that $\pi\phi = f$. A **morphism** of coCartesian fibrations over $\mathcal{C}$ from $\pi : \mathcal{D} \longrightarrow \mathcal{C}$ to $\pi' : \mathcal{D}' \longrightarrow \mathcal{C}$ is a functor $\mathcal{F} : \mathcal{D} \longrightarrow \mathcal{D}'$ over $\mathcal{C}$ which maps π-coCartesian edges to π'-coCartesian edges.

Given a category $\mathcal{C}$, the collection of coCartesian fibrations $\pi : \mathcal{D} \longrightarrow \mathcal{C}$ and their morphisms can be organized into a $(2,1)$-category $\mathrm{coCar}(\mathcal{C})$ (using natural isomorphisms of functors over $\mathcal{C}$ as 2-morphisms). We then have the following classical theorem:

Theorem 3 (Grothendieck's correspondence)**.** *Let $\mathcal{C}$ be a small category. Then the formation of Grothendieck's construction induces an equivalence of $(2,1)$-categories*

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}) \xrightarrow{\simeq} \mathrm{coCar}(\mathcal{C})$$

The property of having the unique relative extension property can be easily dualized to obtain the **unique relative lifting property**. Morphisms satisfying this condition are called **π-Cartesian morphisms**. If for every morphism $f : A \longrightarrow B$ and every $Y \in \mathcal{D}$ such that $\pi(Y) = B$ there exists a π-Cartesian map $\phi : X \longrightarrow Y$ lifting f then we say that π is a **Cartesian fibration**. Using a suitable dual variant of the Grothendieck construction one can form a similar equivalence between Cartesian fibrations over $\mathcal{C}$ and functors $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Cat}$, i.e., contravariant functors from $\mathcal{C}$ to Cat. We note that if $\pi : \mathcal{D} \longrightarrow \mathcal{C}$ is a coCartesian fibration (corresponding to a functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}$) then $\pi^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \longrightarrow \mathcal{C}^{\mathrm{op}}$ is a Cartesian fibration, and the corresponding functor $\mathcal{F}' : (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C} \longrightarrow \mathrm{Cat}$ is given by $\mathcal{F}'(A) = \mathcal{F}(A)^{\mathrm{op}}$. A similar implication holds in the opposite direction.

(co)Cartesian fibrations of ∞-categories. Having encountered the same phenomenon for families of sets parametrized by a space, families of spaces parametrized by a space, and families of categories parametrized by a category, we are naturally led to consider the possibility that all of these can be unified into a global picture concerning families of ∞-categories parametrized by an ∞-category. In order to generalize the definition of a coCartesian edge to the ∞-categorical setting it will be useful to formulate the unique relative extension property in terms of the associated map of nerves $\pi : \mathrm{N}(\mathcal{D}) \longrightarrow \mathrm{N}(\mathcal{C})$ (which we also denote by π). Indeed, a pair of maps of the form $\varphi : X \longrightarrow Y, \psi : X \longrightarrow Z$ in $\mathcal{D}$ can be encoded as a map of simplicial sets $\varphi \vee \psi : \Lambda^2_0 \longrightarrow \mathrm{N}(\mathcal{D})$, and an extension of $\pi\psi$ along $\pi\varphi$ can be encoded as a map of simplicial sets $\Delta^2 \longrightarrow \mathrm{N}(\mathcal{C})$. If we drop the uniqueness condition, then the mere existence of a relative extension amounts to the existence of a dotted lift in the resulting square

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "\varphi\vee\psi"] \arrow[d] & \mathrm{N}(\mathcal{D}) \arrow[d, "\pi"] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathrm{N}(\mathcal{C}) \end{tikzcd}$$

Somewhat surprisingly, the uniqueness can also be phrased as a similar lifting condition using the horn inclusion $\Lambda^3_0 \hookrightarrow \Delta^3$.

Exercise. Suppose that the morphism $\varphi : X \longrightarrow Y$ in $\mathcal{D}$ has the (non-unique) relative extension property. Then φ has the unique extension property if and only if a dotted lift exists in any square of the form

$$\begin{tikzcd} \Lambda^3_0 \arrow[r, "\sigma"] \arrow[d] & \mathrm{N}(\mathcal{D}) \arrow[d, "\pi"] \\ \Delta^3 \arrow[r] \arrow[ur, dashed] & \mathrm{N}(\mathcal{C}) \end{tikzcd}$$

in which σ sends the edge $\Delta^{\{0,1\}} \subseteq \Lambda^3_0$ to φ.

Definition 4. Let $\pi : X \longrightarrow S$ be a map of simplicial sets and let $\varphi : x \longrightarrow y$ be an edge in X. We will say that φ is **π-coCartesian** if for every $n \geq 2$ a dotted lift exists in every square of the form

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\sigma"] \arrow[d] & X \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{1}$$

in which σ maps $\Delta^{\{0,1\}} \subseteq \Lambda^n_0$ to φ. Similarly, we will say that φ is **π-Cartesian** if the same holds when we replace $\sigma : \Lambda^n_0 \longrightarrow X$ in 1 by a map $\tau : \Lambda^n_n \longrightarrow X$ such that τ maps $\Delta^{\{n-1,n\}}$ to φ.

Definition 5. Let $\pi : X \longrightarrow S$ be a map of simplicial sets. We will say that $\pi : X \longrightarrow S$ is a **coCartesian fibration** (resp. **Cartesian fibration**) if the following conditions hold:

(1) π is an **inner fibration**, i.e., π satisfies the right lifting property with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ with $0 < i < n$ (this condition is automatic when X and S are nerves of discrete categories).
(2) For every edge $f : a \longrightarrow b$ in S and every $x \in X$ such that $\pi(x) = a$ (resp. every $y \in X$ such that $\pi(y) = b$) there exists a π-coCartesian (resp. π-Cartesian) edge $\varphi : x \longrightarrow y$ such that $\pi(\varphi) = f$.

Remark 6. The terminal map $\pi : X \longrightarrow *$ is a (co)Cartesian fibration if and only if X is an ∞-category, in which case the π-(co)Cartesian edges are exactly the equivalences.

Remark 7. By definition the property of being a (co)Cartesian fibration is invariant under base change. In other words, if $\pi : X \longrightarrow S$ is a (co)Cartesian fibration and $T \longrightarrow S$ is any map then $X \times_S T \longrightarrow T$ is a (co)Cartesian fibration. In particular, if $\pi : X \longrightarrow S$ is a (co)Cartesian fibration then the fiber $\pi^{-1}(s)$ is an ∞-category for every $s \in S$.

The Lurie-Grothendieck correspondence. As in the case of discrete categories the collection of coCatersian fibrations over S can be organized into an ∞-category $\mathrm{coCar}(S)$ whose mapping spaces are the subspaces of maps over S which preserve coCartesian edges. We then have the following higher categorical analogue of Theorem 3

Theorem 8 (The Lurie-Grothendieck correspondence). *For $S \in \mathrm{Set}_\Delta$ there exists an equivalence of ∞-categories*

$$\mathrm{Un}^\infty_S : \mathrm{Fun}(S, \mathrm{Cat}_\infty) \xrightarrow{\simeq} \mathrm{coCar}(S)$$

which is compatible with base change and which reduces to the tautological identification

$$\mathrm{coCar}(*) = \mathrm{Cat}_\infty \xrightarrow{\mathrm{Id}} \mathrm{Cat}_\infty = \mathrm{Fun}(*, \mathrm{Cat}_\infty)$$

for $S = *$. Furthermore, if $S = \mathrm{N}(\mathcal{C})$ is the nerve of a discrete category and $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}_\infty$ factors through the full $(2,1)$-subcategory $\mathrm{Cat} \subseteq \mathrm{Cat}_\infty$, then $\mathrm{Un}^\infty_S(\mathcal{F})$ is naturally equivalent to $\int_{\mathcal{C}} \mathcal{F}$. The same claim holds if we replace $\mathrm{coCar}(S)$ by $\mathrm{Car}(S)$ and $\mathrm{Fun}(S, \mathrm{Cat}_\infty)$ by $\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Cat}_\infty)$.

If $\pi : X \longrightarrow S$ is a coCartesian fibration and $\mathcal{F} : S \longrightarrow \mathrm{Cat}_\infty$ is such that $\mathrm{Un}^\infty_S(\mathcal{F}) \simeq (\pi : X \longrightarrow S)$ then we will say that $\pi : X \longrightarrow S$ is **classified** by $\mathcal{F}$. By **compatibility with base change** we mean that if $f : T \longrightarrow S$ is a map of simplicial sets then under the Lurie-Grothendieck correspondence the functor $f^* : \mathrm{coCar}(S) \longrightarrow \mathrm{coCar}(T)$ given by $(X \longrightarrow S) \mapsto (X \times_S T \longrightarrow T)$ corresponds to the restriction functor $\mathrm{Fun}(S, \mathrm{Cat}_\infty) \longrightarrow \mathrm{Fun}(T, \mathrm{Cat}_\infty)$. Combined with the "normalization" condition for $S = *$ this means that $\pi : X \longrightarrow S$ is a coCartesian fibration classified by a functor $\mathcal{F} : S \longrightarrow \mathrm{Cat}_\infty$ then for every $s \in S$ the ∞-category $\mathcal{F}(s)$ is equivalent to the fiber $\pi^{-1}(s)$.

Let us now say a few words about the proof of the Lurie-Grothendieck correspondence (see [Lu09, §3]). The main idea consists of finding suitable model categories which model the ∞-categories on both sides and then constructing explicit Quillen equivalence between them. This is done by introducing the category Set^+_Δ is **marked simplicial sets**. A marked simplicial set is a pair (X, M) where X is a simplicial set and $M \subseteq X_1$ is a collection of edges containing all the degenerate edges. Given a simplicial set X one denotes by $X^\sharp = (X, X_1)$ the marked simplicial set in which all edges are marked. The path to the proof of the Lurie-Grothendieck correspondence passes through the following steps:

(1) There exists a model structure on Set^+_Δ whose underlying ∞-category is Cat_∞.
(2) Let $\mathcal{C} = \mathfrak{C}[S]$ be the simplicial category generated from S. Then the model structure of (1) induces a model structure on the functor category $(\mathrm{Set}^+_\Delta)^{\mathcal{C}}$ whose underlying ∞-category is $\mathrm{Fun}(S, \mathrm{Cat}_\infty)$.

(3) The category $(\mathrm{Set}_\Delta^+)_{/S^\sharp}$ of marked simplicial set over $S^\sharp$ can be endowed with a model structure whose underlying ∞-category is $\mathrm{coCar}(S)$.
(4) There exists a Quillen equivalence

$$\mathrm{St}_S^+ : (\mathrm{Set}_\Delta^+)_{/S^\sharp} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathrm{Set}_\Delta^+)^{\mathcal{C}} : \mathrm{Un}_S^+$$

which is suitably compatible with base change, and such that for $S = *$ the resulting Quillen equivalence $\mathrm{St}_*^+ \dashv \mathrm{Un}_*^+$ is naturally equivalent to the identity. Furthermore, if $S = \mathrm{N}(\mathcal{C})$ is the nerve of a discrete category and $\mathcal{F} : \mathcal{C} \longrightarrow \mathrm{Cat}_\infty$ factors through $\mathrm{Cat} \subseteq \mathrm{Cat}_\infty$, then $\mathrm{Un}_S^\infty(\mathcal{F})$ is naturally equivalent to the nerve of $\int_{\mathcal{C}} \mathcal{F}$ (with the marked edges being the coCartesian ones).

The functors St_S^+ and Un_S^+ are known as the **straightening and unstraightening** functors. The exact form of these functors will not be important for us, mainly because we will take the position that if S is a simplicial set then coCartesian fibrations $X \longrightarrow S$ are a much more convenient and accessible object then functors $S \longrightarrow \mathrm{Cat}_\infty$, which are typically very hard to construct explicitly. A framework which allows one to work as much as possible within the realm of coCartesian fibrations, without ever having to explicitly straighten or unstraighten any functor, is hence highly desirable, if not indispensable.

Examples and constructions. Let us begin by singling out an important particular case of coCartesian fibrations.

Claim 9. *Let $p : X \longrightarrow S$ be a (co)Cartesian fibration. Then the following conditions are equivalent:*
(1) Every edge of X is p-(co)Cartesian.
(2) For every $s \in S$ the ∞-category $\pi^{-1}(s)$ is an ∞-groupoid.
(3) The functor classifying π takes values in the full subcategory $\mathrm{Grp}_\infty \subseteq \mathrm{Cat}_\infty$ spanned by ∞-groupoids.

A coCartesian fibration $\pi : X \longrightarrow S$ satisfying the equivalent conditions of Claim 9 is called a **left fibration**, and a Cartesian fibration satisfying these conditions is called a **right fibration**. Combing condition (1) of Claim 9 with Condition (1) of Definition 5 we see that an arbitrary map $\pi : X \longrightarrow S$ is a left (resp. right) fibration if and only if π has the right lifting property with respect to all horn inclusion of the form $\Lambda_i^n \subseteq \Delta^n$ for $0 \leq i < n$ (resp. $0 < i \leq n$). The Lurie-Grothendieck correspondence then descends to an equivalence between the ∞-category of left (resp. right) fibrations over S and the ∞-category of functors $S \longrightarrow \mathrm{Grp}_\infty$ (resp. $S^{\mathrm{op}} \longrightarrow \mathrm{Grp}_\infty$).

Let us now review a few examples of how to express everyday common functors as (co)Cartesian fibrations. We begin with (co)representable functors. Let $\mathcal{C}$ be an ∞-category. The most basic type of a functor $\mathcal{C} \longrightarrow \mathrm{Cat}_\infty$ is the functor which associates to an object $y \in \mathcal{C}$ the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathrm{Grp}_\infty \subseteq \mathrm{Cat}_\infty$ out of a fixed object of $\mathcal{C}$. Such functors are called **corepresentable functors**. The corresponding coCartesian fibration, which is in particular a left fibration, is denoted by $\mathcal{C}_{x/} \longrightarrow \mathcal{C}$ and can be constructed explicitly as follows: the n-simplices of $\mathcal{C}_{x/}$ are given by the $(n+1)$-simplices $\sigma : \Delta^{n+1} \longrightarrow \mathcal{C}$ such that σ sends $\Delta^{\{0\}} \subseteq \Delta^{n+1}$ to x. The ∞-category $\mathcal{C}_{x/}$ is known as the **coslice** ∞-category. Loosely speaking, this is the ∞-category whose objects are pairs (y, f) where y is an object of $\mathcal{C}$ and $f : x \longrightarrow y$ is a map. On the dual side, we may also consider the right fibration

$\mathcal{C}_{/x} \longrightarrow \mathcal{C}$ where $\mathcal{C}_{/x}$ is the simplicial set whose n simplices are given by the $(n+1)$-simplices $\sigma : \Delta^{n+1} \longrightarrow \mathcal{C}$ such that σ sends $\Delta^{\{n+1\}} \subseteq \Delta^{n+1}$ to x. The right fibration $\mathcal{C}_{/x} \longrightarrow \mathcal{C}$ models the representable (contravariant) functor $y \mapsto \mathrm{Map}_{\mathcal{C}}(y, x)$.

The collection of all representable functors can be organized into a single canonical functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Cat}_\infty$ which sends (x, y) to $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathrm{Grp}_\infty \subseteq \mathrm{Cat}_\infty$. Let us now see how we can construct this functor as a right fibration in an intrinsic way. If I is a linearly ordered set, we let I^{op} denote the same set with the opposite ordering. If I and J are linearly ordered sets, we let $I * J$ denote the coproduct $I \coprod J$ equipped with the unique linear ordering which restricts to the given linear orderings of I and J, and satisfies $i \leq j$ for $i \in I$ and $j \in J$. Let Δ denote the category of combinatorial simplices. The construction $I * I^{\mathrm{op}}$ determines a functor $Q : \Delta \longrightarrow \Delta$ to itself, given on objects by $Q([n]) = [2n+1]$. If $\mathcal{C}$ is an ∞-category then we let $\mathrm{TwArr}(\mathcal{C})$ denote the ∞-category whose n-simplices are given by the $Q([n])$-simplices of of $\mathcal{C}$. By construction, the vertices of $\mathrm{TwArr}(\mathcal{C})$ are edges $f : x \longrightarrow y$ in $\mathcal{C}$. More generally, the n-simplices of $\mathrm{TwArr}(\mathcal{C})$ are given by $(2n+1)$-simplices of $\mathcal{C}$, which we may depict as diagrams

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & x_1 \arrow[r] \arrow[d] & \ldots \arrow[r] & x_n \arrow[d] \\ y_0 & y_1 \arrow[l] & \ldots \arrow[l] & y_n \arrow[l] \end{tikzcd}$$

The natural maps $I \longrightarrow I * I^{\mathrm{op}}$ and $I^{\mathrm{op}} \longrightarrow I * I^{op}$ induce a map $\mathrm{TwArr}(\mathcal{C}) \longrightarrow \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

Claim 10 ([Lu09, Proposition 5.2.1.3])**.** *The map* $\mathrm{TwArr}(\mathcal{C}) \longrightarrow \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ *is a right fibration classifying the contravariant functor* $(x, y) \mapsto \mathrm{Map}_{\mathcal{C}}(x, y)$.

Let us now review some more elaborated funtoriality properties of (co)Cartesian fibrations. Let $f : X \longrightarrow S$ be a map of simplicial sets. The functor $f^* : (\mathrm{Set}_\Delta)_{/S} \longrightarrow (\mathrm{Set}_\Delta)_{/X}$ has both a left adjoint and a right adjoint. Then left adjoint $f_! : (\mathrm{Set}_\Delta)_{/X} \longrightarrow (\mathrm{Set}_\Delta)_{/S}$ is given by sending $Y \longrightarrow X$ to the composed map $Y \longrightarrow X \longrightarrow S$. The right adjoint $f_* : (\mathrm{Set}_\Delta)_{/X} \longrightarrow (\mathrm{Set}_\Delta)_{/S}$ is given by sending $Y \longrightarrow X$ to the simplicial set $f_*Y \longrightarrow S$ over S defined by the property that if $\sigma : \Delta^n \longrightarrow S$ is a simplex then $\mathrm{Hom}_S(\sigma, f_*Y) = \mathrm{Hom}_T(X \times_S \Delta^n, Y)$. In general, neither $f_!$ nor f_* send (co)Cartesian fibrations to (co)Cartesian fibrations. However, there are special cases where this does hold:

Proposition 11 ([Lu09],[Lu14, §B.4])**.**

*(1) If $f : X \longrightarrow S$ is a **coCartesian** fibration then $f_!$ sends coCartesian fibrations over X to coCartesian fibrations over S.*

*(2) If $f : X \longrightarrow S$ is a **Cartesian** fibration then f_* sends coCartesian fibration over X to coCartesian fibration over S.*

In the cases described in Proposition 11 the operation $f_!$ corresponds under the Lurie-Grothendieck correspondence to a type of **lax left Kan extension** $\mathrm{Fun}(X, \mathrm{Cat}_\infty) \longrightarrow \mathrm{Fun}(S, \mathrm{Cat}_\infty)$. Similarly, the operation f_* corresponds to a type of **lax right Kan extension**. It is useful to single out the following instances of Proposition 11:

Corollary 12 ([Lu09, Corollary 3.2.2.12])**.** *Let $p : Y \longrightarrow S$ be a coCartesian fibration classified by $\mathcal{F} : S \longrightarrow \mathrm{Cat}_\infty$ and let $f : X \longrightarrow S$ be a map. Then:*

(1) If f is a coCatesian fibration classified by the functor $\mathcal{G} : S \longrightarrow \mathrm{Cat}_\infty$ then $f_! f^* Y \longrightarrow S$ is a coCartesian fibration classified by the functor $s \mapsto \mathcal{G}(s) \times \mathcal{F}(s)$.
(2) If f is a Catesian fibration classified by the functor $\mathcal{G} : S^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$ then $f_* f^* Y \longrightarrow S$ is a coCartesian fibration classified by the functor $s \mapsto \mathrm{Fun}(\mathcal{G}(s), \mathcal{F}(s))$.

The Hinich Comparison theorem. Recall that a relative category is a category $\mathcal{C}$ equipped with a collection of morphisms $\mathcal{W}$ which contains all identities, and which are usually referred to as **weak equivalences**. The **∞-localization** $\mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ of $\mathcal{C}$ by $\mathcal{W}$ is an ∞-category under $\mathcal{C}$ satisfying the following universal property: for every ∞-category $\mathcal{D}$, the restriction map

$$\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully-faithful and its essential image consists of those functors $\mathcal{C} \longrightarrow \mathcal{D}$ which send every morphism in $\mathcal{W}$ to an equivalence. When $\mathcal{W}$ is implied we will also denote $\mathcal{C}[\mathcal{W}^{-1}]$ by $\mathcal{C}_\infty$. A functor $\mathcal{F} : (\mathcal{C}, \mathcal{W}) \longrightarrow (\mathcal{D}, \mathcal{U})$ is called a **Dwyer-Kan equivalence** if it induces an equivalence of ∞-categories after ∞-localization. Let us denote by RelCat the category of (small) relative categories and let DK denote the collection of Dwyer-Kan equivalences. It is then well-known that $\mathrm{RelCat}[\mathrm{DK}^{-1}] \simeq \mathrm{Cat}_\infty$, and so we can consider $(\mathrm{RelCat}, \mathrm{DK})$ as a model for the theory of ∞-categories. A typical scenario in which families of ∞-categories do come with an explicit parametrization is when they come from a family of relative categories via ∞-localization. More specifically consider the situation where we have a functor of relative categories $\mathcal{F} : (\mathcal{C}, \mathcal{W}) \longrightarrow (\mathrm{RelCat}, \mathrm{DK})$, i.e., we have a family of relative categories parametrized by $\mathcal{C}$, such that morphisms in $\mathcal{W}$ are sent to Dwyer-Kan equivalences. By the universal property of ∞-localizations $\mathcal{F}$ descends to an essentially unique functor

$$\mathcal{F}_\infty : \mathcal{C}_\infty \longrightarrow \mathrm{Cat}_\infty$$

and we can model $\mathcal{F}_\infty$ by a suitable coCartesian fibration $p : \mathrm{Un}^\infty_{\mathcal{C}_\infty}(\mathcal{F}_\infty) \longrightarrow \mathcal{C}_\infty$. On the other hand, we may form the classical Grothendieck construction $\int_{\mathcal{C}} \mathcal{F} \longrightarrow \mathcal{C}$ and promote it to a relative category by declaring that a map $(f, \varphi) : (A, X) \longrightarrow (B, Y)$ in $\int_{\mathcal{C}} \mathcal{F}$ is a weak equivalence if $f : A \longrightarrow B$ is a weak equivalence in $\mathcal{C}$ and $\varphi : f_! X \longrightarrow Y$ is a weak equivalence in $\mathcal{F}(B)$. The projection $\pi : \int_{\mathcal{C}} \mathcal{F} \longrightarrow \mathcal{C}$ then become a functor of relative categories. By the compatibility of unstraightening with the Grothendieck construction and with base change we obtain a natural commutative diagram

$$\begin{tikzcd} \int_{\mathcal{C}} \mathcal{F} \arrow[r] \arrow[d, "\pi"] & \mathrm{Un}^\infty_{\mathcal{C}_\infty}(\mathcal{F}_\infty) \arrow[d, "p"] \\ \mathcal{C} \arrow[r] & \mathcal{C}_\infty \end{tikzcd}$$

We then have the following theorem of Hinich (see [Hi13]):

Theorem 13 (The Hinich comparison theorem)**.** *The induced functor*

$$\begin{tikzcd} (\int_{\mathcal{C}} \mathcal{F})_\infty \arrow[rr] \arrow[dr] & & \mathrm{Un}^\infty_{\mathcal{C}_\infty}(\mathcal{F}_\infty) \arrow[dl] \\ & \mathcal{C}_\infty & \end{tikzcd}$$

is an equivalence of ∞-categories over $\mathcal{C}_\infty$.

References

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