2-Fold Complete Segal Spaces

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June 24, 2013

Introduction

Recall that the lecture yesterday where we have learned of the notion of an ∞-category, which models the appropriate weak version of a category enriched in spaces. ∞-categories themselves form an ∞-category which admits various models, in particular the **complete Segal space** combinatorial model category. The ∞-category of ∞-categories has all limits and colimits, and in particular all products. Hence one can try to consider a notion of $(\infty, 2)$-categories, which should be the appropriate weak version of ∞-categories **enriched in ∞-categories**.

There are several ways to access the notion of enriched categories. Let us consider first the discrete case. Let $\mathcal{M}$ be a category and assume for simplicity that it admits all small limits and colimits. One can then consider the notion of categories **enriched in** $\mathcal{M}$ (with respect to the cartesian monoidal structure).

One way to approach this object, which will be relevant to this workshop, is to compare it with the notion of **categories internal to** $\mathcal{M}$. This means that we look at a(small) category as a certain diagram in sets involving the set of objects O, the set of morphisms (or arrows) A, the source and target maps $s, t : A \longrightarrow O$, the identity map $O \longrightarrow A$ and the composition map

$$A \times_O A \longrightarrow A$$

which is required to be associative and unital (all conditions that be expressed by the commutativity of certain obvious diagrams). This "diagramatic" definition makes sense in any category $\mathcal{M}$ which has fiber products (and in particular in our $\mathcal{M}$ which has all limits), by taking A and O to be objects of $\mathcal{M}$ and all the maps to be morphisms in $\mathcal{M}$. This gives the notion of a category internal to $\mathcal{M}$.

The notion of a category enriched in $\mathcal{M}$ is obviously a different one. However, in some cases we can make a connection between the two. The first difference one encounters is that in a (small) category $\mathcal{C}$ the objects $\mathrm{Ob}(\mathcal{C})$ are always a set, and not an object of $\mathcal{M}$. Hence in order to make any connection between enriched and internal categories one needs to be able to interpret sets as objects in $\mathcal{M}$. Note that that since $\mathcal{M}$ has coproducts and a terminal object one can construct a functor

$$c : \mathrm{Set} \longrightarrow \mathcal{M}$$

by setting

$$c(A) = \coprod_A *$$

where $* \in \mathcal{M}$ is the terminal object. We can then make sense of an ”object of objects” by setting

$$\mathcal{O} \overset{\text{def}}{=} c(\mathrm{Ob}(C))$$

We shall call objects in the essential image of c **set-like** objects. Similarly, one can define an ”object of arrows” by setting

$$\mathcal{E} \overset{\text{def}}{=} \coprod_{X,Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathcal{M}$$

Note that $\mathcal{E}$ admits two natural maps $t, s : \mathcal{E} \longrightarrow \mathcal{O}$. However, this is still not enough in order to identify enriched categories as a particular kind of internal categories. The reason is that the composition rule of $\mathcal{C}$ is only defined for each triple of objects separately, in we need to be able to write it as a map in $\mathcal{M}$ of the form

$$\mathcal{E} \times_{\mathcal{O}} \mathcal{E} \longrightarrow \mathcal{E}$$

Furthermore, we will want functors of enriched categories to be given by maps of the corresponding object of objects and object of morphisms. To obtain a sensible theory we will hence need some strong conditions:

Definition 0.0.1. Let $\mathcal{M}$ be a complete and cocomplete category and let $c : \mathrm{Set} \longrightarrow \mathcal{M}$ be as above. We shall say that $\mathcal{M}$ is an **absolute distributer** if

1. The functor c is fully-faithful and respects all limits and colimits (the colimits part is actually automatic as c admits a right adjoint represented by $*$).

2. For each set A, the adjoint functors $\mathcal{M}^A \longrightarrow \mathcal{M}_{/c(A)}$ and $\mathcal{M}_{/c(A)} \longrightarrow \mathcal{M}^A$ given by

$$(X_a)_{a \in A} \mapsto \left[\coprod_{a \in A} X_a \longrightarrow c(a) \right]$$

and

$$[X \longrightarrow c(A)] \longrightarrow (X \times_{c(A)} c(\{a\}))_{a \in A}$$

are categorical inverses (i.e. the unit and counit maps are natural equivalences).

The categories of sets, the category of categories and the category of topological spaces are all absolute distributers.

Now let us assume that $\mathcal{M}$ is an absolute distributer. From condition (2) in the definition of absolute distributer it follows that the fiber product

$$\begin{array}{ccc} \mathcal{E} \times_{\mathcal{O}} \mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow{\scriptstyle s} \\ \mathcal{E} & \xrightarrow{t} & \mathcal{O} \end{array}$$

can be identified with

$$\coprod_{X,Y,Z\in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X,Y)\times \mathrm{Hom}_{\mathcal{C}}(Y,Z)$$

and hence we can write the composition rule as a map

$$\mathcal{E}\times_{\mathcal{O}}\mathcal{E}\longrightarrow \mathcal{E}$$

These constructions gives a functor

$$\mathcal{F}:\mathrm{Cat}_{\mathcal{M}-enriched}\longrightarrow \mathrm{Cat}_{\mathcal{M}-internal}$$

from the category of categories enriched in $\mathcal{M}$ to categories internal to M. Furthermore, from the definition of absolute distributor it follows that this functor is fully-faithful and its essential image is given by the internal category objects whose object of objects is set-like.

The discussion above can be generalized to the ∞-setting. Given an ∞-category $\mathcal{M}$ with (homotopy) limits and colimits one can consider two different notions: ∞-categories internal to $\mathcal{M}$ and ∞-categories enriched in $\mathcal{M}$.

Both notions are a bit more subtle then the corresponding discrete analogue. However, the notion of internal categories is much more straightforward to establish in the ∞-setting - one needs to work with **simplicial objects** in $\mathcal{M}$ satisfying the Segal condition (which makes sense as $\mathcal{M}$ has fiber products), and possibly also some completeness condition. In any case, this is arguably simpler than the enriched notion, and so it is worth while to check when one can formalize the notion of enriched ∞-categories via internal ∞-categories, similarly to the discrete case discussed above.

The first difference one encounters when passing to the ∞-case is that an ∞-category $\mathcal{C}$ naturally has a **space**, or an ∞-groupoid, of objects, as opposed to just a set. This ∞-groupoid is the underlying (or maximal) ∞-groupoid of $\mathcal{C}$. Hence when we define an ∞-category enriched in $\mathcal{M}$ we should first have an ∞-groupoid $\mathcal{O}$ and then add a morphism object $\mathrm{Map}_{\mathcal{C}}(X,Y)\in\mathcal{M}$ for each $X,Y\in\mathcal{O}$ which is **functorial** in $\mathcal{O}\times\mathcal{O}^{\mathrm{op}}$.

In particular, we see that in order to generalize the discussion above to the ∞-setting one should replace the category of sets by that of ∞-groupoids. Note that when $\mathcal{M}$ has colimits and a terminal object $*$ then there is an essentially unique colimit preserving functor

$$c:\mathrm{Grp}_{\infty}\longrightarrow\mathcal{M}$$

which sends the trivial ∞-groupoid to $*$.

We can call objects in essential image of c **space-like** objects of $\mathcal{M}$. Then if $\mathcal{C}$ is a category enriched in $\mathcal{M}$ then $c(\mathcal{O})\in\mathcal{M}$ can serve as our "object of objects". As for the object of morphisms, one can take the colimit

$$\operatorname*{colim}_{X,Y\in\mathcal{O}\times\mathcal{O}^{\mathrm{op}}}\mathrm{Map}_{\mathcal{C}}(X,Y)\in\mathcal{M}$$

which is analogous to the coproduct we took before. This construction can be naturally extended to give a simplicial object in M. However, if we want this simplicial object to satisfy the Segal condition we will need to put strong assumptions on M amounting to an ∞-version of the notion of absolute distributor.

Note that the first condition in Definition 0.0.1 generalizes immediately to the ∞-setting. As for the second condition, note that the condition we wrote corresponded to a given set A, together with the construction of A as a coproduct of its elements. In order to generalize to the ∞-case, one needs to replace A with an arbitrary ∞-groupoid, and the coproduct of singletons with an arbitrary presentation of A as a colimit of ∞-groupoids. We refer the reader to [Lur] for the exact definition.

The important fact which makes this discussion relevant to our needs is the following:

Theorem 0.0.2 (Lurie)**.** *The ∞-category Cat_∞ is an absolute distributor.*

This means that we can study ∞-categories enriched in Cat_∞ (i.e. $(\infty, 2)$-categories) as special kinds of ∞-categories internal to ∞-categories (where the special kinds means having a space-like object of objects). This results in the definition of 2-fold complete Segal spaces.

References

[Lur] Lurie, J. *(∞,2)-Categories and the Goodwillie Calculus I*, `http://www.math.harvard.edu/~lurie/papers/GoodwillieI.pdf`.