Complex Cobordism and Formal Group Laws

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1 Cobordism Homology Theories

In this section we will explain how to construct generalized homology and cohomology theories which are based on cobordism classes of manifolds (with various extra structure). With begin with some basic terminology.

Let X be a topological space and $E \longrightarrow X$ a (real) vector bundle equipped with an inner product. Then one can form the corresponding disc bundle $D(E) \longrightarrow X$ and sphere bundle $S(E) \longrightarrow X$ by replacing each fiber with its corresponding (closed) unit disc or unit sphere. We define the **Thom** space X^E of E to be the quotient space

$$X^E = D(X)/E(X)$$

We will always consider X^E as a **pointed** space with the base point being the collapsed $E(X)$. In particular we will employ the convention that

$$X^{\underline{\mathbb{R}}^0} = X_+ \overset{\text{df}}{=} X \coprod \{*\}$$

where $\underline{\mathbb{R}}^0$ is the trivial 0-dimensional vector bundle. Note that X^E can also be identified with the **cone** of the map $S(E) \longrightarrow X$.

The construction of Thom spaces can be thought of as a **twisted** form of suspension. In particular, if $E = \underline{\mathbb{R}}^k$ is the trivial k-vector bundle then X^E can be naturally identified with the n-fold suspension

$$\Sigma^n(X_+) = S^n \wedge X_+$$

More generally, if $E \longrightarrow X$ and $F \longrightarrow Y$ are two inner product vector bundles then we have a natural vector bundle

$$D = p_X^* E \oplus p_Y^* F \longrightarrow X \times Y$$

One can then identify the Thom space $(X \times Y)^D$ with the smash product

$$X^E \wedge Y^F$$

Taking $Y = *$, for example, one gets the homeomorphism

$$X^{E \oplus \underline{\mathbb{R}}^k} \cong S^k \wedge X^E = \Sigma^k X^E$$

Now recall that if X is a reasonably nice space then k-dimensional vector bundles with inner product $E \longrightarrow X$ are classified by maps

$$c : X \longrightarrow \mathrm{BO}(k)$$

where $\mathrm{O}(k) \subseteq \mathrm{GL}_k(\mathbb{R})$ is the group of orthogonal k-matrices and $\mathrm{BO}(k)$ is the classifying space of $\mathrm{O}(k)$. More explicitly there exists a **universal** vector bundle $E_k \longrightarrow \mathrm{BO}(k)$ and for each (inner product) vector bundle $E \longrightarrow X$ there exists a classifying map $c : X \longrightarrow \mathrm{BO}(k)$ and an isomorphism of inner product vector bundles

$$\psi : E \xrightarrow{\simeq} c^* E_k$$

Further more, given the vector bundle E, the pair (c, ψ) becomes essentially unique - under a natural topology the space of pairs (c, ψ) is **contractible**.

Remark 1.1. Given a vector bundle $E \longrightarrow X$ on a reasonably nice space X the space of inner products on E is contractible. Hence there is no essential loss of generality in assuming that all vector bundles in sight carry an inner product. This claim can also be seen as the observation that the inclusion of topological groups $\mathrm{O}(k) \longrightarrow \mathrm{GL}_k(\mathbb{R})$ is a homotopy equivalence and so we could have replaced in the above discussion $\mathrm{BO}(k)$ by $\mathrm{BGL}_k(\mathbb{R})$ (which is the space classifying vector bundles without inner product).

Now let $\mathrm{MO}(k)$ be the Thom space of the universal bundle $E_k \longrightarrow \mathrm{BO}(k)$. The main idea is that the homotopy groups of the spaces $\mathrm{MO}(k)$ are closely related to (unoriented) cobordism classes of manifolds. In order to see this we first need to understand how to organize all the $\mathrm{MO}(k)$'s together. Note that for each k, the vector bundle $E_k \oplus \underline{\mathbb{R}}$ is classified by the inclusion

$$\iota_k : \mathrm{BO}(k) \hookrightarrow \mathrm{BO}(k+1)$$

which is induced by the "top-left" inclusion of $\mathrm{O}(k)$ inside $\mathrm{O}(k+1)$. In particular we have a natural isomorphism of $(k+1)$-vector bundles

$$\psi_k : E_k \oplus \underline{\mathbb{R}} \longrightarrow \iota_k^* E_{k+1}$$

inducing a map

$$\Sigma\, \mathrm{MO}(k) \cong \mathrm{BO}(k)^{E_k \oplus \underline{\mathbb{R}}} \longrightarrow \mathrm{BO}(k+1)^{E_{k+1}} = \mathrm{MO}(k+1)$$

The sequence of space

$$* = M(0), M(1), M(2), ...$$

together with the maps $\Sigma M(k) \longrightarrow M(k+1)$ form a **spectrum**, which is denoted by MO and is called the **unoriented Thom spectrum**. The key observation of Thom (in his PhD thesis from 1954) was that this spectrum encodes the information of (unoriented) cobordism classes of manifolds. In particular, the homotopy groups of this spectrum are isomorphic to the unoriented cobordism groups.

In order to see how this works suppose that we are given a closed n-manifold M. Then it is known that M can be (smoothly) embedded in Euclidean space $\mathbb{R}^{n+k}$ for large k. In fact, when taking k to infinity we obtain an **essentially unique** embedding, i.e. the direct limit of the sequence

$$\mathrm{Emb}(M, \mathbb{R}^n) \hookrightarrow \mathrm{Emb}(M, \mathbb{R}^{n+1}) \hookrightarrow \mathrm{Emb}(M, \mathbb{R}^{n+2}) \hookrightarrow ...$$

is **contractible** (where each $\mathbb{R}^{n+k}$ is included in $\mathbb{R}^{n+k+1}$ via the first $n+k$ coordinates and the spaces $\mathrm{Emb}(M, \mathbb{R}^{n+k})$ carry a certain natural topology).

Now suppose that we have chosen an embedding $\iota : M \hookrightarrow \mathbb{R}^{n+k}$ for some k. Using the standard inner product on $\mathbb{R}^{n+k}$ we can define the **normal bundle** of M inside $\mathbb{R}^{n+k}$. This is the vector bundle $N^\iota \longrightarrow M$ whose fiber over $m \in M$ is the subspace

$$N^\iota_m \subseteq T_{\iota(m)}\mathbb{R}^{n+k}$$

of vectors which are orthogonal to the tangent space $\iota_* T_m M \subseteq T_{\iota(m)}\mathbb{R}^{n+k}$.

Remark 1.2. Since the choice of embedding $\iota : M \hookrightarrow \mathbb{R}^{n+k}$ is essentially unique with $k \longrightarrow \infty$ we see that the stable equivalence class of the vector bundle $N^\iota \longrightarrow M$ is independent of ι. This class is called the **stable normal bundle** of M. Note that the direct sum $N^\iota \oplus TM$ is isomorphic to the trivial vector bundle $\iota^* T\mathbb{R}^{n+k} = \underline{\mathbb{R}}^{n+k}$, which means that the stable class $[N^\iota]$ is the inverse of the stable class $[TM]$ in the K-group $\widetilde{K}^0(M)$.

Now the vector bundle N^ι is classified by a map $c : M \longrightarrow \mathrm{BO}(k)$ - i.e. there is an essentially unique choice of a pair (c, ψ) where $c : M \longrightarrow \mathrm{BO}(k)$ is a map and

$$\psi : N^\iota \longrightarrow c^* E_k$$

is an isomorphism of inner product vector bundles. In particular ψ induces an isomorphism of disc bundles

$$\psi_D : D(N) \longrightarrow D(c^* E_k)$$

Now for a small enough $\varepsilon > 0$ the map $D(N) \longrightarrow \mathbb{R}^{n+k}$ given by

$$(m, v) \mapsto \iota(m) + \varepsilon v \in \mathbb{R}^{n+k}$$

is a topological embedding. Choosing such an ε (again a contractible menu of choices) we can identify the open unit disc bundle $D^o(N)$ with a small neighborhood U of M inside $\mathbb{R}^{n+k}$ and the sphere bundle $S(N)$ with the boundary

∂U of U inside $\mathbb{R}^{n+k}$. Then the pair (c, ψ) gives us a map $\overline{U} \longrightarrow D(E_k)$ which sends the boundary ∂U to $S(E_k)$. This gives us a map

$$\overline{U} \longrightarrow \mathrm{MO}(k)$$

which sends ∂U to the base point, and so can be extended to map

$$\mathbb{R}^{n+k} \longrightarrow \mathrm{MO}(k)$$

which sends all of $\mathbb{R}^{n+k} \setminus U$ to the base point. Identifying S^{n+k} with the one point compactification $\mathbb{R}^{n+k} \cup \{\infty\}$ of $\mathbb{R}^{n+k}$ we obtain a pointed map

$$S^{n+k} \longrightarrow \mathrm{MO}(k)$$

Now, when $k \longrightarrow \infty$ all the choices we made were essentially unique (i.e. whenever we had to choose we had a contractible space of choices), and so we conclude that the resulting element

$$\alpha \in \pi_n(\mathrm{MO}) = \lim_k \pi_{n+k}(\mathrm{MO}(k))$$

is well defined. We then have Thom's theorem:

Theorem 1.3 (Thom)**.** *Let Ω_n^{un} denote the group of unoriented n-dimensional cobordism classes. Then the construction above induces an isomorphism*

$$\Omega_n^{\mathrm{un}} \xrightarrow{\simeq} \pi_n(\mathrm{MO})$$

Now the spectrum MO gives a **generalized homology theory**

$$\mathrm{MO}_n(X) = \varinjlim_k \pi_{n+k}(X_+ \wedge \mathrm{MO}(k))$$

The group $\mathrm{MO}_n(X)$ can also be described in terms of unoriented coboridsm - the elements of this group are represented by **manifolds over** X, i.e. pairs (M, f) where M is a closed n-manifold and $f : M \longrightarrow X$ is a map. A cobordism from (M, f) to (M', f') is a cobordism W from M to M' together with a map $F : W \longrightarrow X$ extending $f \coprod f'$. The cobordism classes of manifolds over X form a group $\Omega_n^{\mathrm{un}}(X)$ under disjoint union. By the same construction as above one can construct a natural isomorphism

$$\Omega_n^{\mathrm{un}}(X) \cong \mathrm{MO}_n(X)$$

Remark 1.4. One can also construct the corresponding **cohomology groups** by setting

$$\mathrm{MO}^n(X) \cong \varinjlim_k \left[\Sigma^k X_+, \mathrm{MO}(n+k)\right]_*$$

In some cases one can find geometric description of these groups via cobordisms of manifolds (like for the homology groups).

Remark 1.5. There exists natural maps

$$\mathrm{BO}(n) \times \mathrm{BO}(m) \longrightarrow \mathrm{BO}(n+m)$$

which encode the operation of direct sum of vector bundles. These maps induce maps

$$\mathrm{MO}(n) \wedge \mathrm{MO}(m) \longrightarrow \mathrm{MO}(n+m)$$

which in turn induce a map

$$\mathrm{MO} \wedge \mathrm{MO} \longrightarrow \mathrm{MO}$$

This map can be used to endow MO with the structure of a coherently associative and commutative ring spectrum (i.e. an $\mathbf{E}_\infty$-ring). In particular the cohomology theory $\mathrm{MO}^\bullet(-)$ is **multiplicative** (i.e. admits a notion of a cup product).

When $X = *$ the cohomology groups $\mathrm{MO}^{-n}(*)$ can be identified with the homology groups $\mathrm{MO}_n(*)$, i.e. with cobordism classes of n-dimensional manifolds. In this case the cup product admits a simple geometric description - it corresponds to Cartesian product of manifolds.

2 Stable Structures and Complex Cobordism

In many situations one is interested in vector bundles which admit some extra structure, such as an orientation, a complex structure, a trivialization, etc. In particular, one can try to attach such a structure to the tangent bundle of a manifold. In certain cases, one will be able to make sense of cobordism classes of manifolds with this extra structure by requiring the cobordism to carry similar extra structure in a compatible way. One can then try to construct a variant of the homology theory $\mathrm{MO}_\bullet(-)$ by considering cobordism classes of manifolds over X carrying this additional structure.

It turns out that in order for this new set of invariants to form a **generalized homology theory** one needs to consider the structure not on the tangent bundle itself but rather on the stable class if it.

A rather general setting for such "stable extra structure" is the following. A **stable structure** consists of the following pieces of data:

1. An infinite set $A \subseteq \mathbb{N}$ of indices considered as a poset with the linear order induced from $\mathbb{N}$.

2. A functor $k \mapsto G(k)$ from the poset A to topological groups, i.e. for each $k < m$ we have a homomorphism of topological groups $G(k) \longrightarrow G(m)$ which are compatible with each other.

3. A natural transformation $F : G(\bullet) \longrightarrow \mathrm{O}(\bullet)$ where $\mathrm{O}(\bullet)$ is the functor which is assigns to each k the orthogonal group $\mathrm{O}(k)$ and to each pair $k < m$ the top-left inclusion $\mathrm{O}(k) \hookrightarrow \mathrm{O}(m)$.

Now let $(A, G(\bullet), F_\bullet)$ be a stable structure. Recall the universal inner product vector bundles

$$E_k \longrightarrow \mathrm{BO}(k)$$

Then for each $k \in A$ we can pull the universal bundle to $\mathrm{B}\,G(k)$ via the induced map

$$\mathrm{B}\,F_k : \mathrm{B}\,G(k) \longrightarrow \mathrm{BO}(k)$$

We will denote by

$$E_k^G \stackrel{\mathrm{df}}{=} (\mathrm{B}\,F_k)^* E_k \longrightarrow \mathrm{B}\,G(k)$$

the pulled back universal bundle.

Now let X be a space and let $E \longrightarrow X$ an inner product k-vector bundle for some $k \in A$. An **unstable $G(k)$-structure** on E is a map

$$d : X \longrightarrow \mathrm{B}\,G(k)$$

together with an isomorphism of inner product vector bundles (over X)

$$\psi : E \xrightarrow{\simeq} d^* E_k^G$$

As for the stable case, suppose that $E \longrightarrow X$ is an m-vector bundle for some m, not necessarily in A. A **stable G-structure** on E is an integer $m \leq k \in A$, a map

$$d : X \longrightarrow \mathrm{B}\,G(k)$$

and an isomorphism

$$\psi : E \oplus \underline{\mathbb{R}}^{k-m} \xrightarrow{\simeq} d^* E_k^G$$

One can define an **equivalence** of stable G-structure in a straight forward (although slightly tidies) way. This notion will endow the set of stable G-structure on a E with the structure of a **topological groupoid**.

Alternatively, one can described the classifying space of this topological groupoid rather explicitly. If we fix a map $c : X \longrightarrow \mathrm{BO}(m)$ which classifies E then the stable class $[E]$ can be encoded as the composed map

$$X \xrightarrow{c} \mathrm{BO}(m) \hookrightarrow \operatorname*{colim}_i \mathrm{BO}(i) \stackrel{\mathrm{df}}{=} \mathrm{BO}$$

which we can denote by $\widetilde{c} : X \longrightarrow \mathrm{BO}$. Giving a stable structure on E is then equivalent to giving a map $\widetilde{d} : E \longrightarrow \mathrm{B}\,G$ and a homotopy making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{d}} & \mathrm{B}\,G \\ & \searrow^{\widetilde{c}} & \downarrow \\ & & \mathrm{BO} \end{array}$$

commute. In other words, the **classifying space of stable G-structures** on E can be identified with the homotopy fiber of the map

$$[X, \mathrm{B}\,G] \longrightarrow [X, \mathrm{BO}]$$

over the point $\widetilde{c} \in [X, \mathrm{BO}]$.

Examples:

1. The **unoriented** stable structure is given simply by setting $G(k) = \mathrm{O}(k)$ for every k (so that $A = \mathbb{N}$). Every inner product vector bundle carries a natural unoriented structure (both stably and unstably).

2. The **oriented** stable structure is given by setting $G(k) = \mathrm{SO}(k)$ for every k. Note that in this case the notions of stable oriented structure and unstable oriented structure coincide. Such a structure is classically known as an orientation.

3. The **trivial** stable structure is given by setting $G(k) = \{1\}$ for every k. Then a G-structure on a vector bundle E consists of a **trivialization** of $E \oplus \underline{\mathbb{R}}^k$ for large enough k.

4. The **complex** (or **unitary**) stable structure is given by taking $A = 2\mathbb{N}$ to be the even numbers and setting $G(2n) = \mathrm{U}(n)$ where $\mathrm{U}(n)$ is the complex unitary group (which admits a natural map $\mathrm{U}(n) \hookrightarrow \mathrm{O}(2n)$).

5. The **special complex** (or **special unitary**) stable structure is given by taking $A \subseteq \mathbb{N}$ to be the even numbers and setting $G(2n) = \mathrm{SU}(n)$.

6. The **quaternionic stable structure** is given by taking $A = 4\mathbb{N}$ and $G(4n) = \mathrm{Sp}(n)$ where $\mathrm{Sp}(n)$ is the group of $(n \times n)$-matrices over $\mathbb{H}$ which preserve the quarternionic norm.

7. The **spin** stable structure is given by $G(n) = \mathrm{Spin}(n)$ where $\mathrm{Spin}(n)$ is the universal cover of $\mathrm{SO}(n)$.

8. The **string** stable structure is given by $G(n) = \mathrm{String}(n)$ where $\mathrm{String}(n)$ is the 2-connected cover of $\mathrm{Spin}(n)$.

Now let $G(\bullet)$ be a stable structure. A stable G-manifold is a pair (M, d) where M is a manifold and d is a stable G-structure on the tangent bundle TM. Given two stable n-dimensional closed G-manifolds $(M, d), (M', d')$, a stable G-cobordism from (M, d) to (M', d') is an $(n+1)$-dimensional G-manifold (W, D) together with

1. A diffeomorphism
$$B : M \coprod M' \xrightarrow{\simeq} \partial W$$

2. A nowhere vanishing section s of the 1-dimensional normal bundle of ∂W inside W which is inward pointing on M and outward pointing on M' (such a section is unique up to a contractible space of choices).

3. An equivalence of stable G-structures on $M \coprod M'$ between $d \coprod d'$ and the stable G-structure induced by d via B and s (note that B and s provide an isomorphism of bundles $T(M \coprod M') \oplus \underline{\mathbb{R}} \simeq B^*TW$).

The set of stable G-cobordism classes of closed n-dimensional G-manifolds can be endowed with an abelian group structure via disjoint union. This group is called the G-cobordism group and is denote by Ω_n^G.

Now given a space X we can make the above construction relative by considering triplets (M, d, f) where (M, d) is a closed n-dimensional stable G-manifold and $f : M \longrightarrow X$ is a map. A stable G-cobordism from (M, d, f) to (M', d', f') is a G-cobordism (W, D) from (M, d) to (M', d') together with a map $F : W \longrightarrow X$ extending $f \coprod f'$. The group of stable G-cobordism classes of stable G-manifolds over X is denoted by $\Omega_n^G(X)$. As above, one can show that the functors $\Omega_n^G(X)$ actually form a generalized homology theory by constructing a spectrum which represents them. This functor can be constructed in a similar way to MO by letting $\mathrm{M}\,G(k)$ denote the Thom space of the vector bundle

$$E_k^G = (\mathrm{B}\,F_k)^* E_k \longrightarrow \mathrm{B}\,G(k)$$

Given two $k \leq m \in A$ one has a natural map

$$\Sigma^{m-k}\,\mathrm{M}\,G(k) \longrightarrow \mathrm{M}\,G(m)$$

This information is enough to construct a **spectrum** $\mathrm{M}\,G$, which is called the Thom spectrum of G. By similar technics to above one can show that the spectrum $\mathrm{M}\,G$ represents the functors Ω_n^G, i.e. that for each space X one has

$$\Omega_n^G(X) \cong \mathrm{M}\,G_n(X) = \operatorname*{colim}_{k \in A} \pi_{n+k}\left(X_+ \wedge \mathrm{M}\,G(k)\right)$$

One can also form the G-cobordism **cohomology groups** by setting

$$\mathrm{M}\,G^n(X) \cong \operatorname*{colim}_{n \leq k \in A} \left[\Sigma^{k-n} X_+, \mathrm{M}\,G(k)\right]_*$$

Remark 2.1. Since the stable normal bundle is the inverse of stable class $[TM]$ one can equivalently think of a stable G-manifold as a manifold together with a stable structure on its **stable normal bundle**. This view point can be exploited in order to translate a G-manifold M into a map $S^{n+k} \longrightarrow MG(k)$ for large k, yielding the desired isomorphism above.

Now taking the stable structure $G(2n) = \mathrm{U}(n)$ one obtains the notion of **stable complex cobordism**, which will be of special interest for us. For the sake of brevity we will omit from now on the label "stable" and simply say **complex cobordism**. We will denote the Thom spectrum of $\mathrm{U}(\bullet)$ by MU.

Now note that we have natural maps

$$\mathrm{BU}(n) \times \mathrm{BU}(m) \hookrightarrow \mathrm{BU}(n+m)$$

which encode the direct sum operation of complex vector bundles. These maps induce maps

$$\mathrm{MO}(n) \wedge \mathrm{MO}(m) \longrightarrow \mathrm{MO}(n+m)$$

which in turn induce a map

$$\mathrm{MU} \wedge \mathrm{MU} \longrightarrow \mathrm{MU}$$

This map can be used in order to endow MU with the structure of a coherently associative and commutative ring spectrum (i.e. an $\mathbf{E}_\infty$-ring). Hence the associated cohomology theory is multiplicative. In particular the graded group

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{MU}^{-n}(*) = \bigoplus_{n\in\mathbb{Z}} \mathrm{MU}_n(*) \overset{\mathrm{df}}{=} \mathrm{MU}_*$$

has the structure of a graded ring. On the level of manifolds this product can be described by Cartesian products of manifolds (with the associated product stably complex structure).

The coefficient ring MU_* has been calculated as follows:

Theorem 2.2. *1. The ring* MU_* *is isomorphic to the polynomial ring* $\mathbb{Z}[t_1, t_2, ...]$ *where the generator* t_i *has degree* $2i$.

2. The cobordism classes $[\mathbb{C}P^n]$ *freely generate* MU_* *over* $\mathbb{Q}$ *(but not over* $\mathbb{Z}$*), i.e. we have*

$$\mathrm{MU}_* \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], ...]$$

Now in order to better understand the special role of the spectrum MU in stable homotopy theory one needs to understand the notion of a **formal group law**. This will be explained in the next section.

3 Formal Group Laws

Let R be a ring. A **formal group law** over R is a formal power series

$$f(x, y) = \sum_{n,m=0}^{\infty} a_{i,j} x^i y^j$$

satisfying the following properties:

1. $f(x, f(y, z)) = f(f(x, y), z)$.
2. $f(x, 0) = x$ and $f(0, y) = y$.
3. $f(x, y) = f(y, x)$.

Note that the second condition is equivalent to saying $a_{0,0} = 0$ and $a_{1,0} = a_{0,1} = 1$, so that f can be written as

$$f(x, y) = x + y + \sum_{i+j\geq 2} a_{i,j} x^i y^j$$

In order to gain preliminary intuition for this concept one can start by taking R to be the field of real (or complex) numbers and assume that the power series f converges to a function $R \times R \longrightarrow R$. Then the first axiom states that f is an **associative** binary operation. The second axioms states that 0 is a neutral element with respect to this operation. The third axiom claims that this operation is **commutative**. In particular one obtains in this way a structure of a **commutative monoid** on R. However, it is not hard to show (Hensel's lemma) that under the three axioms above there will exist a power series

$$g(x) = \sum_{i=1} b_i x^i$$

over R such that

$$f(x, g(x)) = 0$$

i.e., if we assume g to converge as well we obtain an inverse function $x \mapsto x^{-1}$ for this binary operation, i.e. the binary operation is in fact a **group structure** on R.

What this intuition shows is that a formal group law can be thought of as **infinitesimal** group operation, but in a abstract algebraic setting (without assuming any notion of convergence).

Remark 3.1. By defining an appropriate notion of a **formal scheme** one can make the intuition above more precise - one can consider the formal spectrum $\mathrm{Spf}(R[[x]])$ of the power series ring $R[[x]]$ as the infinitesimal neighborhood of 0 inside the affine line $\mathbb{A}^1_R$ over R. Then specifying a formal group law is equivalent to specifying a group object structure on $\mathrm{Spf}(R[[x]])$.

Examples:

1. The formal power series
$$f(x, y) = x + y$$
is a formal group law over any ring. It is called the **additive** formal group law.

2. The formal power series
$$f(x, y) = x + y + xy$$
is a formal group law over any ring. The last two axioms are clear. In order to see why f is associative write
$$f(t-1, s-1) = (t-1) + (s-1) + (t-1)(s-1) = ts - 1$$
This formal group law is called the **multiplicative** formal group law.

3. Let E be an elliptic curve over a ring R. Then by formally completing the multiplication map $E \times E \longrightarrow E$ along the zero section one will obtain a formal group law.

Now note that if

$$g(t) = \sum_{n=1}^{\infty} b_n t^n$$

is a power series such that b_1 is a unit in R then g admits an inverse

$$g^{-1}(t) = \sum_{n=1}^{\infty} c_n t^n$$

satisfying $g(g^{-1}(t)) = g^{-1}(g(t)) = t$. In this case we say that g is **invertible**. One can think of g as an **infinitesimal coordinate change**. In particular if $f(x,y)$ is a formal group law over R and g is an invertible formal power series over R then one can form a new formal group law $f'(x,y)$ by setting

$$f'(x,y) = g^{-1}\left(f(g(x), g(y))\right)$$

we will say that two formal group laws are **isomorphic** if one is obtain from the other by an invertible coordinate change as above.

Remark 3.2. If R is a $\mathbb{Q}$-algebra then the multiplicative formal group law is isomorphic to the additive formal group via the invertible coordinate changes

$$g(x) = e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

and

$$g^{-1}(x) = \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

In fact, one can show that over $\mathbb{Q}$ every formal group law is isomorphic to the additive one. However if R is not a $\mathbb{Q}$-algebra this need not be the case (for example if $R = \mathbb{Z}$ or R is a field of positive characteristic then the multiplicative formal group law will not isomorphic to the additive one).

For various purposes it is useful to consider also a **graded** variant of the notion of formal group laws. In order to understand how gradings fit in let us start by considering a power series in one variable

$$f(x) = \sum_n c_n x^n$$

We want to think of f as if it was a function $x \mapsto f(x)$. Further more we want it to be an endo-function, i.e. a function from some domain to itself. Hence if we consider x as an object with degree d, then we will want the "output" of f to have degree d as well. This means that c_i should have degree $d(1-n)$. Similarly if we consider a formal power series in two variables

$$f(x,y) = \sum_{i,j} a_{i,j} x^i y^j$$

and we want to think of f as a function which takes two objects of degree d and returns an object of degree d then $a_{i,j}$ should have degree $d(1-i-j)$. When we take $d = 0$ we get the usual ungraded notion. However all the choices $d \neq 0$ are equivalent on the algebraic level (it's just a "rescaling" of all the degrees). For the application we have in mind it will be convenient to take $d = -2$. In particular we have the following definition:

Definition 3.3. Let R be a graded commutative ring. A **graded** formal group law over R is a formal group law

$$f(x,y) = \sum_{i,j=0}^{\infty} a_{i,j} x^i y^j$$

such that each $a_{i,j} \in R$ is homogenous of degree $2(i+j-1)$. In particular if we consider x, y to have degree -2 then f has homogenous degree -2.

Remark 3.4. One can define isomorphisms of graded formal group laws in the obvious way.

3.1 The Universal Formal Group Law

Let R be a graded ring. In order to specify a formal group over R one needs to choose elements $a_{i,j} \in \mathbb{R}$ of homogenous degree $2(i+j-1)$ for all $i, j \geq 0$ such that $i+j \geq 2$ (because $a_{0,0} = 0$ and $a_{1,0} = a_{0,1} = 1$ by axiom 2). The elements $a_{i,j}$ need to satisfy the restriction obtained by the other 2 two axioms of formal group laws. In particular the third axiom is quite simple and translate to the restriction $a_{i,j} = a_{j,i}$. This means that we can content ourselves with choosing the $a_{i,j}$ only for $i \leq j$.

Unfortunately the restriction given by the associativity axiom is less clean. However, it is not hard to verify that the constraints on the $a_{i,j}$ will take the form of **homogenous polynomial equations** (homogenous when taking into account the different degrees of the $a_{i,j}$'s). In particular the set of formal group laws over R can be described as the set of solutions to a set of graded homogenous polynomial equations. We will refer to these equations as the **associativity equations**.

Now each associativity equation can be written as a homogenous element in the graded polynomial ring which is generated over $\mathbb{Z}$ by the set $\{a_{i,j}\}$ (considered as abstract generators). In particular we can taking this polynomial ring and quotient out the ideal generated by the associativity equations. The resulting ring is called the **Lazard ring**, and is denoted by L. The ring L can be thought of as the ring in which the associativity equations admit a natural tautological solution. Further more, for each graded ring R one has a natural identification between formal group laws over R, and homomorphism of rings from L to R (the coefficients of the formal group specify to which element in R one should send each generator of L). We say that L corepresents the functor

$$R \mapsto \mathrm{FGL}(R)$$

where $\mathrm{FGL}(R)$ is the set of formal group laws over R. If one would ignore the grading for a moment, one can consider the spectrum of the underlying ungraded ring of L. This spectrum is the classifying scheme of ungraded formal group laws. The points of $\mathrm{Spec}(L)$ over a ring R are the (ungraded) formal group laws over R.

Now at first sight the Lazard ring looks very complicated - it is generated by infinitely many generators modulu infinitely many relations. However, it turns out the result is not that complicated after all:

Theorem 3.5 (Lazard)**.** *The Lazard ring L is isomorphic to the graded polynomial ring $\mathbb{Z}[t_1, t_2, ...]$ where each t_i has degree $2i$.*

In particular, in order to specify a formal group law on a graded ring R, one simply has to choose (unconstrainedly) a sequence of elements $t_i \in R$ such that t_i has homogenous degree $|2i|$.

4 Complex Orientation and Formal Groups Laws

Now let us go back to algebraic topology. The emergence of formal group laws in algebraic topology begins in the notion of a **complex orientation**. Let $\mathbb{C}P^\infty$ considered with its standard CW structure with one cell in each even dimension. In particular the zero skeleton provides us with a base point.

Let E be a ring spectrum. If (X, x_0) is a pointed space then we can **identify** the reduced E-cohomology groups $\widetilde{E}^n(X)$ with the relative groups $E^n(X, x_0)$ which in turn can be identified with the kernel of the map

$$x_0^* : E^n(X) \longrightarrow E^n(*)$$

In particular the suspension isomorphism can be written as

$$E^n(\Sigma X, x_0) \cong E^{n-1}(X, x_0)$$

We will refer to elements in $E^n(X, x_0) \subseteq E^n(X)$ as **reduced classes**.

Definition 4.1. Let E be a ring spectrum. A **complex orientation** on E is a reduced element

$$t \in E^2(\mathbb{C}P^\infty, x_0) \subseteq E^2(\mathbb{C}P^\infty)$$

such that the restriction of t to

$$E^2\left(S^2, x_0\right) \cong E^0\left(S^0, x_0\right) = E^0(*)$$

is equal to the unit element $1 \in E^0(*)$ (where the map $S^2 \hookrightarrow \mathbb{C}P^\infty$ is the 2-dimensional cell map).

We will say that E is **complex orientable** if there exists a complex orientation $t \in E^2(\mathbb{C}P^\infty, x_0)$.

Examples:

1. The Eilenberg-McLane spectrum HZ admits a unique complex orientation: the restriction map

$$H^2(\mathbb{C}P^\infty, x_0) \longrightarrow H^2(S^2, x_0) \cong \mathbb{Z}$$

   is an isomorphism.

2. The complex K-theory spectrum KU is complex orientable. If we let $L \longrightarrow \mathbb{C}P^{\infty}$ denote the universal line bundle on $\mathbb{C}P^{\infty}$ and use bott periodicity then the class

$$[L] - 1 \in \mathrm{KU}^0\left(\mathbb{C}P^{\infty}\right) = \mathrm{KU}^2\left(\mathbb{C}P^{\infty}\right)$$

is a complex orientation - it vanishes when restricted to the base point (so it is reduced) and gives the standard generator when restricted to

$$\mathrm{KU}^2(S^2, x_0) = \mathrm{KU}^0(S^2, x_0) \cong \mathbb{Z}$$

3. Complex cobordism is complex orientable: recall that MU(1) is the Thom space of the universal line bundle $L \longrightarrow \mathbb{C}P^{\infty}$ (considered as a 2-dimensional real vector space). In particular we have a natural (unpointed) map

$$t : \mathbb{C}P^{\infty} \longrightarrow \mathrm{MU}(1)$$

which is actually a homotopy equivalence (because the sphere bundle of the universal line bundle is EU and in particular contractible). We claim that the image of t in

$$\operatorname*{colim}_k \left[\Sigma^{2k-2}\left(\mathbb{C}P^{\infty}_+\right), \mathrm{MU}(k)\right]_* = \mathrm{MU}^2\left(\mathbb{C}P^{\infty}\right)$$

is a complex orientation. First of all note that the restriction of t to the base point $* \in \mathbb{C}P^{\infty}$ yields a pointed map

$$*_+ = S^0 \longrightarrow \mathrm{MU}(1)$$

which is pointed null-homotopic because MU(1) is connected. Hence the class of t in $\mathrm{MU}^2((\mathbb{C}P^{\infty})$ is reduced. Next we need to check that the restriction of t to $S^2 \subseteq \mathbb{C}P^{\infty}$ is the standard generator. Note that the standard generator of $1 \in \mathrm{MU}^0(*) = \mathrm{MU}_0(*)$ corresponds geometrically to the 0-dimensional manifold which is a point with the trivial stable complex structure. Recalling the discussion in the first section we see that this cobordism class is represented by the pointed map $u : S^2 \longrightarrow \mathrm{MU}(1)$ induced by the inclusion of pairs

$$(D^2, S^1) \hookrightarrow (D(L), S(L))$$

given by some fiber of L (which we can assume is the fiber over the old base point $* \in S^2 \subseteq \mathbb{C}P^2$). In order to show that t gives a complex orientation it will be enough to show that the restriction $t|_{S^2}$ is homotopic to u as unpointed maps. Now note that the Thom space of the restricted line bundle $L|_{S^2}$ can be identified with $\mathbb{C}P^4$ (because the corresponding sphere bundle is the Hopf map). Hence we get an inclusion of Thom spaces

$$\mathbb{C}P^4 \hookrightarrow \mathrm{MU}(1)$$

such that the images of both $t|_{S^2}$ and u are contained in $\mathbb{C}P^4$. In fact, both $t|_{S^2}$ and u described smooth inclusions of S^2 in $\mathbb{C}P^4$ which meet transversely in a single point. Observing the simple structure of the cohomology ring $H^*\left(\mathbb{C}P^4\right)$ we deduce that $t|_{S^2}$ is homotopic to $\mathbb{C}P^4$ and we are done.

4. Let E be a ring spectrum such that $\pi_n(E) = 0$ for odd n's (such spectrums will be called **even spectrums**). Then one can use standard obstruction theory in order to show that the map

$$E^2\left(\mathbb{C}P^\infty, x_0\right) \longrightarrow E^2\left(S^2, x_0\right)$$

is **surjective**. This amounts to showing that every pointed map

$$S^2 \longrightarrow \operatorname*{colim}_n \Omega^{2n-2}\,\mathrm{MU}(n)$$

extends to all of $\mathbb{C}P^\infty$. But $\mathbb{C}P^\infty$ has only even cohomologies and X has no odd homotopy groups. In particular we get that **every even ring spectrum is complex orientable**.

Now let E be a complex orientable ring spectrum and let

$$t \in E^2(\mathbb{C}P^\infty, x_0) \subseteq E^2(\mathbb{C}P^\infty)$$

be a complex orientation. Note that for every space X the ring

$$E^*(X) = \prod_n E^n(X)$$

is naturally an algebra over the ring $E^*(*)$, which can be identified with the coefficients ring $E_{-*} = E_{-*}(*)$ after inverting the grading.

Remark 4.2. Note that the convention

$$E^*(X) = \prod_n E^n(X)$$

differs from the familiar convention

$$E^*(X) = \oplus_n E^n(X)$$

In particular we will write

$$H^*\left(\mathbb{C}P^\infty\right) = \mathbb{Z}[[t]]$$

and not

$$H^*\left(\mathbb{C}P^\infty\right) = \mathbb{Z}[t]$$

as is customary. The advantage of the our convention is that of X is an infinite CW complex then $E^*(X)$ will be the inverse limit of $E^*(F)$ where $F \subseteq X$ ranges over all finite sub CW complexes of X.

The following theorem determines the cohomology rings $E^*(\mathbb{C}P^n)$ and $E^*(\mathbb{C}P^\infty)$ as E_{-*} algebras:

Theorem 4.3. *For every n one has a natural isomorphism of rings*

$$E^*(\mathbb{C}P^n) \cong (E_{-*})[t]/t^{n+1}$$

where $t \in E^2(\mathbb{C}P^n)$ is the image of the complex orientation of E (and by abuse of notation it is denoted by the same name). Further more one has

$$E^*(\mathbb{C}P^\infty) \cong \lim_{\overleftarrow{n}} E^*(\mathbb{C}P^n) \cong E_{-*}[[t]]$$

Proof. The computation of $E^*(\mathbb{C}P^n)$ can be done essentially via the Atyah-Hirtzebruch spectral sequence (which is shown to collapse when E is complex orientable). The last equality follows from the fact that each of the projection homomorphisms

$$(E_{-*})[t]/t^{n+1} \longrightarrow (E_{-*})[t]/t^n$$

is surjective, and so there is no $\lim^1$ term. □

A similar computation shows that

$$E^*(\mathbb{C}P^n \times \mathbb{C}P^m) \cong (E_{-*})[x,y]/\left\langle x^{n+1}, y^{m+1}\right\rangle$$

where $x = p_1^* t$ and $y = p_2^* t$ for the two projections

$$p_1, p_2 : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$$

In particular one gets that

$$E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong (E_{-*})[[x,y]]$$

Now recall that $\mathbb{C}P^\infty \cong \mathrm{BU}(1) \cong K(\mathbb{Z},2)$ and so is an infinite loop space. In particular $\mathbb{C}P^\infty$ admits a binary operation

$$\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$$

which is associative and commutative up to coherent homotopy. In fact, this multiplication operation encodes the information of **tensor product of line bundles**, i.e. it classifies the line bundle

$$p_1^* L \otimes p_2^* L$$

on $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$, where $L \longrightarrow \mathbb{C}P^\infty$ is the universal line bundle.

Remark 4.4. One can easily introduce a strictly associative strictly commutative multiplication on $\mathbb{C}P^\infty$ by identifying points in $\mathbb{C}P^n$ with homogenous polynomials of degree n in two variables up to a scalar, in which case the multiplication can be given simply as multiplication of polynomials. Under this multiplication $\mathbb{C}P^\infty$ becomes a strict topological **monoid**, which can be identified with the strict topological monoid generated by the points of $S^2 = \mathbb{C}P^1$ modulu the the submonoid generated by the base point $\infty \in \mathbb{C}P^1$. Note however that this operation does not admits a strict inverse (only up to homotopy). However, the group completion of this topological monoid (which can be identified with the free abelian group on $\mathbb{C}P^1$ modulu the subgroup generated by ∞) is homotopy equivalent to it

Now suppose that E is a ring spectrum with a complex orientation $t \in E^2(\mathbb{C}P^\infty, x_0) \subseteq E^2(\mathbb{C}P^\infty)$. Then the map μ induces a homomorphism of rings

$$(E_{-*})[[t]] = E^*(\mathbb{C}P^\infty) \xrightarrow{\mu^*} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = (E_{-*})[[x, y]]$$

The image of t under this map is a formal power series

$$f(x, y) = \sum_{i,j} a_{i,j} x^i y^j$$

where $a_{i,j} \in E_{2(i+j-1)}(*)$. Since μ is associative, commutative and unital it is quite immediate that f is a graded formal group law over the graded **commutative** ring

$$E_{2*}(X) = \oplus_n E_{2n}(*)$$

Remark 4.5. The situation is a bit more pleasant when E is an even ring spectrum, in which case $E_* = E_{2*}$. This restriction is not that far fetched as we have seen that any even ring spectrum is automatically complex orientable. In particular in this lecture we have absolutely no reason to consider any complex orientable ring spectrum that is not even.

Remark 4.6. In the definition above the formal group law associated with E depends on the choice of complex orientation $t \in E^2(\mathbb{C}P^\infty, x_0)$. However if one would choose another complex orientation $t' \in E^2(\mathbb{C}P^\infty, x_0)$ then there would be a graded power series

$$g(t) = \sum_{n=0}^{\infty} b_n t^n$$

with $b_n \in E_{2n-2}(*)$ such that

$$t' = g(t)$$

Now since t' is a complex orientation we would get that $t'|_{S^2} = t|_{S^2}$ and so g has the form

$$g(t) = t + \sum_{n=2}^{\infty} b_n t^n$$

and so in particular g is invertible. Further more it is immediate that the formal group law of E with respect to t' will be isomorphic to the formal group law of E with respect to t via the invertible coordinate change g. In particular the isomorphism type of the formal group f does not depend on t, and is an invariant of the ring spectrum E alone.

Examples:

1. The Eilenberg-McLane spectrum HZ has the **additive group law**. There is not much choice there because HZ_* contains no elements of non-zero degree (and the only coefficients of degree 0 are $a_{1,0}, a_{0,1}$ which are bound to be 1 since f is a formal group law).

2. The complex K-theory spectrum KU has the **multiplicative group law** (up to isomorphism). In fact if we take the complex orientation

$$L - 1 \in KU^0(\mathbb{C}P^\infty) \cong KU^2(\mathbb{C}P^\infty)$$

then we get it on the nose - to see this recall that the multiplication map

$$\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$$

Classifies the line bundle

$$p_1^*[L] \otimes p_2^*[L]$$

on $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$. This means that

$$\mu^*([L]-1) = p_1^*[L] \otimes p_2^*[L] - 1 =$$

$$p_1^*([L]-1) + p_2^*([L]-1) + p_1^*([L]-1) \otimes p_2^*([L]-1)$$

and so that the formal group law is

$$f(x,y) = \mu^*(t) = x + y + xy$$

Note that the coefficient $a_{1,1}$ looks like it's sitting in $KU_0(*)$ but it is actually in $KU_2(*)$ (where it belongs) and we are looking at everything through the glasses of bott periodicity.

Coming to the exciting end point of the lecture, one can wonder what is the formal group law of the complex cobordism spectrum MU with respect to the canonical complex orientation $t \in \mathrm{MU}^2(\mathbb{C}P^\infty)$. Note that this formal group law is classified by a homomorphism of graded rings

$$L \longrightarrow \mathrm{MU}_*$$

where L is the Lazard ring which was introduced in the previous section.

We then have the following fundamental result

Theorem 4.7 (Quillen)**.** *The classifying map*

$$L \longrightarrow \mathrm{MU}_*$$

is an isomorphism of rings. In particular MU_* *carries the* ***universal formal group law****.*

This observation raises the question of whether the spectrum MU itself is somehow universal among complex oriented ring spectra. This is in fact the case:

Theorem 4.8. *Let $t \in \mathrm{MU}^2(\mathbb{C}P^\infty, x_0)$ be the canonical complex orientation and let E be a ring spectrum. Then the association $\phi \mapsto \phi_*(t)$ induces a bijection between the set of homotopy classes of ring spectrum maps*

$$\phi : \mathrm{MU} \longrightarrow E$$

and the set of complex orientations of E.

We hence consider MU as the **universal complex oriented ring spectrum**.