The Cobordism Hypothesis in Dimension 1

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Abstract

In [Lur1] Lurie published an expository article outlining a proof for a higher version of the cobordism hypothesis conjectured by Baez and Dolan in [BaDo]. In this note we give a proof for the 1-dimensional case of this conjecture. The proof follows most of the outline given in [Lur1], but differs in a few crucial details. In particular, the proof makes use of the theory of quasi-unital ∞-categories as developed by the author in [Har].

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1 Introduction

Let $\mathcal{B}_1^{\mathrm{or}}$ denote the 1-dimensional oriented cobordism ∞-category, i.e. the symmetric monoidal ∞-category whose objects are oriented 0-dimensional closed manifolds and whose morphisms are oriented 1-dimensional cobordisms between them.

Let $\mathcal{D}$ be a symmetric monoidal ∞-category with duals. The 1-dimensional cobordism hypothesis concerns the ∞-category

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$$

of symmetric monoidal functors $\varphi : \mathcal{B}_1^{\mathrm{or}} \longrightarrow \mathcal{D}$. If $X_+ \in \mathcal{B}_1^{\mathrm{or}}$ is the object corresponding to a point with positive orientation then the evaluation map $Z \mapsto Z(X_+)$ induces a functor

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D}) \longrightarrow \mathcal{D}$$

It is not hard to show that since $\mathcal{B}_1^{\mathrm{or}}$ has duals the ∞-category $\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$ is in fact an ∞-groupoid, i.e. every natural transformation between two functors

$F, G : \mathcal{B}_1^{\mathrm{or}} \longrightarrow \mathcal{D}$ is a natural equivalence. This means that the evaluation map $Z \mapsto Z(X_+)$ actually factors through a map

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D}) \longrightarrow \widetilde{\mathcal{D}}$$

where $\widetilde{\mathcal{D}}$ is the maximal ∞-groupoid of $\mathcal{D}$. The cobordism hypothesis then states

Theorem 1.1. *The evaluation map*

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D}) \longrightarrow \widetilde{\mathcal{D}}$$

is an equivalence of ∞-categories.

Remark 1.2. From the consideration above we see that we could have written the cobordism hypothesis as an equivalence

$$\widetilde{\mathrm{Fun}}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D}) \xrightarrow{\simeq} \widetilde{\mathcal{D}}$$

where $\widetilde{\mathrm{Fun}}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$ is the maximal ∞-groupoid of $\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$ (which in this case happens to coincide with $\mathrm{Fun}^{\otimes}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$). This ∞-groupoid is the fundamental groupoid of the space of maps from $\mathcal{B}_1^{\mathrm{or}}$ to $\mathcal{D}$ in the ∞-category $\mathrm{Cat}^{\otimes}$ of symmetric monoidal ∞-categories.

In his paper [Lur1] Lurie gives an elaborate sketch of proof for a higher dimensional generalization of the 1-dimensional cobordism hypothesis. For this one needs to generalize the notion of ∞-categories to (∞, n)-categories. The strategy of proof described in [Lur1] is inductive in nature. In particular in order to understand the $n = 1$ case, one should start by considering the $n = 0$ case.

Let $\mathcal{B}_0^{\mathrm{un}}$ be the 0-dimensional unoriented cobordism category, i.e. the objects of $\mathcal{B}_0^{\mathrm{un}}$ are 0-dimensional closed manifolds (or equivalently, finite sets) and the morphisms are diffeomorphisms (or equivalently, isomorphisms of finite sets). Note that $\mathcal{B}_0^{\mathrm{un}}$ is a (discrete) ∞-groupoid.

Let $X \in \mathcal{B}_0^{\mathrm{un}}$ be the object corresponding to one point. Then the 0-dimensional cobordism hypothesis states that $\mathcal{B}_0^{\mathrm{un}}$ is in fact the free ∞-groupoid (or $(\infty, 0)$-category) on one object, i.e. if $\mathcal{G}$ is any other ∞-groupoid then the evaluation map $Z \mapsto Z(X)$ induces an equivalence of ∞-groupoids

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_0^{\mathrm{un}}, \mathcal{G}) \xrightarrow{\simeq} \mathcal{G}$$

Remark 1.3. At this point one can wonder what is the justification for considering non-oriented manifolds in the $n = 0$ case oriented ones in the $n = 1$ case. As is explained in [Lur1] the desired notion when working in the n-dimensional cobordism (∞, n)-category is that of **n-framed** manifolds. One then observes that 0-framed 0-manifolds are unoriented manifolds, while taking 1-framed 1-manifolds (and 1-framed 0-manifolds) is equivalent to taking the respective manifolds with orientation.

Now the 0-dimensional cobordism hypothesis is not hard to verify. In fact, it holds in a slightly more general context - we do not have to assume that $\mathcal{G}$ is an ∞-groupoid. In fact, if $\mathcal{G}$ is **any symmetric monoidal ∞-category** then the evaluation map induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\otimes}(\mathcal{B}_0^{\mathrm{un}}, \mathcal{G}) \xrightarrow{\simeq} \mathcal{G}$$

and hence also an equivalence of ∞-groupoids

$$\widetilde{\mathrm{Fun}}^{\otimes}(\mathcal{B}_0^{\mathrm{un}}, \mathcal{G}) \xrightarrow{\simeq} \widetilde{\mathcal{G}}$$

Now consider the under-category $\mathrm{Cat}^{\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$ of symmetric monoidal ∞-categories $\mathcal{D}$ equipped with a functor $\mathcal{B}_0^{\mathrm{un}} \longrightarrow \mathcal{D}$. Since $\mathcal{B}_0^{\mathrm{un}}$ is free on one generator this category can be identified with the ∞-category of **pointed** symmetric monoidal ∞-categories, i.e. symmetric monoidal ∞-categories with a chosen object. We will often not distinguish between these two notions.

Now the point of positive orientation $X_+ \in \mathcal{B}_1^{\mathrm{or}}$ determines a functor $\mathcal{B}_0^{\mathrm{un}} \longrightarrow \mathcal{B}_1^{\mathrm{or}}$, i.e. an object in $\mathrm{Cat}^{\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$, which we shall denote by $\mathcal{B}_1^{+}$. The 1-dimensional coborodism hypothesis is then equivalent to the following statement:

Theorem 1.4. *[Cobordism Hypothesis 0-to-1] Let $\mathcal{D} \in \mathrm{Cat}^{\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$ be a pointed symmetric monoidal ∞-category with duals. Then the ∞-groupoid*

$$\widetilde{\mathrm{Fun}}^{\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}(\mathcal{B}_1^{+}, \mathcal{D})$$

*is **contractible**.*

Theorem 1.4 can be considered as the inductive step from the 0-dimensional cobordism hypothesis to the 1-dimensional one. Now the strategy outlines in [Lur1] proceeds to bridge the gap between $\mathcal{B}_0^{\mathrm{un}}$ to $\mathcal{B}_1^{\mathrm{or}}$ by considering an intermediate ∞-category

$$\mathcal{B}_0^{\mathrm{un}} \hookrightarrow \mathcal{B}_1^{\mathrm{ev}} \hookrightarrow \mathcal{B}_1^{\mathrm{or}}$$

This intermediate ∞-category is defined in [Lur1] in terms of framed functions and index restriction. However in the 1-dimensional case one can describe it without going into the theory of framed functors. In particular we will use the following definition:

Definition 1.5. Let $\iota : \mathcal{B}_1^{\mathrm{ev}} \hookrightarrow \mathcal{B}_1^{\mathrm{or}}$ be the subcategory containing all objects and only the cobordisms M in which every connected component $M_0 \subseteq M$ is either an identity segment or an evaluation segment.

Let us now describe how to bridge the gap between $\mathcal{B}_0^{\mathrm{un}}$ and $\mathcal{B}_1^{\mathrm{ev}}$. Let $\mathcal{D}$ be an ∞-category with duals and let

$$\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$$

be a symmetric monoidal functor. We will say that φ is **non-degenerate** if for each $X \in \mathcal{B}_1^{\mathrm{ev}}$ the map

$$\varphi(\mathrm{ev}_X) : \varphi(X) \otimes \varphi(\check{X}) \simeq \varphi(X \otimes \check{X}) \longrightarrow \varphi(1) \simeq 1$$

is **non-degenerate**, i.e. identifies $\varphi(\check{X})$ with a dual of $\varphi(X)$. We will denote by

$$\mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/} \subseteq \mathrm{Cat}^{\otimes}_{\mathcal{B}_1^{\mathrm{ev}}/}$$

the full subcategory spanned by objects $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ such that $\mathcal{D}$ has duals and φ is non-degenerate.

Let $X_+ \in \mathcal{B}_1^{\mathrm{ev}}$ be the point with positive orientation. Then X_+ determines a functor

$$\mathcal{B}_0^{\mathrm{un}} \longrightarrow \mathcal{B}_1^{\mathrm{ev}}$$

The restriction map $\varphi \mapsto \varphi|_{\mathcal{B}_0^{\mathrm{un}}}$ then induces a functor

$$\mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/} \longrightarrow \mathrm{Cat}^{\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$$

Now the gap between $\mathcal{B}_1^{\mathrm{ev}}$ and $\mathcal{B}_0^{\mathrm{un}}$ can be climbed using the following lemma (see [Lur1]):

Lemma 1.6. *The functor*

$$\mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/} \longrightarrow \mathrm{Cat}^{\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$$

is fully faithful.

Proof. First note that if $F : \mathcal{D} \longrightarrow \mathcal{D}'$ is a symmetric monoidal functor where $\mathcal{D}, \mathcal{D}'$ have duals and $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ is non-degenerate then $f \circ \varphi$ will be non-degenerate as well. Hence it will be enough to show that if $\mathcal{D}$ has duals then the restriction map induces an equivalence between the ∞-groupoid of non-degenerate symmetric monoidal functors

$$\mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$$

and the ∞-groupoid of symmetric monoidal functors

$$\mathcal{B}_0^{\mathrm{un}} \longrightarrow \mathcal{D}$$

Now specifying a non-degenerate functor

$$\mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$$

is equivalent to specifying a pair of objects $D_+, D_- \in \mathcal{D}$ (the images of X_+, X_- respectively) and a non-degenerate morphism

$$e : D_+ \otimes D_- \longrightarrow 1$$

which is the image of ev_{X_+}. Since $\mathcal{D}$ has duals the ∞-groupoid of triples (D_+, D_-, e) in which e is non-degenerate is equivalent to the ∞-groupoid of triples $(D_+, \check{D}_-, f)$ where $f : D_+ \longrightarrow \check{D}_-$ is an equivalence. Hence the forgetful map $(D_+, D_-, e) \mapsto D_+$ is an equivalence. □

Now consider the natural inclusion $\iota : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{B}_1^{\mathrm{or}}$ as an object in $\mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/}$. Then by Lemma 1.6 we see that the 1-dimensional cobordism hypothesis will be established once we make the following last step:

Theorem 1.7 (Cobordism Hypothesis - Last Step)**.** *Let $\mathcal{D}$ be a symmetric monoidal ∞-category with duals and let $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ be a **non-degenerate** functor. Then the ∞-groupoid*

$$\widetilde{\mathrm{Fun}}^{\otimes}_{\mathcal{B}_1^{\mathrm{ev}}/}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$$

is contractible.

Note that since $\mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{B}_1^{\mathrm{or}}$ is essentially surjective all the functors in

$$\widetilde{\mathrm{Fun}}^{\otimes}_{\mathcal{B}_1^{\mathrm{ev}}/}(\mathcal{B}_1^{\mathrm{or}}, \mathcal{D})$$

will have the same essential image of φ. Hence it will be enough to prove for the claim for the case where $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ is **essentially surjective**. We will denote by

$$\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}_1^{\mathrm{ev}}/} \subseteq \mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/}$$

the full subcategory spanned by essentially surjective functors $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$. Hence we can phrase Theorem 1.7 as follows:

Theorem 1.8 (Cobordism Hypothesis - Last Step 2)**.** *Let $\mathcal{D}$ be a symmetric monoidal ∞-category with duals and let $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ be an **essentially surjective non-degenerate** functor. Then the space of maps*

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}_1^{\mathrm{ev}}/}}(\iota, \varphi)$$

is contractible.

The purpose of this paper is to provide a formal proof for this last step. This paper is constructed as follows. In § 2 we prove a variant of Theorem 1.8 which we call the quasi-unital cobordism hypothesis (Theorem 2.6). Then in § 3 we explain how to deduce Theorem 1.8 from Theorem 2.6. Section § 3 relies on the notion of **quasi-unital ∞-categories** which is developed rigourously in [Har] (however § 2 is completely independent of [Har]).

2 The Quasi-Unital Cobordism Hypothesis

Let $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ be a non-degenerate functor and let Grp_∞ denote the ∞-category of ∞-groupoids. We can define a lax symmetric functor $M_\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ by setting

$$M_\varphi(X) = \mathrm{Map}_{\mathcal{D}}(1, \varphi(X))$$

We will refer to M_φ as the **fiber functor** of φ. Now if $\mathcal{D}$ has duals and φ is non-degenerate, then one can expect this to be reflected in M_φ somehow. More precisely, we have the following notion:

Definition 2.1. Let $M : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ be a lax symmetric monoidal functor. An object $Z \in M(X \otimes \check{X})$ is called **non-degenerate** if for each object $Y \in \mathcal{B}_1^{\mathrm{ev}}$ the natural map

$$M(Y \otimes \check{X}) \stackrel{Id \times Z}{\longrightarrow} M(Y \otimes \check{X}) \times M(X \otimes \check{X}) \longrightarrow M(Y \otimes \check{X} \otimes X \otimes \check{X}) \stackrel{M(Id \otimes \mathrm{ev} \otimes Id)}{\longrightarrow} M(Y \otimes \check{X})$$

is an equivalence of ∞-groupoids.

Remark 2.2. If a non-degenerate element $Z \in M(X \otimes \check{X})$ exists then it is unique up to a (non-canonical) equivalence.

Example 1. *Let $M : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ be a lax symmetric monoidal functor. The lax symmetric structure of M includes a structure map $1_{\mathrm{Grp}_\infty} \longrightarrow M(1)$ which can be described by choosing an object $Z_1 \in M(1)$. The axioms of lax monoidality then ensure that Z_1 is non-degenerate.*

Definition 2.3. A lax symmetric monoidal functor $M : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ will be called **non-degenerate** if for each object $X \in \mathcal{B}_1^{\mathrm{ev}}$ there exists a non-degenerate object $Z \in M(X \otimes \check{X})$.

Definition 2.4. Let $M_1, M_2 : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ be two non-degenerate lax symmetric monoidal functors. A lax symmetric natural transformation $T : M_1 \longrightarrow M_2$ will be called **non-degenerate** if for each object $X \in \mathrm{Bord}^{\mathrm{ev}}$ and each non-degenerate object $Z \in M(X \otimes \check{X})$ the objects $T(Z) \in M_2(X \otimes \check{X})$ is non-degerate.

Remark 2.5. From remark 2.2 we see that if $T(Z) \in M_2(X \otimes \check{X})$ is non-degenerate for **at least one** non-degenerate $Z \in M_1(X \otimes \check{X})$ then it will be true for all non-degenerate $Z \in M_1(X \otimes \check{X})$.

Now we claim that if $\mathcal{D}$ has duals and $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ is non-degenerate then the fiber functor M_φ will be non-degenerate: for each object $X \in \mathcal{B}_1^{\mathrm{ev}}$ there exists a coevaluation morphism

$$\mathrm{coev}_{\varphi(X)} : 1 \longrightarrow \varphi(X) \otimes \varphi(\check{X}) \simeq \varphi(X \otimes \check{X})$$

which determines an element in $Z_X \in M_\varphi(X \otimes \check{X})$. It is not hard to see that this element is non-degenerate.

Let $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_\infty)$ denote the ∞-category of lax symmetric monoidal functors $\mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ and by

$$\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_\infty) \subseteq \mathrm{Fun}^{\mathrm{lax}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_\infty)$$

the subcategory spanned by non-degenerate functors and non-degenerate natural transformations. Now the construction $\varphi \mapsto M_\varphi$ determines a functor

$$\mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/} \longrightarrow \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_\infty)$$

In particular if $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{C}$ and $\psi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ are non-degenerate then any functor $T : \mathcal{C} \longrightarrow \mathcal{D}$ under $\mathcal{B}_1^{\mathrm{ev}}$ will induce a non-degenerate natural transformation

$$T_* : M_\varphi \longrightarrow M_\psi$$

The rest of this section is devoted to proving the following result, which we call the "quasi-unital cobordism hypothesis":

Theorem 2.6 (Cobordism Hypothesis - Quasi-Unital)**.** *Let $\mathcal{D}$ be a symmetric monoidal ∞-category with duals, let $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{D}$ be a non-degenerate functor and let $\iota : \mathcal{B}_1^{\mathrm{ev}} \hookrightarrow \mathcal{B}_1^{\mathrm{or}}$ be the natural inclusion. Let $M_\iota, M_\varphi \in \mathrm{Fun}_{\mathrm{nd}}^{\mathrm{lax}}$ be the corresponding fiber functors. Them the space of maps*

$$\mathrm{Map}_{\mathrm{Fun}_{\mathrm{nd}}^{\mathrm{lax}}}(M_\iota, M_\varphi)$$

is contractible.

Proof. We start by transforming the lax symmetric monoidal functors M_ι, M_φ to **left fibrations** over $\mathcal{B}_1^{\mathrm{ev}}$ using the symmetric monoidal analogue of Grothendieck's construction, as described in [Lur1], page $67 - 68$.

Let $M : \mathcal{B} \longrightarrow \mathrm{Grp}_\infty$ be a lax symmetric monoidal functor. We can construct a symmetric monoidal ∞-category $\mathrm{Groth}(\mathcal{B}, M)$ as follows:

1. The objects of $\mathrm{Groth}(\mathcal{B}, M)$ are pairs (X, η) where $X \in \mathcal{B}$ is an object and η is an object of $M(X)$.
2. The space of maps from (X, η) to (X', η') in $\mathrm{Groth}(\mathcal{B}, M)$ is defined to be the classifying space of the ∞-groupoid of pairs (f, α) where $f : X \longrightarrow X'$ is a morphism in B and $\alpha : f_*\eta \longrightarrow \eta$ is a morphism in $M(X')$. Composition is defined in a straightforward way.
3. The symmetric monoidal structure on $\mathrm{Groth}(\mathcal{B}, M)$ is obtained by defining

$$(X, \eta) \otimes (X', \eta') = (X \otimes X', \beta_{X,Y}(\eta \otimes \eta'))$$

where $\beta_{X,Y} : M(X) \times M(Y) \longrightarrow M(X \otimes Y)$ is given by the lax symmetric structure of M.

The forgetful functor $(X, \eta) \mapsto X$ induces a **left fibration**

$$\mathrm{Groth}(\mathcal{B}, M) \longrightarrow \mathcal{B}$$

Theorem 2.7. *The association $M \mapsto \mathrm{Groth}(\mathcal{B}, M)$ induces an equivalence between the ∞-category of lax-symmetric monoidal functors $\mathcal{B} \longrightarrow \mathrm{Grp}_\infty$ and the full subcategory of the over ∞-category $\mathrm{Cat}^{\otimes}_{/\mathcal{B}}$ spanned by left fibrations.*

Proof. This follows from the more general statement given in [Lur1] Proposition 3.3.26. Note that any map of left fibrations over $\mathcal{B}$ is in particular a map of coCartesian fibrations because if $p : \mathcal{C} \longrightarrow \mathcal{B}$ is a left fibration then any edge in $\mathcal{C}$ is p-coCartesian. □

Remark 2.8. Note that if $\mathcal{C} \longrightarrow \mathcal{B}$ is a left fibration of symmetric monoidal ∞-categories and $\mathcal{A} \longrightarrow \mathcal{B}$ is a symmetric monoidal functor then the ∞-category

$$\mathrm{Fun}^{\otimes}_{/\mathcal{B}}(\mathcal{A}, \mathcal{C})$$

is actually an ∞-**groupoid**, and by Theorem 2.7 is equivalent to the ∞-groupoid of lax-monoidal natural transformations between the corresponding lax monoidal functors from $\mathcal{B}$ to Grp_∞.

Now set

$$\mathcal{F}_\iota \stackrel{\text{def}}{=} \mathrm{Groth}(\mathcal{B}_1^{\text{ev}}, M_\iota)$$

$$\mathcal{F}_\varphi \stackrel{\text{def}}{=} \mathrm{Groth}(\mathcal{B}_1^{\text{ev}}, M_\varphi)$$

Let

$$\mathrm{Fun}^{\text{nd}}_{/\mathcal{B}_1^{\text{ev}}}(\mathcal{F}_\iota, \mathcal{F}_\varphi) \subseteq \mathrm{Fun}^{\otimes}_{/\mathcal{B}_1^{\text{ev}}}(\mathcal{F}_\iota, \mathcal{F}_\varphi)$$

denote the full sub ∞-groupoid of functors which correspond to **non-degenerate** natural transformations

$$M_\iota \longrightarrow M_\varphi$$

under the Grothendieck construction. Note that $\mathrm{Fun}^{\text{nd}}_{/\mathcal{B}_1^{\text{ev}}}(\mathcal{F}_\iota, \mathcal{F}_\varphi)$ is a union of connected components of the ∞-groupoid $\mathrm{Fun}^{\otimes}_{/\mathcal{B}_1^{\text{ev}}}(\mathcal{F}_\iota, \mathcal{F}_\varphi)$.

We now need to show that the ∞-groupoid

$$\mathrm{Fun}^{\text{nd}}_{/\mathcal{B}_1^{\text{ev}}}(\mathcal{F}_\iota, \mathcal{F}_\varphi)$$

is contractible.

Unwinding the definitions we see that the objects of $\mathcal{F}_\iota$ are pairs (X, M) where $X \in \mathcal{B}_1^{\text{ev}}$ is a 0-manifold and $M \in \mathrm{Map}_{\mathcal{B}_1^{\text{or}}}(\emptyset, X)$ is a cobordism from $\emptyset$ to X. A morphism in φ from (X, M) to (X', M') consists of a morphism in $\mathcal{B}_1^{\text{ev}}$

$$N : X \longrightarrow X'$$

and a diffeomorphism

$$T : M \coprod_X N \cong M'$$

respecting X'. Note that for each $(X, M) \in \mathcal{F}_\iota$ we have an identification $X \simeq \partial M$. Further more the space of morphisms from $(\partial M, M)$ to $(\partial M', M')$ is **homotopy equivalent to the space of orientation-preserving π_0-surjective embeddings of M in M'** (which are not required to respect the boundaries in any way).

Now in order to analyze the symmetric monoidal ∞-category $\mathcal{F}_\iota$ we are going to use the theory of ∞-**operads**, as developed in [Lur2]. Recall that the category $\mathrm{Cat}^\otimes$ of symmetric monoidal ∞-categories admits a forgetful functor

$$\mathrm{Cat}^\otimes \longrightarrow \mathrm{Op}^\infty$$

to the ∞-category of ∞-**operads**. This functor has a left adjoint

$$\mathrm{Env} : \mathrm{Op}^\infty \longrightarrow \mathrm{Cat}^\otimes$$

called the **monoidal envelope** functor (see [Lur2] §2.2.4). In particular, if $\mathcal{C}^\otimes$ is an ∞-operad and $\mathcal{D}$ is a symmetric monoidal ∞-category with corresponding ∞-operad $\mathcal{D}^\otimes \longrightarrow \mathrm{N}(\Gamma_*)$ then there is an **equivalence of ∞-categories**

$$\mathrm{Fun}^\otimes(\mathrm{Env}(\mathcal{C}^\otimes), \mathcal{D}) \simeq \mathrm{Alg}_{\mathcal{C}}(\mathcal{D}^\otimes)$$

Where $\mathrm{Alg}_{\mathcal{C}}(\mathcal{D}^{\otimes}) \subseteq \mathrm{Fun}_{/\,\mathrm{N}(\Gamma_*)}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ denotes the full subcategory spanned by ∞-operad maps (see Proposition 2.2.4.9 of [Lur2]).

Now observing the definition of monoidal envelop (see Remark 2.2.4.3 in [Lur2]) we see that $\mathcal{F}_\iota$ is equivalent to the monoidal envelope of a certain simple ∞-operad

$$F_\iota \simeq \mathrm{Env}\left(\mathcal{OF}^{\otimes}\right)$$

which can be described as follows: the underlying ∞-category $\mathcal{OF}$ of $\mathcal{OF}^{\otimes}$ is the ∞-category of **connected** 1-manifolds (i.e. either the segment or the circle) and the morphisms are **orientation-preserving embeddings** between them. The (active) n-to-1 operations of $\mathcal{OF}$ (for $n \geq 1$) from $(M_1, ..., M_n)$ to M are the orientation-preserving embeddings

$$M_1 \coprod ... \coprod M_n \longrightarrow M$$

and there are no 0-to-1 operations.

Now observe that the induced map $\mathcal{OF}^{\otimes} \longrightarrow (\mathcal{B}_1^{\mathrm{ev}})^{\infty}$ is a fibration of ∞-operads. We claim that $\mathcal{F}_\iota$ is not only the enveloping symmetric monoidal ∞-category of $\mathcal{OF}^{\otimes}$, but that $\mathcal{F}_\iota \longrightarrow \mathcal{B}_1^{\mathrm{ev}}$ is the enveloping **left fibration** of $\mathcal{OF} \longrightarrow \mathcal{B}_1^{\mathrm{ev}}$. More precisely we claim that for any left fibration $\mathcal{D} \longrightarrow \mathcal{B}_1^{\mathrm{ev}}$ of symmetric monoidal ∞-categories the natural map

$$\mathrm{Fun}^{\otimes}_{/\mathcal{B}_1^{\mathrm{ev}}}(F_\iota, \mathcal{D}) \longrightarrow \mathrm{Alg}_{\mathcal{OF}/\mathcal{B}_1^{\mathrm{ev}}}(\mathcal{D}^{\otimes})$$

is an equivalence if ∞-groupoids (where both terms denote mapping objects in the respective **over-categories**). This is in fact not a special property of F_ι:

Lemma 2.9. *Let $\mathcal{O}$ be a symmetric monoidal ∞-category with corresponding ∞-operad $\mathcal{O}^{\otimes} \longrightarrow \mathrm{N}(\Gamma_*)$ and let $p : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of ∞-operads such that the induced map*

$$\overline{p} : \mathrm{Env}\left(\mathcal{C}^{\otimes}\right) \longrightarrow \mathcal{O}$$

is a left fibration. Let $\mathcal{D} \longrightarrow \mathcal{O}$ be some other left fibration of symmetric monoidal categories. Then the natural map

$$\mathrm{Fun}^{\otimes}_{/\mathcal{O}}\left(\mathrm{Env}\left(\mathcal{C}^{\otimes}\right), \mathcal{D}\right) \longrightarrow \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}^{\otimes})$$

is an equivalence of ∞-categories. Further more both sides are in fact ∞-groupoids.

Proof. Consider the diagram

$$\begin{array}{ccc} \mathrm{Fun}^{\otimes}(\mathrm{Env}\left(\mathcal{C}^{\otimes}\right), \mathcal{D}) & \xrightarrow{\simeq} & \mathrm{Alg}_{\mathcal{C}}(\mathcal{D}^{\otimes}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{\otimes}(\mathrm{Env}\left(\mathcal{C}^{\otimes}\right), \mathcal{O}) & \xrightarrow{\simeq} & \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}^{\otimes}) \end{array}$$

Now the vertical maps are left fibrations and by adjunction the horizontal maps are equivalences. By [Lur3] Proposition 3.3.1.5 we get that the induced map on the fibers of p and $\overline{p}$ respectively

$$\mathrm{Fun}^{\otimes}_{/\mathcal{O}}\left(\mathrm{Env}\left(\mathcal{C}^{\otimes}\right), \mathcal{D}\right) \longrightarrow \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}^{\otimes})$$

is a weak equivalence of ∞-groupoids. □

Remark 2.10. In [Lur2] a relative variant $\mathrm{Env}_{\mathcal{B}_1^{\mathrm{ev}}}$ of Env is introduced which sends a fibration of ∞-operads $\mathcal{C}^{\otimes} \longrightarrow (\mathcal{B}_1^{\mathrm{ev}})^{\otimes}$ to its enveloping coCartesin fibration $\mathrm{Env}_{\mathcal{O}}\left(\mathcal{C}^{\otimes}\right) \longrightarrow \mathcal{B}_1^{\mathrm{ev}}$. Note that in our case the map

$$\mathcal{F}_{\iota} \longrightarrow \mathcal{B}_1^{\mathrm{ev}}$$

is **not** the enveloping coCartesian fibration of $\mathcal{OF}^{\otimes} \longrightarrow (\mathcal{B}_1^{\mathrm{ev}})^{\otimes}$. However from Lemma 2.9 it follows that the map

$$\begin{array}{ccc} \mathcal{F}_{\iota} & \longrightarrow & \mathrm{Env}_{\mathcal{B}_1^{\mathrm{ev}}}\left(\mathcal{OF}^{\otimes}\right) \\ & \searrow \quad \swarrow & \\ & \mathcal{B}_1^{\mathrm{ev}} & \end{array}$$

is a **covariant equivalence** over $\mathcal{B}_1^{\mathrm{ev}}$, i.e. induces a weak equivalence of simplicial sets on the fibers (where the fibers on the left are ∞-groupoids and the fibers on the right are ∞-categories). This claim can also be verified directly by unwinding the definition of $\mathrm{Env}_{\mathcal{B}_1^{\mathrm{ev}}}\left(\mathcal{OF}^{\otimes}\right)$.

Summing up the discussion so far we observe that we have a weak equivalence of ∞-groupoids

$$\mathrm{Fun}^{\otimes}_{/\mathcal{B}_1^{\mathrm{ev}}}\left(\mathcal{F}_{\iota}, \mathcal{F}_{\varphi}\right) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{OF}/\mathcal{B}_1^{\mathrm{ev}}}\left(\mathcal{F}_{\varphi}^{\otimes}\right)$$

Let

$$\mathrm{Alg}^{\mathrm{nd}}_{\mathcal{OF}/\mathcal{B}_1^{\mathrm{ev}}}\left(\mathcal{F}_{\varphi}^{\otimes}\right) \subseteq \mathrm{Alg}_{\mathcal{OF}/\mathcal{B}_1^{\mathrm{ev}}}\left(\mathcal{F}_{\varphi}^{\otimes}\right)$$

denote the full sub ∞-groupoid corresponding to

$$\mathrm{Fun}^{\mathrm{nd}}_{/\mathcal{B}_1^{\mathrm{ev}}}(\mathcal{F}_{\iota}, \mathcal{F}_{\varphi}) \subseteq \mathrm{Fun}^{\otimes}_{/\mathcal{B}_1^{\mathrm{ev}}}(\mathcal{F}_{\iota}, \mathcal{F}_{\varphi})$$

under the adjunction. We are now reduced to prove that the ∞-groupoid

$$\mathrm{Alg}^{\mathrm{nd}}_{\mathcal{OF}/\mathcal{B}_1^{\mathrm{ev}}}\left(\mathcal{F}_{\varphi}^{\otimes}\right)$$

is contractible.

Let $\mathcal{OI}^{\otimes} \subseteq \mathcal{OF}^{\otimes}$ be the full sub ∞-operad of $\mathcal{OF}^{\otimes}$ spanned by connected 1-manifolds which are diffeomorphic to the segment (and all n-to-1 operations between them). In particular we see that $\mathcal{OI}^{\otimes}$ is equivalent to the **non-unital associative ∞-operad**.

We begin with the following theorem which reduces the handling of $\mathcal{OF}^{\otimes}$ to $\mathcal{OI}^{\otimes}$.

Theorem 2.11. *Let $q : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a left fibration of ∞-operads. Then the restriction map*

$$\mathrm{Alg}_{\mathcal{OF}/\mathcal{O}}(\mathcal{C}^{\otimes}) \longrightarrow \mathrm{Alg}_{\mathcal{OI}/\mathcal{O}}(\mathcal{C}^{\otimes})$$

is a weak equivalence.

Proof. We will base our claim on the following general lemma:

Lemma 2.12. *Let $\mathcal{A}^{\otimes} \longrightarrow \mathcal{B}^{\otimes}$ be a map of ∞-groupoids and let $q : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be* ***left fibration*** *of ∞-operads. Suppose that for every object $B \in \mathcal{B}$, the category*

$$\mathcal{F}_B = \mathcal{A}^{\otimes}_{\mathrm{act}} \times_{\mathcal{B}^{\otimes}_{\mathrm{act}}} \mathcal{B}^{\otimes}_{/B}$$

is weakly contractible (see [Lur2] for the terminology). Then the natural restriction map

$$\mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C}^{\otimes}) \longrightarrow \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}^{\otimes})$$

is a weak equivalence.

Proof. In [Lur2] §3.1.3 it is explained how under certain conditions the forgetful functor (i.e. restriction map)

$$\mathrm{Alg}_{\mathcal{A}/\mathcal{O}}(\mathcal{C}^{\otimes}) \longrightarrow \mathrm{Alg}_{\mathcal{B}/\mathcal{O}}(\mathcal{C}^{\otimes})$$

admits a left adjoint, called the **free algebra functor**. Since $\mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is a left fibration both these ∞-categories are ∞-groupoids, and so any adjunction between them will be an equivalence. Hence it will suffice to show that the conditions for existence of left adjoint are satisfies in this case.

Since $q : \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is a left fibration q is **compatible with colimits indexed by weakly contractible diagrams** in the sense of [Lur2] Definition 3.1.1.18 (because weakly contractible colimits exists in every ∞-groupoid and are preserved by any functor between ∞-groupoids). Combining Corollary 3.1.3.4 and Proposition 3.1.1.20 of [Lur2] we see that the desired free algebra functor exists. □

In view of Lemma 2.12 it will be enough to check that for every object $M \in \mathcal{OF}$ (i.e. every connected 1-manifolds) the ∞-category

$$\mathcal{F}_M \stackrel{\mathrm{def}}{=} \mathcal{OI}^{\otimes}_{\mathrm{act}} \times_{\mathcal{OF}^{\otimes}_{\mathrm{act}}} \left(\mathcal{OF}^{\otimes}_{\mathrm{act}}\right)_{/M}$$

is weakly contractible.

Unwinding the definitions we see that the objects of $\mathcal{F}_M$ are tuples of 1-manifolds $(M_1, ..., M_n)$ $(n \geq 1)$, such that each M_i is diffeomorphic to a segment, together with an orientation preserving embedding

$$f : M_1 \coprod ... \coprod M_n \hookrightarrow M$$

A morphisms in $\mathcal{F}_M$ from

$$f : M_1 \coprod ... \coprod M_n \hookrightarrow M$$

to

$$g : M_1' \coprod ... \coprod M_m' \hookrightarrow M$$

is a π_0-surjective orientation-preserving embedding

$$T : M_1 \coprod ... \coprod M_n \longrightarrow M_1' \coprod ... \coprod M_m'$$

together with an **isotopy** $g \circ T \sim f$.

Now when M is the segment then $\mathcal{F}_M$ contains a terminal object and so is weakly contractible. Hence we only need to take care of the case of the circle $M = S^1$.

It is not hard to verify that the category F_{S^1} is in fact discrete - the space of self isotopies of any embedding $f : M_1 \coprod ... \coprod M_n \hookrightarrow M$ is equivalent to the loop space of S^1 and hence discrete. In fact one can even describe F_{S^1} in completely combinatorial terms. In order to do that we will need some terminology.

Definition 2.13. Let Λ_∞ be the category whose objects correspond to the natural numbers $1, 2, 3, ...$ and the morphisms from n to m are (weak) order preserving maps $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $f(x + n) = f(x) + m$.

The category Λ_∞ is a model for the the universal fibration over the cyclic category, i.e., there is a left fibration $\Lambda_\infty \longrightarrow \Lambda$ (where Λ is connes' cyclic category) such that the fibers are connected groupoids with a single object having automorphism group $\mathbb{Z}$ (or in other words circles). In particular the category Λ_∞ is known to be weakly contractible. See [Kal] for a detailed introduction and proof (Lemma 4.8).

Let $\Lambda_\infty^{\text{sur}}$ be the sub category of Λ_∞ which contains all the objects and only **surjective** maps between. It is not hard to verify explicitly that the map $\Lambda_\infty^{\text{sur}} \longrightarrow \Lambda_\infty$ is cofinal and so $\Lambda_\infty^{\text{sur}}$ is contractible as well. Now we claim that F_{S^1} is in fact equivalent to $\Lambda_\infty^{\text{sur}}$.

Let $\Lambda_{\text{big}}^{\text{sur}}$ be the category whose objects are linearly ordered sets S with an order preserving automorphisms $\sigma : S \longrightarrow S$ and whose morphisms are surjective order preserving maps which commute with the respective automorphisms. Then $\Lambda_\infty^{\text{sur}}$ can be considered as a full subcategory of $\Lambda_{\text{big}}^{\text{sur}}$ such that n corresponds to the object $(\mathbb{Z}, \sigma_n)$ where $\sigma_n : \mathbb{Z} \longrightarrow \mathbb{Z}$ is the automorphism $x \mapsto x+n$.

Now let $p : \mathbb{R} \longrightarrow S^1$ be the universal covering. We construct a functor $F_{S^1} \longrightarrow \Lambda_{\text{big}}^{\text{sur}}$ as follows: given an object

$$f : M_1 \coprod ... \coprod M_n \hookrightarrow S^1$$

of F_{S^1} consider the fiber product

$$P = \left[M_1 \coprod ... \coprod M_n \right] \times_{S^1} \mathbb{R}$$

note that P is homeomorphic to an infinite union of segments and the projection

$$P \longrightarrow \mathbb{R}$$

is injective (because f is injective) giving us a well defined linear order on P. The automorphism $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ of $\mathbb{R}$ over S^1 given by $x \mapsto x+1$ gives an order preserving automorphism $\widetilde{\sigma} : P \longrightarrow P$.

Now suppose that $((M_1, ..., M_n), f)$ and $((M_1', ..., M_m'), g)$ are two objects and we have a morphism between them, i.e. an embedding

$$T : M_1 \coprod ... \coprod M_n \longrightarrow M_1' \coprod ... \coprod M_m'$$

and an isotopy $\psi : g \circ T \sim f$. Then we see that the pair (T, ψ) determine a well defined order preserving map

$$\left[M_1 \coprod ... \coprod M_n\right] \times_{S^1} \mathbb{R} \longrightarrow \left[M_1' \coprod ... \coprod M_m'\right] \times_{S^1} \mathbb{R}$$

which commutes with the respective automorphisms. Clearly we obtain in this way a functor $u : F_{S^1} \longrightarrow \Lambda^{\text{sur}}_{\text{big}}$ whose essential image is the same as the essential image of $\Lambda^{\text{sur}}_{\infty}$. It is also not hard to see that u is fully faithful. Hence F_{S^1} is equivalent to $\Lambda^{\text{sur}}_{\infty}$ which is weakly contractible. This finishes the proof of the theorem.

□

Let
$$\text{Alg}^{\text{nd}}_{\mathcal{O}\mathcal{I}/\mathcal{B}_1^{\text{ev}}}\left(\mathcal{F}_\varphi^{\otimes}\right) \subseteq \text{Alg}_{\mathcal{O}\mathcal{I}/\mathcal{B}_1^{\text{ev}}}\left(\mathcal{F}_\varphi^{\otimes}\right)$$
denote the full sub ∞-groupoid corresponding to the full sub ∞-groupoid

$$\text{Alg}^{\text{nd}}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{\text{ev}}}\left(\mathcal{F}_\varphi^{\otimes}\right) \subseteq \text{Alg}_{\mathcal{O}\mathcal{F}/\mathcal{B}_1^{\text{ev}}}\left(\mathcal{F}_\varphi^{\otimes}\right)$$

under the equivalence of Theorem 2.11.

Now the last step of the cobordism hypothesis will be complete once we show the following:

Lemma 2.14. *The ∞-groupoid*

$$\text{Alg}^{\text{nd}}_{\mathcal{O}\mathcal{I}/\mathcal{B}_1^{\text{ev}}}\left(\mathcal{F}_\varphi^{\otimes}\right)$$

is contractible.

Proof. Let
$$q : p^*\mathcal{F}_\varphi \longrightarrow \mathcal{O}\mathcal{I}^{\otimes}$$
be the pullback of left fibration $\mathcal{F}_\varphi \longrightarrow \mathcal{B}_1^{\text{ev}}$ via the map $p : \mathcal{O}\mathcal{I}^{\otimes} \longrightarrow B_1^{\text{ev}}$, so that q is a left fibration as well. In particular, since $\mathcal{O}\mathcal{I}^{\otimes}$ is the non-unital associative ∞-operad, we see that q classifies an ∞-groupoid $q^{-1}(\mathcal{O}\mathcal{I})$ with a non-unital monoidal structure. Unwinding the definitions one sees that this ∞-groupoid is the fundamental groupoid of the space

$$\text{Map}_{\mathcal{C}}(1, \varphi(X_+) \otimes \varphi(X_-))$$

where $X_+, X_- \in \mathcal{B}^{\mathrm{ev}_1}$ are the points with positive and negative orientations respectively. The monoidal structure sends a pair of maps

$$f, f' : 1 \longrightarrow \varphi(X_+) \otimes \varphi(X_-)$$

to the composition

$$1 \xrightarrow{f \otimes f'} [\varphi(X_+) \otimes \varphi(X_-)] \otimes [\varphi(X_+) \otimes \varphi(X_-)] \xrightarrow{\simeq}$$

$$\varphi(X_+) \otimes [\varphi(X_-) \otimes \varphi(X_+)] \otimes \varphi(X_-) \xrightarrow{Id \otimes \varphi(\mathrm{ev}) \otimes Id} \varphi(X_+) \otimes \varphi(X_-)$$

Since $\mathcal{C}$ has duals we see that this monoidal ∞-groupoid is equivalent to the fundamental ∞-groupoid of the space

$$\mathrm{Map}_{\mathcal{C}}(\varphi(X_+), \varphi(X_+))$$

with the monoidal product coming from **composition**.

Now

$$\mathrm{Alg}_{\mathcal{OI}/\mathcal{B}_1^{\mathrm{ev}}}(\mathcal{F}_\varphi) \simeq \mathrm{Alg}_{\mathcal{OI}/\mathcal{OI}}(p^*\mathcal{F}_\varphi)$$

classifies $\mathcal{OI}^{\otimes}$-algebra objects in $p^*\mathcal{F}_\varphi$, i.e. non-unital algebra objects in

$$\mathrm{Map}_{\mathcal{C}}(\varphi(X_+), \varphi(X_+))$$

with respect to composition. The full sub ∞-groupoid

$$\mathrm{Alg}^{\mathrm{nd}}_{\mathcal{OI}/\mathcal{B}_1^{\mathrm{ev}}}(\mathcal{F}_\varphi) \subseteq \mathrm{Alg}_{\mathcal{OI}/\mathcal{B}_1^{\mathrm{ev}}}(\mathcal{F}_\varphi)$$

will then classify non-unital algebra objects A which correspond to **self equivalences**

$$\varphi(X_+) \longrightarrow \varphi(X_+)$$

It is left to prove the following lemma:

Lemma 2.15. *Let $\mathcal{C}$ be an ∞-category. Let $X \in \mathcal{C}$ be an object and let $\mathcal{E}_X$ denote the ∞-groupoid of self equivalences $u : X \longrightarrow X$ with the monoidal product induced from composition. Then the ∞-groupoid of non-unital algebra objects in $\mathcal{E}_X$ is contractible.*

Proof. Let $\mathcal{A}ss_{\mathrm{nu}}$ denote the non-unital associative ∞-operad. The identity map $\mathcal{A}ss_{\mathrm{nu}} \longrightarrow \mathcal{A}ss_{\mathrm{nu}}$ which is in particular a left fibration of ∞-operads classifies the terminal non-unital monoidal ∞-groupoid $\mathcal{A}$ which consists of single automorphismless idempotent object $a \in \mathcal{A}$. The non-unital algebra objects in $\mathcal{E}_X$ are then classified by non-unital lax monoidal functors

$$\mathcal{A} \longrightarrow \mathcal{E}_X$$

Since $\mathcal{E}_X$ is an ∞-groupoid this is same as non-unital monoidal functors (without the lax)

$$\mathcal{A} \longrightarrow \mathcal{E}_X$$

Now the forgetful functor from unital to non-unital monoidal ∞-groupoids has a left adjoint. Applying this left adjoint to $\mathcal{A}$ we obtain the ∞-groupoid $\mathcal{U}\mathcal{A}$ with two automorphismless objects

$$\mathcal{U}\mathcal{A} = \{1, a\}$$

such that 1 is the unit of the monoidal structure and a is an idempotent object.

Hence we need to show that the ∞-groupoids of monoidal functors

$$\mathcal{U}\mathcal{A} \longrightarrow \mathcal{E}_X$$

is contractible. Now given a monoidal ∞-groupoid $\mathcal{G}$ we can form the ∞-category $\mathcal{B}(\mathcal{G})$ having a single object with endomorphism space $\mathcal{G}$ (the monoidal structure on $\mathcal{G}$ will then give the composition structure). This construction determines a fully faithful functor from the ∞-category of monoidal ∞-groupoids and the ∞-category of pointed ∞-categories (see [Lur1] Remark 4.4.6 for a much more general statement). In particular it will be enough to show that the ∞-groupoid of **pointed functors**

$$\mathcal{B}(\mathcal{U}\mathcal{A}) \longrightarrow \mathcal{B}(\mathcal{E}_X)$$

is contractible. Since $\mathcal{B}(\mathcal{E}_X)$ is an ∞-groupoid it will be enough to show that $\mathcal{B}(\mathcal{U}\mathcal{A})$ is weakly contractible.

Now the nerve $\mathrm{N}\,\mathcal{B}(\mathcal{U}\mathcal{A})$ of $\mathcal{B}(\mathcal{U}\mathcal{A})$ is the simplicial set in which for each n there exists a single **non-degenerate** n-simplex $\sigma_n \in \mathrm{N}\,\mathcal{B}(\mathcal{U}\mathcal{A})_n$ such that $d_i(\sigma_n) = \sigma_{n-1}$ for all $i = 0, ..., n$. By Van-Kampen it follows that $\mathrm{N}\,\mathcal{B}(\mathcal{U}\mathcal{A})$ is simply connected and by direct computation all the homology groups vanish. $\square$

This finishes the proof of Lemma 2.14. $\square$

This finishes the proof of Theorem 2.6. $\square$

3 From Quasi-Unital to Unital Cobordism Hypothesis

In this section we will show how the quasi-unital cobordism hypothesis (Theorem 2.6) implies the last step in the proof of the 1-dimensional cobordism hypothesis (Theorem 1.8).

Let $M : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathrm{Grp}_\infty$ be a non-degenerate lax symmetric monoidal functor. We can construct a pointed **non-unital** symmetric monoidal ∞-category $\mathcal{C}_M$ as follows:

1. The objects of $\mathcal{C}_M$ are the objects of $\mathcal{B}_1^{\mathrm{ev}}$. The marked point is the object X_+.
2. Given a pair of objects $X, Y \in \mathcal{C}_M$ we define

$$\mathrm{Map}_{\mathcal{C}_M}(X, Y) = M(\check{X} \otimes Y)$$

Given a triple of objects $X, Y, Z \in \mathcal{C}_M$ the composition law

$$\mathrm{Map}_{\mathcal{C}_M}(\check{X}, Y) \times \mathrm{Map}_{\mathcal{C}_M}(\check{Y}, Z) \longrightarrow \mathrm{Map}_{\mathcal{C}_M}(\check{X}, Z)$$

is given by the composition

$$M(\check{X} \otimes Y) \times M(\check{Y} \otimes Z) \longrightarrow M(\check{X} \otimes Y \otimes \check{Y} \otimes Z) \longrightarrow M(\check{X} \otimes Z)$$

where the first map is given by the lax symmetric monoidal structure on the functor M and the second is induced by the evaluation map

$$\mathrm{ev}_Y : \check{Y} \otimes Y \longrightarrow 1$$

in $\mathcal{B}_1^{\mathrm{ev}}$.

3. The symmetric monoidal structure is defined in a straight forward way using the lax monoidal structure of M.

It is not hard to see that if M is non-degenerate then $\mathcal{C}_M$ is **quasi-unital**, i.e. each object contains a morphism which **behaves** like an identity map (see [Har]). This construction determines a functor

$$G : \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_\infty) \longrightarrow \mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$$

where $\mathrm{Cat}^{\mathrm{qu},\otimes}$ is the ∞-category of symmetric monoidal quasi-unital categories (i.e. commutative algebra objects in the ∞-category $\mathrm{Cat}^{\mathrm{qu}}$ of quasi-unital ∞-categories). In [Har] it is proved that the forgetful functor

$$S : \mathrm{Cat} \longrightarrow \mathrm{Cat}^{\mathrm{qu}}$$

From ∞-categories to quasi-unital ∞-categories is an **equivalence** and so the forgetful functor

$$S^{\otimes} : \mathrm{Cat}^{\otimes} \longrightarrow \mathrm{Cat}^{\mathrm{qu},\otimes}$$

is an equivalence as well.

Now recall that

$$\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}_1^{\mathrm{ev}}/} \subseteq \mathrm{Cat}^{\mathrm{nd}}_{\mathcal{B}_1^{\mathrm{ev}}/}$$

is the full subcategory spanned by essentially surjective functors $\varphi : \mathcal{B}_1^{\mathrm{ev}} \longrightarrow \mathcal{C}$. The fiber functor construction $\varphi \mapsto M_\varphi$ induces a functor

$$F : \mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}_1^{\mathrm{ev}}/} \longrightarrow \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}_1^{\mathrm{ev}}, \mathrm{Grp}_\infty)$$

The composition $G \circ F$ gives a functor

$$\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}_1^{\mathrm{ev}}/} \longrightarrow \mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$$

We claim that $G \circ F$ is in fact **equivalent** to the composition

$$\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}_1^{\mathrm{ev}}/} \xrightarrow{T} \mathrm{Cat}^{\otimes}_{\mathcal{B}_0^{\mathrm{un}}/} \xrightarrow{S} \mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}_0^{\mathrm{un}}/}$$

where T is given by the restriction along $X_+ : \mathcal{B}_0^{\text{un}} \hookrightarrow \mathcal{B}_1^{\text{ev}}$ and S is the forgetful functor.

Explicitly, we will construct a natural transformation

$$N : G \circ F \xrightarrow{\simeq} S \circ T$$

In order to construct N we need to construct for each non-degenerate functor $\varphi : \mathcal{B}_1^{\text{ev}} \longrightarrow \mathcal{D}$ a natural pointed functor

$$N_\varphi : \mathcal{C}_{M_\varphi} \longrightarrow \mathcal{D}$$

The functor N_φ will map the objects of $\mathcal{C}_{M_\varphi}$ (which are the objects of $\mathcal{B}_1^{\text{ev}}$) to $\mathcal{D}$ via φ. Then for each $X, Y \in \mathcal{B}_1^{\text{ev}}$ we can map the morphisms

$$\text{Map}_{\mathcal{C}_{M_\varphi}}(X, Y) = \text{Map}_{\mathcal{D}}(1, \check{X} \otimes Y) \longrightarrow \text{Map}_{\mathcal{D}}(X, Y)$$

via the duality structure - to a morphism $f : 1 \longrightarrow \check{X} \otimes Y$ one associates the morphism $\widehat{f} : X \longrightarrow Y$ given as the composition

$$X \stackrel{Id \otimes f}{\longrightarrow} X \otimes \check{X} \otimes Y \stackrel{\varphi(\text{ev}_X) \otimes Y}{\longrightarrow} Y$$

Since $\mathcal{D}$ has duals we get that N_φ is fully faithful and since we have restricted to essentially surjective φ we get that N_φ is essentially surjective. Hence N_φ is an equivalence of quasi-unital symmetric monoidal ∞-categories and N is a natural equivalence of functors.

In particular we have a homotopy commutative diagram:

$$\begin{array}{ccccc} & & \text{Cat}^{\text{sur}}_{\mathcal{B}_1^{\text{ev}}/} & & \\ & \swarrow^{F} & & \searrow^{T} & \\ \text{Fun}^{\text{lax}}_{\text{nd}}(\mathcal{B}_1^{\text{ev}}, \text{Grp}_\infty) & & & & \text{Cat}^{\otimes}_{\mathcal{B}_0^{\text{un}}/} \\ & \searrow_{G} & & \swarrow_{S} & \\ & & \text{Cat}^{\text{qu},\otimes}_{\mathcal{B}_0^{\text{un}}/} & & \end{array}$$

Now from Lemma 1.6 we see that T is fully faithful. Since S is an equivalence of ∞-categories we get

Corollary 3.1. *The functor $G \circ F$ is fully faithful.*

We are now ready to complete the proof of 1.8. Let $\mathcal{D}$ be a symmetric monoidal ∞-category with duals and let $\varphi : \mathcal{B} \longrightarrow \mathcal{D}$ be a non-degenerate functor. We wish to show that the space of maps

$$\text{Map}_{\text{Cat}^{\text{sur}}_{\mathcal{B}_1^{\text{ev}}/}}(\iota, \varphi)$$

is contractible. Consider the sequence

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}^{\mathrm{ev}}_1/}}(\iota, \varphi) \longrightarrow \mathrm{Map}_{\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nd}}(\mathcal{B}^{\mathrm{ev}}_1, \mathrm{Grp}_\infty)}(M_\iota, M_\varphi) \longrightarrow \mathrm{Map}_{\mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}^{\mathrm{un}}_0/}}(\mathcal{B}^{\mathrm{or}}_1, \mathcal{D})$$

By Theorem 2.6 the middle space is contractible and by lemma 3.1 the composition

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}^{\mathrm{ev}}_1/}}(\iota, \varphi) \longrightarrow \mathrm{Map}_{\mathrm{Cat}^{\mathrm{qu},\otimes}_{\mathcal{B}^{\mathrm{un}}_0/}}(\mathcal{B}^{\mathrm{or}}_1, \mathcal{D})$$

is a weak equivalence. Hence we get that

$$\mathrm{Map}_{\mathrm{Cat}^{\mathrm{sur}}_{\mathcal{B}^{\mathrm{ev}}_1/}}(\iota, \varphi)$$

is contractible. This completes the proof of Theorem 1.8.

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